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AN EXTENSION THEOREM FOR SEPARATELY HOLOMORPHIC FUNCTIONS WITH PLURIPOLAR SINGULARITIES

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Abstract. Let $D_j \subset \mathbb{C}^{n_j}$ be a pseudoconvex domain and let $A_j \subset D_j$ be a locally pluriregular set, $j = 1, \dots, N$. Put

$$X := \bigcup_{j=1}^{N} A_1 \times \dots \times A_{j-1} \times D_j \times A_{j+1} \times \dots \times A_N \subset \mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_N} = \mathbb{C}^n.$$

Let $U \subset \mathbb{C}^n$ be an open neighborhood of X and let $M \subset U$ be a relatively closed subset of U. For $j \in \{1, \dots, N\}$ let Σ_j be the set of all $(z', z'') \in (A_1 \times \dots \times A_{j-1}) \times (A_{j+1} \times \dots \times A_N)$ for which the fiber $M_{(z',\cdot,z'')} := \{z_j \in \mathbb{C}^{n_j} : (z', z_j, z'') \in M\}$ is not pluripolar. Assume that $\Sigma_1, \dots, \Sigma_N$ are pluripolar. Put

$$X' := \bigcup_{j=1}^{N} \{(z', z_j, z'') \in (A_1 \times \dots \times A_{j-1}) \times D_j \times (A_{j+1} \times \dots \times A_N) : (z', z'') \notin \Sigma_j\}.$$

Then there exists a relatively closed pluripolar subset $\widehat{M} \subset \widehat{X}$ of the "envelope of holomorphy" $\widehat{X} \subset \mathbb{C}^n$ of X such that:

- $\widehat{M} \cap X' \subset M$,
- for every function f separately holomorphic on $X \setminus M$ there exists exactly one function $\widehat{f}$ holomorphic on $\widehat{X} \setminus \widehat{M}$ with $\widehat{f} = f$ on $X' \setminus M$, and
- $\widehat{M}$ is singular with respect to the family of all functions $\widehat{f}$.

1. Introduction. Main Theorem

Let $N \in \mathbb{N}$, $N \geq 2$, and let

$$\varnothing \neq A_j \subset D_j \subset \mathbb{C}^{n_j},$$

where D_j is a domain, $j = 1, \dots, N$. We define an *N–fold cross*

$$X = \mathbb{X}(A_1, \dots, A_N; D_1, \dots, D_N) := \bigcup_{j=1}^{N} A_1 \times \dots \times A_{j-1} \times D_j \times A_{j+1} \times \dots \times A_N \subset \mathbb{C}^{n_1 + \dots + n_N} = \mathbb{C}^n.$$

Observe that X is connected.

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Let $\Omega \subset \mathbb{C}^n$ be an open set and let $A \subset \Omega$. Put

$$h_{A,\Omega} := \sup\{u : \ u \in \mathcal{PSH}(\Omega), \ u \le 1 \text{ on } \Omega, \ u \le 0 \text{ on } A\},$$

where $\mathcal{PSH}(\Omega)$ denotes the set of all functions plurisubharmonic on Ω. Define

$$\omega_{A,\Omega} := \lim_{k \to +\infty} h^*_{A \cap \Omega_k, \Omega_k},$$

where $(\Omega_k)_{k=1}^\infty$ is a sequence of relatively compact open sets $\Omega_k \subset \Omega_{k+1} \Subset \Omega$ with $\bigcup_{k=1}^\infty \Omega_k = \Omega$ (h^* denotes the upper semicontinuous regularization of h). Observe that the definition is independent of the exhausting sequence $(\Omega_k)_{k=1}^\infty$. Moreover, $\omega_{A,\Omega} \in \mathcal{PSH}(\Omega)$. Recall that if Ω is bounded, then $\omega_{A,\Omega} = h^*_{A,\Omega}$.

For an N–fold cross $X = \mathbb{X}(A_1, \dots, A_N; D_1, \dots, D_N)$ put

$$\widehat{X} := \{(z_1, \dots, z_N) \in D_1 \times \dots \times D_N : \sum_{j=1}^N \omega_{A_j, D_j}(z_j) < 1\}.$$

Observe that if $D_1, \dots, D_N$ are pseudoconvex, then $\widehat{X}$ is a pseudoconvex open set in $\mathbb{C}^n$.

We say that a subset $\varnothing \ne A \subset \mathbb{C}^n$ is *locally pluriregular* if $h^*_{A \cap \Omega, \Omega}(a) = 0$ for any $a \in A$ and for any open neighborhood Ω of a (in particular, $A \cap \Omega$ is non-pluripolar).

Note that if $A_1, \dots, A_N$ are locally pluriregular, then $X \subset \widehat{X}$ and $\widehat{X}$ is connected ([8], Lemma 4).

Let U be an open neighborhood of X and let $M \subset U$ be a relatively closed set. We say that a function $f : X \setminus M \longrightarrow \mathbb{C}$ is *separately holomorphic* ($f \in \mathcal{O}_s(X \setminus M)$) if for any $(a_1, \dots, a_N) \in A_1 \times \dots \times A_N$ and $j \in \{1, \dots, N\}$ the function $f(a_1, \dots, a_{j-1}, \cdot, a_{j+1}, \dots, a_N)$ is holomorphic in the open set

$$D_j \setminus M_{(a_1, \dots, a_{j-1}, \cdot, a_{j+1}, \dots, a_N)},$$

where

$$M_{(a_1, \dots, a_{j-1}, \cdot, a_{j+1}, \dots, a_N)} := \{z_j \in \mathbb{C}^{n_j} : (a_1, \dots, a_{j-1}, z_j, a_{j+1}, \dots, a_N) \in M\}.$$

Suppose that $S_j \subset A_1 \times \dots \times A_{j-1} \times A_{j+1} \times \dots \times A_N$, $j = 1, \dots, N$, and define the *generalized N–fold cross*

$$\begin{aligned} &T = \mathbb{T}(A_1, \dots, A_N; D_1, \dots, D_N; S_1, \dots, S_N) \\ &:= \bigcup_{j=1}^N \{(z', z_j, z'') \in (A_1 \times \dots \times A_{j-1}) \times D_j \times (A_{j+1} \times \dots \times A_N) : (z', z'') \notin S_j\}. \end{aligned}$$

It is clear that $T \subset X$. Observe that

$$\mathbb{X}(A_1, \dots, A_N; D_1, \dots, D_N) = \mathbb{T}(A_1, \dots, A_N; D_1, \dots, D_N; \varnothing, \dots, \varnothing).$$

Moreover, if $N = 2$, then $\mathbb{T}(A_1, A_2; D_1, D_2; S_1, S_2) = \mathbb{X}(A_1 \setminus S_2, A_2 \setminus S_1; D_1, D_2)$. Consequently, any generalized 2–fold cross is a 2–fold cross.

Let $S \subset \Omega$ be a relatively closed pluripolar subset of an open set $\Omega \subset \mathbb{C}^n$. Let $\mathcal{F} \subset \mathcal{O}(\Omega \setminus S)$. We say that S is *singular with respect to* $\mathcal{F}$ if for each point $a \in S$ there exists a function $f_a \in \mathcal{F}$ that is not holomorphically extendible to a neighborhood of a (cf. [5], § 3.4). Equivalently: the set S is minimal in the sense that there is no relatively closed set $S' \subsetneq S$ such that any function from $\mathcal{F}$ extends holomorphically to $\Omega \setminus S'$. It is clear that for any relatively closed pluripolar set $S \subset \Omega$ and for any family $\mathcal{F} \subset \mathcal{O}(\Omega \setminus S)$ there exists a relatively closed set $S' \subset S$

such that any function $f \in \mathcal{F}$ extends to an $f' \in \mathcal{O}(\Omega \setminus S')$ and S' is singular with respect to the family $\{f' : f \in \mathcal{F}\}$.

The main result of our paper is the following extension theorem for separately holomorphic functions.

Main Theorem. *Let $D_j \subset \mathbb{C}^{n_j}$ be a pseudoconvex domain, let $A_j \subset D_j$ be a locally pluriregular set, $j = 1, \dots, N$, and let U be an open neighborhood of the N–fold cross*

$$X := \mathbb{X}(A_1, \dots, A_N; D_1, \dots, D_N).$$

Let $M \subset U$ be a relatively closed subset of U such that for each $j \in \{1, \dots, N\}$ the set

$$\begin{aligned}&\Sigma_j = \Sigma_j(A_1, \dots, A_N; M)\\ &:= \{(z', z'') \in (A_1 \times \dots \times A_{j-1}) \times (A_{j+1} \times \dots \times A_N) : M_{(z',\cdot,z'')} \text{ is not pluripolar}\}\end{aligned}$$

is pluripolar. Put

$$X' := \mathbb{T}(A_1, \dots, A_N; D_1, \dots, D_N; \Sigma_1, \dots, \Sigma_N).$$

Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{X}$ such that:

- $\widehat{M} \cap X' \subset M$,
- *for every $f \in \mathcal{O}_s(X \setminus M)$ there exists exactly one $\widehat{f} \in \mathcal{O}(\widehat{X} \setminus \widehat{M})$ with $\widehat{f} = f$ on $X' \setminus M$,*
- *$\widehat{M}$ is singular with respect to the family $\{\widehat{f} : f \in \mathcal{O}_s(X \setminus M)\}$, and*
- *$\widehat{X} \setminus \widehat{M}$ is pseudoconvex.*

In particular, $\widehat{X} \setminus \widehat{M}$ is the envelope of holomorphy of $X \setminus M$ with respect to the space of separately holomorphic functions.

Notice that if $M \subset U$ is a pluripolar set, then $\Sigma_1, \dots, \Sigma_N$ are always pluripolar (cf. Lemma 8(a)).

The case where $N = 2$, $n_1 = n_2 = 1$, $D_1 = D_2 = \mathbb{C}$ was studied in [7], Theorem 2.

Corollary 1. *Let D_j, A_j, $j = 1, \dots, N$, X, and U be as in the Main Theorem. Assume that $M \subset U$ is a relatively closed set such that for any $(a_1, \dots, a_N) \in A_1 \times \dots \times A_N$ and $j \in \{1, \dots, N\}$ the fiber $M_{(a_1,\dots,a_{j-1},\cdot,a_{j+1},\dots,a_N)}$ is pluripolar.*[1] *Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{X}$ such that:*

- $\widehat{M} \cap X \subset M$,
- *for every $f \in \mathcal{O}_s(X \setminus M)$ there exists exactly one $\widehat{f} \in \mathcal{O}(\widehat{X} \setminus \widehat{M})$ with $\widehat{f} = f$ on $X \setminus M$, and*
- *the domain $\widehat{X} \setminus \widehat{M}$ is pseudoconvex.*

The case where $N = 2$, $D_2 = \mathbb{C}^{n_2}$, and A_2 is open was studied in [4] (for $n_2 = 1$) and in [9] (for arbitrary n_2).

The proof of the Main Theorem will be presented in Sections 3 (for $N = 2$) and 4 (for arbitrary N).

The following two examples illustrate the role played by the sets Σ_j and show that the assertion of the Main Theorem is in some sense optimal.

[1] That is, $\Sigma_1 = \dots = \Sigma_N = \varnothing$.

Example 2. Let $n_1 = n_2 = 1$, $D_1 = D_2 = \mathbb{C}$, $A_1 = E :=$ the unit disc.

(a) Let $A_2 := E$, $X := \mathbb{X}(E, E; \mathbb{C}, \mathbb{C}) = (E \times \mathbb{C}) \cup (\mathbb{C} \times E)$, and $M := \{0\} \times \overline{E}$. Then $\Sigma_1 = \varnothing$, $\Sigma_2 = \{0\}$, $X' = \mathbb{X}(E \setminus \{0\}, E; \mathbb{C}, \mathbb{C})$, $\widehat{M} = \{0\} \times \mathbb{C}$.

Put $f_0(z, w) := 1/z$, $z \neq 0$, and $f_0(0, w) = 1$, $|w| > 1$. Then $f_0 \in \mathcal{O}_s(X \setminus M)$ and $\widehat{M}$ is singular with respect to f_0.

(b) Let $A_2 := E \setminus r\overline{E}$, $X := \mathbb{X}(E, E \setminus r\overline{E}; \mathbb{C}, \mathbb{C})$, and $M := \{0\} \times \{|w| = r\}$ for some $0 < r < 1$. Then $\Sigma_1 = \varnothing$, $\Sigma_2 = \{0\}$, $X' = \mathbb{X}(E \setminus \{0\}, A_2; \mathbb{C}, \mathbb{C})$, $\widehat{M} = \varnothing$.

Put

$$f_0(z, w) := \begin{cases} w & \text{if } z \neq 0 \text{ or } (z = 0 \text{ and } |w| > r), \\ 0 & \text{if } \qquad z = 0 \text{ and } |w| < r, \end{cases} \qquad (z, w) \in X \setminus M.$$

Then $f_0 \in \mathcal{O}_s(X \setminus M)$, $\widehat{f}_0(z, w) \equiv w$, and $\widehat{f}_0(0, w) \neq f(0, w)$, $0 < |w| < r$.

2. Auxiliary Results

In the case $M = \varnothing$ the problem of extension of separately holomorphic functions was studied by many authors (under various assumptions on $(D_j, A_j)_{j=1}^N$), e.g. [17], [20], [18], [16], [12], [10], [1] (for $N = 2$), and [18], [13], [8] (for arbitrary N).

Theorem 3 ([13], [1]). *Let $(D_j, A_j)_{j=1}^N$ and X be as in the Main Theorem. Then any function from $\mathcal{O}_s(X)$ extends holomorphically to the pseudoconvex domain $\widehat{X}$.*

The case where M is analytic was studied in [14], [15], [19], [6]. The problem was completely solved in [8].

Theorem 4 ([7]). *Let $(D_j, A_j)_{j=1}^N$ and X be as in the Main Theorem. Let $M \subsetneq U$ be an analytic subset of an open connected neighborhood U of X. Then there exists an analytic set $\widehat{M} \subset \widehat{X}$ such that:*

- *$\widehat{M} \cap U_0 \subset M$ for an open neighborhood U_0 of X, $U_0 \subset U$,*
- *for every $f \in \mathcal{O}_s(X \setminus M)$ there exists exactly one $\widehat{f} \in \mathcal{O}(\widehat{X} \setminus \widehat{M})$ with $\widehat{f} = f$ on $X \setminus M$, and*
- *the domain $\widehat{X} \setminus \widehat{M}$ is pseudoconvex.*

Remark 5. It is a natural idea to try to obtain Theorem 4 from the Main Theorem. More precisely, let $(D_j, A_j)_{j=1}^N$, X, U, and M be as in Theorem 4. Then, by the Main Theorem, there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{X}$ which has all the properties listed in the Main Theorem. We would like to know whether there is a direct argument showing that $\widehat{M}$ must be analytic.

The following two results will play the fundamental role in the sequel.

Theorem 6 ([3]). *Let $D \subset \mathbb{C}^n$ be a domain and let $\widehat{D}$ be the envelope of holomorphy of D. Assume that S is a relatively closed pluripolar subset of D. Then there exists a relatively closed pluripolar subset $\widehat{S}$ of $\widehat{D}$ such that $\widehat{S} \cap D \subset S$ and $\widehat{D} \setminus \widehat{S}$ is the envelope of holomorphy of $D \setminus S$.*

Theorem 7 ([7]). *Let $A \subset E^{n-1}$ be locally pluriregular, let*

$$X := \mathbb{X}(A, E; E^{n-1}, \mathbb{C})$$

(notice that $\widehat{X} = E^{n-1} \times \mathbb{C}$), and let $U \subset E^{N-1} \times \mathbb{C}$ be an open neighborhood of X. Let $M \subset U$ be a relatively closed set such that $M \cap E^n = \varnothing$ and for any

$a \in A$ the fiber $M_{(a,\cdot)}$ is polar. Then there exists a relatively closed pluripolar set $S \subset E^{n-1} \times \mathbb{C}$ such that

- *$S \cap X \subset M$,*
- *any function from $\mathcal{O}_s(X \setminus M)$ extends holomorphically to $E^{n-1} \times \mathbb{C} \setminus S$, and*
- *$E^{n-1} \times \mathbb{C} \setminus S$ [2] is pseudoconvex.*

Notice that the above result is a special case of our Main Theorem with $N = 2$, $n_1 = n - 1$, $D_1 = E^{n-1}$, $A_1 = A$, $n_2 = 1$, $D_2 = \mathbb{C}$, $A_2 = E$, $\Sigma_1 = \Sigma_2 = \varnothing$.

Proof. It is known (cf. [4]) that each function $f \in \mathcal{O}_s(X \setminus M)$ has the univalent domain of existence $G_f \subset E^{n-1} \times \mathbb{C}$.[3] Let G denote the connected component of $\operatorname{int} \bigcap_{f \in \mathcal{O}_s(X \setminus M)} G_f$ that contains E^n and let $S := E^{n-1} \times \mathbb{C} \setminus G$. It remains to show that S is pluripolar.

Take $(a, b) \in A \times \mathbb{C} \setminus M$. Since $M_{(a,\cdot)}$ is polar, there exists a curve $\gamma : [0, 1] \longrightarrow \mathbb{C} \setminus M_{(a,\cdot)}$ such that $\gamma(0) = 0$, $\gamma(1) = b$. Take an $\varepsilon > 0$ so small that

$$\Delta_a(\varepsilon) \times (\gamma([0, 1]) + \Delta_0(\varepsilon)) \subset U \setminus M,$$

where $\Delta_{z_0}(r) = \Delta^k_{z_0}(r) \subset \mathbb{C}^k$ denotes the polydisc with center $z_0 \in \mathbb{C}^k$ and radius $r > 0$. Put $V_b := E \cup (\gamma([0, 1]) + \Delta_0(\varepsilon))$ and consider the cross

$$Y := \mathbb{X}(A \cap \Delta_a(\varepsilon), E; \Delta_a(\varepsilon), V_b).$$

Then $f \in \mathcal{O}_s(Y)$ for any $f \in \mathcal{O}_s(X \setminus M)$. Consequently, by Theorem 3, we get $\widehat{Y} \subset G_f$, $f \in \mathcal{O}_s(X \setminus M)$. Hence $\widehat{Y} \subset G$. In particular, we conclude that $\{a\} \times (\mathbb{C} \setminus M_{(a,\cdot)}) \subset G$.

Thus $S_{(a,\cdot)} \subset M_{(a,\cdot)}$ for all $a \in A$. Consequently, by Lemma 5 from [4], S is pluripolar. $\square$

Lemma 8. (a) *Let $S \subset \mathbb{C}^p \times \mathbb{C}^q$ be pluripolar. Then the set*

$$A := \{z \in \mathbb{C}^p : S_{(z,\cdot)} \text{ is not pluripolar}\}$$

is pluripolar.

(b) *Let $M \subset \mathbb{C}^p \times \mathbb{C}^q$ be such that for each $a \in \mathbb{C}^p$ the fiber $M_{(a,\cdot)}$ is pluripolar. Let $C \subset \mathbb{C}^p \times \mathbb{C}^q$ be such that the set $\{z \in \mathbb{C}^p : C_{(z,\cdot)}$ is not pluripolar$\}$ is not pluripolar (e.g. $C = C' \times C''$, where $C' \subset \mathbb{C}^p$, $C'' \subset \mathbb{C}^q$ are nonpluripolar). Then $C \setminus M$ is nonpluripolar.*

(c) *Let $M \subset \mathbb{C}^p \times \mathbb{C}^q$ be such that for each $a \in \mathbb{C}^p$ the fiber $M_{(a,\cdot)}$ is pluripolar. Let $A \subset \mathbb{C}^p$ be locally pluriregular. Let $C := \{(a, b') \in A \times \mathbb{C}^{q-1} : M_{(a,b',\cdot)}$ is polar$\}$. Then C is locally pluriregular.*

Proof. (a) Let $v \in \mathcal{PSH}(\mathbb{C}^{p+q})$, $v \not\equiv -\infty$, be such that $S \subset v^{-1}(-\infty)$. Define

$$u(z) := \sup\{v(z, w) : w \in \overline{E}^q\}, \quad z \in \mathbb{C}^p.$$

Then $A \subset u^{-1}(-\infty)$. Moreover, $u \in \mathcal{PSH}(\mathbb{C}^p)$ and $u \not\equiv -\infty$.

(b) Suppose that $C \setminus M$ is pluripolar. Then, by (a), there exists a pluripolar set $A \subset \mathbb{C}^p$ such that the fiber $(C \setminus M)_{(a,\cdot)}$ is pluripolar, $a \in \mathbb{C}^p \setminus A$. Consequently, the fiber $C_{(a,\cdot)}$ is pluripolar, $a \in \mathbb{C}^p \setminus A$, a contradiction.

[2] Here and in the sequel, to simplify notation we write $P_1 \times \cdots \times P_k \setminus Q$ instead of $(P_1 \times \cdots \times P_k) \setminus Q$.

[3] We like to thank Professor Evgeni Chirka for explaining to us some details of the proof of Theorem 1 in [4].

(c) Fix a point $(a_0,b_0')\in C$ and a neighborhood $U:=\Delta_{(a_0,b_0')}(r)$. We have to show that $h^*_{C\cap U,U}(a_0,b_0')=0$. First we show that

$$(*)\qquad h^*_{C\cap U,U}(a_0,b_0')\le h^*_{(A\cap\Delta_{a_0}(r))\times\Delta_{b_0'}(r),U}(a_0,b_0').$$

Indeed, let $u\in\mathcal{PSH}(U)$ be such that $u\le 1$ and $u\le 0$ on $C\cap U$. Then for any $a\in A\cap\Delta_{a_0}(r)$ the function $u(a,\cdot)$ is plurisubharmonic on $\Delta_{b_0'}(r)$, and $u(a,\cdot)\le 0$ on the set

$$(C\cap U)_{(a,\cdot)}=\{b'\in\Delta_{b_0'}(r):(M_{(a,\cdot)})_{(b',\cdot)}\text{ is polar}\}.$$

By (a) (applied to the set $M_{(a,\cdot)}$), the set $\Delta_{b_0'}(r)\setminus(C\cap U)_{(a,\cdot)}$ is pluripolar. Hence $u(a,\cdot)\le 0$ on $\Delta_{b_0'}(r)$. Consequently, $u\le 0$ on $(A\cap\Delta_{a_0}(r))\times\Delta_{b_0'}(r)$, which implies that $h_{C\cap U,U}\le h_{(A\cap\Delta_{a_0}(r))\times\Delta_{b_0'}(r),U}$, and finally, $h^*_{C\cap U,U}(a_0,b_0')\le h^*_{(A\cap\Delta_{a_0}(r))\times\Delta_{b_0'},U}(a_0,b_0')$.

Now, by virtue of the product property of the relative extremal function (cf. [11]), using (*) and the fact that A is locally pluriregular, we get

$$\begin{aligned}h^*_{C\cap U,U}(a_0,b_0')&\le h^*_{(A\cap\Delta_{a_0}(r))\times\Delta_{b_0'}(r),U}(a_0,b_0')\\&=\max\left\{h^*_{A\cap\Delta_{a_0}(r),\Delta_{a_0}(r)}(a_0),\ h^*_{\Delta_{b_0'}(r),\Delta_{b_0'}(r)}(b_0')\right\}\\&=h^*_{A\cap\Delta_{a_0}(r),\Delta_{a_0}(r)}(a_0)=0.\end{aligned}$$

□

Lemma 9. *Let D_j, A_j, $j=1,\dots,N$, and X be as in the Main Theorem. Let*

$$S_j\subset A_1\times\dots\times A_{j-1}\times A_{j+1}\times\dots\times A_N$$

be pluripolar, $j=1,\dots,N$. Put

$$T:=\mathbb{T}(A_1,\dots,A_N;D_1,\dots,D_N;S_1,\dots,S_N).$$

Then any function $f\in\mathcal{O}_s(T)\cap\mathcal{C}(T)$ [4] *extends holomorphically to $\widehat{X}$.*

If $N=2$, then the result is true for any function $f\in\mathcal{O}_s(T)$ (see the proof). In the case where $N\ge 3$ we do not know whether the result is true for arbitrary $f\in\mathcal{O}_s(T)$.

Proof. We apply induction on N. The case $N=2$ follows from Theorem 3 and the fact that $\widehat{X}=\widehat{T}$ (recall that if $N=2$, then T is a 2–fold cross). Moreover, if $N=2$, then the result is true for any $f\in\mathcal{O}_s(T)$.

Assume that the result is true for $N-1\ge 2$. Take an $f\in\mathcal{O}_s(T)\cap\mathcal{C}(T)$. Let Q denote the set of all $z_N\in A_N$ for which there exists a $j\in\{1,\dots,N-1\}$ such that the fiber $(S_j)_{(\cdot,z_N)}$ is not pluripolar. Then, by Lemma 8(a), Q is pluripolar. Take a $z_N\in A_N\setminus Q$ and define

$$T_{z_N}:=\mathbb{T}(A_1,\dots,A_{N-1};D_1,\dots,D_{N-1};(S_1)_{(\cdot,z_N)},\dots,(S_{N-1})_{(\cdot,z_N)}).$$

Then $f(\cdot,z_N)\in\mathcal{O}_s(T_{z_N})\cap\mathcal{C}(T_{z_N})$. By the inductive assumption, the function $f(\cdot,z_N)$ extends to an $\widehat{f}_{z_N}\in\mathcal{O}(\widehat{Y})$, where $Y=\mathbb{X}(A_1,\dots,A_{N-1};D_1,\dots,D_{N-1})$.

Let $A':=A_1\times\dots\times A_{N-1}$. Consider the 2–fold cross

$$Z:=\mathbb{T}(A',A_N;\widehat{Y},D_N;S_N,Q)=((A'\setminus S_N)\times D_N)\cup(\widehat{Y}\times(A_N\setminus Q)).$$

[4] We say that a function $f:T\longrightarrow\mathbb{C}$ is *separately holomorphic* if for any $j\in\{1,\dots,N\}$ and $(a',a'')\in(A_1\times\dots\times A_{j-1})\times(A_{j+1}\times\dots\times A_N)\setminus S_j$ the function $f(a',\cdot,a'')$ is holomorphic in D_j.

Let $g : Z \longrightarrow \mathbb{C}$ be given by the formulae

$$g(z', z_N) := f(z', z_N), (z', z_N) \in (A' \setminus S_N) \times D_N,$$
$$g(z', z_N) := \widehat{f}_{z_N}(z'), (z', z_N) \in \widehat{Y} \times (A_N \setminus Q).$$

Observe that g is well-defined.

Indeed, let $(z', z_N) \in ((A' \setminus S_N) \times D_N) \cap (\widehat{Y} \times (A_N \setminus Q))$. If $z' \in T_{z_N}$, then obviously $\widehat{f}_{z_N}(z') = f(z', z_N)$. Suppose that $z' \notin T_{z_N}$. Then

$$z' \in P_{z_N} := \bigcap_{j=1}^{N-1} \{(w', w_j, w'') \in (A_1 \times \cdots \times A_{j-1}) \times A_j \times (A_{j+1} \times \cdots \times A_{N-1}) : (w', w'') \in (S_j)_{(\cdot, z_N)}\};$$

P_{z_N} is pluripolar. Take a sequence $A' \setminus (S_N \cup P_{z_N}) \ni z'^{\nu} \longrightarrow z'$. Then $z'^{\nu} \in T_{z_N}$. Thus $\widehat{f}_{z_N}(z'^{\nu}) = f(z'^{\nu}, z_N)$. Hence, by continuity, $\widehat{f}_{z_N}(z') = f(z', z_N)$. [5]

Moreover, $g \in \mathcal{O}_s(Z)$. Put $V := \mathbb{X}(A', A_N; \widehat{Y}, D_N) \supset Z$. Since the result is true for $N = 2$ (without the continuity), we get a holomorphic extension of g to $\widehat{V}$. It remains to observe that $\widehat{V} = \widehat{X}$; cf. [8], the proof of Step 3. $\square$

Lemma 10. *Let $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$ be pseudoconvex domains, let $A \subset D$, $B \subset G$ be locally pluriregular, and let $M \subset U$ be a relatively closed subset of an open neighborhood U of the cross $X := \mathbb{X}(A, B; D, G)$. Let $A' \subset A$, $B' \subset B$ be such that $A \setminus A'$, $B \setminus B'$ are pluripolar and for any $(a, b) \in A' \times B'$ the fibers $M_{(a,\cdot)}$, $M_{(\cdot,b)}$ are pluripolar. Let $(D_j)_{j=1}^{\infty}$, $(G_j)_{j=1}^{\infty}$ be sequences of pseudoconvex domains, $D_j \Subset D$, $G_j \Subset G$, with $D_j \nearrow D$, $G_j \nearrow G$, such that $A'_j := A' \cap D_j \neq \varnothing$, $B'_j := B' \cap G_j \neq \varnothing$, $j \in \mathbb{N}$. We assume that for each $j \in \mathbb{N}$, $a \in A'_j$, and $b \in B'_j$, there exist:*

- *polydiscs $\varDelta_a(r_{a,j}) \subset D_j$, $\varDelta_b(s_{b,j}) \subset G_j$ and*
- *relatively closed pluripolar sets $S_{a,j} \subset \varDelta_a(r_{a,j}) \times G_j$, $S^{b,j} \subset D_j \times \varDelta_b(s_{b,j})$ such that:*
- *$(\varDelta_a(r_{a,j}) \times G_j) \cup (D_j \times \varDelta_b(s_{b,j})) \subset U \cap \widehat{X}$,*
- *$((A' \cap \varDelta_a(r_{a,j})) \times G_j) \cap S_{a,j} \subset M$, $(D_j \times (B' \cap \varDelta_b(s_{b,j}))) \cap S^{b,j} \subset M$,*
- *for any $f \in \mathcal{O}_s(X \setminus M)$ there exist functions $f_{a,j} \in \mathcal{O}(\varDelta_a(r_{a,j}) \times G_j \setminus S_{a,j})$, $f^{b,j} \in \mathcal{O}(D_j \times \varDelta_b(s_{b,j}) \setminus S^{b,j})$ with*

$$f_{a,j} = f \quad \text{on } (A' \cap \varDelta_a(r_{a,j})) \times G_j \setminus M,$$
$$f^{b,j} = f \quad \text{on } D_j \times (B' \cap \varDelta_b(s_{b,j})) \setminus M,$$

- *$S_{a,j}$ is singular with respect to the family $\{f_{a,j} : f \in \mathcal{O}_s(X \setminus M)\}$, while $S^{b,j}$ is singular with respect to the family $\{f^{b,j} : f \in \mathcal{O}_s(X \setminus M)\}$.*

Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{X}$ such that:

- *$\widehat{M} \cap X' \subset M$, where $X' := \mathbb{X}(A', B'; D, G)$,*
- *for any $f \in \mathcal{O}_s(X \setminus M)$ there exists exactly one $\widehat{f} \in \mathcal{O}(\widehat{X} \setminus \widehat{M})$ with $\widehat{f} = f$ on $X' \setminus M$, and*
- *the set $\widehat{M}$ is singular with respect to the family $\{\widehat{f} : f \in \mathcal{O}_s(X \setminus M)\}$.*

[5] Here is the only place where the continuity of f is used.

Proof. Fix a $j \in \mathbb{N}$. Put

$$\begin{aligned}\widetilde{U}_j &:= \bigcup_{a\in A_j',\, b\in B_j'} (\varDelta_a(r_{a,j}) \times G_j) \cup (D_j \times \varDelta_b(s_{b,j})),\\ X_j &:= ((A\cap D_j)\times G_j) \cup (D_j \times (B\cap G_j)),\\ X_j' &:= (A_j' \times G_j) \cup (D_j \times B_j').\end{aligned}$$

Note that $X_j' \subset \widetilde{U}_j$. Take an $f \in \mathcal{O}_s(X \setminus M)$. We want to glue the sets $(S_{a,j})_{a\in A_j'}$, $(S^{b,j})_{b\in B_j'}$ and the functions $(f_{a,j})_{a\in A_j'}$, $(f^{b,j})_{b\in B_j'}$ to obtain a global holomorphic function $f_j := \bigcup_{a\in A_j',\, b\in B_j'} f_{a,j} \cup f^{b,j}$ on $\widetilde{U}_j \setminus S_j$ where $S_j := \bigcup_{a\in A_j',\, b\in B_j'} S_{a,j} \cup S^{b,j}$.

Let $a \in A_j'$, $b \in B_j'$. Observe that

$$\begin{aligned} f_{a,j} &= f \ \text{ on } (A' \cap \varDelta_a(r_{a,j})) \times G_j \setminus M,\\ f^{b,j} &= f \ \text{ on } D_j \times (B' \cap \varDelta_b(s_{b,j})) \setminus M.\end{aligned}$$

Thus $f_{a,j} = f^{b,j}$ on the non-pluripolar set $(A' \cap \varDelta_a(r_{a,j})) \times (B' \cap \varDelta_b(s_{b,j})) \setminus M$ (cf. Lemma 8(b)). Hence

$$f_{a,j} = f^{b,j} \ \text{ on } \varDelta_a(r_{a,j}) \times \varDelta_b(s_{b,j}) \setminus (S_{a,j} \cup S^{b,j}).$$

Using the minimality of $S_{a,j}$ and $S^{b,j}$, we conclude that

$$S_{a,j} \cap (\varDelta_a(r_{a,j}) \times \varDelta_b(s_{b,j})) = S^{b,j} \cap (\varDelta_a(r_{a,j}) \times \varDelta_b(s_{b,j})).$$

Now let $a', a'' \in A_j'$ be such that $C := \varDelta_{a'}(r_{a',j}) \cap \varDelta_{a''}(r_{a'',j}) \neq \varnothing$. Fix a $b \in B_j'$. We know that $f_{a',j} = f^{b,j} = f_{a'',j}$ on $C \times \varDelta_b(r_{b,j}) \setminus (S_{a',j} \cup S^{b,j} \cup S_{a'',j})$. Hence, by the identity principle, we conclude that $f_{a',j} = f_{a'',j}$ on $C \times G_j \setminus (S_{a',j} \cup S_{a'',j})$ and, moreover,

$$S_{a',j} \cap (C \times G_j) = S_{a'',j} \cap (C \times G_j).$$

The same argument works for $b', b'' \in B' \cap G_j$.

Let U_j be the connected component of $\widetilde{U}_j \cap \widehat{X}_j'$ with $X_j' \subset U_j$. We have constructed a relatively closed pluripolar set $S_j \subset U_j$ such that:

- $S_j \cap X_j' \subset M$, and
- for any $f \in \mathcal{O}_s(X \setminus M)$ there exists (exactly one) $f_j \in \mathcal{O}(U_j \setminus S_j)$ with $f_j = f$ on $X_j' \setminus M$.

Recall that $X_j' \subset U_j \subset \widehat{X}_j'$. Hence the envelope of holomorphy $\widehat{U}_j$ coincides with $\widehat{X}_j'$ (cf. [7], the proof of Step 4).

Applying the Chirka theorem (Theorem 6), we find a relatively closed pluripolar set $\widehat{M}_j \subset \widehat{X}_j'$ such that:

- $\widehat{M}_j \cap U_j \subset S_j$,
- for any $f \in \mathcal{O}_s(X \setminus M)$ there exists (exactly one) function $\widehat{f}_j \in \mathcal{O}(\widehat{X}_j' \setminus \widehat{M}_j)$ with $\widehat{f}_j = f_j$ on $U_j \setminus S_j$ (in particular, $\widehat{f}_j = f$ on $X_j' \setminus M$), and
- the set $\widehat{M}_j$ is singular with respect to the family $\{\widehat{f}_j : f \in \mathcal{O}_s(X \setminus M)\}$.

Since $A \setminus A'$, $B \setminus B'$ are pluripolar, we get

$$\begin{aligned}\widehat{X}_j' &= \{(z,w) \in D_j \times G_j : h^*_{A'\cap D_j, D_j}(z) + h^*_{B'\cap G_j, G_j}(w) < 1\}\\ &= \{(z,w) \in D_j \times G_j : h^*_{A\cap D_j, D_j}(z) + h^*_{B\cap G_j, G_j}(w) < 1\} = \widehat{X}_j.\end{aligned}$$

So, in fact, $\widehat{f}_j \in \mathcal{O}(\widehat{X}_j \setminus \widehat{M}_j)$. Observe that $\bigcup_{j=1}^{\infty} X_j = X$, $\widehat{X}_j \subset \widehat{X}_{j+1}$, and $\bigcup_{j=1}^{\infty} \widehat{X}_j = \widehat{X}$. Using again the minimality of the $\widehat{M}_j$'s (and gluing the $\widehat{f}_j$'s), we get a relatively closed pluripolar set $\widehat{M} \subset \widehat{X}$ which satisfies all the required conditions. □

Lemma 11. *Let $A \subset E^{n-1}$ be locally pluriregular, let $G \subset \mathbb{C}$ be a domain with $E \Subset G$, let $X := \mathbb{X}(A, E; E^{n-1}, G)$, and let $U \subset E^{n-1} \times G$ be an open neighborhood of X. Let $M \subset U$ be a relatively closed set such that $M \cap E^n = \varnothing$ and for any $a \in A$ the fiber $M_{(a,\cdot)}$ is polar. Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{X}$ such that:*
- *$\widehat{M} \cap X \subset M$,*
- *for any $f \in \mathcal{O}_s(X \setminus M)$ there exists exactly one $\widehat{f} \in \mathcal{O}(\widehat{X} \setminus \widehat{M})$ with $\widehat{f} = f$ on $X \setminus M$, and*
- *the set $\widehat{M}$ is singular with respect to the family $\{\widehat{f} : f \in \mathcal{O}_s(X \setminus M)\}$.*

Notice that the above result is a special case of our Main Theorem with $N = 2$, $n_1 = n - 1$, $D_1 = E^{n-1}$, $A_1 = A$, $n_2 = 1$, $D_2 = G$, $A_2 = E$, $\Sigma_1 = \Sigma_2 = \varnothing$.

Proof. By Lemma 10, it suffices to show that for any $a_0 \in A$ and for any domain $G' \Subset G$ with $E \Subset G'$ there exist $r > 0$ and a relatively closed pluripolar set $S \subset \varDelta_{a_0}(r) \times G' \subset U$ such that:
- $S \cap X \subset M$, and
- any function from $\mathcal{O}_s(X \setminus M)$ extends holomorphically to $\varDelta_{a_0}(r) \times G' \setminus S$.

Fix a_0 and G'. For $b \in G$, let $\rho = \rho_b > 0$ be such that $\varDelta_b(\rho) \Subset G$ and $M_{(a_0,\cdot)} \cap \partial\varDelta_b(\rho) = \varnothing$ (cf. [2], Th. 7.3.9). Take $\rho^- = \rho_b^- > 0$, $\rho^+ = \rho_b^+ > 0$ such that $\rho^- < \rho < \rho^+$, $\varDelta_b(\rho^+) \Subset G$, and $M_{(a_0,\cdot)} \cap \overline{P} = \varnothing$, where

$$P = P_b := \{w \in \mathbb{C} : \rho^- < |w| < \rho^+\}.$$

Let $\gamma : [0,1] \longrightarrow G \setminus M_{(a_0,\cdot)}$ be a curve such that $\gamma(0) = 0$ and $\gamma(1) \in \partial\varDelta_b(\rho)$. There exists an $\varepsilon = \varepsilon_b > 0$ such that

$$\varDelta_{a_0}(\varepsilon) \times ((\gamma([0,1]) + \varDelta_0(\varepsilon)) \cup P) \subset U \setminus M.$$

Put $V = V_b := E \cup \big(\gamma([0,1]) + \varDelta_0(\varepsilon)\big) \cup P$ and consider the cross

$$Y = Y_b := \mathbb{X}(A \cap \varDelta_{a_0}(\varepsilon), E; \varDelta_{a_0}(\varepsilon), V).$$

Then $f \in \mathcal{O}_s(Y)$ for any $f \in \mathcal{O}_s(X \setminus M)$. Consequently, by Theorem 3, any function from $\mathcal{O}_s(X \setminus M)$ extends holomorphically to $\widehat{Y} \supset \{a_0\} \times V$. Shrinking ε and V, we may assume that any function $f \in \mathcal{O}_s(X \setminus M)$ extends to a function $\widetilde{f} = \widetilde{f}_b \in \mathcal{O}(\varDelta_{a_0}(\varepsilon) \times W)$, where

$$W = W_b := \varDelta_0(1-\varepsilon) \cup (\gamma([0,1]) + \varDelta_0(\varepsilon)) \cup P.$$

In particular, $\widetilde{f}$ is holomorphic in $\varDelta_{a_0}(\varepsilon) \times P$, and therefore may be represented by the Hartogs–Laurent series

$$\widetilde{f}(z,w) = \sum_{k=0}^{\infty} \widetilde{f}_k(z)(w-b)^k + \sum_{k=1}^{\infty} \widetilde{f}_{-k}(z)(w-b)^{-k} =: \widetilde{f}^+(z,w) + \widetilde{f}^-(z,w),$$
$$(z,w) \in \varDelta_{a_0}(\varepsilon) \times P,$$

where $\widetilde{f}^+ \in \mathcal{O}(\varDelta_{a_0}(\varepsilon) \times \varDelta_b(\rho^+))$ and $\widetilde{f}^- \in \mathcal{O}(\varDelta_{a_0}(\varepsilon) \times (\mathbb{C} \setminus \overline{\varDelta}_b(\rho^-)))$. Recall that for any $a \in A \cap \varDelta_{a_0}(\varepsilon)$ the function $\widetilde{f}(a,\cdot)$ extends holomorphically to $G \setminus M_{(a,\cdot)}$.

Consequently, for any $a \in A \cap \mathit{\Delta}_{a_0}(\varepsilon)$ the function $\widetilde{f}^-(a,\cdot)$ extends holomorphically to $\mathbb{C} \setminus (M_{(a,\cdot)} \cap \overline{\mathit{\Delta}}_b(\rho^-))$. Now, by Theorem 7, there exists a relatively closed pluripolar set $S = S_b \subset \mathit{\Delta}_{a_0}(\varepsilon) \times \overline{\mathit{\Delta}}_b(\rho^-)$ such that:

- $S \cap ((A \cap \mathit{\Delta}_{a_0}(\varepsilon)) \times \overline{\mathit{\Delta}}_b(\rho^-)) \subset M$, and
- any function $\widetilde{f}^-$ extends holomorphically to a function $\widetilde{\widetilde{f}}^- \in \mathcal{O}(\mathit{\Delta}_{a_0}(\varepsilon) \times \mathbb{C} \setminus S)$.

Since $\widetilde{f} = \widetilde{f}^+ + \widetilde{f}^-$, the function $\widetilde{f}$ extends holomorphically to a function $\widehat{f} = \widehat{f}_b \in \mathcal{O}(\mathit{\Delta}_{a_0}(\varepsilon) \times \mathit{\Delta}_b(\rho^+) \setminus S)$. We may assume that the set S is singular with respect to the family $\{\widehat{f} : f \in \mathcal{O}_s(X \setminus M)\}$.

Using the identity principle and the minimality of the S_b's, one can easily show that for $b', b'' \in G$, if $B := \mathit{\Delta}_{b'}(\rho^+_{b'}) \cap \mathit{\Delta}_{b''}(\rho^+_{b''}) \neq \varnothing$, then

$$S_{b'} \cap (\mathit{\Delta}_{a_0}(\eta) \times B) = S_{b''} \cap (\mathit{\Delta}_{a_0}(\eta) \times B), \quad \widehat{f}_{b'} = \widehat{f}_{b''} \text{ on } \mathit{\Delta}_{a_0}(\eta) \times B,$$

where $\eta := \min\{\varepsilon_{b'}, \varepsilon_{b''}\}$. Thus the functions $\widehat{f}_{b'}$, $\widehat{f}_{b''}$ and sets $S_{b'}$, $S_{b''}$ may be glued together.

Now, select $b_1, \dots, b_k \in G$ so that $G' \subset \bigcup_{j=1}^k \mathit{\Delta}_{b_j}(\rho^+_{b_j})$. Put

$$r := \min\{\varepsilon_{b_j} : j = 1, \dots, k\}.$$

Then $S := (\mathit{\Delta}_{a_0}(r) \times G') \cap \bigcup_{j=1}^k S_{b_j}$ gives the required relatively closed pluripolar subset of $\mathit{\Delta}_{a_0}(r) \times G'$ such that $S \cap X \subset M$ and for any $f \in \mathcal{O}_s(X \setminus M)$, the function $\widehat{f} := \bigcup_{j=1}^k \widehat{f}_{b_j}$ extends holomorphically f to $\mathit{\Delta}_{a_0}(r) \times G' \setminus S$. $\square$

Lemma 12. *Let $A \subset E^p$ be locally pluriregular, let $R > 1$, let*

$$X := \mathbb{X}(A, E^q; E^p, \mathit{\Delta}_0^q(R)),$$

and let $U \subset E^p \times \mathit{\Delta}_0^q(R)$ be an open neighborhood of X. Let $M \subset U$ be a relatively closed set such that $M \cap E^{p+q} = \varnothing$ and for any $a \in A$ the fiber $M_{(a,\cdot)}$ is pluripolar. Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{X}$ such that:

- *$\widehat{M} \cap X \subset M$,*
- *for any $f \in \mathcal{O}_s(X \setminus M)$ there exists exactly one $\widehat{f} \in \mathcal{O}(\widehat{X} \setminus \widehat{M})$ with $\widehat{f} = f$ on $X \setminus M$, and*
- *the set $\widehat{M}$ is singular with respect to the family $\{\widehat{f} : f \in \mathcal{O}_s(X \setminus M)\}$.*

Notice that the above result is a special case of our Main Theorem with $N = 2$, $n_1 = p$, $D_1 = E^p$, $A_1 = A$, $n_2 = q$, $D_2 = \mathit{\Delta}_0^q(R)$, $A_2 = E^q$, $\Sigma_1 = \Sigma_2 = \varnothing$.

Proof. The case $q = 1$ follows from Lemma 11. Thus assume that $q \geq 2$. By Lemma 10, it suffices to show that for any $a_0 \in A$ and for any $R' \in (1, R)$ there exist $r = r_{R'} > 0$ and a relatively closed pluripolar set $S = S_{R'} \subset \mathit{\Delta}_{a_0}(r) \times \mathit{\Delta}_0^q(R') \subset U$ such that

- $S \cap X \subset M$, and
- any function from $\mathcal{O}_s(X \setminus M)$ extends holomorphically to $\mathit{\Delta}_{a_0}(r) \times \mathit{\Delta}_0^q(R') \setminus S$.

Fix an $a_0 \in A$ and let R'_0 be the supremum of all $R' \in (0, R)$ such that $r_{R'}$ and $S_{R'}$ exist. Note that $1 \leq R'_0 \leq R$. It suffices to show that $R'_0 = R$.

Suppose that $R'_0 < R$. Fix $R'_0 < R'' < R$ and choose $R' \in (0, R'_0)$ such that $\sqrt[q]{R'^{q-1} R''} > R'_0$. Let $r := r_{R'}$, $S := S_{R'}$.

Write $w = (w', w_q) \in \mathbb{C}^q = \mathbb{C}^{q-1} \times \mathbb{C}$. Let C denote the set of all $(a, b') \in (A \cap \mathit{\Delta}_{a_0}(r)) \times \mathit{\Delta}_0^{q-1}(R')$ such that the fiber $(M \cup S)_{(a,b',\cdot)}$ is polar. By Lemma 8(a,c),

C is pluriregular. Now, by Lemma 11 applied to the cross

$$Y_q := \mathbb{X}(C, \varDelta_0(R'); \varDelta_{a_0}(r) \times \varDelta_0^{q-1}(R'), \varDelta_0(R))$$

and the set $M_q := M \cup S$, we conclude that there exists a closed pluripolar set $S_q \subset \widehat{Y}_q$ such that $S_q \cap Y_q \subset M_q$ and any function $f \in \mathcal{O}_s(X \setminus M)$ extends holomorphically to $\widehat{Y}_q \setminus S_q$. Using the product property of the relative extremal function (cf. [11]), we get

$$\begin{aligned}\widehat{Y}_q &= \{(z, w', w_q) \in \varDelta_{a_0}(r) \times \varDelta_0^{q-1}(R') \times \varDelta_0(R) : \\ &\qquad\qquad h^*_{C, \varDelta_{a_0}(r) \times \varDelta_0^{q-1}(R')}(z, w') + h^*_{\varDelta_0(R'), \varDelta_0(R)}(w_q) < 1\} \\ &= \{(z, w', w_q) \in \varDelta_{a_0}(r) \times \varDelta_0^{q-1}(R') \times \varDelta_0(R) : \\ &\qquad h^*_{(A \cap \varDelta_{a_0}(r)) \times \varDelta_0^{q-1}(R'), \varDelta_{a_0}(r) \times \varDelta_0^{q-1}(R')}(z, w') + h^*_{\varDelta_0(R'), \varDelta_0(R)}(w_q) < 1\} \\ &= \{(z, w', w_q) \in \varDelta_{a_0}(r) \times \varDelta_0^{q-1}(R') \times \varDelta_0(R) : \\ &\quad \max\{h^*_{A \cap \varDelta_{a_0}(r), \varDelta_{a_0}(r)}(z),\ h^*_{\varDelta_0^{q-1}(R'), \varDelta_0^{q-1}(R')}(w')\} + h^*_{\varDelta_0(R'), \varDelta_0(R)}(w_q) < 1\} \\ &= \{(z, w', w_q) \in \varDelta_{a_0}(r) \times \varDelta_0^{q-1}(R') \times \varDelta_0(R) : \\ &\qquad\qquad\qquad h^*_{A \cap \varDelta_{a_0}(r), \varDelta_{a_0}(r)}(z) + h^*_{\varDelta_0(R'), \varDelta_0(R)}(w_q) < 1\}.\end{aligned}$$

Since $R'' < R$, we find an $r_q \in (0, r]$ such that any function $f \in \mathcal{O}_s(X \setminus M)$ extends holomorphically to a function $\widetilde{f}_q$ on $\varDelta_{a_0}(r_q) \times \varDelta_0^{q-1}(R') \times \varDelta_0(R'') \setminus S_q$. We may assume that S_q is singular with respect to the family $\{\widetilde{f}_q : f \in \mathcal{O}_s(X \setminus M)\}$.

Repeating the above argument for the coordinates w_ν, $\nu = 1, \dots, q-1$, and gluing the obtained sets, we find an $r_0 \in (0, r]$ and a relatively closed pluripolar set $S_0 := \bigcup_{j=1}^q S_j$ such that any function $f \in \mathcal{O}_s(X \setminus M)$ extends holomorphically to a function $\widetilde{f}_0 := \bigcup_{j=1}^q \widetilde{f}_j$ holomorphic in $\varDelta_{a_0}(r_0) \times \varOmega \setminus S_0$, where

$$\varOmega := \bigcup_{\nu=1}^{q} \varDelta_0^{j-1}(R') \times \varDelta_0(R'') \times \varDelta_0^{q-j}(R').$$

Let $\widehat{\varOmega}$ denote the envelope of holomorphy of $\varOmega$. Applying the Chirka theorem (Theorem 6), we find a relatively closed pluripolar subset $\widehat{S}_0$ of $\varDelta_{a_0}(r_0) \times \widehat{\varOmega}$ such that any function $f \in \mathcal{O}_s(X \setminus M)$ extends to a function $\widehat{f}$ holomorphic on $\varDelta_{a_0}(r_0) \times \widehat{\varOmega} \setminus \widehat{S}_0$. Let $R''' := \sqrt[q]{R'^{q-1} R''}$. Observe that $\varDelta_0(R''') \subset \widehat{\varOmega}$. Recall that $R''' > R'_0$. We may assume that $\widehat{M}$ is singular with respect to the family $\{\widehat{f} : f \in \mathcal{O}_s(X \setminus M)\}$. To get a contradiction it suffices to show that $\widehat{M} \cap X \subset M$. We argue as in the proof of Lemma 11:

Take $(a, b) \in (A \cap \varDelta_{a_0}(r_0)) \times \varDelta_0^q(R''') \setminus M$. Since $M_{(a,\cdot)}$ is pluripolar, there exists a curve $\gamma : [0, 1] \longrightarrow \varDelta_0(R''') \setminus M_{(a,\cdot)}$ such that $\gamma(0) = 0$, $\gamma(1) = b$. Take an $\varepsilon > 0$ so small that

$$\varDelta_a(\varepsilon) \times (\gamma([0, 1]) + \varDelta_0^q(\varepsilon)) \subset \varDelta_{a_0}(r) \times \varDelta_0^q(R''') \setminus M.$$

Put $V_b := E^q \cup (\gamma([0, 1]) + \varDelta_0^q(\varepsilon))$ and consider the cross

$$Y := \mathbb{X}(A \cap \varDelta_a(\varepsilon), E^q; \varDelta_a(\varepsilon), V_b).$$

Then $f \in \mathcal{O}_s(Y)$ for any $f \in \mathcal{O}_s(X \setminus M)$. Consequently, by Theorem 3, $\widehat{Y} \subset \varDelta_{a_0}(r) \times \varDelta_0^q(R''') \setminus \widehat{M}$, which implies that $\widehat{M}_{(a,\cdot)} \cap \varDelta_0^q(R''') \subset M_{(a,\cdot)}$. $\square$

3. Proof of the Main Theorem for $N = 2$

To simplify notation, put $p := n_1$, $D := D_1$, $A := A_1$, $A' := A \setminus \Sigma_2$, $q := n_2$, $G := D_2$, $B := A_2$, $B' := B \setminus \Sigma_1$.

It suffices to verify the assumptions of Lemma 10. Let $(D_j)_{j=1}^{\infty}$, $(G_j)_{j=1}^{\infty}$ be approximation sequences: $D_j \Subset D_{j+1} \Subset D$, $G_j \Subset G_{j+1} \Subset G$, $D_j \nearrow D$, $G_j \nearrow G$, $A' \cap D_j \neq \varnothing$, and $B' \cap G_j \neq \varnothing$, $j \in \mathbb{N}$.

Fix $j \in \mathbb{N}$, $a \in A' \cap D_j$, and let Ω_j be the set of all $b \in G_{j+1}$ such that there exist a polydisc $\Delta_{(a,b)}(r_b) \subset D_j \times G_{j+1}$ and a relatively closed pluripolar set $S_b \subset \Delta_{(a,b)}(r_b)$ such that:

- $S_b \cap ((A' \cap \Delta_a(r_b)) \times \Delta_b(r_b)) \subset M$,
- any function $f \in \mathcal{O}_s(X \setminus M)$ extends to a function $\widetilde{f}_b \in \mathcal{O}(\Delta_{(a,b)}(r_b) \setminus S_b)$ with $\widetilde{f}_b = f$ on $(A' \cap \Delta_a(r_b)) \times \Delta_b(r_b) \setminus M$, and
- S_b is singular with respect to the family $\{\widetilde{f}_b : f \in \mathcal{O}_s(X \setminus M)\}$.

It is clear that Ω_j is open. Observe that $\Omega_j \neq \varnothing$. Indeed, since $B \cap G_j \setminus M_{(a,\cdot)} \neq \varnothing$, we find a point $b \in B \cap G_j \setminus M_{(a,\cdot)}$. Therefore there is a polydisc $\Delta_{(a,b)}(r) \subset D_j \times G_j \setminus M$. Put

$$Y := \mathbb{X}(A \cap \Delta_a(r), B \cap \Delta_b(r); \Delta_a(r), \Delta_b(r)).$$

By Theorem 3, we find an $r_b \in (0, r)$ such that any function $f \in \mathcal{O}_s(X \setminus M)$ extends to $\widetilde{f}_b \in \mathcal{O}(\Delta_{(a,b)}(r_b))$ with $\widetilde{f}_b = f$ on $\Delta_{(a,b)}(r_b) \cap Y \supset (A \cap \Delta_a(r_b)) \times \Delta_b(r_b)$. Consequently, $b \in \Omega_j$.

Moreover, Ω_j is relatively closed in G_{j+1}. Indeed, let c be an accumulation point of Ω_j in G_{j+1} and let $\Delta_c(3R) \subset G_{j+1}$. Take a point $b \in \Omega_j \cap \Delta_c(R) \setminus M_{(a,\cdot)}$ and let $r \in (0, r_b]$, $r < 2R$, be such that $\Delta_{(a,b)}(r) \cap M = \varnothing$. Observe that $\widetilde{f}_b \in \mathcal{O}(\Delta_{(a,b)}(r))$ and $\widetilde{f}_b(z, \cdot) = f(z, \cdot) \in \mathcal{O}(\Delta_b(2R) \setminus M_{(z,\cdot)})$ for any $z \in A' \cap \Delta_a(r)$. Hence, by Lemma 12 (with $R' := R$), there exists a relatively closed pluripolar set $S \subset \Delta_a(\rho') \times \Delta_b(R)$ with $\rho' \in (0, r)$ such that any f has an extension $\widehat{\widetilde{f}}_b \in \mathcal{O}(\Delta_a(\rho') \times \Delta_b(R) \setminus S)$. Take an $r_c > 0$ so small that $\Delta_{(a,c)}(r_c) \subset \Delta_a(\rho') \times \Delta_b(R)$, and put $S_c := S \cap \Delta_{(a,c)}(r_c)$, $\widetilde{f}_c := \widehat{\widetilde{f}}_b$ on $\Delta_{(a,c)}(r_c) \setminus S_c$. Obviously $\widetilde{f}_c = \widehat{\widetilde{f}}_b = f$ on $(A' \cap \Delta_a(r_c)) \times \Delta_c(r_c) \setminus M$. Hence $c \in \Omega_j$.

Thus $\Omega_j = G_{j+1}$. There exists a finite set $T \subset \overline{G}_j$ such that

$$\overline{G}_j \subset \bigcup_{b \in T} \Delta_b(r_b).$$

Define $r_{a,j} := \min\{r_b : b \in T\}$. Take $b', b'' \in T$ with $C := \Delta_{b'}(r_{b'}) \cap \Delta_{b''}(r_{b''}) \neq \varnothing$. Then $\widetilde{f}_{b'} = f = \widetilde{f}_{b''}$ on $(A' \cap \Delta_a(r_{a,j})) \times (\Delta_{b'}(r_{b'}) \cap \Delta_{b''}(r_{b''})) \setminus M$. Consequently, $\widetilde{f}_{b'} = \widetilde{f}_{b''}$ on $\Delta_a(r_{a,j}) \times C \setminus (S_{b'} \cup S_{b''})$. In particular, using the minimality of the sets $S_{b'}$ and $S_{b''}$, we conclude that they coincide on $\Delta_a(r_{a,j}) \times C$ and that the functions $f_{b'}$ and $f_{b''}$ glue together. Thus we get a relatively closed pluripolar set $S_{a,j} \subset \Delta_a(r_{a,j}) \times G_j$ such that $S_{a,j} \cap ((A' \cap \Delta_a(r_{a,j})) \times G_j) \subset M$ and any function $f \in \mathcal{O}_s(X \setminus M)$ extends holomorphically to an $f_{a,j} \in \mathcal{O}(\Delta_a(r_{a,j}) \times G_j \setminus S_{a,j})$ with $f_{a,j} = f$ on $(A' \cap \Delta_a(r_{a,j})) \times G_j \setminus M$.

Changing the roles of z and w, we get $S^{b,j}$ and $f^{b,j}$, $b \in B' \cap G_j$. $\square$

The above proof of the Main Theorem for $N = 2$ shows that the following generalization of Lemma 12 is true.

Theorem 13. *Let $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$ be pseudoconvex domains, let $A \subset D$ be locally pluriregular, let $B \subset G$ be open and nonempty, and let $M \subset U$ be a relatively closed subset of an open neighborhood U of the cross $X := \mathbb{X}(A, B; D, G)$ such that $M \cap (D \times B) = \varnothing$ and for any $a \in A$ the fiber $M_{(a,\cdot)}$ is pluripolar. Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{X}$ such that:*
- *$\widehat{M} \cap X \subset M$,*
- *for any $f \in \mathcal{O}_s(X \setminus M)$ there exists exactly one $\widehat{f} \in \mathcal{O}(\widehat{X} \setminus \widehat{M})$ with $\widehat{f} = f$ on $X \setminus M$, and*
- *the set $\widehat{M}$ is singular with respect to the family $\{\widehat{f} : f \in \mathcal{O}_s(X \setminus M)\}$.*

Observe that if $G = \mathbb{C}^q$, then $\widehat{X} = D \times \mathbb{C}^q$. Consequently, Theorem 13 also generalizes Theorem 7.

Proof. We apply Lemma 10 (as in the proof of the Main Theorem for $N = 2$). The functions $f_{a,j}$ are constructed exactly as in that proof (with $A' = A$). The functions $f^{b,j}$ are simply given as $f^{b,j} := f|_{D_j \times \varDelta_b(s_{b,j})}$ with $\varDelta_b(s_{b,j}) \subset B \cap D_j$ ($S^{b,j} := \varnothing$). $\square$

4. Proof of the Main Theorem

First observe that, by Lemma 8(b), the set $X' \setminus M$ is not pluripolar. Consequently, the function $\widehat{f}$ is uniquely determined.

We proceed by induction on N. The case $N = 2$ is proved.

Let $D_{j,k} \nearrow D_j$, $D_{j,k} \Subset D_{j,k+1} \Subset D_j$, where the $D_{j,k}$ are pseudoconvex domains with $A_{j,k} := A_j \cap D_{j,k} \neq \varnothing$, $j = 1, \dots, N$. Put

$$X_k := \mathbb{X}(A_{1,k}, \dots, A_{N,k}; D_{1,k}, \dots, D_{N,k}) \subset X,$$
$$\Sigma_{j,k} := (A_{1,k} \times \dots \times A_{j-1,k} \times A_{j+1,k} \times \dots \times A_{N,k}) \cap \Sigma_j, \quad j = 1, \dots, N,$$
$$X'_k := \mathbb{T}(A_{1,k}, \dots, A_{N,k}; D_{1,k}, \dots, D_{N,k}; \Sigma_{1,k}, \dots, \Sigma_{N,k}) \subset X_k.$$

It suffices to show that for each $k \in \mathbb{N}$ the following condition (*) holds.

(*) There exist a domain U_k, $X'_k \subset U_k \subset \widehat{X}_k$, and a relatively closed pluripolar set $M_k \subset U_k$, such that:
- $M_k \cap X'_k \subset M$, and
- for any $f \in \mathcal{O}_s(X \setminus M)$ there exists an $\widetilde{f}_k \in \mathcal{O}(U_k \setminus M_k)$ with $\widetilde{f}_k = f$ on $X'_k \setminus M$.

Indeed, fix a $k \in \mathbb{N}$ and observe that, by Lemma 9, $\widehat{X}_k$ is the envelope of holomorphy of U_k. Hence, by virtue of the Chirka theorem (Theorem 6), there exists a relatively closed pluripolar set $\widehat{M}_k$ of $\widehat{X}_k$, $\widehat{M}_k \cap U_k \subset M_k$, such that $\widehat{X}_k \setminus \widehat{M}_k$ is the envelope of holomorphy of $U_k \setminus M_k$. In particular, for each $f \in \mathcal{O}_s(X \setminus M)$ there exists an $\widehat{f}_k \in \mathcal{O}(\widehat{X}_k \setminus \widehat{M}_k)$ with $\widehat{f}_k|_{U_k \setminus M_k} = \widetilde{f}_k$. We may assume that $\widehat{M}_k$ is singular with respect to the family $\{\widehat{f}_k : f \in \mathcal{O}_s(X \setminus M)\}$.

In particular, $\widehat{M}_{k+1} \cap \widehat{X}_k = \widehat{M}_k$. Consequently:
- $\widehat{M} := \bigcup_{k=1}^{\infty} \widehat{M}_k$ is a relatively closed pluripolar subset of $\widehat{X}$ with $\widehat{M} \cap X' \subset M$,
- for each $f \in \mathcal{O}_s(X \setminus M)$, the function $\widehat{f} := \bigcup_{k=1}^{\infty} \widehat{f}_k$ is holomorphic on $\widehat{X} \setminus \widehat{M}$ with $\widehat{f} = f$ on $X' \setminus M$, and
- $\widehat{M}$ is singular with respect to the family $\{\widehat{f} : f \in \mathcal{O}_s(X \setminus M)\}$.

It remains to prove (*). Fix a $k \in \mathbb{N}$. For any

$$a = (a_1, \dots, a_N) \in A_{1,k} \times \dots \times A_{N,k} \setminus M$$

let $\tau = \tau_k(a)$ be such that $\mathit{\Delta}_a(\tau) \subset D_{1,k} \times \cdots \times D_{N,k} \setminus M$. Consider the N–fold cross

$$Y_a := \mathbb{X}(A_1 \cap \mathit{\Delta}_{a_1}(\tau), \dots, A_N \cap \mathit{\Delta}_{a_N}(\tau); \mathit{\Delta}_{a_1}(\tau), \dots, \mathit{\Delta}_{a_N}(\tau)).$$

Observe that any function from $\mathcal{O}_s(X \setminus M)$ belongs to $\mathcal{O}_s(Y_a)$. Consequently, by Theorem 3, any function from $\mathcal{O}_s(X \setminus M)$ extends holomorphically to $\widehat{Y}_a$. Let $\rho = \rho_k(a) \in (0, \tau]$ be such that $\mathit{\Delta}_a(\rho) \subset \widehat{Y}_a$.

If $N \geq 4$, then we additionally define $(N-2)$–fold crosses

$$\begin{aligned} Y_{k,\mu,\nu} := \mathbb{X}(&A_{1,k}, \dots, A_{\mu-1,k}, A_{\mu+1,k}, \dots, A_{\nu-1,k}, A_{\nu+1,k}, \dots, A_{N,k};\\ &D_{1,k}, \dots, D_{\mu-1,k}, D_{\mu+1,k}, \dots, D_{\nu-1,k}, D_{\nu+1,k}, \dots, D_{N,k}),\\ &\qquad\qquad 1 \leq \mu < \nu \leq N, \end{aligned}$$

and we assume that ρ is so small that

$$\mathit{\Delta}_{(a_1,\dots,a_{\mu-1},a_{\mu+1},\dots,a_{\nu-1},a_{\nu+1},\dots,a_N)}(\rho) \subset \widehat{Y}_{k,\mu,\nu}, \quad 1 \leq \mu < \nu \leq N.$$

For $j \in \{1, \dots, N\}$, define the 2–fold crosses

$$\begin{aligned} Z'_{k,a,j} := \Big\{(z', z_j, z'') \in ((A_1 \cap \mathit{\Delta}_{a_1}(\rho)) \times \cdots \times (A_{j-1} \cap \mathit{\Delta}_{a_{j-1}}(\rho))) \times D_{j,k+1}\\ \times ((A_{j+1} \cap \mathit{\Delta}_{a_{j+1}}(\rho)) \times \cdots \times (A_N \cap \mathit{\Delta}_{a_N}(\rho))) : (z', z'') \notin \Sigma_j\Big\} \cup \mathit{\Delta}_a(\rho), \end{aligned}$$

$$\begin{aligned} Z_{k,a,j} := \Big((A_1 \cap \mathit{\Delta}_{a_1}(\rho)) \times \cdots \times (A_{j-1} \cap \mathit{\Delta}_{a_{j-1}}(\rho)) \times D_{j,k+1}\\ \times (A_{j+1} \cap \mathit{\Delta}_{a_{j+1}}(\rho)) \times \cdots \times (A_N \cap \mathit{\Delta}_{a_N}(\rho)\Big) \cup \mathit{\Delta}_a(\rho). \end{aligned}$$

Now, we apply Theorem 13 to the 2–fold cross $Z'_{k,a,j}$ and the set M. We find a relatively closed pluripolar set $S_{k,a,j} \subset \widehat{Z}'_{k,a,j} = \widehat{Z}_{k,a,j}$ such that:

- $S_{k,a,j} \cap Z'_{k,a,j} \subset M$,
- for any function $f \in \mathcal{O}_s(X \setminus M)$ there exists an $\widetilde{f}_{k,a,j} \in \mathcal{O}(\widehat{Z}_{k,a,j} \setminus S_{k,a,j})$ such that $\widetilde{f}_{k,a,j} = f$ on $Z'_{k,a,j} \setminus M$, and
- $S_{k,a,j}$ is singular with respect to the space $\{\widetilde{f}_{k,a,j} : f \in \mathcal{O}_s(X \setminus M)\}$.

Observe that $\{(a_1, \dots, a_{j-1})\} \times \overline{D}_{j,k} \times \{(a_{j+1}, \dots, a_N)\} \Subset \widehat{Z}_{k,a,j}$. Consequently, we find $r = r_k(a) \in (0, \rho]$ such that

$$V_{k,a,j} := \mathit{\Delta}_{(a_1,\dots,a_{j-1})}(r) \times D_{j,k} \times \mathit{\Delta}_{(a_{j+1},\dots,a_N)}(r) \subset \widehat{Z}_{k,a,j}, \quad j = 1, \dots, N.$$

Let

$$V_k := \bigcup_{\substack{a \in A_{1,k} \times \cdots \times A_{N,k} \setminus M\\ j \in \{1,\dots,N\}}} V_{k,a,j}.$$

Note that $X'_k \subset V_k$. Let U_k be the connected component of $V_k \cap \widehat{X}_k$ that contains X_k.

It remains to glue the sets $S_{k,a,j}$ and functions $\widetilde{f}_{k,a,j}$. Then

$$S_k := \bigcup_{\substack{a \in A_{1,k} \times \cdots \times A_{N,k} \setminus M\\ j \in \{1,\dots,N\}}} S_{k,a,j} \cap U_k, \quad \widetilde{f}_k := \bigcup_{\substack{a \in A_{1,k} \times \cdots \times A_{N,k} \setminus M\\ j \in \{1,\dots,N\}}} \widetilde{f}_{k,a,j}|_{V_{k,a,j} \cap U_k \setminus S_k}$$

will satisfy (*).

To check that the gluing process is possible, let $a, b \in A_{1,k} \times \cdots \times A_{N,k} \setminus M$, $i, j \in \{1, \dots, N\}$ be such that $V_{k,a,i} \cap V_{k,b,j} \neq \varnothing$. We have the following two cases:

(a) $i \neq j$: We may assume that $i = N-1$, $j = N$. Write $w = (w', w'') \in \mathbb{C}^{n_1+\dots+n_{N-2}} \times \mathbb{C}^{n_{N-1}+n_N}$. Observe that

$$V_{k,a,N-1} \cap V_{k,b,N} = \Big(\varDelta_{a'}(r_k(a)) \cap \varDelta_{b'}(r_k(b))\Big) \times \varDelta_{b_{N-1}}(r_k(b)) \times \varDelta_{a_N}(r_k(a)).$$

We consider the following three subcases:

$N = 2$ (cf. the proof of Lemma 10): Then $V_{k,a,1} \cap V_{k,b,2} = \varDelta_{b_1}(r_k(b)) \times \varDelta_{a_2}(r_k(a))$. We know that $\widetilde{f}_{k,a,1} = \widetilde{f}_{k,b,2}$ on the non-pluripolar set

$$(A_1 \cap \varDelta_{b_1}(r_k(b)) \setminus \Sigma_2) \times (A_2 \cap \varDelta_{a_2}(r_k(a)) \setminus \Sigma_1) \setminus M;$$

cf. Lemma 8(b). Hence, by the identity principle, $\widetilde{f}_{k,a,1} = \widetilde{f}_{k,b,2}$ on $V_{k,a,1} \cap V_{k,b,2} \setminus (S_{k,a,1} \cup S_{k,b,2})$. Consequently, the sets $S_{k,a,1}$, $S_{k,b,2}$ and the functions $\widetilde{f}_{k,a,1}$, $\widetilde{f}_{k,b,2}$ glue together.

$N = 3$: Then $V_{k,a,2} \cap V_{k,b,3} = (\varDelta_{a_1}(r_k(a)) \cap \varDelta_{b_1}(r_k(b)) \times \varDelta_{b_2}(r_k(b)) \times \varDelta_{a_3}(r_k(a))$. Let

$$C'' := (A_2 \cap \varDelta_{b_2}(r_k(b))) \times (A_3 \cap \varDelta_{a_3}(r_k(a))) \setminus \Sigma_1.$$

Recall that for any $c'' \in C''$ the fiber $M_{(\cdot,c'')}$ is pluripolar. We have $\widetilde{f}_{k,a,2}(\cdot, c'') = f(\cdot, c'') = \widetilde{f}_{k,b,3}(\cdot, c'')$ on $\varDelta_{a_1}(r_k(a)) \cap \varDelta_{b_1}(r_k(b)) \setminus M_{(\cdot,c'')}$.

Now, let C' denote the set of all $c' \in \varDelta_{a_1}(r_k(a)) \cap \varDelta_{b_1}(r_k(b))$ such that the fiber $(S_{k,a,2} \cup S_{k,b,3})_{(c',\cdot)}$ is pluripolar. Recall that the complement of C' is pluripolar (Lemma 8(a)). If $c' \in C'$, then $\widetilde{f}_{k,a,2}(c', \cdot) = \widetilde{f}_{k,b,3}(c', \cdot)$ on $C'' \setminus (S_{k,a,2} \cup S_{k,b,3})_{(c',\cdot)}$. Consequently, by the identity principle, $\widetilde{f}_{k,a,2}(c', \cdot) = \widetilde{f}_{k,b,3}(c', \cdot)$ on $\varDelta_{b_2}(r_k(b)) \times \varDelta_{a_3}(r_k(a)) \setminus (S_{k,a,2} \cup S_{k,b,3})_{(c',\cdot)}$, $c' \in C'$. Finally, $\widetilde{f}_{k,a,2} = \widetilde{f}_{k,b,3}$ on $V_{k,a,2} \cap V_{k,b,3} \setminus (S_{k,a,2} \cup S_{k,b,3})$. Consequently, the sets $S_{k,a,2}$, $S_{k,b,3}$ and the functions $\widetilde{f}_{k,a,2}$, $\widetilde{f}_{k,b,3}$ glue together.

If $N \in \{2, 3\}$, then we jump directly to (b), and we conclude that the Main Theorem is true for $N \in \{2, 3\}$.

$N \geq 4$: Here is the only place where the induction over N is used. We assume that the Main Theorem is true for $N - 1 \geq 3$.

Let

$$\begin{aligned} C'' := \{c'' \in (A_{N-1} \cap \varDelta_{b_{N-1}}(r_k(b))) \times (A_N \cap \varDelta_{a_N}(r_k(a))) : \\ (\Sigma_s)_{(\cdot,c'')} \text{ is pluripolar, } s = 1, \dots, N-2\}; \end{aligned}$$

note that, by Lemma 8(a), C'' is not pluripolar. For any $c'' \in C''$ the function $f_{c''} := f(\cdot, c'')$ is separately holomorphic on $Y_{k,N-1,N} \setminus M_{(\cdot,c'')}$. Moreover, the set $M_{(\cdot,c'')}$ satisfies all the assumptions of the Main Theorem. Indeed,

$$\begin{aligned} \Sigma_s(A_{1,k}, \dots, A_{N-2,k}; M_{(\cdot,c'')}) = (\Sigma_s(A_{1,k}, \dots, A_{N,k}; M))_{(\cdot,c'')} \subset (\Sigma_s)_{(\cdot,c'')}, \\ s = 1, \dots, N-2. \end{aligned}$$

By the inductive assumption, the function $f_{c''}$ extends to a function

$$\widehat{f}_{c''} \in \mathcal{O}(\widehat{Y}_{k,N-1,N} \setminus \widehat{M}(c'')),$$

where $\widehat{M}(c'')$ is a relatively closed pluripolar subset of $\widehat{Y}_{k,N-1,N}$ such that $\widehat{M}(c'') \cap Y'_{k,N-1,N} \subset M_{(\cdot,c'')}$. Recall that

$$\varDelta_{a'}(r_k(a)) \cup \varDelta_{b'}(r_k(b)) \subset \widehat{Y}_{k,N-1,N}.$$

Since $\widetilde{f}_{k,a,N-1}(\cdot, c'') = f_{c''}$ on $\Delta_{a'}(r_k(a)) \cap Y'_{k,N-1,N} \setminus M_{(\cdot,c'')}$ and $\widetilde{f}_{k,b,N}(\cdot, c'') = f_{c''}$ on $\Delta_{b'}(r_k(b)) \cap Y'_{k,N-1,N} \setminus M_{(\cdot,c'')}$, we conclude that $\widetilde{f}_{k,a,N-1}(\cdot, c'') = \widehat{f}_{c''} = \widetilde{f}_{k,b,N}(\cdot, c'')$ on $\Delta_{a'}(r_k(a)) \cap \Delta_{b'}(r_k(b)) \setminus M_{(\cdot,c'')}$.

Let $c' \in \Delta_{a'}(r_k(a)) \cap \Delta_{b'}(r_k(b))$ be such that the fiber $(S_{k,a,N-1} \cup S_{k,b,N})_{(c',\cdot)}$ is pluripolar. Then $\widetilde{f}_{k,a,N-1}(c', \cdot) = \widetilde{f}_{k,b,N}(c', \cdot)$ on $C'' \setminus (S_{k,a,N-1} \cup S_{k,b,N})_{(\cdot,c')}$. Consequently, by the identity principle, $\widetilde{f}_{k,a,N-1}(c', \cdot) = \widetilde{f}_{k,b,N}(c', \cdot)$ on $(\Delta_{b_{N-1}}(r_k(b)) \times \Delta_{a_N}(r_k(a))) \setminus (S_{k,a,N-1} \cup S_{k,b,N})_{c'}$ and, finally, $\widetilde{f}_{k,a,N-1} = \widetilde{f}_{k,b,N}$ on $(V_{k,a,N-1} \cap V_{k,b,N}) \setminus (S_{k,a,N-1} \cup S_{k,b,N})$. Consequently, the sets $S_{k,a,N-1}$, $S_{k,b,N}$ and the functions $\widetilde{f}_{k,a,N-1}$, $\widetilde{f}_{k,b,N}$ glue together.

(b) $i = j$: We may assume that $i = j = N$. Observe that

$$V_{k,a,N} \cap V_{k,b,N} = \Big(\Delta_{(a_1,\dots,a_{N-1})}(r_k(a)) \cap \Delta_{(b_1,\dots,b_{N-1})}(r_k(b))\Big) \times D_{N,k}.$$

By (a) we know that

$$\widetilde{f}_{k,a,N} = \widetilde{f}_{k,a,N-1} \quad \text{on } V_{k,a,N} \cap V_{k,a,N-1} \setminus (S_{k,a,N} \cup S_{k,a,N-1}),$$
$$\widetilde{f}_{k,a,N-1} = \widetilde{f}_{k,b,N} \quad \text{on } V_{k,a,N-1} \cap V_{k,b,N} \setminus (S_{k,a,N-1} \cup S_{k,b,N}).$$

Hence (we write $w = (w', w_N) \in \mathbb{C}^{n_1+\dots+n_{N-1}} \times \mathbb{C}^{n_N}$)

$$\widetilde{f}_{k,a,N} = \widetilde{f}_{k,b,N}$$

on

$$\begin{aligned} V_{k,a,N} \cap V_{k,a,N-1} &\cap V_{k,b,N} \setminus (S_{k,a,N-1} \cup S_{k,a,N} \cup S_{k,b,N}) \\ &= \Big(\Delta_{a'}(r_k(a)) \cap \Delta_{b'}(r_k(b))\Big) \times \Delta_{a_N}(r_k(a)) \setminus (S_{k,a,N-1} \cup S_{k,a,N} \cup S_{k,b,N}), \end{aligned}$$

and finally, by the identity principle,

$$\widetilde{f}_{k,a,N} = \widetilde{f}_{k,b,N} \quad \text{on } V_{k,a,N} \cap V_{k,b,N} \setminus (S_{k,a,N} \cup S_{k,b,N}).$$

Consequently, the sets $S_{k,a,N}$, $S_{k,b,N}$ and the functions $\widetilde{f}_{k,a,N}$, $\widetilde{f}_{k,b,N}$ glue together.

The proof of the Main Theorem is completed. □

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