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HOMOLOGICAL PROPERTIES OF BALANCED COHEN-MACAULAY ALGEBRAS

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ABSTRACT. A balanced Cohen-Macaulay algebra is a connected algebra A having a balanced dualizing complex $\omega_A[d]$ in the sense of Yekutieli (1992) for some integer d and some graded A-A bimodule ω_A. We study some homological properties of a balanced Cohen-Macaulay algebra. In particular, we will prove the following theorem:

Theorem 0.1. *Let A be a Noetherian balanced Cohen-Macaulay algebra, and M a nonzero finitely generated graded left A-module. Then:*

1. *M has a finite resolution of the form*
$$0 \to \bigoplus_{j=1}^{r_m} \omega_A(-l_{mj}) \to \cdots \to \bigoplus_{j=1}^{r_1} \omega_A(-l_{1j}) \to H \to M \to 0,$$
where H is a finitely generated maximal Cohen-Macaulay graded left A-module.
2. *M has finite injective dimension if and only if M has a finite resolution of the form*
$$0 \to \bigoplus_{j=1}^{r_m} \omega_A(-l_{mj}) \to \cdots \to \bigoplus_{j=1}^{r_1} \omega_A(-l_{1j})$$
$$\to \bigoplus_{j=1}^{r_0} \omega_A(-l_{0j}) \to M \to 0.$$

As a corollary, we will have the following characterizations of AS Gorenstein algebras and AS regular algebras:

Corollary 0.2. *Let A be a Noetherian balanced Cohen-Macaulay algebra.*

1. *A is AS Gorenstein if and only if ω_A has finite projective dimension as a graded left A-module.*
2. *A is AS regular if and only if every finitely generated maximal Cohen-Macaulay graded left A-module is free.*

1. HYPERHOMOLOGICAL ALGEBRAS

Throughout this paper, we fix a field k. A connected algebra is a graded algebra of the form $A = k \oplus A_1 \oplus A_2 \oplus \cdots$. The augmentation ideal of A is defined by $\mathfrak{m} = A_1 \oplus A_2 \oplus \cdots$. In this first section, we will fix terminology and notation, and collect some elementary results on hyperhomological algebras over connected algebras.

Let A, B, C be connected algebras. The category of graded left A-modules and graded left A-module homomorphisms of degree 0 is denoted by $\operatorname{GrMod} A$. For

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$M, N \in \operatorname{GrMod} A$, the set of graded left A-module homomorphisms $M \to N$ of degree 0 is denoted by $\operatorname{Hom}_A(M, N)$, which has a natural k-vector space structure. The full subcategory of $\operatorname{GrMod} A$ consisting of finitely generated graded left A-modules is denoted by $\operatorname{grmod} A$. The category of graded right A-modules is denoted by $\operatorname{GrMod} A^o$, where A^o is the opposite algebra of A. The category of graded A-B bimodules is denoted by $\operatorname{GrMod}(A \otimes B^o)$. In particular, the category of graded A-A bimodules is denoted by $\operatorname{GrMod} A^e$, where $A^e = A \otimes A^o$. The natural restriction functors are denoted by

$$\operatorname{res}_A : \operatorname{GrMod}(A \otimes B^o) \to \operatorname{GrMod} A$$

and

$$\operatorname{res}_{B^o} : \operatorname{GrMod}(A \otimes B^o) \to \operatorname{GrMod} B^o.$$

We write $k = A/\mathfrak{m}$, viewed as an object in $\operatorname{GrMod} A, \operatorname{GrMod} A^o$, or $\operatorname{GrMod} A^e$, depending on the context.

A graded left A-module $M \in \operatorname{GrMod} A$ is right bounded (resp. left bounded) if $M_i = 0$ for all $i \gg 0$ (resp. $i \ll 0$), and bounded if it is both right bounded and left bounded. We say that M is locally finite if the M_i are finite dimensional over k for all i. For each integer n, the shift of M is denoted by $M(n) \in \operatorname{GrMod} A$, so that $M(n)_i = M_{n+i}$. For $M \in \operatorname{GrMod}(A \otimes B^o)$ and $N \in \operatorname{GrMod}(A \otimes C^o)$, we define

$$\underline{\operatorname{Ext}}^i_A(M, N) = \bigoplus_{n=-\infty}^{\infty} \operatorname{Ext}^i_A(M, N(n)),$$

which has a natural graded B-C bimodule structure for each i. Similarly, for $M \in \operatorname{GrMod}(B \otimes A^o)$ and $N \in \operatorname{GrMod}(A \otimes C^o)$, $\operatorname{Tor}^A_i(M, N)$ has a natural graded B-C bimodule structure for each i. For $M \in \operatorname{GrMod}(A \otimes B^o)$, the Matlis dual of M is defined by $M' = \underline{\operatorname{Hom}}_k(M, k)$, which has a natural graded B-A bimodule structure. If M is locally finite, then $M'' \cong M$ in $\operatorname{GrMod}(A \otimes B^o)$.

Let X, Y be cochain complexes of graded left A-modules. The ith cohomology of X is denoted by $h^i(X)$. We say that a cochain map $f : X \to Y$ is a quasi-isomorphism if the induced maps $h^i(f) : h^i(X) \to h^i(Y)$ are isomorphisms in $\operatorname{GrMod} A$ for all i. The derived category of graded left A-modules is denoted by $\mathcal{D}(A)$, so that a cochain map $f : X \to Y$ is a quasi-isomorphism if and only if it induces an isomorphism $f : X \to Y$ in $\mathcal{D}(A)$. We define $\mathcal{D}_{fg}(A)$ (resp. $\mathcal{D}_{lf}(A)$) to be the full subcategory of $\mathcal{D}(A)$ consisting of complexes whose cohomologies are all finitely generated (resp. locally finite) graded left A-modules.

For $X \in \mathcal{D}(A)$, we define

$$\sup X = \sup\{i \mid h^i(X) \neq 0\}$$

and

$$\inf X = \inf\{i \mid h^i(X) \neq 0\}.$$

If $X \cong 0$ in $\mathcal{D}(A)$, then we define $\sup X = -\infty$ and $\inf X = \infty$.

A complex $X \in \mathcal{D}(A)$ is bounded above (resp. bounded below) if $\sup X < \infty$ (resp. $\inf X > -\infty$), and bounded if it is both bounded above and bounded below. The full subcategory of $\mathcal{D}(A)$ consisting of bounded (resp. bounded above, resp. bounded below) complexes is denoted by $\mathcal{D}^b(A)$ (resp. $\mathcal{D}^-(A)$, resp. $\mathcal{D}^+(A)$).

The right derived functor of

$$\underline{\operatorname{Hom}}_A(-,-) : \mathcal{D}^-(A \otimes B^o) \times \mathcal{D}^+(A \otimes C^o) \to \mathcal{D}(B \otimes C^o)$$

is denoted by $R\underline{\mathrm{Hom}}_A(-,-)$, and its cohomologies are denoted by

$$\underline{\mathrm{Ext}}^i_A(-,-) = h^i(R\underline{\mathrm{Hom}}_A(-,-)).$$

The left derived functor of

$$-\otimes_A - : \mathcal{D}^-(B\otimes A^o)\times\mathcal{D}^-(A\otimes C^o)\to\mathcal{D}(B\otimes C^o)$$

is denoted by $-\otimes^L_A -$, and its cohomologies are denoted by

$$\mathrm{Tor}^A_{-i}(-,-) = h^i(-\otimes^L_A -).$$

Let $X\in\mathcal{D}(A)$. For each integer n, the twist of X is denoted by $X[n]\in\mathcal{D}(A)$, so that $(X[n])^i = X^{n+i}$. Note that $h^i(X)=0$ for all $i\neq n$ if and only if $X\cong h^n(X)[-n]$ in $\mathcal{D}(A)$. If $X\in\mathcal{D}^-(A\otimes B^o)$ and $Y\in\mathcal{D}^+(A\otimes C^o)$, then

$$R\underline{\mathrm{Hom}}_A(X[n],Y)\cong R\underline{\mathrm{Hom}}_A(X,Y[-n])\cong R\underline{\mathrm{Hom}}_A(X,Y)[-n]$$

in $\mathcal{D}(B\otimes C^o)$ for each n. If $X\in\mathcal{D}^-(B\otimes A^o)$ and $Y\in\mathcal{D}^-(A\otimes C^o)$, then

$$(X[n])\otimes^L_A Y\cong X\otimes^L_A(Y[n])\cong(X\otimes^L_A Y)[n]$$

in $\mathcal{D}(B\otimes C^o)$ for each n.

Definition 1.1. Let A be a connected algebra.

1. A free resolution of $X\in\mathcal{D}^-(A)$ is a complex F of free graded left A-modules such that $F\cong X$ in $\mathcal{D}(A)$. A complex F of free graded left A-modules is called minimal if the differentials in $\underline{\mathrm{Hom}}_A(F,k)$ are all zero.
2. A projective resolution of $X\in\mathcal{D}^-(A)$ is a complex P of projective graded left A-modules such that $P\cong X$ in $\mathcal{D}(A)$. We define the projective dimension of X by
$$\mathrm{pd}_A(X) = \inf_P(-\inf\{i\mid P^i\neq 0\}),$$
where the infimum is taken over all projective resolutions P of X.
3. An injective resolution of $X\in\mathcal{D}^+(A)$ is a complex E of injective graded left A-modules such that $E\cong X$ in $\mathcal{D}(A)$. We define the injective dimension of X by
$$\mathrm{id}_A(X) = \inf_E(\sup\{i\mid E^i\neq 0\}),$$
where the infimum is taken over all injective resolutions E of X.
4. A flat resolution of $X\in\mathcal{D}^-(A)$ is a complex F of flat graded left A-modules such that $F\cong X$ in $\mathcal{D}(A)$. We define the flat dimension of X by
$$\mathrm{fd}_A(X) = \inf_F(-\inf\{i\mid F^i\neq 0\}),$$
where the infimum is taken over all flat resolutions F of X.

Lemma 1.2. *Let A be a connected algebra.*

1. *For $X\in\mathcal{D}^+(A)$,*
$$\begin{aligned}\mathrm{id}_A(X) &= \sup(\{\sup R\underline{\mathrm{Hom}}_A(M,X)\mid M\in\mathrm{GrMod}\,A\})\\ &= \sup(\{\sup R\underline{\mathrm{Hom}}_A(M,X)\mid M\in\mathrm{grmod}\,A\}).\end{aligned}$$
2. *For $X\in\mathcal{D}^-(A)$,*
$$\begin{aligned}\mathrm{fd}_A(X) &= \sup(\{-\inf(N\otimes^L_A X)\mid N\in\mathrm{GrMod}\,A^o\})\\ &= \sup(\{-\inf(N\otimes^L_A X)\mid N\in\mathrm{grmod}\,A^o\}).\end{aligned}$$

3. *If $X \in \mathcal{D}^-(A)$ has a minimal free resolution, then*
$$\operatorname{pd}_A(X) = \sup R\underline{\operatorname{Hom}}(X, k) = -\inf(k \otimes^L_A X) = \operatorname{fd}_A(X).$$

Proof. These are direct consequences of [6, Propositions 1.7, 1.8, 1.9]. □

The following lemma is standard (cf. [8, Lemma 1.8]).

Lemma 1.3. *Let A be a connected algebra.*

1. *If $X \in \mathcal{D}^-(A \otimes B^o)$ and $Y \in \mathcal{D}^+(A \otimes C^o)$, then*
$$\inf R\underline{\operatorname{Hom}}_A(X, Y) \geq \inf Y - \sup X.$$
2. *If $X \in \mathcal{D}^-(B \otimes A^o)$ and $Y \in \mathcal{D}^-(A \otimes C^o)$, then*
$$\sup(X \otimes^L_A Y) \leq \sup X + \sup Y.$$
Moreover, if $h^{\sup X}(X), h^{\sup Y}(Y) \in \operatorname{GrMod} A$ are left bounded, then
$$\sup(X \otimes^L_A Y) = \sup X + \sup Y.$$

2. Foxby equivalence

Definition 2.1. Let A, B be connected algebras, and let $\mathfrak{m} = A_{\geq 1}$ be the augmentation ideal of A. We define the functor $\Gamma_{\mathfrak{m}} : \mathcal{D}(A \otimes B^o) \to \mathcal{D}(A \otimes B^o)$ by
$$\Gamma_{\mathfrak{m}}(-) = \lim_{n\to\infty} \underline{\operatorname{Hom}}_A(A/A_{\geq n}, -).$$
The right derived functor of $\Gamma_{\mathfrak{m}}$ is denoted by $R\Gamma_{\mathfrak{m}}$, and its cohomologies are denoted by
$$H^i_{\mathfrak{m}}(-) = h^i(R\Gamma_{\mathfrak{m}}(-)) = \lim_{n\to\infty} \underline{\operatorname{Ext}}^i_A(A/A_{\geq n}, -).$$
Similarly, we define the functor $\Gamma_{\mathfrak{m}^o} : \mathcal{D}(B \otimes A^o) \to \mathcal{D}(B \otimes A^o)$ by
$$\Gamma_{\mathfrak{m}^o}(-) = \lim_{n\to\infty} \underline{\operatorname{Hom}}_{A^o}(A/A_{\geq n}, -).$$
The right derived functor of $\Gamma_{\mathfrak{m}^o}$ is denoted by $R\Gamma_{\mathfrak{m}^o}$, and its cohomologies are denoted by
$$H^i_{\mathfrak{m}^o}(-) = h^i(R\Gamma_{\mathfrak{m}^o}(-)) = \lim_{n\to\infty} \underline{\operatorname{Ext}}^i_{A^o}(A/A_{\geq n}, -).$$

Let us recall the following definition from [13].

Definition 2.2. Let A be a Noetherian connected algebra. A complex $D \in \mathcal{D}^b(A^e)$ is called dualizing if

- $\operatorname{res}_A D \in \mathcal{D}^b_{fg}(A), \operatorname{res}_{A^o} D \in \mathcal{D}^b_{fg}(A^o)$,
- $\operatorname{id}_A(D) < \infty, \operatorname{id}_{A^o}(D) < \infty$, and
- the natural morphisms $A \to R\underline{\operatorname{Hom}}_A(D, D)$ and $A \to R\underline{\operatorname{Hom}}_{A^o}(D, D)$ are isomorphisms in $\mathcal{D}(A^e)$.

A dualizing complex $D \in \mathcal{D}(A^e)$ is called balanced if

- $R\Gamma_{\mathfrak{m}}(D) \cong R\Gamma_{\mathfrak{m}^o}(D) \cong A'$ in $\mathcal{D}(A^e)$.

By [13, Proposition 3.5], if D is a dualizing complex, then the functor
$$R\underline{\operatorname{Hom}}_A(-, D) : \mathcal{D}(A) \to \mathcal{D}(A^o)$$
and the functor
$$R\underline{\operatorname{Hom}}_{A^o}(-, D) : \mathcal{D}(A^o) \to \mathcal{D}(A)$$
define a duality between $\mathcal{D}^b_{fg}(A)$ and $\mathcal{D}^b_{fg}(A^o)$, that is,
$$R\underline{\operatorname{Hom}}_A(X, D) \in \mathcal{D}^b_{fg}(A^o) \text{ and } R\underline{\operatorname{Hom}}_{A^o}(R\underline{\operatorname{Hom}}_A(X, D), D) \cong X \text{ in } \mathcal{D}(A)$$

for all $X \in \mathcal{D}^b_{fg}(A)$, and

$$R\underline{\mathrm{Hom}}_{A^o}(Y,D) \in \mathcal{D}^b_{fg}(A) \text{ and } R\underline{\mathrm{Hom}}_A(R\underline{\mathrm{Hom}}_{A^o}(Y,D),D) \cong Y \text{ in } \mathcal{D}(A^o)$$

for all $Y \in \mathcal{D}^b_{fg}(A^o)$. In this section, we study another type of equivalence, known as Foxby equivalence.

Proposition 2.3. *Let A, B be Noetherian connected algebras.*

1. *Let*
$$X \in \mathcal{D}^-(B \otimes A^o), \quad Y \in \mathcal{D}^b(A^e), \quad Z \in \mathcal{D}^+(A)$$
be such that $\mathrm{res}_{A^o} X \in \mathcal{D}^-_{fg}(A^o)$. If either $\mathrm{pd}_{A^o}(X) < \infty$ or $\mathrm{id}_A(Z) < \infty$, then there is a natural isomorphism
$$X \otimes^L_A R\underline{\mathrm{Hom}}_A(Y,Z) \cong R\underline{\mathrm{Hom}}_A(R\underline{\mathrm{Hom}}_{A^o}(X,Y),Z)$$
in $\mathcal{D}(B)$.
2. *Let*
$$X \in \mathcal{D}^-(A \otimes B^o), \quad Y \in \mathcal{D}^b(A^e), \quad Z \in \mathcal{D}^-(A)$$
be such that $\mathrm{res}_A X \in \mathcal{D}^-_{fg}(A)$. If either $\mathrm{pd}_A(X) < \infty$ or $\mathrm{id}_A(Z) < \infty$, then there is a natural isomorphism
$$R\underline{\mathrm{Hom}}_A(X,Y) \otimes^L_A Z \cong R\underline{\mathrm{Hom}}_A(X, Y \otimes^L_A Z)$$
in $\mathcal{D}(B)$.

Proof. By [8, Theorem 1.4] and [6, Proposition 2.1], if X, Y, Z are as above, then the evaluation morphisms

$$\theta_{XYZ} : X \otimes^L_A R\underline{\mathrm{Hom}}_A(Y,Z) \to R\underline{\mathrm{Hom}}_A(R\underline{\mathrm{Hom}}_{A^o}(X,Y),Z)$$

and

$$\omega_{XYZ} : R\underline{\mathrm{Hom}}_A(X,Y) \otimes^L_A Z \to R\underline{\mathrm{Hom}}_A(X, Y \otimes^L_A Z)$$

defined in [3, Notation 4.3] are isomorphisms in $\mathcal{D}(k)$. We will leave it to the reader to check that θ_{XYZ} and ω_{XYZ} are in fact induced by maps of complexes of graded left B-modules. □

Definition 2.4. Let A be a connected algebra. We define $\widehat{\mathcal{I}}(A)$ to be the full subcategory of $\mathcal{D}^b(A)$ consisting of complexes having finite injective dimension, and $\widehat{\mathcal{F}}(A)$ to be the full subcategory of $\mathcal{D}^b(A)$ consisting of complexes having finite flat dimension.

Now Foxby equivalence is stated as follows:

Theorem 2.5. *Let A be a Noetherian connected algebra. If $D \in \mathcal{D}^b(A^e)$ is a dualizing complex, then the functors $D \otimes^L_A - : \mathcal{D}^b(A) \to \mathcal{D}^-(A)$ and $R\underline{\mathrm{Hom}}_A(D,-) : \mathcal{D}^b(A) \to \mathcal{D}^+(A)$ define inverse equivalences between $\widehat{\mathcal{F}}(A)$ and $\widehat{\mathcal{I}}(A)$. They also define inverse equivalences between $\widehat{\mathcal{F}}_{fg}(A)$ and $\widehat{\mathcal{I}}_{fg}(A)$.*

Proof. If $X \in \widehat{\mathcal{F}}(A)$, then

$$R\underline{\mathrm{Hom}}_A(M, D \otimes^L_A X) \cong R\underline{\mathrm{Hom}}_A(M,D) \otimes^L_A X$$

in $\mathcal{D}(k)$ for all $M \in \operatorname{grmod} A$ by Proposition 2.3(2). By Lemma 1.2(1) and Lemma 1.3(2),

$$\begin{aligned}\operatorname{id}_A(D \otimes_A^L X) &= \sup\{\sup R\underline{\operatorname{Hom}}_A(M, D \otimes_A^L X) \mid M \in \operatorname{grmod} A\}\\ &= \sup\{\sup(R\underline{\operatorname{Hom}}_A(M, D) \otimes_A^L X) \mid M \in \operatorname{grmod} A\}\\ &\le \sup\{\sup R\underline{\operatorname{Hom}}_A(M, D) + \sup X \mid M \in \operatorname{grmod} A\}\\ &= \sup\{\sup R\underline{\operatorname{Hom}}_A(M, D) \mid M \in \operatorname{grmod} A\} + \sup X\\ &= \operatorname{id}_A(D) + \sup X < \infty,\end{aligned}$$

and hence $D \otimes_A^L X \in \widehat{\mathcal{I}}(A)$. Moreover,

$$R\underline{\operatorname{Hom}}_A(D, D \otimes_A^L X) \cong R\underline{\operatorname{Hom}}_A(D, D) \otimes_A^L X \cong X$$

in $\mathcal{D}(A)$ by Proposition 2.3(2).

If $X \in \widehat{\mathcal{I}}(A)$, then

$$N \otimes_A^L R\underline{\operatorname{Hom}}_A(D, X) \cong R\underline{\operatorname{Hom}}_A(R\underline{\operatorname{Hom}}_{A^o}(N, D), X)$$

in $\mathcal{D}(k)$ for all $N \in \operatorname{grmod} A^o$ by Proposition 2.3(1). By Lemma 1.2(2) and Lemma 1.3(1),

$$\begin{aligned}\operatorname{fd}_A(R\underline{\operatorname{Hom}}_A(D, X)) &= \sup\{-\inf(N \otimes L_A R\underline{\operatorname{Hom}}_A(D, X)) \mid N \in \operatorname{grmod} A^o\}\\ &= \sup\{-\inf(R\underline{\operatorname{Hom}}_A(R\underline{\operatorname{Hom}}_{A^o}(N, D), X)) \mid N \in \operatorname{grmod} A^o\}\\ &\le \sup\{\sup(R\underline{\operatorname{Hom}}_{A^o}(N, D)) - \inf X \mid N \in \operatorname{grmod} A^o\}\\ &= \sup\{\sup(R\underline{\operatorname{Hom}}_{A^o}(N, D)) \mid N \in \operatorname{grmod} A^o\} - \inf X\\ &= \operatorname{id}_{A^o}(D) - \inf X < \infty,\end{aligned}$$

and hence $R\underline{\operatorname{Hom}}_A(D, X) \in \widehat{\mathcal{F}}(A)$. Moreover,

$$D \otimes_A^L R\underline{\operatorname{Hom}}_A(D, X) \cong R\underline{\operatorname{Hom}}_A(R\underline{\operatorname{Hom}}_{A^o}(D, D), X) \cong X$$

in $\mathcal{D}(A)$ by Proposition 2.3(1); hence the functors $D \otimes_A^L -$ and $R\underline{\operatorname{Hom}}_A(D, -)$ define inverse equivalences between $\widehat{\mathcal{F}}(A)$ and $\widehat{\mathcal{I}}(A)$.

If $X \in \mathcal{D}_{fg}^b(A)$, then

$$\begin{aligned}R\underline{\operatorname{Hom}}_A(D, X) &\cong R\underline{\operatorname{Hom}}_A(D, R\underline{\operatorname{Hom}}_{A^o}(R\underline{\operatorname{Hom}}_A(X, D), D))\\ &\cong R\underline{\operatorname{Hom}}_{A^o}(R\underline{\operatorname{Hom}}_A(X, D), R\underline{\operatorname{Hom}}_A(D, D))\\ &\cong R\underline{\operatorname{Hom}}_{A^o}(R\underline{\operatorname{Hom}}_A(X, D), A)\end{aligned}$$

in $\mathcal{D}(A)$ by [8, Theorem 1.2]. Since $R\underline{\operatorname{Hom}}_A(X, D) \in \mathcal{D}_{fg}^b(A^o)$, it follows that $R\underline{\operatorname{Hom}}_A(D, X) \in \mathcal{D}_{fg}^+(A)$, and hence the functors $D \otimes_A^L -$ and $R\underline{\operatorname{Hom}}_A(D, -)$ define inverse equivalences between $\widehat{\mathcal{F}}_{fg}(A)$ and $\widehat{\mathcal{I}}_{fg}(A)$. $\square$

I thank the referee for comments on how to improve the above theorem.

3. Cohen-Macaulay algebras

In this section, we define a Cohen-Macaulay algebra and list some elementary properties of such an algebra.

Definition 3.1. Let A be a connected algebra, and $M \in \operatorname{GrMod} A$. We define

- $\operatorname{depth} M = \inf R\underline{\operatorname{Hom}}_A(k, M) = \inf\{i \mid \underline{\operatorname{Ext}}_A^i(k, M) \neq 0\}$,
- $\operatorname{ldim} M = \sup R\Gamma_{\mathfrak{m}}(M) = \sup\{i \mid H_{\mathfrak{m}}^i(M) \neq 0\}$, and

- $\operatorname{lcd}(A) = \sup\{\operatorname{ldim} M \mid M \in \operatorname{GrMod} A\}$.

We say that M is Cohen-Macaulay if $\operatorname{depth} M = \operatorname{ldim} M < \infty$, and maximal Cohen-Macaulay if $\operatorname{depth} M = \operatorname{ldim} M = \operatorname{ldim} A < \infty$. We say that A is Cohen-Macaulay on the left if A is Cohen-Macaulay as a graded left A-module.

If A is a connected algebra and $M \in \operatorname{GrMod} A$, then

$$\operatorname{depth} M = \inf R\Gamma_{\mathfrak{m}}(M) = \inf\{i \mid H^i_{\mathfrak{m}}(M) \neq 0\}$$

by [11, Chapter 11, Lemma 4.1]. So M is Cohen-Macaulay with $\operatorname{depth} M = m$ if and only if $R\Gamma_{\mathfrak{m}}(M) \cong H^m_{\mathfrak{m}}(M)[-m] \neq 0$ in $\mathcal{D}(A)$.

Definition 3.2. Let A be a connected algebra with $\operatorname{ldim} A = d < \infty$, and let $\mathfrak{m} = A_{\geq 1}$ be the augmentation ideal. A left canonical module ω_A is a graded A-A bimodule such that for every $M \in \operatorname{GrMod} A$,

$$R\underline{\operatorname{Hom}}_A(M, \omega_A) \cong R\Gamma_{\mathfrak{m}}(M)'[-d]$$

in $\mathcal{D}(A^o)$, that is, there are functorial isomorphisms

$$\underline{\operatorname{Ext}}^i_A(M, \omega_A) \cong H^{d-i}_{\mathfrak{m}}(M)'$$

in $\operatorname{GrMod} A^o$ for all i.

If A has a left canonical module ω_A, then $M \in \operatorname{GrMod} A$ is Cohen-Macaulay with $\operatorname{depth} M = m$ if and only if $R\underline{\operatorname{Hom}}_A(M, \omega_A) \cong \underline{\operatorname{Ext}}^{d-m}_A(M, \omega_A)[m-d] \neq 0$ in $\mathcal{D}(A^o)$. In particular, $M \in \operatorname{GrMod} A$ is maximal Cohen-Macaulay if and only if $R\underline{\operatorname{Hom}}_A(M, \omega_A) \cong \underline{\operatorname{Hom}}_A(M, \omega_A) \neq 0$ in $\mathcal{D}(A^o)$, that is, $\underline{\operatorname{Ext}}^i_A(M, \omega_A) = 0$ for all $i \neq 0$ and $\underline{\operatorname{Hom}}_A(M, \omega_A) \neq 0$.

Definition 3.3 ([12])**.** Let A be a connected algebra. We say that A is Ext-finite if the $\underline{\operatorname{Ext}}^i_A(k, k)$ are finite dimensional over k for all i.

Since $\underline{\operatorname{Ext}}^i_A(k, k)' \cong \operatorname{Tor}^i_A(k, k) \cong \underline{\operatorname{Ext}}^i_{A^o}(k, k)'$ as graded k-vector spaces, A is Ext-finite if and only if A^o is Ext-finite. In particular, if A is either left Noetherian or right Noetherian, then A is Ext-finite.

Theorem 3.4. *Let A be a connected algebra. If A has a left canonical module, then A is Cohen-Macaulay on the left. Conversely, if A is an* Ext*-finite Cohen-Macaulay algebra on the left such that* $\operatorname{lcd}(A) < \infty$*, then A has a left canonical module $\omega_A = H^d_{\mathfrak{m}}(A)'$, where* $d = \operatorname{ldim} A < \infty$.

Proof. If A has a left canonical module ω_A, then

$$H^i_{\mathfrak{m}}(A)' \cong \underline{\operatorname{Ext}}^{d-i}_A(A, \omega_A) \cong \begin{cases} 0 & \text{if } i \neq d, \\ \omega_A & \text{if } i = d, \end{cases}$$

in $\operatorname{GrMod} A^o$. So A is Cohen-Macaulay on the left with $\operatorname{depth} A = \operatorname{ldim} A = d < \infty$.

Conversely, suppose that A is an Ext-finite Cohen-Macaulay algebra on the left with $\operatorname{depth} A = \operatorname{ldim} A = d < \infty$. Since $R\Gamma_{\mathfrak{m}}(A) \cong H^d_{\mathfrak{m}}(A)[-d]$ in $\mathcal{D}(A^e)$, we have

$$\begin{aligned} R\underline{\operatorname{Hom}}_A(M, H^d_{\mathfrak{m}}(A)') &\cong \underline{\operatorname{Hom}}_A(M, R\Gamma_{\mathfrak{m}}(A)'[-d]) \\ &\cong R\underline{\operatorname{Hom}}_A(M, R\Gamma_{\mathfrak{m}}(A)')[-d] \\ &\cong R\Gamma_{\mathfrak{m}}(M)'[-d] \end{aligned}$$

in $\mathcal{D}(A^o)$ for every $M \in \operatorname{GrMod} A$ by [12, Theorem 5.1]. So A has a left canonical module $\omega_A = H^d_{\mathfrak{m}}(A)'$. $\square$

If A is an Ext-finite connected algebra, then k has a finitely generated minimal free resolution F in $\operatorname{GrMod} A^o$. Since F' is an injective resolution of k in $\operatorname{GrMod} A$ and $(F')^i \cong (F^{-i})'$ is a finite direct sum of shifts of A' that is torsion for each i, it follows that

$$H^i_{\mathfrak{m}}(k) = h^i(R\Gamma_{\mathfrak{m}}(k)) \cong h^i(\Gamma_{\mathfrak{m}}(F')) \cong h^i(F') \cong \begin{cases} k & \text{if } i = 0, \\ 0 & \text{if } i \neq 0, \end{cases}$$

that is, $R\Gamma_{\mathfrak{m}}(k) \cong k$ in $\mathcal{D}(A)$. Using this fact, we can prove the following lemma (cf. [11, Chapter 11, Lemma 5.6]):

Lemma 3.5. *Let A be a Cohen-Macaulay algebra on the left with* $\operatorname{ldim} A = d < \infty$, *and let ω_A be a left canonical module. Then*:

1. $\operatorname{lcd}(A) = d < \infty$. *In particular, $M \in \operatorname{GrMod} A$ is maximal Cohen-Macaulay if and only if* $\operatorname{depth} M = d$.
2. $\operatorname{id}_A(\omega_A) = d < \infty$.
3. *If A is* Ext-*finite, then $R\Gamma_{\mathfrak{m}}(\omega_A) \cong A'[-d]$ in $\mathcal{D}(A)$, that is,*

$$H^i_{\mathfrak{m}}(\omega_A) \cong \begin{cases} 0 & \text{if } i \neq d, \\ A' & \text{if } i = d, \end{cases}$$

   in $\operatorname{GrMod} A$. *In particular, ω_A is maximal Cohen-Macaulay as a graded left A-module.*
4. *If A is* Ext-*finite, then $R\underline{\operatorname{Hom}}_A(\omega_A, \omega_A) \cong A$ in $\mathcal{D}(A^o)$, that is,*

$$\underline{\operatorname{Ext}}^i_A(\omega_A, \omega_A) \cong \begin{cases} 0 & \text{if } i \neq 0, \\ A & \text{if } i = 0, \end{cases}$$

   in $\operatorname{GrMod} A^o$.

Theorem 3.6. *Let A be a Cohen-Macaulay algebra on the left, and let ω_A be a left canonical module. If $M \in \operatorname{GrMod} A$ with $0 \leq m = \operatorname{depth} A - \operatorname{depth} M < \infty$, then M has a finite resolution of the form*

$$0 \to H \to F^{-m-1} \to \cdots \to F^0 \to M \to 0$$

in $\operatorname{GrMod} A$, *where $F^{-i} \in \operatorname{GrMod} A$ are free, and $H \in \operatorname{GrMod} A$ is maximal Cohen-Macaulay.*

Proof. Let F be a free resolution of M and let H^{-i} be the i-th syzygy of M, that is,

$$0 \to H^{-i-1} \to F^{-i} \to H^{-i} \to 0$$

is an exact sequence in $\operatorname{GrMod} A$ for each $i \geq 0$, where $H^0 = M$. By the long exact sequence of $\underline{\operatorname{Ext}}^i_A(k, -)$,

$$\begin{aligned} \operatorname{depth} H^{-m} &= \inf R\underline{\operatorname{Hom}}_A(k, H^{-m}) \\ &= \inf R\underline{\operatorname{Hom}}_A(k, H^{-m+1}) + 1 \\ &= \cdots \\ &= \inf R\underline{\operatorname{Hom}}_A(k, H^0) + m \\ &= \operatorname{depth} M + m = d. \end{aligned}$$

So $H^{-m} \in \operatorname{GrMod} A$ is maximal Cohen-Macaulay by Lemma 3.5(1). $\square$

4. Balanced Cohen-Macaulay algebras

Definition 4.1. Let A be a connected algebra. A graded A-A bimodule ω_A is called a dualizing module if

- $\operatorname{res}_A \omega_A \in \operatorname{grmod} A, \operatorname{res}_{A^o} \omega_A \in \operatorname{grmod} A^o$,
- $\operatorname{id}_A(\omega_A) < \infty, \operatorname{id}_{A^o}(\omega_A) < \infty$, and
- the natural morphisms $A \to R\underline{\operatorname{Hom}}_A(\omega_A, \omega_A)$ and $A \to R\underline{\operatorname{Hom}}_{A^o}(\omega_A, \omega_A)$ are isomorphisms in $\mathcal{D}(A^e)$.

A dualizing module ω_A is called balanced if $R\Gamma_{\mathfrak{m}}(\omega_A) \cong R\Gamma_{\mathfrak{m}^o}(\omega_A) \cong A'[-d]$ in $\mathcal{D}(A^e)$ for some integer d.

A connected algebra A is called balanced Cohen-Macaulay if A has a balanced dualizing module.

Let A be a Noetherian connected algebra. Then a graded A-A bimodule ω_A is a dualizing module if and only if ω_A is a dualizing complex, viewed as an object in $\mathcal{D}(A^e)$. Moreover, ω_A is a balanced dualizing module if and only if $\omega_A[d] \in \mathcal{D}(A^e)$ is a balanced dualizing complex for some integer d. In particular, a balanced dualizing module ω_A is unique up to isomorphisms in $\operatorname{GrMod} A^e$ by [13].

Definition 4.2. Let A be a connected algebra and $M \in \operatorname{GrMod} A$. We say that χ holds for M if $\underline{\operatorname{Ext}}^i_A(k, M)$ are bounded for all i. We say that A satisfies χ on the left if χ holds for all $M \in \operatorname{grmod} A$.

We say that A is AS Gorenstein if A satisfies χ on both sides and $\operatorname{id}_A(A) = \operatorname{id}_{A^o}(A) < \infty$. We say that A is AS regular if A satisfies χ on both sides and $\operatorname{gldim} A < \infty$.

Clearly, every AS regular algebra is AS Gorenstein. By [5, Theorem 1.2], if A is a Noetherian AS Gorenstein algebra, then A has a balanced dualizing complex $A_\alpha(-l)[d]$ for some graded algebra automorphism α of A, some integer l, and $d = \operatorname{id}_A(A) = \operatorname{id}_{A^o}(A)$. So $A_\alpha(-l)$ is a balanced dualizing module and A is balanced Cohen-Macaulay. In fact, let A be a Noetherian Cohen-Macaulay algebra on the left. Then A is balanced Cohen-Macaulay if and only if A is a graded quotient algebra of a Noetherian AS Gorenstein algebra, by [7, Theorem 1.6].

The following characterization of a balanced Cohen-Macaulay algebra is immediate from [12, Theorem 6.3].

Theorem 4.3. *Let A be a Noetherian connected algebra. Then A is balanced Cohen-Macaulay if and only if A is Cohen-Macaulay satisfying χ on both sides. If A is a Noetherian balanced Cohen-Macaulay algebra, then a balanced dualizing module is given by $\omega_A \cong H^d_{\mathfrak{m}}(A)' \cong H^d_{\mathfrak{m}^o}(A)'$ in $\operatorname{GrMod} A^e$, where $d = \operatorname{ldim}_A A = \operatorname{ldim}_{A^o} A < \infty$. In particular, ω_A is a left and right canonical module.*

Definition 4.4. Let A be a balanced Cohen-Macaulay algebra and ω_A a balanced dualizing module. If $M \in \operatorname{GrMod} A$, then we define $M^\dagger = \underline{\operatorname{Hom}}_A(M, \omega_A) \in \operatorname{GrMod} A^o$. Similarly, if $N \in \operatorname{GrMod} A^o$, then we define $N^\dagger = \underline{\operatorname{Hom}}_{A^o}(N, \omega_A) \in \operatorname{GrMod} A$. We say that $M \in \operatorname{GrMod} A$ is totally ω_A-reflexive if $\underline{\operatorname{Ext}}^i_A(M, \omega_A) = \underline{\operatorname{Ext}}^i_A(M^\dagger, \omega_A) = 0$ for all $i \neq 0$ and $M^{\dagger\dagger} \cong M$ in $\operatorname{GrMod} A$.

Let $\mathcal{A}$ be an abelian category and $\mathcal{B} \subset \mathcal{A}$ a full subcategory. A $\mathcal{B}$-resolution of an object $M \in \mathcal{A}$ is an exact sequence

$$\cdots \to B^{-i} \to \cdots \to B^{-1} \to B^0 \to M \to 0$$

in $\mathcal{A}$, where $B^{-i} \in \mathcal{B}$ for all $i \geq 0$.

Definition 4.5. We define $\mathcal{H}$ to be the full subcategory of $\operatorname{GrMod} A$ consisting of totally ω_A-reflexive modules, and $\mathcal{H}_{fg}$ to be the full subcategory of $\operatorname{grmod} A$ consisting of totally ω_A-reflexive modules.

For $M \in \operatorname{GrMod} A$, we define

$$\operatorname{Hdim} M = \inf_H(-\inf\{i \mid H^i \neq 0\}),$$

where the infimum is taken over all $\mathcal{H}$-resolutions H of X, and

$$\operatorname{hdim} M = \inf_H(-\inf\{i \mid H^i \neq 0\}),$$

where the infimum is taken over all $\mathcal{H}_{fg}$-resolutions H of X.

Lemma 4.6. *Let A be a Noetherian balanced Cohen-Macaulay algebra and ω_A a balanced dualizing module. For $M \in \operatorname{grmod} A$, the following are equivalent*:

1. $M \in \mathcal{H}_{fg}$;
2. $\underline{\operatorname{Ext}}^i_A(M, \omega_A) = 0$ *for all* $i \neq 0$;
3. M *is either a maximal Cohen-Macaulay module or the zero module.*

Proof. (1) $\Rightarrow$ (2): By definition.

(2) $\Rightarrow$ (3): Suppose that $\underline{\operatorname{Ext}}^i_A(M, \omega_A) = 0$ for all $i \neq 0$. If $\underline{\operatorname{Hom}}_A(M, \omega_A) \neq 0$, then $M \in \operatorname{grmod} A$ is maximal Cohen-Macaulay. Since $\omega_A \in \mathcal{D}^b(A^e)$ is a dualizing complex, if $\underline{\operatorname{Hom}}_A(M, \omega_A) = 0$, then $M \cong R\underline{\operatorname{Hom}}_{A^o}(R\underline{\operatorname{Hom}}_A(M, \omega_A), \omega_A) = 0$.

(3) $\Rightarrow$ (1): If $M \in \operatorname{grmod} A$ is maximal Cohen-Macaulay, then $\underline{\operatorname{Ext}}^i_A(M, \omega_A) = 0$ for all $i \neq 0$, that is, $R\underline{\operatorname{Hom}}_A(M, \omega_A) \cong \underline{\operatorname{Hom}}_A(M, \omega_A) = M^\dagger$ in $\mathcal{D}(A^o)$. Since $\omega_A \in \mathcal{D}(A^e)$ is a dualizing complex,

$$R\underline{\operatorname{Hom}}_{A^o}(M^\dagger, \omega_A) \cong R\underline{\operatorname{Hom}}_{A^o}(R\underline{\operatorname{Hom}}_A(M, \omega_A), \omega_A) \cong M$$

in $\mathcal{D}(A)$. It follows that $\underline{\operatorname{Ext}}^i_{A^o}(M^\dagger, \omega_A) = 0$ for all $i \neq 0$, and

$$M^{\dagger\dagger} \cong \underline{\operatorname{Hom}}_{A^o}(M^\dagger, \omega_A) \cong M$$

in $\operatorname{GrMod} A$. Clearly, $0 \in \mathcal{H}_{fg}$. □

In particular, if A is a Noetherian balanced Cohen-Macaulay algebra and ω_A is a balanced dualizing module, then $\operatorname{hdim} A = \operatorname{hdim} \omega_A = 0$.

Let A be a left Noetherian connected algebra and $M \in \operatorname{grmod} A$. If A satisfies χ on the left and $\operatorname{pd}(M) < \infty$, then Jörgensen proved the Auslander-Buchsbaum formula

$$\operatorname{pd}(M) + \operatorname{depth} M = \operatorname{depth} A$$

in [6, Theorem 3.2] (he actually proved the formula for $X \in \mathcal{D}^b_{fg}(A)$, using the obvious definition of $\operatorname{depth} X$). The following theorem can be regarded as a version of the Auslander-Buchsbaum formula for hdim.

Theorem 4.7. *Let A be a Noetherian balanced Cohen-Macaulay algebra and ω_A a balanced dualizing module. If $0 \neq M \in \operatorname{grmod} A$, then*

$$\operatorname{hdim} M = \sup R\underline{\operatorname{Hom}}_A(M, \omega_A) = \operatorname{depth} A - \operatorname{depth} M \leq \operatorname{depth} A < \infty.$$

In particular, if $\operatorname{pd}(M) < \infty$*, then* $\operatorname{hdim} M = \operatorname{pd}(M)$.

Proof. Let $d = \operatorname{depth} A = \operatorname{ldim} A < \infty$. For $M \in \operatorname{GrMod} A$,

$$\sup R\underline{\operatorname{Hom}}_A(M, \omega_A) = \sup R\Gamma_{\mathfrak{m}}(M)'[-d] = d - \inf R\Gamma_{\mathfrak{m}}(M) = d - \operatorname{depth} M \leq d.$$

We will now prove that $\operatorname{hdim} M = \sup R\underline{\operatorname{Hom}}_A(M, \omega_A)$ using induction on $m = \sup R\underline{\operatorname{Hom}}_A(M, \omega_A)$. If $m = 0$, then M is maximal Cohen-Macaulay; so $\operatorname{hdim} M =$

0 by Lemma 4.6. Suppose that $m \geq 1$. Let $F \to M \to 0$ be a finitely generated minimal free resolution, and $0 \to N \to F^0 \to M \to 0$ be an exact sequence in $\operatorname{GrMod} A$. Since $\underline{\operatorname{Ext}}_A^i(F^0, \omega_A) = 0$ for all $i \geq 1$, from the long exact sequence of $\underline{\operatorname{Ext}}_A^i(-, \omega_A)$, we have an epimorphism $\underline{\operatorname{Hom}}_A(N, \omega_A) \to \underline{\operatorname{Ext}}_A^1(M, \omega_A)$ and isomorphisms $\underline{\operatorname{Ext}}_A^i(N, \omega_A) \cong \underline{\operatorname{Ext}}^{i+1}(M, \omega_A)$ for all $i \geq 1$. So $\sup R\underline{\operatorname{Hom}}_A(N, \omega_A) = m-1$. By induction, $\operatorname{hdim} M \leq \operatorname{hdim} N + 1 = \sup R\underline{\operatorname{Hom}}_A(N, \omega_A) + 1 = m$.

Let $H \to M \to 0$ be an $\mathcal{H}_{fg}$-resolution of minimal length, and $0 \to N \to H^0 \to M \to 0$ an exact sequence in $\operatorname{GrMod} A$. By a similar argument, we can show that $\sup R\underline{\operatorname{Hom}}_A(N, \omega_A) = m - 1$, and

$$\operatorname{hdim} M = \operatorname{hdim} N + 1 = \sup R\underline{\operatorname{Hom}}_A(N, \omega_A) + 1 = m.$$

Since A satisfies χ on the left by Theorem 4.3, if $\operatorname{pd}(M) < \infty$, then

$$\operatorname{pd}(M) = \operatorname{depth} A - \operatorname{depth} M = \operatorname{hdim} A$$

by the classical Auslander-Buchsbaum formula. $\square$

Corollary 4.8. *Let A be a Noetherian balanced Cohen-Macaulay algebra. Then every finitely generated maximal Cohen-Macaulay module having finite projective dimension is free.*

Proof. If $M \in \operatorname{grmod} A$ is maximal Cohen-Macaulay having finite projective dimension, then $\operatorname{pd}(M) = \operatorname{hdim} M = 0$ by Lemma 4.6 and Theorem 4.7; so M is free. $\square$

5. Maximal Cohen-Macaulay approximations

In this section, we will prove the theorem proposed in the abstract, using maximal Cohen-Macaulay approximations and Foxby equivalence.

Throughout this section, we assume that A is a Noetherian balanced Cohen-Macaulay algebra with $\operatorname{ldim}_A A = \operatorname{ldim}_{A^o} A = d < \infty$, and that ω_A is a balanced dualizing module.

Let $\mathcal{A}$ be an abelian category and $\mathcal{B} \subset \mathcal{A}$ a full subcategory. We define $\widehat{\mathcal{B}}$ to be the full subcategory of $\mathcal{A}$ consisting of objects $M \in \mathcal{A}$ having $\mathcal{B}$-resolutions of finite length.

A full additive subcategory $\mathcal{B} \subset \mathcal{A}$ is called additively closed if $\mathcal{B}$ is closed under finite direct sums in $\mathcal{A}$ and closed under direct summands in $\mathcal{A}$. A full additive subcategory $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$ is called a cogenerator for $\mathcal{B}$ if for every object $M \in \mathcal{B}$, there is an exact sequence

$$0 \to M \to C \to B \to 0$$

in $\mathcal{A}$, where $C \in \mathcal{C}$ and $B \in \mathcal{B}$.

Theorem 5.1 ([2, Theorem 1.1]). *Let $\mathcal{A}$ be an abelian category, $\mathcal{B} \subset \mathcal{A}$ an additively closed subcategory that is closed under extensions in $\mathcal{A}$, and $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$ an additively closed subcategory that is a cogenerator for $\mathcal{B}$. For every $M \in \widehat{\mathcal{B}}$, there are exact sequences*

$$0 \to \widehat{C}_M \to B_M \to M \to 0$$

and

$$0 \to M \to \widehat{C}^M \to B^M \to 0$$

in $\mathcal{A}$, where $B_M, B^M \in \mathcal{B}$ and $\widehat{C}_M, \widehat{C}^M \in \widehat{\mathcal{C}}$.

Using Theorem 5.1, we will prove maximal Cohen-Macaulay approximations for Noetherian balanced Cohen-Macaulay algebras (cf. [2, Example 3]).

Definition 5.2. We define $\mathcal{J}$ to be the full subcategory of $\mathcal{H}$ consisting of all modules $M \in \mathcal{H}$ having finite injective dimension.

Proposition 5.3. *For every $M \in \operatorname{grmod} A$, there are exact sequences*

$$0 \to \widehat{J}_M \to H_M \to M \to 0$$

and

$$0 \to M \to \widehat{J}^M \to H^M \to 0$$

in $\operatorname{GrMod} A$*, where $H_M, H^M \in \mathcal{H}_{fg}$ and $\widehat{J}_M, \widehat{J}^M \in \widehat{\mathcal{J}}_{fg}$.*

Proof. We will apply Theorem 5.1 to $\mathcal{A} = \operatorname{grmod} A, \mathcal{B} = \mathcal{H}_{fg}$, and $\mathcal{C} = \mathcal{J}_{fg}$. By Theorem 4.7, $\widehat{\mathcal{H}}_{fg} = \operatorname{grmod} A$.

Clearly, $\mathcal{H}_{fg}$ and $\mathcal{J}_{fg}$ are closed under finite direct sums in $\operatorname{grmod} A$. Let $M \in \operatorname{grmod} A$, and let $N \in \operatorname{grmod} A$ be a direct summand of M. If $M \in \mathcal{H}_{fg}$, then $\underline{\operatorname{Ext}}^i_A(N, \omega_A) \subset \underline{\operatorname{Ext}}^i_A(M, \omega_A) = 0$ for all $i \neq 0$; so $N \in \mathcal{H}_{fg}$ by Lemma 4.6. If $M \in \mathcal{J}_{fg}$, then

$$\begin{aligned} \operatorname{id}(N) &= \sup\{\sup R\underline{\operatorname{Hom}}_A(L, N) \mid L \in \operatorname{grmod} A\} \\ &\leq \sup\{\sup R\underline{\operatorname{Hom}}_A(L, M) \mid L \in \operatorname{grmod} A\} \\ &= \operatorname{id}(M) < \infty \end{aligned}$$

by Lemma 1.2(1); so $N \in \mathcal{J}_{fg}$. It follows that $\mathcal{H}_{fg}$ and $\mathcal{J}_{fg}$ are additively closed in $\operatorname{grmod} A$. Let

$$0 \to L \to M \to N \to 0$$

be an exact sequence in $\operatorname{grmod} A$. If $L, N \in \mathcal{H}_{fg}$, then $\underline{\operatorname{Ext}}^i_A(M, \omega_A) = 0$ for all $i \neq 0$ from the long exact sequence of $\underline{\operatorname{Ext}}^i_A(-, \omega_A)$, and so $M \in \mathcal{H}_{fg}$ by Lemma 4.6; hence $\mathcal{H}_{fg}$ is closed under extensions.

Finally, we will prove that $\mathcal{J}_{fg}$ is a cogenerator for $\mathcal{H}_{fg}$. Let $0 \neq M \in \mathcal{H}_{fg}$. Since $M^\dagger = \underline{\operatorname{Hom}}_A(M, \omega_A) \in \operatorname{grmod} A^o$, there is an exact sequence

$$0 \to N \to F \to M^\dagger \to 0$$

in $\operatorname{grmod} A^o$, where $F \in \operatorname{grmod} A^o$ is free. Since $F, M^\dagger \in \operatorname{grmod} A^o$ are maximal Cohen-Macaulay, it follows that $M^{\dagger\dagger} \cong M$ in $\operatorname{grmod} A$ by Lemma 4.6. So there is an exact sequence

$$0 \to M^{\dagger\dagger} \cong M \to F^\dagger \to N^\dagger \to \underline{\operatorname{Ext}}^1_{A^o}(M^\dagger, \omega_A) = 0$$

in $\operatorname{grmod} A$. Since $F^\dagger = \underline{\operatorname{Hom}}_{A^o}(F, \omega_A)$ is a finite direct sum of shifts of ω_A in $\operatorname{grmod} A$, it follows that $F^\dagger \in \mathcal{J}_{fg}$. Since $M, F^\dagger \in \mathcal{H}_{fg}$, it follows that $\underline{\operatorname{Ext}}^i_A(N^\dagger, \omega_A) = 0$ for all $i \geq 2$ from the long exact sequence of $\underline{\operatorname{Ext}}^i_A(-, \omega_A)$. Since $F \in \operatorname{grmod} A^o$ is maximal Cohen-Macaulay, we have $F^{\dagger\dagger} \cong F$ in $\operatorname{grmod} A^o$ by Lemma 4.6. So there is an exact sequence

$$0 \to \underline{\operatorname{Hom}}_A(N^\dagger, \omega_A) \to F^{\dagger\dagger} \cong F \to M^\dagger \to \underline{\operatorname{Ext}}^1_A(N^\dagger, \omega_A) \to \underline{\operatorname{Ext}}^1_A(F^\dagger, \omega_A) = 0$$

in $\operatorname{grmod} A^o$. It follows that $\underline{\operatorname{Hom}}_A(N^\dagger, \omega_A) \cong N$ in $\operatorname{grmod} A^o$ and $\underline{\operatorname{Ext}}^1_A(N^\dagger, \omega_A) = 0$; so $N^\dagger \in \mathcal{H}_{fg}$ by Lemma 4.6. This shows that $\mathcal{J}_{fg}$ is a cogenerator for $\mathcal{H}_{fg}$. $\square$

Now we will return to Foxby equivalence discussed in section 2, and apply it to Noetherian balanced Cohen-Macaulay algebras.

Lemma 5.4. *Let $0 \neq X \in \widehat{\mathcal{I}}(A)$. If $R\underline{\mathrm{Hom}}_A(\omega_A, X) \in \widehat{\mathcal{F}}(A)$ has a minimal free resolution, then*

$$\mathrm{pd}(R\underline{\mathrm{Hom}}(\omega_A, X)) = \mathrm{fd}(R\underline{\mathrm{Hom}}(\omega_A, X)) = \operatorname{depth} A - \operatorname{depth} X.$$

Proof. Since $X \in \widehat{\mathcal{I}}(A)$,

$$\begin{aligned} k \otimes^L_A R\underline{\mathrm{Hom}}_A(\omega_A, X) &\cong R\underline{\mathrm{Hom}}_A(R\underline{\mathrm{Hom}}_{A^o}(k, \omega_A), X) \\ &\cong R\underline{\mathrm{Hom}}_A(k[-d], X) \\ &\cong R\underline{\mathrm{Hom}}_A(k, X)[d] \end{aligned}$$

in $\mathcal{D}(k)$ by Proposition 2.3(1). If $R\underline{\mathrm{Hom}}_A(\omega_A, X) \in \widehat{\mathcal{F}}(A)$ has a minimal free resolution, then, by Lemma 1.2(3),

$$\begin{aligned} \mathrm{pd}(R\underline{\mathrm{Hom}}_A(\omega_A, X)) &= \mathrm{fd}(R\underline{\mathrm{Hom}}_A(\omega_A, X)) \\ &= -\inf(k \otimes^L_A R\underline{\mathrm{Hom}}_A(\omega_A, X)) \\ &= -\inf(R\underline{\mathrm{Hom}}_A(k, X)[d]) \\ &= -\inf(R\underline{\mathrm{Hom}}_A(k, X)) + d \\ &= \operatorname{depth} A - \operatorname{depth} X. \end{aligned}$$

□

Proposition 5.5. *If $M \in \operatorname{grmod} A$ has finite injective dimension, then*

$$R\underline{\mathrm{Hom}}_A(\omega_A, M) \cong \underline{\mathrm{Hom}}_A(\omega_A, M)$$

in $\mathcal{D}_{fg}(A)$; that is, $\underline{\mathrm{Ext}}^i_A(\omega_A, M) = 0$ for all $i \neq 0$, and $\underline{\mathrm{Hom}}_A(\omega_A, M) \in \operatorname{grmod} A$.
Moreover, if $M \neq 0$, then

$$\mathrm{pd}(\underline{\mathrm{Hom}}_A(\omega_A, M)) = \operatorname{depth} A - \operatorname{depth} M = \operatorname{hdim} M.$$

Proof. Suppose that $M \in \operatorname{grmod} A$ has finite injective dimension. Since ω_A and $\underline{\mathrm{Ext}}^i_A(\omega_A, M) \in \operatorname{GrMod} A$ are left bounded for all i by [1, Proposition 3.1(1)], it follows that

$$\begin{aligned} \sup(R\underline{\mathrm{Hom}}_A(\omega_A, M)) &= \sup \omega_A + \sup(R\underline{\mathrm{Hom}}_A(\omega_A, M)) \\ &= \sup(\omega_A \otimes^L_A R\underline{\mathrm{Hom}}_A(\omega_A, M)) \\ &= \sup M = 0 \end{aligned}$$

by Lemma 1.3(2). So $R\underline{\mathrm{Hom}}_A(\omega_A, M) \cong \underline{\mathrm{Hom}}_A(\omega_A, M)$ in $\mathcal{D}(A)$. Since $M \in \widehat{\mathcal{I}}_{fg}(A)$, by Theorem 2.5, $R\underline{\mathrm{Hom}}_A(\omega_A, M) \cong \underline{\mathrm{Hom}}_A(\omega_A, M) \in \widehat{\mathcal{F}}_{fg}(A)$ has a minimal free resolution. By Lemma 5.4 and Theorem 4.7,

$$\mathrm{pd}(\underline{\mathrm{Hom}}_A(\omega_A, M)) = \operatorname{depth} A - \operatorname{depth} M = \operatorname{hdim} M.$$

□

Corollary 5.6. *Every finitely generated maximal Cohen-Macaulay module having finite injective dimension is a finite direct sum of shifts of ω_A.*

Proof. Let $M \in \operatorname{grmod} A$ be maximal Cohen-Macaulay having finite injective dimension. By Proposition 5.5, $R\underline{\mathrm{Hom}}_A(\omega_A, M) \cong \underline{\mathrm{Hom}}_A(\omega_A, M)$ in $\mathcal{D}(A)$. Since

$$\omega_A \otimes^L_A \underline{\mathrm{Hom}}_A(\omega_A, M) \cong \omega_A \otimes^L_A R\underline{\mathrm{Hom}}_A(\omega_A, M) \cong M$$

in $\mathcal{D}(A)$ by Theorem 2.5, $\omega_A \otimes_A \underline{\mathrm{Hom}}_A(\omega_A, M) \cong M$ in $\operatorname{GrMod} A$. By Proposition 5.5, $\mathrm{pd}(\underline{\mathrm{Hom}}_A(\omega_A, M)) = \operatorname{depth} A - \operatorname{depth} M = 0$. So $\underline{\mathrm{Hom}}_A(\omega_A, M) \in \operatorname{grmod} A$ is finitely generated free; hence $M \cong \omega_A \otimes_A \underline{\mathrm{Hom}}_A(\omega_A, M) \in \operatorname{grmod} A$ is a finite direct sum of shifts of ω_A. □

Remark 5.7. It would be much nicer if we could prove the dual statement of Proposition 5.5 as in the commutative case [4, Corollary 3.6], namely, A has the following property (P): "if $M \in \operatorname{grmod} A$ has finite flat dimension, then $\omega_A \otimes_A^L M \cong \omega_A \otimes_A M$ in $\mathcal{D}_{fg}(A)$, so that $\omega_A \otimes_A M \in \operatorname{grmod} A$ has finite injective dimension." This property (P) implies an important property of ω_A discussed in [7] and [9], namely, "if $x \in A$ is regular on A, then x is regular on ω_A from both sides." In fact, suppose that A has the property (P). Let $x \in A$ be a homogeneous regular element on A of degree l, and let

$$0 \to A(-l) \xrightarrow{\cdot x} A \to M \to 0$$

be an exact sequence in $\operatorname{GrMod} A$. Since $M \in \operatorname{grmod} A$ has finite flat dimension, it follows that $\operatorname{Tor}_1^A(\omega_A, M) = 0$ by the property (P). So

$$0 \to \omega_A(-l) \xrightarrow{\cdot x} \omega_A \to \omega_A \otimes_A M \to 0$$

is an exact sequence in $\operatorname{GrMod} A$. It follows that x is regular on ω_A from the right. By symmetry, x is regular on ω_A from the left.

Let $M \in \operatorname{grmod} A$. By Theorem 3.6, M has a finite resolution of the form

$$0 \to H \to \bigoplus_{j=1}^{r_{m-1}} A(-l_{m-1\,j}) \to \cdots \to \bigoplus_{j=1}^{r_0} A(-l_{0j}) \to M \to 0$$

in $\operatorname{GrMod} A$, where $H \in \mathcal{H}_{fg}$ and $m = \operatorname{depth} A - \operatorname{depth} M = \operatorname{hdim} M < \infty$. It is well known that M has finite projective dimension if and only if M has a finite resolution of the form

$$0 \to \bigoplus_{j=1}^{r_m} A(-l_{mj}) \to \bigoplus_{j=1}^{r_{m-1}} A(-l_{m-1\ j}) \to \cdots \to \bigoplus_{j=1}^{r_0} A(-l_{0j}) \to M \to 0$$

in $\operatorname{GrMod} A$, where $m = \operatorname{pd}(M) = \operatorname{hdim} M < \infty$. The following theorem can be compared with these facts.

Theorem 5.8. *Let* $M \in \operatorname{grmod} A$. *Then*:

1. M *has a finite resolution of the form*

$$0 \to \bigoplus_{j=1}^{r_m} \omega_A(-l_{mj}) \to \cdots \to \bigoplus_{j=1}^{r_1} \omega_A(-l_{1j}) \to H \to M \to 0$$

in $\operatorname{GrMod} A$, *where* $H \in \mathcal{H}_{fg}$;

2. M *has finite injective dimension if and only if* M *has a finite resolution of the form*

$$(*) \qquad 0 \to \bigoplus_{j=1}^{r_m} \omega_A(-l_{mj}) \to \cdots \to \bigoplus_{j=1}^{r_1} \omega_A(-l_{1j}) \to \bigoplus_{j=1}^{r_0} \omega_A(-l_{0j}) \to M \to 0$$

in $\operatorname{GrMod} A$, *where* $m = \operatorname{hdim} M < \infty$.

Proof. (1): By Proposition 5.3, there is an exact sequence

$$0 \to \widehat{J} \to H \to M \to 0$$

in $\operatorname{GrMod} A$, where $H \in \mathcal{H}_{fg}$ and $\widehat{J} \in \widehat{\mathcal{J}}_{fg}$. Since every $J \in \mathcal{J}_{fg}$ is a finite direct sum of shifts of ω_A or the zero module by Corollary 5.6, $\widehat{J}$ has a finite resolution of the form

$$0 \to \bigoplus_{j=1}^{r_m} \omega_A(-l_{mj}) \to \cdots \to \bigoplus_{j=1}^{r_1} \omega_A(-l_{1j}) \to \widehat{J} \to 0,$$

in $\operatorname{GrMod} A$, hence the result.

(2): Suppose that $M \in \operatorname{grmod} A$ has finite injective dimension. By Proposition 5.5, $\underline{\operatorname{Hom}}_A(\omega_A, M) \in \operatorname{grmod} A$, and $\operatorname{pd}(\underline{\operatorname{Hom}}_A(\omega_A, M)) = \operatorname{hdim} M = m < \infty$. So $R\underline{\operatorname{Hom}}_A(\omega_A, M) \cong \underline{\operatorname{Hom}}_A(\omega_A, M) \in \mathcal{D}^b_{fg}(A)$ has a finitely generated minimal free resolution F of length m. By Theorem 2.5,

$$\omega_A \otimes_A F \cong \omega_A \otimes^L_A R\underline{\operatorname{Hom}}_A(\omega_A, M) \cong M$$

in $\mathcal{D}(A)$. So $\omega_A \otimes_A F$ is a resolution of the form $(*)$.

Conversely, if M has a resolution of the form $(*)$, then clearly $\operatorname{id}(M) < \infty$. □

As corollaries, we have the following characterizations of AS Gorenstein algebras and AS regular algebras.

Corollary 5.9. *Let A be a Noetherian balanced Cohen-Macaulay algebra. Then the following are equivalent*:

1. *A is AS Gorenstein*;
2. $\operatorname{id}_A(A) < \infty$;
3. $\operatorname{pd}_A(\omega_A) < \infty$;
4. *for every $M \in \operatorname{grmod} A$, $\operatorname{id}(M) < \infty$ if and only if $\operatorname{pd}(M) < \infty$.*

Proof. (4) $\Rightarrow$ (3): Suppose that A has the property (4). Since $\operatorname{id}_A(\omega_A) < \infty$, it follows that $\operatorname{pd}_A(\omega_A) < \infty$.

(3) $\Rightarrow$ (2): If $\operatorname{pd}_A(\omega_A) < \infty$, then ω_A is free by Corollary 4.8. Since $\operatorname{id}_A(\omega_A) < \infty$, it follows that $\operatorname{id}_A(A) < \infty$.

(2) $\Rightarrow$ (1): This follows from [8, Corollary 4.6].

(1) $\Rightarrow$ (4): If A is AS Gorenstein, then $\omega_A \cong A(-l)$ in $\operatorname{GrMod} A$ for some integer l by [5, Theorem 1.2]. The result follows from Theorem 5.8. □

Remark 5.10. The direction (1) $\Rightarrow$ (4) of the above corollary was proved by Zhang, using a spectral sequence [11, Chapter 1, Proposition 6.7].

Corollary 5.11. *Let A be a Noetherian balanced Cohen-Macaulay algebra. Then the following are equivalent*:

1. *A is AS regular*;
2. *every $H \in \mathcal{H}_{fg}$ has finite projective dimension*;
3. *every nonzero $H \in \mathcal{H}_{fg}$ is free*;
4. *every $H \in \mathcal{H}_{fg}$ has finite injective dimension*;
5. *every nonzero $H \in \mathcal{H}_{fg}$ is a finite direct sum of shifts of ω_A.*

Proof. (2) $\Leftrightarrow$ (3) by Corollary 4.8, and (4) $\Leftrightarrow$ (5) by Corollary 5.6.

If A is AS regular, then clearly every $M \in \operatorname{GrMod} A$ has finite projective dimension and finite injective dimension; so (1) $\Rightarrow$ (2), (4).

If every $H \in \mathcal{H}_{fg}$ has finite projective dimension, then every $M \in \operatorname{grmod} A$ has finite projective dimension by Theorem 5.8; so (2) $\Rightarrow$ (1). Similarly, if every $H \in \mathcal{H}_{fg}$ has finite injective dimension, then every $M \in \operatorname{grmod} A$ has finite injective dimension by Theorem 5.8; so (4) $\Rightarrow$ (1). □

6. An application to the intersection multiplicity

We will end the paper by an application of Theorem 5.8 to the intersection multiplicity discussed in [10].

Definition 6.1. For $V \in \operatorname{GrMod} k$ locally finite, we define the Hilbert series of V by

$$H_V(t) = \sum_{i=-\infty}^{\infty} \dim_k V_i t^i \in \mathbb{Z}[[t, t^{-1}]].$$

If $H_V(t)$ is a rational function over $\mathbb{C}$, then we define $\operatorname{GKdim} V$ to be the order of the pole of $H_V(t)$ at $t = 1$, and we define the multiplicity of V by

$$e(V) = \lim_{t \to 1} (1-t)^{\operatorname{GKdim} V} H_V(t).$$

For $X \in \mathcal{D}^b_{lf}(k)$, we define the Hilbert series of X by

$$H_X(t) = \sum_{i=-\infty}^{\infty} (-1)^i H_{h^i(X)}(t) \in \mathbb{Z}[[t, t^{-1}]].$$

Let A be a Cohen-Macaulay algebra on the left and ω_A a left canonical module. If $M \in \operatorname{grmod} A$ has a finite ω_A-resolution of the form

$$(*) \qquad 0 \to \bigoplus_{j=1}^{r_m} \omega_A(-l_{mj}) \to \cdots \to \bigoplus_{j=1}^{r_0} \omega_A(-l_{0j}) \to M \to 0$$

in $\operatorname{GrMod} A$, then the ω_A-characteristic polynomial of M is defined by

$$r_M(t) := \sum_{i=0}^{m} (-1)^i \sum_{j=1}^{r_i} t^{l_{ij}} \in \mathbb{Z}[t, t^{-1}].$$

Note that if M has a finite ω_A-resolution of the form $(*)$, then

$$H_M(t) = \sum_{i=0}^{m} (-1)^i \sum_{j=1}^{r_i} H_{\omega_A(-l_{ij})}(t) = \sum_{i=0}^{m} (-1)^i \sum_{j=1}^{r_i} t^{l_{ij}} H_{\omega_A}(t) = r_M(t) H_{\omega_A}(t).$$

Definition 6.2 ([9], [10])**.** Let A be a connected algebra, and let $M \in \operatorname{GrMod} A$ be locally finite. We say that M is rational if

- $R\Gamma_{\mathfrak{m}}(M) \in \mathcal{D}^b_{lf}(A)$;
- $H_M(t)$ and $H_{R\Gamma_{\mathfrak{m}}}(t)$ are both rational functions over $\mathbb{C}$;
- $H_M(t) = H_{R\Gamma_{\mathfrak{m}}(M)}(t)$ as rational functions over $\mathbb{C}$.

We say that A is universally rational, if every $M \in \operatorname{grmod} A$ is rational.

Lemma 6.3. *Let A be a universally rational, Noetherian balanced Cohen-Macaulay algebra, and $M, N \in \operatorname{grmod} A$. If N has finite injective dimension, then*

$$H_{R\underline{\operatorname{Hom}}_A(M,N)}(t) = H_M(t^{-1}) H_N(t) / H_A(t^{-1}).$$

Proof. Since $N \in \operatorname{grmod} A$ has finite injective dimension, N has a finite ω_A-resolution of the form

$$0 \to \bigoplus_{j=1}^{r_n} \omega_A(-l_{nj}) \to \cdots \to \bigoplus_{j=1}^{r_0} \omega_A(-l_{0j}) \to N \to 0,$$

by Theorem 5.8.

Since A is universally rational, Noetherian balanced Cohen-Macaulay, we have

$$H_{\omega_A}(t) = H_{R\Gamma_{\mathfrak{m}}(A)'[-d]}(t) = (-1)^d H_{R\Gamma_{\mathfrak{m}}(A)}(t^{-1}) = (-1)^d H_A(t^{-1}).$$

So

$$\begin{aligned} H_{R\underline{\mathrm{Hom}}_A(M,N)}(t) &= \sum_{i=0}^{n}(-1)^i \sum_{j=1}^{r_i} H_{R\underline{\mathrm{Hom}}_A(M,\omega_A(-l_{ij}))}(t) \\ &= \sum_{i=0}^{n}(-1)^i \sum_{j=1}^{r_i} t^{l_{ij}} H_{R\underline{\mathrm{Hom}}_A(M,\omega_A)}(t) \\ &= r_N(t) H_{R\Gamma_{\mathfrak{m}}(M)'[-d]}(t) \\ &= (-1)^d H_{R\Gamma_{\mathfrak{m}}(M)}(t^{-1}) r_N(t) \\ &= (-1)^d H_M(t^{-1}) H_N(t)/H_{\omega_A}(t) \\ &= H_M(t^{-1}) H_N(t)/H_A(t^{-1}). \end{aligned}$$

□

Let A be a connected algebra. For $M, N \in \operatorname{grmod} A$, we define the intersection multiplicity of M and N by

$$M \cdot N = (-1)^{\operatorname{GKdim} N} \sum_{i=0}^{\infty} (-1)^i \dim_k \underline{\mathrm{Ext}}_A^i(M,N).$$

It is well defined if $\underline{\mathrm{Ext}}_A^i(M,N) = 0$ for all $i \gg 0$, and $\dim_k \underline{\mathrm{Ext}}_A^i(M,N) < \infty$ for all $i \geq 0$. We can then prove a version of Serre's multiplicity conjectures as in [10].

Theorem 6.4. *Let A be a universally rational, Noetherian balanced Cohen-Macaulay algebra, and $M, N \in \operatorname{grmod} A$. Suppose that N has finite injective dimension, and $M \cdot N$ is well defined. Then*

1. (Dimension) $\operatorname{GKdim} M + \operatorname{GKdim} N \leq \operatorname{GKdim} A$.
2. (Vanishing) *If* $\operatorname{GKdim} M + \operatorname{GKdim} N < \operatorname{GKdim} A$, *then* $M \cdot N = 0$.
3. (Positivity) *If* $\operatorname{GKdim} M + \operatorname{GKdim} N = \operatorname{GKdim} A$, *then*

$$M \cdot N = e(M)e(N)/e(A) > 0.$$

Proof. Using Lemma 6.3, exactly the same proof as in [10, Theorem 3.9] goes through. □

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