HOMOLOGICAL PROPERTIES OF BALANCED COHEN-MACaulAY ALGEBRAS

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Abstract. A balanced Cohen-Macaulay algebra is a connected algebra $A$ having a balanced dualizing complex $\omega_A[d]$ in the sense of Yekutieli (1992) for some integer $d$ and some graded $A$-$A$ bimodule $\omega_A$. We study some homological properties of a balanced Cohen-Macaulay algebra. In particular, we will prove the following theorem:

Theorem 0.1. Let $A$ be a Noetherian balanced Cohen-Macaulay algebra, and $M$ a nonzero finitely generated graded left $A$-module. Then:

1. $M$ has a finite resolution of the form

$$0 \to \bigoplus_{j=1}^{r_m} \omega_A(-l_{mj}) \to \cdots \to \bigoplus_{j=1}^{r_1} \omega_A(-l_{1j}) \to H \to M \to 0,$$

where $H$ is a finitely generated maximal Cohen-Macaulay graded left $A$-module.

2. $M$ has finite injective dimension if and only if $M$ has a finite resolution of the form

$$0 \to \bigoplus_{j=1}^{r_0} \omega_A(-l_{0j}) \to \cdots \to \bigoplus_{j=1}^{r_1} \omega_A(-l_{1j}) \to \bigoplus_{j=1}^{r_m} \omega_A(-l_{mj}) \to M \to 0.$$

As a corollary, we will have the following characterizations of AS Gorenstein algebras and AS regular algebras:

Corollary 0.2. Let $A$ be a Noetherian balanced Cohen-Macaulay algebra.

1. $A$ is AS Gorenstein if and only if $\omega_A$ has finite projective dimension as a graded left $A$-module.

2. $A$ is AS regular if and only if every finitely generated maximal Cohen-Macaulay graded left $A$-module is free.

1. Hyperhomological algebras

Throughout this paper, we fix a field $k$. A connected algebra is a graded algebra of the form $A = k \oplus A_1 \oplus A_2 \oplus \cdots$. The augmentation ideal of $A$ is defined by $m = A_1 \oplus A_2 \oplus \cdots$. In this first section, we will fix terminology and notation, and collect some elementary results on hyperhomological algebras over connected algebras.

Let $A, B, C$ be connected algebras. The category of graded left $A$-modules and graded left $A$-module homomorphisms of degree 0 is denoted by GrMod$A$. For

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$M, N \in \text{GrMod}_A$, the set of graded left $A$-module homomorphisms $M \to N$ of degree 0 is denoted by $\text{Hom}_A(M, N)$, which has a natural $k$-vector space structure. The full subcategory of $\text{GrMod}_A$ consisting of finitely generated graded left $A$-modules is denoted by $\text{grmod}_A$. The category of graded right $A$-modules is denoted by $\text{GrMod}_A$, where $A^o$ is the opposite algebra of $A$. The category of graded $A$-$B$ bimodules is denoted by $\text{GrMod}(A \otimes B^o)$. In particular, the category of graded $A$-$A$ bimodules is denoted by $\text{GrMod}_A$, where $A^e = A \otimes A^o$. The natural restriction functors are denoted by

$$\text{res}_A : \text{GrMod}(A \otimes B^o) \to \text{GrMod}_A$$

and

$$\text{res}_{B^o} : \text{GrMod}(A \otimes B^o) \to \text{GrMod}_{B^o}.$$  

We write $k = A/m$, viewed as an object in $\text{GrMod}_A, \text{GrMod}_A^e$, or $\text{GrMod}_A^e$, depending on the context.

A graded left $A$-module $M \in \text{GrMod}_A$ is right bounded (resp. left bounded) if $M_i = 0$ for all $i > 0$ (resp. $i < 0$), and bounded if it is both right bounded and left bounded. We say that $M$ is locally finite if the $M_i$ are finite dimensional over $k$ for all $i$. For each integer $n$, the shift of $M$ is denoted by $M(n) \in \text{GrMod}_A$, so that $M(n)_i = M_{n+i}$. For $M \in \text{GrMod}(A \otimes B^o)$ and $N \in \text{GrMod}(A \otimes C^o)$, we define

$$\text{Ext}^i_A(M, N) = \bigoplus_{n=-\infty}^{\infty} \text{Ext}^i_A(M(n), N),$$

which has a natural graded $B$-$C$ bimodule structure for each $i$. Similarly, for $M \in \text{GrMod}(B \otimes A^o)$ and $N \in \text{GrMod}(A \otimes C^o)$, $\text{Tor}^i_A(M, N)$ has a natural graded $B$-$C$ bimodule structure for each $i$. For $M \in \text{GrMod}(A \otimes B^o)$, the Matlis dual of $M$ is defined by $M' = \text{Hom}_B(M, k)$, which has a natural graded $B$-$A$ bimodule structure. If $M$ is locally finite, then $M'' \cong M$ in $\text{GrMod}(A \otimes B^o)$.

Let $X, Y$ be cochain complexes of graded left $A$-modules. The $i$th cohomology of $X$ is denoted by $h^i(X)$. We say that a cochain map $f : X \to Y$ is a quasi-isomorphism if the induced maps $h^i(f) : h^i(X) \to h^i(Y)$ are isomorphisms in $\text{GrMod}_A$ for all $i$. The derived category of graded left $A$-modules is denoted by $\mathcal{D}(A)$, so that a cochain map $f : X \to Y$ is a quasi-isomorphism if and only if it induces an isomorphism $f : X \to Y$ in $\mathcal{D}(A)$. We define $\mathcal{D}_{fg}(A)$ (resp. $\mathcal{D}_{lf}(A)$) to be the full subcategory of $\mathcal{D}(A)$ consisting of complexes whose cohomologies are all finitely generated (resp. locally finite) graded left $A$-modules.

For $X \in \mathcal{D}(A)$, we define

$$\sup X = \{i \mid h^i(X) \neq 0\}$$

and

$$\inf X = \{i \mid h^i(X) \neq 0\}.$$  

If $X \cong 0$ in $\mathcal{D}(A)$, then we define $\sup X = -\infty$ and $\inf X = \infty$.

A complex $X \in \mathcal{D}(A)$ is bounded above (resp. bounded below) if $\sup X < \infty$ (resp. $\inf X > -\infty$), and bounded if it is both bounded above and bounded below. The full subcategory of $\mathcal{D}(A)$ consisting of bounded (resp. bounded above, resp. bounded below) complexes is denoted by $\mathcal{D}^b(A)$ (resp. $\mathcal{D}^+(A)$, resp. $\mathcal{D}^+(A)$).

The right derived functor of

$$\text{Hom}_A(-, -) : \mathcal{D}^-(A \otimes B^o) \times \mathcal{D}^+(A \otimes C^o) \to \mathcal{D}(B \otimes C^o)$$

is denoted by $\mathcal{R}^A(-, -) : \mathcal{D}^-(A \otimes B^o) \times \mathcal{D}^+(A \otimes C^o) \to \mathcal{D}(B \otimes C^o)$. The natural restriction functors are denoted by

$$\text{res}_A : \mathcal{D}(A \otimes B^o) \to \mathcal{D}_A$$

and

$$\text{res}_{B^o} : \mathcal{D}(A \otimes B^o) \to \mathcal{D}_{B^o}.$$  

We write $k = A/m$, viewed as an object in $\mathcal{D}_A, \mathcal{D}_A^e$, or $\mathcal{D}_A^e$, depending on the context.
is denoted by $R\text{Hom}_A(-, -)$, and its cohomologies are denoted by

$$\text{Ext}^i_A(-, -) = h^i(R\text{Hom}_A(-, -)).$$

The left derived functor of

$$- \otimes_A - : \mathcal{D}^- (B \otimes A^o) \times \mathcal{D}^- (A \otimes C^o) \to \mathcal{D} (B \otimes C^o)$$

is denoted by $- \otimes^L_A -$, and its cohomologies are denoted by

$$\text{Tor}^A_i(-, -) = h^i(- \otimes^L_A -).$$

Let $X \in \mathcal{D}(A)$. For each integer $n$, the twist of $X$ is denoted by $X[n] \in \mathcal{D}(A)$, so that $(X[n])^i = X^{n+i}$. Note that $h^i(X) = 0$ for all $i \neq n$ if and only if $X \cong h^n(X)[-n]$ in $\mathcal{D}(A)$. If $X \in \mathcal{D}^- (A \otimes B^o)$ and $Y \in \mathcal{D}^+ (A \otimes C^o)$, then

$$R\text{Hom}_A(X[n], Y) \cong R\text{Hom}_A(X, Y[-n]) \cong R\text{Hom}_A(X, Y)[-n]$$

in $\mathcal{D}(B \otimes C^o)$ for each $n$. If $X \in \mathcal{D}^- (B \otimes A^o)$ and $Y \in \mathcal{D}^- (A \otimes C^o)$, then

$$(X[n]) \otimes_A^L Y \cong X \otimes_A^L (Y[n]) \cong (X \otimes_A^L Y)[n]$$

in $\mathcal{D}(B \otimes C^o)$ for each $n$.

**Definition 1.1.** Let $A$ be a connected algebra.

1. A free resolution of $X \in \mathcal{D}^- (A)$ is a complex $F$ of free graded left $A$-modules such that $F \cong X$ in $\mathcal{D}(A)$. A complex $F$ of free graded left $A$-modules is called minimal if the differentials in $\text{Hom}_A(F, k)$ are all zero.

2. A projective resolution of $X \in \mathcal{D}^- (A)$ is a complex $P$ of projective graded left $A$-modules such that $P \cong X$ in $\mathcal{D}(A)$. We define the projective dimension of $X$ by

$$\text{pd}_A(X) = \inf_P (- \inf \{i \mid P^i \neq 0\}),$$

where the infimum is taken over all projective resolutions $P$ of $X$.

3. An injective resolution of $X \in \mathcal{D}^+ (A)$ is a complex $E$ of injective graded left $A$-modules such that $E \cong X$ in $\mathcal{D}(A)$. We define the injective dimension of $X$ by

$$\text{id}_A(X) = \inf_E (\sup \{i \mid E^i \neq 0\}),$$

where the infimum is taken over all injective resolutions $E$ of $X$.

4. A flat resolution of $X \in \mathcal{D}^- (A)$ is a complex $F$ of flat graded left $A$-modules such that $F \cong X$ in $\mathcal{D}(A)$. We define the flat dimension of $X$ by

$$\text{fd}_A(X) = \inf_F (- \inf \{i \mid F^i \neq 0\}),$$

where the infimum is taken over all flat resolutions $F$ of $X$.

**Lemma 1.2.** Let $A$ be a connected algebra.

1. For $X \in \mathcal{D}^+ (A)$,

$$\text{id}_A(X) = \sup \{\sup \text{RHom}_A(M, X) \mid M \in \text{GrMod} A\} = \sup \{\sup \text{RHom}_A(M, X) \mid M \in \text{gmod} A\}.$$

2. For $X \in \mathcal{D}^- (A)$,

$$\text{fd}_A(X) = \sup \{- \inf (N \otimes_A^L X) \mid N \in \text{GrMod} A^o\} = \sup \{- \inf (N \otimes_A^L X) \mid N \in \text{gmod} A^o\}.$$
Lemma 1.3. Let $A$ be a connected algebra.

1. If $X \in \mathcal{D}^{-}(A \otimes B^{o})$ and $Y \in \mathcal{D}^{+}(A \otimes C^{o})$, then
   \[
   \inf R\text{Hom}_{A}(X,Y) \geq \inf Y - \sup X.
   \]

2. If $X \in \mathcal{D}^{-}(B \otimes A^{o})$ and $Y \in \mathcal{D}^{-}(A \otimes C^{o})$, then
   \[
   \sup(X \otimes_{A}^{L} Y) \leq \sup X + \sup Y.
   \]

Moreover, if $h^{\sup X}(X), h^{\sup Y}(Y) \in \text{GrMod} A$ are left bounded, then
\[
\sup(X \otimes_{A}^{L} Y) = \sup X + \sup Y.
\]

2. Foxby equivalence

Definition 2.1. Let $A, B$ be connected algebras, and let $m = A_{\geq 1}$ be the augmentation ideal of $A$. We define the functor $\Gamma_{m} : \mathcal{D}(A \otimes B^{o}) \to \mathcal{D}(A \otimes B^{o})$ by
\[
\Gamma_{m}(-) = \lim_{n \to \infty} \text{Hom}_{A}(A/_{A_{\geq n}}, -).
\]

The right derived functor of $\Gamma_{m}$ is denoted by $R\Gamma_{m}$, and its cohomologies are denoted by
\[
H^{i}_{m}(-) = h^{i}(R\Gamma_{m}(-)) = \lim_{n \to \infty} \text{Ext}^{i}_{A}(A/_{A_{\geq n}}, -).
\]

Similarly, we define the functor $\Gamma_{m^{o}} : \mathcal{D}(B \otimes A^{o}) \to \mathcal{D}(B \otimes A^{o})$ by
\[
\Gamma_{m^{o}}(-) = \lim_{n \to \infty} \text{Hom}_{A^{o}}(A/_{A_{\geq n}}, -).
\]

The right derived functor of $\Gamma_{m^{o}}$ is denoted by $R\Gamma_{m^{o}}$, and its cohomologies are denoted by
\[
H^{i}_{m^{o}}(-) = h^{i}(R\Gamma_{m^{o}}(-)) = \lim_{n \to \infty} \text{Ext}^{i}_{A^{o}}(A/_{A_{\geq n}}, -).
\]

Let us recall the following definition from [13].

Definition 2.2. Let $A$ be a Noetherian connected algebra. A complex $D \in \mathcal{D}^{b}(A^{e})$ is called dualizing if
- $\text{res}_{A} D \in \mathcal{D}_{f,g}^{b}(A)$, $\text{res}_{A^{e}} D \in \mathcal{D}_{f,g}^{b}(A^{e})$,
- $\text{id}_{A}(D) < \infty$, $\text{id}_{A^{e}}(D) < \infty$, and
- the natural morphisms $A \to R\text{Hom}_{A}(D, D)$ and $A \to R\text{Hom}_{A^{e}}(D, D)$ are isomorphisms in $\mathcal{D}(A^{e})$.

A dualizing complex $D \in \mathcal{D}(A^{e})$ is called balanced if
- $R\Gamma_{m}(D) \cong R\Gamma_{m^{o}}(D) \cong A^{e}$ in $\mathcal{D}(A^{e})$.

By [13] Proposition 3.5, if $D$ is a dualizing complex, then the functor
\[
R\text{Hom}_{A}(\cdot, D) : \mathcal{D}(A) \to \mathcal{D}(A^{o})
\]
and the functor
\[
R\text{Hom}_{A^{e}}(\cdot, D) : \mathcal{D}(A^{e}) \to \mathcal{D}(A)
\]
define a duality between $\mathcal{D}_{f,g}^{b}(A)$ and $\mathcal{D}_{f,g}^{b}(A^{e})$, that is,
\[
R\text{Hom}_{A}(X, D) \in \mathcal{D}_{f,g}^{b}(A^{e}) \text{ and } R\text{Hom}_{A^{e}}(R\text{Hom}_{A}(X, D), D) \cong X \text{ in } \mathcal{D}(A)
\]
for all $X \in \mathcal{D}_{\mathcal{F}_g}(A)$, and
\[
R\text{Hom}_{A^e}(Y, D) \in \mathcal{D}_{\mathcal{F}_g}(A) \quad \text{and} \quad R\text{Hom}_{A^e}(R\text{Hom}_{A^e}(Y, D), D) \cong Y \text{ in } \mathcal{D}(A^e)
\]
for all $Y \in \mathcal{D}_{\mathcal{F}_g}(A^e)$. In this section, we study another type of equivalence, known as Foxby equivalence.

**Proposition 2.3.** Let $A, B$ be Noetherian connected algebras.

1. Let
\[
X \in \mathcal{D}^{-}(B \otimes A^e), \quad Y \in \mathcal{D}^{b}(A^e), \quad Z \in \mathcal{D}^{+}(A)
\]
be such that $\text{res}_{A^e} X \in \mathcal{D}_{\mathcal{F}_g}^{-}(A^e)$. If either $\text{pd}_{A^e}(X) < \infty$ or $\text{id}_{A}(Z) < \infty$, then there is a natural isomorphism
\[
X \otimes_{A^e}^{L} \text{RHom}_{A^e}(Y, Z) \cong \text{RHom}_{A^e}(\text{RHom}_{A^e}(X, Y), Z)
\]
in $\mathcal{D}(B)$.

2. Let
\[
X \in \mathcal{D}^{-}(A \otimes B^e), \quad Y \in \mathcal{D}^{b}(A^e), \quad Z \in \mathcal{D}^{-}(A)
\]
be such that $\text{res}_{A} X \in \mathcal{D}_{\mathcal{F}_g}^{-}(A)$. If either $\text{pd}_{A}(X) < \infty$ or $\text{id}_{A}(Z) < \infty$, then there is a natural isomorphism
\[
\text{RHom}_{A}(X, Y) \otimes_{A^e}^{L} Z \cong \text{RHom}_{A}(X, Y \otimes_{A^e}^{L} Z)
\]
in $\mathcal{D}(B)$.

**Proof.** By [8] Theorem 1.4] and [6] Proposition 2.1], if $X, Y, Z$ are as above, then the evaluation morphisms
\[
\theta_{XYZ} : X \otimes_{A^e}^{L} \text{RHom}_{A^e}(Y, Z) \to \text{RHom}_{A^e}(\text{RHom}_{A^e}(X, Y), Z)
\]
and
\[
\omega_{XYZ} : \text{RHom}_{A}(X, Y) \otimes_{A^e}^{L} Z \to \text{RHom}_{A}(X, Y \otimes_{A^e}^{L} Z)
\]
defined in [8] Notation 4.3] are isomorphisms in $\mathcal{D}(k)$. We will leave it to the reader to check that $\theta_{XYZ}$ and $\omega_{XYZ}$ are in fact induced by maps of complexes of graded left $B$-modules. \qed

**Definition 2.4.** Let $A$ be a connected algebra. We define $\tilde{\mathcal{F}}(A)$ to be the full subcategory of $\mathcal{D}^{b}(A)$ consisting of complexes having finite injective dimension, and $\tilde{\mathcal{F}}(A)$ to be the full subcategory of $\mathcal{D}^{b}(A)$ consisting of complexes having finite flat dimension.

Now Foxby equivalence is stated as follows:

**Theorem 2.5.** Let $A$ be a Noetherian connected algebra. If $D \in \mathcal{D}^{b}(A^e)$ is a dualizing complex, then the functors $D \otimes_{A^e}^{L} : \mathcal{D}^{b}(A) \to \mathcal{D}^{+}(A)$ and $\text{RHom}_{A}(D, -) : \mathcal{D}^{b}(A) \to \mathcal{D}^{+}(A)$ define inverse equivalences between $\tilde{\mathcal{F}}(A)$ and $\tilde{\mathcal{F}}(A)$. They also define inverse equivalences between $\tilde{\mathcal{F}}_{fg}(A)$ and $\tilde{\mathcal{F}}_{fg}(A)$.

**Proof.** If $X \in \tilde{\mathcal{F}}(A)$, then
\[
\text{RHom}_{A}(M, D \otimes_{A^e}^{L} X) \cong \text{RHom}_{A}(M, D) \otimes_{A^e}^{L} X
\]
in $D(k)$ for all $M \in \text{grmod} A$ by Proposition 2.3(2). By Lemma 1.3(1) and Lemma 1.3(2),

$$\text{id}_A(D \otimes_A^L X) = \sup \{\sup R\text{Hom}_A(M, D \otimes_A^L X) \mid M \in \text{grmod} A\}$$

$$= \sup \{\sup (R\text{Hom}_A(M, D) \otimes_A^L X) \mid M \in \text{grmod} A\}$$

$$\leq \sup \{\sup R\text{Hom}_A(M, D) + \sup X \mid M \in \text{grmod} A\}$$

$$= \sup \{\sup R\text{Hom}_A(M, D) \mid M \in \text{grmod} A\} + \sup X$$

$$= \text{id}_A(D) + \sup X < \infty,$$

and hence $D \otimes_A^L X \in \hat{\mathcal{I}}(A)$. Moreover,

$$R\text{Hom}_A(D, D \otimes_A^L X) \cong R\text{Hom}_A(D, D) \otimes_A^L X \cong X$$

in $D(A)$ by Proposition 2.3(2).

If $X \in \hat{\mathcal{I}}(A)$, then

$$N \otimes_A^L R\text{Hom}_A(D, X) \cong R\text{Hom}_A(R\text{Hom}_A(N, D), X)$$

in $D(k)$ for all $N \in \text{grmod} A^o$ by Proposition 2.3(1). By Lemma 1.3(2) and Lemma 1.3(1),

$$\text{fd}_A(R\text{Hom}_A(D, X)) = \sup \{−\inf (N \otimes_A R\text{Hom}_A(D, X)) \mid N \in \text{grmod} A^o\}$$

$$= \sup \{−\inf (R\text{Hom}_A(R\text{Hom}_A(N, D), X)) \mid N \in \text{grmod} A^o\}$$

$$\leq \sup \{\sup (R\text{Hom}_A(N, D)) - \inf X \mid N \in \text{grmod} A^o\}$$

$$= \sup \{\sup (R\text{Hom}_A(N, D)) \mid N \in \text{grmod} A^o\} - \inf X$$

$$= \text{id}_{A^o}(D) - \inf X < \infty,$$

and hence $R\text{Hom}_A(D, X) \in \hat{\mathcal{I}}(A)$. Moreover,

$$D \otimes_A^L R\text{Hom}_A(D, X) \cong R\text{Hom}_A(R\text{Hom}_{A^o}(D, D), X) \cong X$$

in $D(A)$ by Proposition 2.3(1); hence the functors $D \otimes_A^L -$ and $R\text{Hom}_A(D, -)$ define inverse equivalences between $\hat{\mathcal{F}}(A)$ and $\hat{\mathcal{I}}(A)$.

If $X \in D^b_f(A)$, then

$$R\text{Hom}_A(D, X) \cong R\text{Hom}_A(D, R\text{Hom}_{A^o}(R\text{Hom}_A(X, D), D))$$

$$\cong R\text{Hom}_{A^o}(R\text{Hom}_A(X, D), R\text{Hom}_A(D, D))$$

$$\cong R\text{Hom}_{A^o}(R\text{Hom}_A(X, D), A)$$

in $D(A)$ by [8, Theorem 1.2]. Since $R\text{Hom}_A(X, D) \in D^b_f(A^o)$, it follows that $R\text{Hom}_A(D, X) \in D^+_f(A)$, and hence the functors $D \otimes_A^L -$ and $R\text{Hom}_A(D, -)$ define inverse equivalences between $\hat{\mathcal{F}}_f(A)$ and $\hat{\mathcal{I}}_f(A)$. \hfill $\square$

I thank the referee for comments on how to improve the above theorem.

3. Cohen-Macaulay algebras

In this section, we define a Cohen-Macaulay algebra and list some elementary properties of such an algebra.

**Definition 3.1.** Let $A$ be a connected algebra, and $M \in \text{GrMod} A$. We define

- $\text{depth} M = \inf R\text{Hom}_A(k, M) = \inf \{i \mid \text{Ext}_A^i(k, M) \neq 0\}$,
- $\text{idim} M = \sup R\Gamma_m(M) = \sup \{i \mid H_m^i(M) \neq 0\}$, and
\[ \text{lc}(A) = \sup \{ \ldim M \mid M \in \text{GrMod} A \} \]

We say that \( M \) is Cohen-Macaulay if depth \( M = \ldim M < \infty \), and maximal Cohen-Macaulay if depth \( M = \ldim M = \ldim A < \infty \). We say that \( A \) is Cohen-Macaulay on the left if \( A \) is Cohen-Macaulay as a graded left \( A \)-module.

If \( A \) is a connected algebra and \( M \in \text{GrMod} A \), then
\[
\text{depth} M = \inf \text{RHom}_{A}(M) = \inf \{ i \mid H^i_{m}(M) \neq 0 \}
\]
by \([11, \text{Chapter 11, Lemma 4.1}]\). So \( M \) is Cohen-Macaulay with depth \( m \) if and only if \( \text{RHom}_{A}(M) \cong H^m_{m}(M)[-m] \neq 0 \) in \( D(A) \).

**Definition 3.2.** Let \( A \) be a connected algebra with \( \ldim A = d < \infty \), and let \( m = A_{\geq 1} \) be the augmentation ideal. A left canonical module \( \omega_A \) is a graded \( A \)-\( A \) bimodule such that for every \( M \in \text{GrMod} A \),
\[
\text{RHom}_{A}(M, \omega_A) \cong \text{RHom}_{A}(M)[-d]
\]
in \( D(A^\omega) \), that is, there are functorial isomorphisms
\[
\text{Ext}^i_{A}(M, \omega_A) \cong H^{d-i}_{m}(M)^{\prime}
\]
in \( \text{GrMod} A^\omega \) for all \( i \).

If \( A \) has a left canonical module \( \omega_A \), then \( M \in \text{GrMod} A \) is Cohen-Macaulay with depth \( M = m \) if and only if \( \text{RHom}_{A}(M, \omega_A) \cong \text{Ext}^{d-m}_{A}(M, \omega_A)[m-d] \neq 0 \) in \( D(A^\omega) \). In particular, \( M \in \text{GrMod} A \) is maximal Cohen-Macaulay if and only if \( \text{RHom}_{A}(M, \omega_A) \cong \text{Hom}_{A}(M, \omega_A) \neq 0 \) in \( D(A^\omega) \), that is, \( \text{Ext}^i_{A}(M, \omega_A) = 0 \) for all \( i \neq 0 \) and \( \text{Hom}_{A}(M, \omega_A) \neq 0 \).

**Definition 3.3** \([12]\). Let \( A \) be a connected algebra. We say that \( A \) is Ext-finite if the \( \text{Ext}^i_{A}(k, k) \) are finite dimensional over \( k \) for all \( i \).

Since \( \text{Ext}^i_{A}(k, k)^{\prime} \cong \text{Tor}^i_{A}(k, k) \cong \text{Ext}^{i-\omega}_{A}(k, k)^{\prime} \) as graded \( k \)-vector spaces, \( A \) is Ext-finite if and only if \( A^\omega \) is Ext-finite. In particular, if \( A \) is either left Noetherian or right Noetherian, then \( A \) is Ext-finite.

**Theorem 3.4.** Let \( A \) be a connected algebra. If \( A \) has a left canonical module, then \( A \) is Cohen-Macaulay on the left. Conversely, if \( A \) is an Ext-finite Cohen-Macaulay algebra on the left such that \( \text{lc}(A) < \infty \), then \( A \) has a left canonical module \( \omega_A = H^{d}_{m}(A)^{\prime} \), where \( d = \ldim A < \infty \).

**Proof.** If \( A \) has a left canonical module \( \omega_A \), then
\[
H^{i}_{m}(A)^{\prime} \cong \text{Ext}^{d-i}_{A}(A, \omega_A) \cong \begin{cases} 0 & \text{if } i \neq d, \\ \omega_A & \text{if } i = d, \end{cases}
\]
in \( \text{GrMod} A^\omega \). So \( A \) is Cohen-Macaulay on the left with depth \( A = \ldim A = d < \infty \).

Conversely, suppose that \( A \) is an Ext-finite Cohen-Macaulay algebra on the left with depth \( A = \ldim A = d < \infty \). Since \( \text{RHom}_{m}(A) \cong H^{d}_{m}(A)[-d] \) in \( D(A^\omega) \), we have
\[
\text{RHom}_{A}(M, H^{d}_{m}(A)^{\prime}) \cong \text{Hom}_{A}(M, \text{RHom}_{m}(A)^{\prime}[-d])
\cong \text{RHom}_{A}(M, \text{RHom}_{m}(A)^{\prime}[-d])
\cong \text{RHom}_{A}(M, \text{RHom}_{m}(A)^{\prime}[-d])
\cong \text{RHom}_{A}(M, \text{RHom}_{m}(A)^{\prime}[-d])
\]
in \( D(A^\omega) \) for every \( M \in \text{GrMod} A \) by \([12, \text{Theorem 5.1}]\). So \( A \) has a left canonical module \( \omega_A = H^{d}_{m}(A)^{\prime} \). \( \square \)
If \( A \) is an Ext-finite connected algebra, then \( k \) has a finitely generated minimal free resolution \( F \) in \( \text{GrMod} A^o \). Since \( F' \) is an injective resolution of \( k \) in \( \text{GrMod} A \) and \((F')^i \cong (F^i)'\) is a finite direct sum of shifts of \( A' \) that is torsion for each \( i \), it follows that

\[
H^i_m(k) = h^i(R\Gamma_m(k)) \cong h^i(\Gamma_m(F')) \cong h^i(F') \cong \begin{cases} k & \text{if } i = 0, \\ 0 & \text{if } i \neq 0, \end{cases}
\]

that is, \( R\Gamma_m(k) \cong k \) in \( \mathcal{D}(A) \). Using this fact, we can prove the following lemma (cf. [11, Chapter 11, Lemma 5.6]):

**Lemma 3.5.** Let \( A \) be a Cohen-Macaulay algebra on the left with \( \text{idim } A = d < \infty \), and let \( \omega_A \) be a left canonical module. Then:

1. \( \text{lcd}(A) = d < \infty \). In particular, \( M \in \text{GrMod } A \) is maximal Cohen-Macaulay if and only if \( \text{depth } M = d \).
2. \( \text{id}_A(\omega_A) = d < \infty \).
3. If \( A \) is Ext-finite, then \( R\Gamma_m(\omega_A) \cong A'[−d] \) in \( \mathcal{D}(A) \), that is,

\[
H^i_m(\omega_A) \cong \begin{cases} 0 & \text{if } i \neq d, \\ A' & \text{if } i = d, \end{cases}
\]

in \( \text{GrMod } A \). In particular, \( \omega_A \) is maximal Cohen-Macaulay as a graded left \( A \)-module.
4. If \( A \) is Ext-finite, then \( R\text{Hom}_A(\omega_A, \omega_A) \cong A \) in \( \mathcal{D}(A^o) \), that is,

\[
\text{Ext}^i_A(\omega_A, \omega_A) \cong \begin{cases} 0 & \text{if } i \neq 0, \\ A & \text{if } i = 0, \end{cases}
\]

in \( \text{GrMod } A^o \).

**Theorem 3.6.** Let \( A \) be a Cohen-Macaulay algebra on the left, and let \( \omega_A \) be a left canonical module. If \( M \in \text{GrMod } A \) with \( 0 \leq m = \text{depth } A − \text{depth } M < \infty \), then \( M \) has a finite resolution of the form

\[
0 \rightarrow H \rightarrow F^{−m−1} \rightarrow \cdots \rightarrow F^0 \rightarrow M \rightarrow 0
\]

in \( \text{GrMod } A \), where \( F^{−i} \in \text{GrMod } A \) are free, and \( H \in \text{GrMod } A \) is maximal Cohen-Macaulay.

**Proof.** Let \( F \) be a free resolution of \( M \) and let \( H^{−i} \) be the \( i \)-th syzygy of \( M \), that is,

\[
0 \rightarrow H^{−i−1} \rightarrow F^{−i} \rightarrow H^{−i} \rightarrow 0
\]

is an exact sequence in \( \text{GrMod } A \) for each \( i \geq 0 \), where \( H^0 = M \). By the long exact sequence of \( \text{Ext}_A^i(k, −) \),

\[
\text{depth } H^{−m} = \inf \text{RHom}_A(k, H^{−m})
\]

\[
= \inf \text{RHom}_A(k, H^{−m+1}) + 1
\]

\[
= \cdots
\]

\[
= \inf \text{RHom}_A(k, H^0) + m
\]

\[
= \text{depth } M + m = d.
\]

So \( H^{−m} \in \text{GrMod } A \) is maximal Cohen-Macaulay by Lemma [11, Chapter 11, Lemma 5.6].

Definition 4.1. Let $A$ be a connected algebra. A graded $A$-$A$ bimodule $\omega_A$ is called a dualizing module if

- $\text{res}_A \omega_A \in \text{grmod} A$, $\text{res}_{A^e} \omega_A \in \text{grmod} A^e$,
- $\text{id}_A(\omega_A) < \infty$, $\text{id}_{A^e}(\omega_A) < \infty$, and
- the natural morphisms $A \to R\text{Hom}_A(\omega_A, \omega_A)$ and $A \to R\text{Hom}_{A^e}(\omega_A, \omega_A)$ are isomorphisms in $D(A^\bullet)$.

A dualizing module $\omega_A$ is called balanced if $R\Gamma_m(\omega_A) \cong R\Gamma_m^e(\omega_A) \cong A'[d]$ in $D(A^\bullet)$ for some integer $d$.

A connected algebra $A$ is called balanced Cohen-Macaulay if $A$ has a balanced dualizing module.

Let $A$ be a Noetherian connected algebra. Then a graded $A$-$A$ bimodule $\omega_A$ is a dualizing module if and only if $\omega_A$ is a dualizing complex, viewed as an object in $D(A^\bullet)$. Moreover, $\omega_A$ is a balanced dualizing module if and only if $\omega_A[d] \in D(A^\bullet)$ is a balanced dualizing complex for some integer $d$. In particular, a balanced dualizing module $\omega_A$ is unique up to isomorphisms in $\text{GrMod} A^e$ by [13].

Definition 4.2. Let $A$ be a connected algebra and $M \in \text{GrMod} A$. We say that $\chi$ holds for $M$ if $\text{Ext}^i_A(k, M)$ are bounded for all $i$. We say that $A$ satisfies $\chi$ on the left if $\chi$ holds for all $M \in \text{grmod} A$.

We say that $A$ is AS Gorenstein if $A$ satisfies $\chi$ on both sides and $\text{id}_A(A) = \text{id}_{A^e}(A) < \infty$. We say that $A$ is AS regular if $A$ satisfies $\chi$ on both sides and $\text{gldim} A < \infty$.

Clearly, every AS regular algebra is AS Gorenstein. By [5, Theorem 1.2], if $A$ is a Noetherian AS Gorenstein algebra, then $A$ has a balanced dualizing complex $A_\omega(i)[-l][d]$ for some graded algebra automorphism $\omega$ of $A$, some integer $l$, and $d = \text{id}_A(A) = \text{id}_{A^e}(A)$. So $A_\omega(i)[-l]$ is a balanced dualizing module and $A$ is balanced Cohen-Macaulay. In fact, let $A$ be a Noetherian Cohen-Macaulay algebra on the left. Then $A$ is balanced Cohen-Macaulay if and only if $A$ is a graded quotient algebra of a Noetherian AS Gorenstein algebra, by [7, Theorem 1.6].

The following characterization of a balanced Cohen-Macaulay algebra is immediate from [12, Theorem 6.3].

Theorem 4.3. Let $A$ be a Noetherian connected algebra. Then $A$ is balanced Cohen-Macaulay if and only if $A$ is Cohen-Macaulay satisfying $\chi$ on both sides. If $A$ is a Noetherian balanced Cohen-Macaulay algebra, then a balanced dualizing module is given by $\omega_A \cong H^d_m(A)' \cong H^d_{m^e}(A)'$ in $\text{GrMod} A^\bullet$, where $d = \text{ldim}_A A = \text{ldim}_{A^e} A < \infty$. In particular, $\omega_A$ is a left and right canonical module.

Definition 4.4. Let $A$ be a balanced Cohen-Macaulay algebra and $\omega_A$ a balanced dualizing module. If $M \in \text{GrMod} A$, then we define $M^\dagger = \text{Hom}_A(M, \omega_A) \in \text{GrMod} A^\bullet$. Similarly, if $N \in \text{GrMod} A^\bullet$, then we define $N^\dagger = \text{Hom}_{A^e}(N, \omega_A) \in \text{GrMod} A$. We say that $M \in \text{GrMod} A$ is totally $\omega_A$-reflexive if $\text{Ext}^i_A(M, \omega_A) = \text{Ext}^i_A(M^\dagger, \omega_A) = 0$ for all $i \neq 0$ and $M^\dagger \cong M$ in $\text{GrMod} A$.

Let $\mathcal{A}$ be an abelian category and $B \subset A$ a full subcategory. A $B$-resolution of an object $M \in A$ is an exact sequence

$$\cdots \to B^{-i} \to \cdots \to B^{-1} \to B^0 \to M \to 0$$

in $\mathcal{A}$, where $B^{-i} \in B$ for all $i \geq 0$. 

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**Definition 4.5.** We define $\mathcal{H}$ to be the full subcategory of $\text{GrMod } A$ consisting of totally $\omega_A$-reflexive modules, and $\mathcal{H}_{fg}$ to be the full subcategory of $\text{grmod } A$ consisting of totally $\omega_A$-reflexive modules.

For $M \in \text{GrMod } A$, we define
\[
\text{Hdim } M = \inf_H (- \inf_i \{ H^i \neq 0 \}),
\]
where the infimum is taken over all $\mathcal{H}$-resolutions $H$ of $X$, and
\[
\text{hdim } M = \inf_H (- \inf_i \{ H^i \neq 0 \}),
\]
where the infimum is taken over all $\mathcal{H}_{fg}$-resolutions $H$ of $X$.

**Lemma 4.6.** Let $A$ be a Noetherian balanced Cohen-Macaulay algebra and $\omega_A$ a balanced dualizing module. For $M \in \text{grmod } A$, the following are equivalent:
1. $M \in \mathcal{H}_{fg}$;
2. $\text{Ext}^i_A(M, \omega_A) = 0$ for all $i \neq 0$;
3. $M$ is either a maximal Cohen-Macaulay module or the zero module.

**Proof.** (1) $\Rightarrow$ (2): By definition.

(2) $\Rightarrow$ (3): Suppose that $\text{Ext}^i_A(M, \omega_A) = 0$ for all $i \neq 0$. If $\text{Hom}^{\dagger}_A(M, \omega_A) \neq 0$, then $M \in \text{grmod } A$ is maximal Cohen-Macaulay. Since $\omega_A \in D^b(A^\circ)$ is a dualizing complex, if $\text{Hom}^{\dagger}_A(M, \omega_A) = 0$, then $M \cong \text{RHom}^A(\text{RHom}_A(M, \omega_A), \omega_A) = 0$.

(3) $\Rightarrow$ (1): If $M \in \text{grmod } A$ is maximal Cohen-Macaulay, then $\text{Ext}^i_A(M, \omega_A) = 0$ for all $i \neq 0$, that is, $\text{RHom}^A(M, \omega_A) \cong \text{Hom}^A(M, \omega_A) = M^{\dagger}$ in $D(A^\circ)$. Since $\omega_A \in D(A^\circ)$ is a dualizing complex,
\[
\text{RHom}^A(M^{\dagger}, \omega_A) \cong \text{RHom}^A(\text{RHom}^A(M, \omega_A), \omega_A) \cong M
\]
in $\text{GrMod } A$. It follows that $\text{Ext}^i_A(M^{\dagger}, \omega_A) = 0$ for all $i \neq 0$, and $M^{\dagger} \cong \text{Hom}^A(M^{\dagger}, \omega_A) \cong M$
in $\text{GrMod } A$. Clearly, $0 \in \mathcal{H}_{fg}$. 

In particular, if $A$ is a Noetherian balanced Cohen-Macaulay algebra and $\omega_A$ is a balanced dualizing module, then $\text{hdim } A = \text{hdim } \omega_A = 0$.

Let $A$ be a left Noetherian connected algebra and $M \in \text{grmod } A$. If $A$ satisfies $\chi$ on the left and $\text{pd}(M) < \infty$, then Jörgensen proved the Auslander-Buchsbaum formula
\[
\text{pd}(M) + \text{depth } M = \text{depth } A
\]
in $[6]$ Theorem 3.2] (he actually proved the formula for $X \in D^b_{fg}(A)$, using the obvious definition of depth $X$). The following theorem can be regarded as a version of the Auslander-Buchsbaum formula for $\text{hdim}$.

**Theorem 4.7.** Let $A$ be a Noetherian balanced Cohen-Macaulay algebra and $\omega_A$ a balanced dualizing module. If $0 \neq M \in \text{grmod } A$, then
\[
\text{hdim } M = \sup \text{RHom}^A(M, \omega_A) = \text{depth } A - \text{depth } M \leq \text{depth } A < \infty.
\]
In particular, if $\text{pd}(M) < \infty$, then $\text{hdim } M = \text{pd}(M)$.

**Proof.** Let $d = \text{depth } A = \text{ldim } A < \infty$. For $M \in \text{GrMod } A$,
\[
\sup \text{RHom}^A(M, \omega_A) = \sup \text{RHom}^{\dagger}_A(M^{\dagger}, \omega_A) = \text{depth } A - \text{depth } M \leq d.
\]
We will now prove that $\text{hdim } M = \sup \text{RHom}^A(M, \omega_A)$ using induction on $m = \sup \text{RHom}^A(M, \omega_A)$. If $m = 0$, then $M$ is maximal Cohen-Macaulay; so $\text{hdim } M = \text{depth } A$. If $m > 0$, then $M$ is reflexive. Let $M' = M/mM$. Then $M'$ is reflexive, and $\text{hdim } M = \text{hdim } M' + m$. By induction, $\text{hdim } M' = \sup \text{RHom}^A(M', \omega_A) = \text{depth } A - \text{depth } M' \leq m - 1$. Hence $\text{hdim } M = \text{hdim } M' + m$. 

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Suppose that \( m \geq 1 \). Let \( F \rightarrow M \rightarrow 0 \) be a finitely generated minimal free resolution, and \( 0 \rightarrow N \rightarrow F^0 \rightarrow M \rightarrow 0 \) be an exact sequence in \( \text{GrMod} A \). Since \( \text{Ext}^1_A(F^0, \omega_A) = 0 \) for all \( i \geq 1 \), from the long exact sequence of \( \text{Ext}^i_A(\cdot, \omega_A) \), we have an epimorphism \( \text{Hom}_A(N, \omega_A) \rightarrow \text{Ext}^1_A(M, \omega_A) \) and isomorphisms \( \text{Ext}^i_A(N, \omega_A) \cong \text{Ext}^{i+1}_A(M, \omega_A) \) for all \( i \geq 1 \). So \( \sup R\text{Hom}_A(N, \omega_A) = m - 1 \).

By induction, \( \text{hdim} M \leq \text{hdim} N + 1 = \sup R\text{Hom}_A(N, \omega_A) + 1 = m \).

Let \( H \rightarrow M \rightarrow 0 \) be an \( H_{fg} \)-resolution of minimal length, and \( 0 \rightarrow N \rightarrow H^0 \rightarrow M \rightarrow 0 \) an exact sequence in \( \text{GrMod} A \). By a similar argument, we can show that \( \sup R\text{Hom}_A(N, \omega_A) = m - 1 \), and \( \text{hdim} M = \text{hdim} N + 1 = \sup R\text{Hom}_A(N, \omega_A) + 1 = m \).

Since \( A \) satisfies \( \chi \) on the left by Theorem 4.3, if \( \text{pd}(M) < \infty \), then \( \text{pd}(M) = \text{depth} A - \text{depth} M = \text{hdim} A \)

by the classical Auslander-Buchsbaum formula. \( \square \)

**Corollary 4.8.** Let \( A \) be a Noetherian balanced Cohen-Macaulay algebra. Then every finitely generated maximal Cohen-Macaulay module having finite projective dimension is free.

**Proof.** If \( M \in \text{grmod} A \) is maximal Cohen-Macaulay having finite projective dimension, then \( \text{pd}(M) = \text{hdim} M = 0 \) by Lemma 4.6 and Theorem 4.7 so \( M \) is free. \( \square \)

## 5. Maximal Cohen-Macaulay approximations

In this section, we will prove the theorem proposed in the abstract, using maximal Cohen-Macaulay approximations and Foxby equivalence.

Throughout this section, we assume that \( A \) is a Noetherian balanced Cohen-Macaulay algebra with \( \text{idim}_A A = \text{idim}_{A^e} A = d < \infty \), and that \( \omega_A \) is a balanced dualizing module.

Let \( \mathcal{A} \) be an abelian category and \( \mathcal{B} \subset \mathcal{A} \) a full subcategory. We define \( \hat{\mathcal{B}} \) to be the full subcategory of \( \mathcal{A} \) consisting of objects \( M \in \mathcal{A} \) having \( \mathcal{B} \)-resolutions of finite length.

A full additive subcategory \( \mathcal{B} \subset \mathcal{A} \) is called additively closed if \( \mathcal{B} \) is closed under finite direct sums in \( \mathcal{A} \) and closed under direct summands in \( \mathcal{A} \). A full additive subcategory \( \mathcal{C} \subset \mathcal{B} \subset \mathcal{A} \) is called a cogenerator for \( \mathcal{B} \) if for every object \( M \in \mathcal{B} \), there is an exact sequence

\[
0 \rightarrow M \rightarrow C \rightarrow B \rightarrow 0
\]

in \( \mathcal{A} \), where \( C \in \mathcal{C} \) and \( B \in \mathcal{B} \).

**Theorem 5.1 ([2, Theorem 1.1]).** Let \( \mathcal{A} \) be an abelian category, \( \mathcal{B} \subset \mathcal{A} \) an additively closed subcategory that is closed under extensions in \( \mathcal{A} \), and \( \mathcal{C} \subset \mathcal{B} \subset \mathcal{A} \) an additively closed subcategory that is a cogenerator for \( \mathcal{B} \). For every \( M \in \hat{\mathcal{B}} \), there are exact sequences

\[
0 \rightarrow \hat{C}_M \rightarrow B_M \rightarrow M \rightarrow 0
\]

and

\[
0 \rightarrow M \rightarrow \hat{C}^M \rightarrow B^M \rightarrow 0
\]

in \( \mathcal{A} \), where \( B_M, B^M \in \mathcal{B} \) and \( \hat{C}_M, \hat{C}^M \in \hat{\mathcal{C}} \).
Using Theorem \ref{thm:main} we will prove maximal Cohen-Macaulay approximations for Noetherian balanced Cohen-Macaulay algebras (cf. \cite{2} Example 3]).

**Definition 5.2.** We define $\mathcal{J}$ to be the full subcategory of $\mathcal{H}$ consisting of all modules $M \in \mathcal{H}$ having finite injective dimension.

**Proposition 5.3.** For every $M \in \text{grmod } A$, there are exact sequences

$$0 \to \tilde{J}_M \to H_M \to M \to 0$$

and

$$0 \to M \to \tilde{M} \to H_M^M \to 0$$

in $\text{GrMod } A$, where $H_M, H_M^M \in \mathcal{H}_fg$ and $\tilde{J}_M, \tilde{M} \in \tilde{\mathcal{J}}_fg$.

**Proof.** We will apply Theorem 5.1 to $\text{grmod } A \in \text{grmod } A, \mathcal{B} = \mathcal{H}_fg$, and $\mathcal{C} = \tilde{\mathcal{J}}_fg$. By Theorem 4.7, $\tilde{\mathcal{H}}_fg = \text{grmod } A$.

Clearly, $\mathcal{H}_fg$ and $\tilde{\mathcal{J}}_fg$ are closed under finite direct sums in $\text{grmod } A$. Let $M \in \text{grmod } A$, and let $N \in \text{grmod } A$ be a direct summand of $M$. If $M \in \mathcal{H}_fg$, then $\text{Ext}_{A}(N, \omega_A) \subseteq \text{Ext}_{A}^1(M, \omega_A) = 0$ for all $i \neq 0$; so $N \in \mathcal{H}_fg$ by Lemma 4.6. If $M \in \tilde{\mathcal{J}}_fg$, then

$$\text{id}(N) = \sup \{ \text{sup } \text{Hom}_{A}(L, N) \mid L \in \text{grmod } A \}$$

$$\leq \sup \{ \text{sup } \text{Hom}_{A}(L, M) \mid L \in \text{grmod } A \}$$

$$= \text{id}(M) < \infty$$

by Lemma 1.3); so $N \in \tilde{\mathcal{J}}_fg$. It follows that $\mathcal{H}_fg$ and $\tilde{\mathcal{J}}_fg$ are additively closed in $\text{grmod } A$. Let

$$0 \to L \to M \to N \to 0$$

be an exact sequence in $\text{grmod } A$. If $L, N \in \mathcal{H}_fg$, then $\text{Ext}_{A}^k(M, \omega_A) = 0$ for all $i \neq 0$ from the long exact sequence of $\text{Ext}_{A}^k(-, \omega_A)$, and so $M \in \mathcal{H}_fg$ by Lemma 1.4. Hence $\mathcal{H}_fg$ is closed under extensions.

Finally, we will prove that $\tilde{\mathcal{J}}_fg$ is a cogenerator for $\mathcal{H}_fg$. Let $0 \neq M \in \mathcal{H}_fg$. Since $M^\dagger = \text{Hom}_{A^\dagger}(M, \omega_A) \in \text{grmod } A^\circ$, there is an exact sequence

$$0 \to N \to F \to M^\dagger \to 0$$

in $\text{grmod } A^\circ$, where $F \in \text{grmod } A^\circ$ is free. Since $F, M^\dagger \in \text{grmod } A^\circ$ are maximal Cohen-Macaulay, it follows that $M^{\dagger \dagger} \cong M$ in $\text{grmod } A$ by Lemma 4.6. So there is an exact sequence

$$0 \to M^{\dagger \dagger} \cong M \to F^\dagger \to N^\dagger \to \text{Ext}_{A^\circ}^1(M^{\dagger \dagger}, \omega_A) = 0$$

in $\text{grmod } A$. Since $F^\dagger = \text{Hom}_{A^\circ}(F, \omega_A)$ is a finite direct sum of shifts of $\omega_A$ in $\text{grmod } A$, it follows that $F^\dagger \in \tilde{\mathcal{J}}_fg$. Since $M, F^\dagger \in \mathcal{H}_fg$, it follows that $\text{Ext}_{A^\circ}^k(N^{\dagger \dagger}, \omega_A) = 0$ for all $i \geq 2$ from the long exact sequence of $\text{Ext}_{A^\circ}^k(-, \omega_A)$. Since $F \in \text{grmod } A^\circ$ is maximal Cohen-Macaulay, we have $F^{\dagger \dagger} \cong F$ in $\text{grmod } A^\circ$ by Lemma 4.6. So there is an exact sequence

$$0 \to \text{Hom}_{A^\circ}(N^{\dagger \dagger}, \omega_A) \to F^{\dagger \dagger} \cong F \to M^\dagger \to \text{Ext}_{A^\circ}^1(N^{\dagger \dagger}, \omega_A) \to \text{Ext}_{A^\circ}^1(F^{\dagger \dagger}, \omega_A) = 0$$

in $\text{grmod } A^\circ$. It follows that $\text{Hom}_{A^\circ}(N^{\dagger \dagger}, \omega_A) \cong N$ in $\text{grmod } A^\circ$ and $\text{Ext}_{A^\circ}^1(N^{\dagger \dagger}, \omega_A) = 0$; so $N^\dagger \in \mathcal{H}_fg$ by Lemma 4.6. This shows that $\tilde{\mathcal{J}}_fg$ is a cogenerator for $\mathcal{H}_fg$. □

Now we will return to Foxby equivalence discussed in section 2, and apply it to Noetherian balanced Cohen-Macaulay algebras.
Lemma 5.4. Let \( 0 \neq X \in \mathcal{I}(A) \). If \( R\text{Hom}_A(\omega_A, X) \in \mathcal{F}(A) \) has a minimal free resolution, then
\[
\text{pd}(R\text{Hom}(\omega_A, X)) = \text{fd}(R\text{Hom}(\omega_A, X)) = \text{depth } A - \text{depth } X.
\]

Proof. Since \( X \in \mathcal{I}(A) \),
\[
k \otimes^A_A R\text{Hom}_A(\omega_A, X) \cong R\text{Hom}_A(R\text{Hom}_A(k, \omega_A), X)
\cong R\text{Hom}_A(k[-d], X)
\cong R\text{Hom}_A(k, X)[d]
\]
in \( \mathcal{D}(k) \) by Proposition 2.3(1). If \( R\text{Hom}_A(\omega_A, X) \in \mathcal{F}(A) \) has a minimal free resolution, then, by Lemma 1.2(3),
\[
\text{pd}(R\text{Hom}_A(\omega_A, X)) = \text{fd}(R\text{Hom}_A(\omega_A, X))
= - \inf (k \otimes^A_A R\text{Hom}_A(\omega_A, X))
= - \inf (R\text{Hom}_A(k, X)[d])
= - \inf (R\text{Hom}_A(k, X)) + d
= \text{depth } A - \text{depth } X.
\]

Proposition 5.5. If \( M \in \text{grmod } A \) has finite injective dimension, then
\[
R\text{Hom}_A(\omega_A, M) \cong \text{Hom}_A(\omega_A, M)
\]
in \( \mathcal{D}_{fg}(A) \); that is, \( \text{Ext}^i_A(\omega_A, M) = 0 \) for all \( i \neq 0 \), and \( \text{Hom}_A(\omega_A, M) \in \text{grmod } A \). Moreover, if \( M \neq 0 \), then
\[
\text{pd}(\text{Hom}_A(\omega_A, M)) = \text{depth } A - \text{depth } M = \text{hdim } M.
\]

Proof. Suppose that \( M \in \text{grmod } A \) has finite injective dimension. Since \( \omega_A \) and \( \text{Ext}^i_A(\omega_A, M) \in \text{GrMod } A \) are left bounded for all \( i \) by [1, Proposition 3.1(1)], it follows that
\[
\sup(R\text{Hom}_A(\omega_A, M)) = \sup \omega_A + \sup(R\text{Hom}_A(\omega_A, M))
= \sup(\omega_A \otimes^A_A R\text{Hom}_A(\omega_A, M))
= \sup M = 0
\]
by Lemma 1.3(2). So \( R\text{Hom}_A(\omega_A, M) \cong \text{Hom}_A(\omega_A, M) \) in \( \mathcal{D}(A) \). Since \( M \in \mathcal{I}_{fg}(A) \), by Theorem 2.5 \( R\text{Hom}_A(\omega_A, M) \cong \text{Hom}_A(\omega_A, M) \in \mathcal{F}_{fg}(A) \) has a minimal free resolution. By Lemma 5.4 and Theorem 4.7
\[
\text{pd}(\text{Hom}_A(\omega_A, M)) = \text{depth } A - \text{depth } M = \text{hdim } M.
\]

Corollary 5.6. Every finitely generated maximal Cohen-Macaulay module having finite injective dimension is a finite direct sum of shifts of \( \omega_A \).

Proof. Let \( M \in \text{grmod } A \) be maximal Cohen-Macaulay having finite injective dimension. By Proposition 5.5 \( R\text{Hom}_A(\omega_A, M) \cong \text{Hom}_A(\omega_A, M) \) in \( \mathcal{D}(A) \). Since
\[
\omega_A \otimes^A_A \text{Hom}_A(\omega_A, M) \cong \omega_A \otimes^A_A R\text{Hom}_A(\omega_A, M) \cong M
\]
in \( \mathcal{D}(A) \) by Theorem 2.5 \( \omega_A \otimes^A_A \text{Hom}_A(\omega_A, M) \cong M \) in \( \text{GrMod } A \). By Proposition 5.5 \( \text{pd}(\text{Hom}_A(\omega_A, M)) = \text{depth } A - \text{depth } M = 0 \). So \( \text{Hom}_A(\omega_A, M) \in \text{grmod } A \) is finitely generated free; hence \( M \cong \omega_A \otimes_A \text{Hom}_A(\omega_A, M) \in \text{grmod } A \) is a finite direct sum of shifts of \( \omega_A \).
Remark 5.7. It would be much nicer if we could prove the dual statement of Proposition 5.5 as in the commutative case [4 Corollary 3.6], namely, $A$ has the following property (P): "if $M \in \grmod A$ has finite flat dimension, then $\omega_A \otimes_A^L M \cong \omega_A \otimes_A M$ in $\mathcal{D}_{fg}(A)$, so that $\omega_A \otimes_A M \in \grmod A$ has finite injective dimension." This property (P) implies an important property of $\omega_A$ discussed in [7] and [9], namely, "if $x \in A$ is regular on $A$, then $x$ is regular on $\omega_A$ from both sides." In fact, suppose that $A$ has the property (P). Let $x \in A$ be a homogeneous regular element on $A$ of degree $l$, and let

$$0 \to A(-l) \xrightarrow{x} A \to M \to 0$$

be an exact sequence in $\grmod A$. Since $M \in \grmod A$ has finite flat dimension, it follows that $\operatorname{Tor}^A_1(\omega_A, M) = 0$ by the property (P). So

$$0 \to \omega_A(-l) \xrightarrow{x} \omega_A \to \omega_A \otimes_A M \to 0$$

is an exact sequence in $\grmod A$. It follows that $x$ is regular on $\omega_A$ from the right. By symmetry, $x$ is regular on $\omega_A$ from the left.

Let $M \in \grmod A$. By Theorem 5.6, $M$ has a finite resolution of the form

$$0 \to H \to \bigoplus_{j=1}^{r_m} A(-l_{m-1}j) \to \cdots \to \bigoplus_{j=1}^{r_0} A(-l_{0j}) \to M \to 0$$

in $\grmod A$, where $H \in \mathcal{H}_{fg}$ and $m = \operatorname{depth} A - \operatorname{depth} M = \operatorname{hdim} M < \infty$. It is well known that $M$ has finite projective dimension if and only if $M$ has a finite resolution of the form

$$0 \to \bigoplus_{j=1}^{r_m} A(-l_{mj}) \to \bigoplus_{j=1}^{r_{m-1}} A(-l_{m-1}j) \to \cdots \to \bigoplus_{j=1}^{r_0} A(-l_{0j}) \to M \to 0$$

in $\grmod A$, where $m = \operatorname{pd}(M) = \operatorname{hdim} M < \infty$. The following theorem can be compared with these facts.

Theorem 5.8. Let $M \in \grmod A$. Then:

1. $M$ has a finite resolution of the form

$$0 \to \bigoplus_{j=1}^{r_m} \omega_A(-l_{mj}) \to \cdots \to \bigoplus_{j=1}^{r_1} \omega_A(-l_{1j}) \to H \to M \to 0$$

in $\grmod A$, where $H \in \mathcal{H}_{fg}$;

2. $M$ has finite injective dimension if and only if $M$ has a finite resolution of the form

$$(*) \quad 0 \to \bigoplus_{j=1}^{r_m} \omega_A(-l_{mj}) \to \cdots \to \bigoplus_{j=1}^{r_1} \omega_A(-l_{1j}) \to \bigoplus_{j=1}^{r_0} \omega_A(-l_{0j}) \to M \to 0$$

in $\grmod A$, where $m = \operatorname{hdim} M < \infty$.

Proof. (1): By Proposition 5.5 there is an exact sequence

$$0 \to \tilde{J} \to H \to M \to 0$$

in $\grmod A$, where $H \in \mathcal{H}_{fg}$ and $\tilde{J} \in \mathcal{J}_{fg}$. Since every $J \in \mathcal{J}_{fg}$ is a finite direct sum of shifts of $\omega_A$ or the zero module by Corollary 5.6, $\tilde{J}$ has a finite resolution
of the form
\[ 0 \rightarrow r_m \bigoplus_{j=1}^{r_m} \omega_A(-l_{mj}) \rightarrow \cdots \rightarrow r_1 \bigoplus_{j=1}^{r_1} \omega_A(-l_{1j}) \rightarrow \hat{J} \rightarrow 0, \]
in $\text{GrMod}_A$, hence the result.

(2): Suppose that $M \in \text{grmod } A$ has finite injective dimension. By Proposition 5.5, $\text{Hom}_A(\omega_A, M) \in \text{grmod } A$, and $\text{pd}(\text{Hom}_A(\omega_A, M)) = \text{hdim } M = m < \infty$. So $R\text{Hom}_A(\omega_A, M) \cong \text{Hom}_A(\omega_A, M) \in D^b_{fg}(A)$ has a finitely generated minimal free resolution $F$ of length $m$. By Theorem 2.5,
\[ \omega_A \otimes_A F \cong \omega_A \otimes_A \bigoplus_{i} R\text{Hom}_A(\omega_A, M) \cong M \]
in $D(A)$. So $\omega_A \otimes_A F$ is a resolution of the form (8).

Conversely, if $M$ has a resolution of the form (8), then clearly $\text{id}(M) < \infty$. □

As corollaries, we have the following characterizations of AS Gorenstein algebras and AS regular algebras.

**Corollary 5.9.** Let $A$ be a Noetherian balanced Cohen-Macaulay algebra. Then the following are equivalent:

1. $A$ is AS Gorenstein;
2. $\text{id}_A(A) < \infty$;
3. $\text{pd}_A(\omega_A) < \infty$;
4. for every $M \in \text{grmod } A$, $\text{id}(M) < \infty$ if and only if $\text{pd}(M) < \infty$.

**Proof.** (4) $\Rightarrow$ (3): Suppose that $A$ has the property (4). Since $\text{id}_A(\omega_A) < \infty$, it follows that $\text{pd}_A(\omega_A) < \infty$.

(3) $\Rightarrow$ (2): If $\text{pd}_A(\omega_A) < \infty$, then $\omega_A$ is free by Corollary 4.8. Since $\text{id}_A(\omega_A) < \infty$, it follows that $\text{id}_A(A) < \infty$.

(2) $\Rightarrow$ (1): This follows from [8, Corollary 4.6].

(1) $\Rightarrow$ (4): If $A$ is AS Gorenstein, then $\omega_A \cong A(-l)$ in $\text{GrMod } A$ for some integer $l$ by [8, Theorem 1.2]. The result follows from Theorem 5.8. □

**Remark 5.10.** The direction (1) $\Rightarrow$ (4) of the above corollary was proved by Zhang, using a spectral sequence [11, Chapter 1, Proposition 6.7].

**Corollary 5.11.** Let $A$ be a Noetherian balanced Cohen-Macaulay algebra. Then the following are equivalent:

1. $A$ is AS regular;
2. every $H \in \mathcal{H}_{fg}$ has finite projective dimension;
3. every nonzero $H \in \mathcal{H}_{fg}$ is free;
4. every $H \in \mathcal{H}_{fg}$ has finite injective dimension;
5. every nonzero $H \in \mathcal{H}_{fg}$ is a finite direct sum of shifts of $\omega_A$.

**Proof.** (2) $\Leftrightarrow$ (3) by Corollary 4.8 and (4) $\Leftrightarrow$ (5) by Corollary 5.6.

If $A$ is AS regular, then clearly every $M \in \text{grmod } A$ has finite projective dimension and finite injective dimension; so (1) $\Rightarrow$ (2), (4).

If every $H \in \mathcal{H}_{fg}$ has finite projective dimension, then every $M \in \text{grmod } A$ has finite projective dimension by Theorem 5.8 so (2) $\Rightarrow$ (1). Similarly, if every $H \in \mathcal{H}_{fg}$ has finite injective dimension, then every $M \in \text{grmod } A$ has finite injective dimension by Theorem 5.8 so (4) $\Rightarrow$ (1). □
6. An application to the intersection multiplicity

We will end the paper by an application of Theorem 5.8 to the intersection multiplicity discussed in [10].

**Definition 6.1.** For $V \in \text{GrMod}k$ locally finite, we define the Hilbert series of $V$ by

$$H_V(t) = \sum_{i=-\infty}^{\infty} \dim_k V_i t^i \in \mathbb{Z}[\![t, t^{-1}]\!] .$$

If $H_V(t)$ is a rational function over $\mathbb{C}$, then we define $\text{GKdim} V$ to be the order of the pole of $H_V(t)$ at $t = 1$, and we define the multiplicity of $V$ by

$$\epsilon(V) = \lim_{t \to 1} (1 - t)^{\text{GKdim} V} H_V(t).$$

For $X \in \mathcal{D}^b_f(k)$, we define the Hilbert series of $X$ by

$$H_X(t) = \sum_{i=-\infty}^{\infty} (-1)^i H_Ki(X)(t) \in \mathbb{Z}[\![t, t^{-1}]\!] .$$

Let $A$ be a Cohen-Macaulay algebra on the left and $\omega_A$ a left canonical module. If $M \in \text{grmod} A$ has a finite $\omega_A$-resolution of the form

$$(*) \quad 0 \to \bigoplus_{j=1}^{r_m} \omega_A(-l_{mj}) \to \cdots \to \bigoplus_{j=1}^{r_0} \omega_A(-l_{0j}) \to M \to 0$$

in $\text{GrMod} A$, then the $\omega_A$-characteristic polynomial of $M$ is defined by

$$r_M(t) := \sum_{i=0}^{m} (-1)^i \sum_{j=1}^{r_i} t^{i_j} \in \mathbb{Z}[t, t^{-1}] .$$

Note that if $M$ has a finite $\omega_A$-resolution of the form $(*)$, then

$$H_M(t) = \sum_{i=0}^{m} (-1)^i \sum_{j=1}^{r_i} H_{\omega_A(-l_{ij})}(t) = \sum_{i=0}^{m} (-1)^i \sum_{j=1}^{r_i} t^{i_j} H_{\omega_A}(t) = r_M(t) H_{\omega_A}(t) .$$

**Definition 6.2** ([9], [11]). Let $A$ be a connected algebra, and let $M \in \text{GrMod} A$ be locally finite. We say that $M$ is rational if

- $\text{RT}_m(M) \in \mathcal{D}^b_f(A)$;
- $H_M(t)$ and $H_{\text{RT}_m(M)}(t)$ are both rational functions over $\mathbb{C}$;
- $H_M(t) = H_{\text{RT}_m(M)}(t)$ as rational functions over $\mathbb{C}$.

We say that $A$ is universally rational, if every $M \in \text{grmod} A$ is rational.

**Lemma 6.3.** Let $A$ be a universally rational, Noetherian balanced Cohen-Macaulay algebra, and $M, N \in \text{grmod} A$. If $N$ has finite injective dimension, then

$$H_{\text{RHom}_A(M, N)}(t) = H_M(t^{-1}) H_N(t) / H_A(t^{-1}) .$$

**Proof.** Since $N \in \text{grmod} A$ has finite injective dimension, $N$ has a finite $\omega_A$-resolution of the form

$$0 \to \bigoplus_{j=1}^{r_n} \omega_A(-l_{nj}) \to \cdots \to \bigoplus_{j=1}^{r_0} \omega_A(-l_{0j}) \to N \to 0 ,$$

by Theorem 5.8.
Since $A$ is universally rational, Noetherian balanced Cohen-Macaulay, we have 
\[ H_{\omega_A}(t) = H_{\text{RHom}_A}(M,N)[t-d](t) = (-1)^d H_{\text{RHom}_A}(A)(t^{-1}) = (-1)^d H_A(t^{-1}). \]

So 
\[ H_{\text{RHom}_A}(M,N)(t) = \sum_{i=0}^{n} (-1)^i \sum_{j=1}^{r_i} H_{\text{RHom}_A}(M\omega_A(-i_j))(t) \]
\[ = \sum_{i=0}^{n} (-1)^i \sum_{j=1}^{r_i} t^{i_j} H_{\text{RHom}_A}(M\omega_A)(t) \]
\[ = r_N(t) H_{\text{RHom}_A}(M)[t-d](t) \]
\[ = (-1)^d H_{\text{RHom}_A}(M)(t^{-1}) r_N(t) \]
\[ = (-1)^d H_M(t^{-1}) H_N(t)/H_A(t) \]
\[ = H_M(t^{-1}) H_N(t)/H_A(t^{-1}). \]

Let $A$ be a connected algebra. For $M, N \in \text{grmod} A$, we define the intersection multiplicity of $M$ and $N$ by
\[ M \cdot N = (-1)^{GKdim N} \sum_{i=0}^{\infty} (-1)^i \dim_k \text{Ext}^i_A(M,N). \]

It is well defined if $\text{Ext}^i_A(M,N) = 0$ for all $i > 0$, and $\dim_k \text{Ext}^i_A(M,N) < \infty$ for all $i \geq 0$. We can then prove a version of Serre’s multiplicity conjectures as in [10].

**Theorem 6.4.** Let $A$ be a universally rational, Noetherian balanced Cohen-Macaulay algebra, and $M, N \in \text{grmod} A$. Suppose that $N$ has finite injective dimension, and $M \cdot N$ is well defined. Then

1. (Dimension) $GKdim M + GKdim N \leq GKdim A$.
2. (Vanishing) If $GKdim M + GKdim N < GKdim A$, then $M \cdot N = 0$.
3. (Positivity) If $GKdim M + GKdim N = GKdim A$, then
\[ M \cdot N = e(M)e(N)/e(A) > 0. \]

**Proof.** Using Lemma 6.3, exactly the same proof as in [10, Theorem 3.9] goes through. \(\square\)

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