AN EXTENSION THEOREM
FOR SEPARATELY HOLOMORPHIC FUNCTIONS
WITH PLURIPOLAR SINGULARITIES

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Abstract. Let $D_j \subset \mathbb{C}^{n_j}$ be a pseudoconvex domain and let $A_j \subset D_j$ be a locally pluriregular set, $j = 1, \ldots, N$. Put

$$X := \bigcup_{j=1}^{N} A_1 \times \cdots \times A_{j-1} \times D_j \times A_{j+1} \times \cdots \times A_N \subset \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_N} = \mathbb{C}^n.$$ 

Let $U \subset \mathbb{C}^n$ be an open neighborhood of $X$ and let $M \subset U$ be a relatively closed subset of $U$. For $j \in \{1, \ldots, N\}$ let $\Sigma_j$ be the set of all $(z', z''_j) \in (A_1 \times \cdots \times A_{j-1}) \times (A_{j+1} \times \cdots \times A_N)$ for which the fiber $M(z', z''_j) := \{z_j \in \mathbb{C}^{n_j} : (z', z_j, z'') \in M\}$ is not pluripolar. Assume that $\Sigma_1, \ldots, \Sigma_N$ are pluripolar. Put

$$X' := \bigcup_{j=1}^{N} \{(z', z_j, z'') \in (A_1 \times \cdots \times A_{j-1}) \times D_j \times (A_{j+1} \times \cdots \times A_N) : (z', z'') \notin \Sigma_j\}.$$ 

Then there exists a relatively closed pluripolar subset $\hat{M} \subset \hat{X}$ of the “envelope of holomorphy” $\hat{X} \subset \mathbb{C}^n$ of $X$ such that:

- $\hat{M} \cap X' \subset M$,
- for every function $f$ separately holomorphic on $X \setminus M$ there exists exactly one function $\hat{f}$ holomorphic on $\hat{X} \setminus \hat{M}$ with $\hat{f} = f$ on $X' \setminus M$, and
- $\hat{M}$ is singular with respect to the family of all functions $\hat{f}$.

1. Introduction. Main Theorem

Let $N \in \mathbb{N}$, $N \geq 2$, and let

$$\varnothing \neq A_j \subset D_j \subset \mathbb{C}^{n_j},$$

where $D_j$ is a domain, $j = 1, \ldots, N$. We define an $N$-fold cross

$$X = \hat{X}(A_1, \ldots, A_N; D_1, \ldots, D_N)$$

$$:= \bigcup_{j=1}^{N} A_1 \times \cdots \times A_{j-1} \times D_j \times A_{j+1} \times \cdots \times A_N \subset \mathbb{C}^{n_1+\cdots+n_N} = \mathbb{C}^n.$$ 

Observe that $X$ is connected.
Let $\Omega \subset \mathbb{C}^n$ be an open set and let $A \subset \Omega$. Put
\[
h_{A,\Omega} := \sup\{u : u \in \mathcal{P}SH(\Omega), u \leq 1 \text{ on } \Omega, u \leq 0 \text{ on } A\},
\]
where $\mathcal{P}SH(\Omega)$ denotes the set of all plurisubharmonic functions on $\Omega$. Define
\[
\omega_{A,\Omega} := \lim_{k \to +\infty} h^*_{A \cap \Omega_k, \Omega_k},
\]
where $(\Omega_k)_{k=1}^{\infty}$ is a sequence of relatively compact open sets $\Omega_k \subset \Omega_{k+1} \Subset \Omega$ with $\bigcup_{k=1}^{\infty} \Omega_k = \Omega$ ($h^*$ denotes the upper semicontinuous regularization of $h$). Observe that the definition is independent of the exhausting sequence $(\Omega_k)_{k=1}^{\infty}$.

Moreover, if $\omega_{A,\Omega} \in \mathcal{P}SH(\Omega)$. Recall that if $\Omega$ is bounded, then $\omega_{A,\Omega} = h^*_{A,\Omega}$.

For an $N$–fold cross $X = \mathcal{X}(A_1, \ldots, A_N; D_1, \ldots, D_N)$ put
\[
\widehat{X} := \{(z_1, \ldots, z_N) \in D_1 \times \cdots \times D_N : \sum_{j=1}^{N} \omega_{A_j, D_j}(z_j) < 1\}.
\]
Observe that if $D_1, \ldots, D_N$ are pseudoconvex, then $\widehat{X}$ is a pseudoconvex open set in $\mathbb{C}^n$.

We say that a subset $\emptyset \neq A \subset \mathbb{C}^n$ is \textit{locally pluriregular} if $h^*_{A \cap \Omega, \Omega}(a) = 0$ for any $a \in A$ and for any open neighborhood $\Omega$ of $a$ (in particular, $A \cap \Omega$ is non-pluripolar).

Note that if $A_1, \ldots, A_N$ are locally pluriregular, then $X \subset \widehat{X}$ and $\widehat{X}$ is connected (§3, Lemma 4).

Let $U$ be an open neighborhood of $X$ and let $M \subset U$ be a relatively closed set. We say that a function $f : X \setminus M \longrightarrow \mathbb{C}$ is \textit{separately holomorphic} ($f \in \mathcal{O}_s(X \setminus M)$) if for any $(a_1, \ldots, a_N) \in A_1 \times \cdots \times A_N$ and $j \in \{1, \ldots, N\}$ the function $f(a_1, \ldots, a_{j-1},', a_{j+1}, \ldots, a_N)$ is holomorphic in the open set
\[
D_j \setminus M(a_1, \ldots, a_{j-1},', a_{j+1}, \ldots, a_N),
\]
where
\[
M(a_1, \ldots, a_{j-1},', a_{j+1}, \ldots, a_N) := \{z \in \mathbb{C}^n : (a_1, \ldots, a_{j-1}, z, a_{j+1}, \ldots, a_N) \in M\}.
\]

Suppose that $S_j \subset A_1 \times \cdots \times A_{j-1} \times A_{j+1} \times \cdots \times A_N$, $j = 1, \ldots, N$, and define the \textit{generalized $N$–fold cross}
\[
T = T(A_1, \ldots, A_N; D_1, \ldots, D_N; S_1, \ldots, S_N)
\]
\[
:= \bigcup_{j=1}^{N} \{(z', z_j, z'') \in (A_1 \times \cdots \times A_{j-1}) \times D_j \times (A_{j+1} \times \cdots \times A_N) : (z', z'') \notin S_j\}.
\]

It is clear that $T \subset X$. Observe that
\[
\mathcal{X}(A_1, \ldots, A_N; D_1, \ldots, D_N) = T(A_1, \ldots, A_N; D_1, \ldots, D_N; \emptyset, \ldots, \emptyset).
\]
Moreover, if $N = 2$, then $T(A_1, A_2; D_1, D_2; S_1, S_2) = \mathcal{X}(A_1 \setminus S_2, A_2 \setminus S_1; D_1, D_2)$. Consequently, any generalized 2–fold cross is a 2–fold cross.

Let $S \subset \Omega$ be a relatively closed pluripolar subset of an open set $\Omega \subset \mathbb{C}^n$. Let $\mathcal{F} \subset \mathcal{O}(\Omega \setminus S)$. We say that $S$ is \textit{singular with respect to $\mathcal{F}$} if for each point $a \in S$ there exists a function $f_a \in \mathcal{F}$ that is not holomorphically extendible to a neighborhood of $a$ (cf. [5], § 3.4). Equivalently: the set $S$ is minimal in the sense that there is no relatively closed set $S' \subset S$ such that any function from $\mathcal{F}$ extends holomorphically to $\Omega \setminus S'$.

Let $\mathcal{F} \subset \mathcal{O}(\Omega \setminus S)$, we say that $S$ is \textit{singular with respect to $\mathcal{F}$} if for each point $a \in S$ there exists a function $f_a \in \mathcal{F}$ that is not holomorphically extendible to a neighborhood of $a$ (cf. [5], § 3.4). Equivalently: the set $S$ is minimal in the sense that there is no relatively closed set $S' \subset S$ such that any function from $\mathcal{F}$ extends holomorphically to $\Omega \setminus S'$. It is clear that for any relatively closed pluripolar set $S \subset \Omega$ and for any family $\mathcal{F} \subset \mathcal{O}(\Omega \setminus S)$ there exists a relatively closed set $S' \subset S$
such that any function \( f \in \mathcal{F} \) extends to an \( f' \in \mathcal{O}(\Omega \setminus S') \) and \( S' \) is singular with respect to the family \( \{ f' : f \in \mathcal{F} \} \).

The main result of our paper is the following extension theorem for separately holomorphic functions.

**Main Theorem.** Let \( D_j \subset \mathbb{C}^n_j \) be a pseudoconvex domain, let \( A_j \subset D_j \) be a locally pluriregular set, \( j = 1, \ldots, N \), and let \( U \) be an open neighborhood of the \( N \)-fold cross

\[
X := X(A_1, \ldots, A_N; D_1, \ldots, D_N).
\]

Let \( M \subset U \) be a relatively closed subset of \( U \) such that for each \( j \in \{1, \ldots, N\} \) the set

\[
\Sigma_j = \Sigma_j(A_1, \ldots, A_N; M) := \{(z', z'') \in (A_1 \times \cdots \times A_{j-1}) \times (A_{j+1} \times \cdots \times A_N) : M_{(z', z'')} \text{ is not pluripolar}\}
\]

is pluripolar. Put

\[
X' := T(A_1, \ldots, A_N; D_1, \ldots, D_N; \Sigma_1, \ldots, \Sigma_N).
\]

Then there exists a relatively closed pluripolar set \( \widetilde{M} \subset \widehat{X} \) such that:

- \( \widetilde{M} \cap X' \subset M \),
- for every \( f \in \mathcal{O}_s(X \setminus M) \) there exists exactly one \( \widehat{f} \in \mathcal{O}(\widehat{X} \setminus \widehat{M}) \) with \( \widehat{f} = f \) on \( X' \setminus M \),
- \( \widetilde{M} \) is singular with respect to the family \( \{ \widehat{f} : f \in \mathcal{O}_s(X \setminus M) \} \), and
- \( \widehat{X} \setminus \widehat{M} \) is pseudoconvex.

In particular, \( \widehat{X} \setminus \widehat{M} \) is the envelope of holomorphy of \( X \setminus M \) with respect to the space of separately holomorphic functions.

Notice that if \( M \subset U \) is a pluripolar set, then \( \Sigma_1, \ldots, \Sigma_N \) are always pluripolar (cf. Lemma 8(a)).

The case where \( N = 2, n_1 = n_2 = 1, D_1 = D_2 = \mathbb{C} \) was studied in [7], Theorem 2.

**Corollary 1.** Let \( D_j, A_j, j = 1, \ldots, N, X, \) and \( U \) be as in the Main Theorem. Assume that \( M \subset U \) is a relatively closed set such that for any \( (a_1, \ldots, a_N) \in A_1 \times \cdots \times A_N \) and \( j \in \{1, \ldots, N\} \) the fiber \( M_{(a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_N)} \) is pluripolar.

Then there exists a relatively closed pluripolar set \( \widetilde{M} \subset \widehat{X} \) such that:

- \( \widetilde{M} \cap X' \subset M \),
- for every \( f \in \mathcal{O}_s(X \setminus M) \) there exists exactly one \( \widehat{f} \in \mathcal{O}(\widehat{X} \setminus \widehat{M}) \) with \( \widehat{f} = f \) on \( X' \setminus M \), and
- the domain \( \widehat{X} \setminus \widehat{M} \) is pseudoconvex.

The case where \( N = 2, D_2 = \mathbb{C}^{n_2} \), and \( A_2 \) is open was studied in [4] (for \( n_2 = 1 \)) and in [9] (for arbitrary \( n_2 \)).

The proof of the Main Theorem will be presented in Sections 3 (for \( N = 2 \)) and 4 (for arbitrary \( N \)).

The following two examples illustrate the role played by the sets \( \Sigma_j \) and show that the assertion of the Main Theorem is in some sense optimal.

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1 That is, \( \Sigma_1 = \cdots = \Sigma_N = \emptyset \).
Example 2. Let $n_1 = n_2 = 1$, $D_1 = D_2 = \mathbb{C}$, $A_1 = E :=$ the unit disc.

(a) Let $A_2 := E$, $X := \mathcal{X}(E, E; \mathbb{C}, \mathbb{C}) = (E \times \mathbb{C}) \cup (\mathbb{C} \times E)$, and $M := \{0\} \times \overline{\mathbb{C}}$.

Then $\Sigma_1 = \emptyset$, $\Sigma_2 = \{0\}$, $X' = \mathcal{X}(E \setminus \{0\}, E; \mathbb{C}, \mathbb{C})$, $\widehat{M} = \{0\} \times \mathbb{C}$.

Put $f_0(z, w) := 1/z$, $z \neq 0$, and $f_0(0, w) = 1$, $|w| > 1$. Then $f_0 \in \mathcal{O}_s(X \setminus M)$ and $\widehat{M}$ is singular with respect to $f_0$.

(b) Let $A_2 := E \setminus rE$, $X := \mathcal{X}(E, E \setminus rE; \mathbb{C}, \mathbb{C})$, and $M := \{0\} \times \{|w| = r\}$ for some $0 < r < 1$. Then $\Sigma_1 = \emptyset$, $\Sigma_2 = \{0\}$, $X' = \mathcal{X}(E \setminus \{0\}, A_2; \mathbb{C}, \mathbb{C})$, $\widehat{M} = \emptyset$.

Put

$$f_0(z, w) := \begin{cases} w & \text{if } z \neq 0 \text{ or } (z = 0 \text{ and } |w| > r), \\ 0 & \text{if } z = 0 \text{ and } |w| < r, \end{cases} \quad (z, w) \in X \setminus M.$$ 

Then $f_0 \in \mathcal{O}_s(X \setminus M)$, $\widehat{f}_0(z, w) \equiv w$, and $\widehat{f}_0(0, w) \neq f(0, w)$, $0 < |w| < r$.

2. Auxiliary Results

In the case $M = \emptyset$ the problem of extension of separately holomorphic functions was studied by many authors (under various assumptions on $(D_j, A_j)_{j=1}^N$), e.g. [17], [20], [18], [16], [12], [10], [1] (for $N = 2$), and [18], [13], [8] (for arbitrary $N$).

Theorem 3 ([13], [1]). Let $(D_j, A_j)_{j=1}^N$ and $X$ be as in the Main Theorem. Then any function from $\mathcal{O}_s(X)$ extends holomorphically to the pseudoconvex domain $\widehat{X}$.

The case where $M$ is analytic was studied in [14], [15], [19], [6]. The problem was completely solved in [8].

Theorem 4 ([8]). Let $(D_j, A_j)_{j=1}^N$ and $X$ be as in the Main Theorem. Let $M \subset U$ be an analytic subset of an open connected neighborhood $U$ of $X$. Then there exists an analytic set $\widehat{M} \subset \widehat{X}$ such that:

- $\widehat{M} \cap U_0 \subset M$ for an open neighborhood $U_0$ of $X$, $U_0 \subset U$,
- for every $f \in \mathcal{O}_s(X \setminus M)$ there exists exactly one $\widehat{f} \in \mathcal{O}(\widehat{X} \setminus \widehat{M})$ with $\widehat{f} = f$ on $X \setminus M$, and
- the domain $\widehat{X} \setminus \widehat{M}$ is pseudoconvex.

Remark 5. It is a natural idea to try to obtain Theorem 4 from the Main Theorem. More precisely, let $(D_j, A_j)_{j=1}^N$, $X$, $U$, and $M$ be as in Theorem 4. Then, by the Main Theorem, there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{X}$ which has all the properties listed in the Main Theorem. We would like to know whether there is a direct argument showing that $\widehat{M}$ must be analytic.

The following two results will play the fundamental role in the sequel.

Theorem 6 ([3]). Let $D \subset \mathbb{C}^n$ be a domain and let $\widehat{D}$ be the envelope of holomorphy of $D$. Assume that $S$ is a relatively closed pluripolar subset of $D$. Then there exists a relatively closed pluripolar subset $\widehat{S}$ of $\widehat{D}$ such that $\widehat{S} \cap D \subset S$ and $\widehat{D} \setminus \widehat{S}$ is the envelope of holomorphy of $D \setminus S$.

Theorem 7 ([3]). Let $A \subset E^{n-1}$ be locally pluriregular, let

$$X := \mathcal{X}(A, E; E^{n-1}, \mathbb{C})$$

(notice that $\widehat{X} = E^{n-1} \times \mathbb{C}$), and let $U \subset E^{N-1} \times \mathbb{C}$ be an open neighborhood of $X$. Let $M \subset U$ be a relatively closed set such that $M \cap E^n = \emptyset$ and for any
a ∈ A the fiber $M_{(a, \cdot)}$ is polar. Then there exists a relatively closed pluripolar set $S \subset E^{n-1} \times \mathbb{C}$ such that

- $S \cap X \subset M$,
- any function from $\mathcal{O}_s(X \setminus M)$ extends holomorphically to $E^{n-1} \times \mathbb{C} \setminus S$, and
- $E^{n-1} \times \mathbb{C} \setminus S$ is pseudoconvex.

Notice that the above result is a special case of our Main Theorem with $N = 2$, $n_1 = n - 1$, $D_1 = E^{n-1}$, $A_1 = A$, $n_2 = 1$, $D_2 = \mathbb{C}$, $A_2 = E$, $\Sigma_1 = \Sigma_2 = \emptyset$.

**Proof.** It is known (cf. [4]) that each function $f \in \mathcal{O}_s(X \setminus M)$ has the univalent domain of existence $G_f \subset E^{n-1} \times \mathbb{C}$ 

Let $G$ denote the connected component of $\operatorname{int} \bigcap_{f \in \mathcal{O}_s(X \setminus M)} G_f$ that contains $E^n$ and let $S := E^{n-1} \times \mathbb{C} \setminus G$. It remains to show that $S$ is pluripolar.

Take $(a, b) \in A \times \mathbb{C} \setminus M$. Since $M_{(a, \cdot)}$ is polar, there exists a curve $\gamma : [0, 1] \to \mathbb{C} \setminus M_{(a, \cdot)}$ such that $\gamma(0) = 0$, $\gamma(1) = b$. Take an $\varepsilon > 0$ so small that

$$\Delta_a(\varepsilon) \times (\gamma([0, 1]) + \Delta_0(\varepsilon)) \subset U \setminus M,$$

where $\Delta_z(\varepsilon) = \Delta^k_z(\varepsilon) \subset \mathbb{C}^k$ denotes the polydisc with center $z_0 \in \mathbb{C}^k$ and radius $r > 0$. Put $V_\varepsilon := E \cup (\gamma([0, 1]) + \Delta_0(\varepsilon))$ and consider the cross

$$Y := X(A \cap \Delta_a(\varepsilon), E; \Delta_a(\varepsilon), V_\varepsilon).$$

Then $f \in \mathcal{O}_s(Y)$ for any $f \in \mathcal{O}_s(X \setminus M)$. Consequently, by Theorem 3, we get $Y \subset G_f$, $f \in \mathcal{O}_s(X \setminus M)$. Hence $Y \subset G$. In particular, we conclude that $\{a\} \times (\mathbb{C} \setminus M_{(a, \cdot)}) \subset G$.

Thus $S_{(a, \cdot)} \subset M_{(a, \cdot)}$ for all $a \in A$. Consequently, by Lemma 5 from [4], $S$ is pluripolar.

**Lemma 8.** (a) Let $S \subset \mathbb{C}^p \times \mathbb{C}^q$ be pluripolar. Then the set

$$A := \{ z \in \mathbb{C}^p : S(z, \cdot) \text{ is not pluripolar} \}$$

is pluripolar.

(b) Let $M \subset \mathbb{C}^p \times \mathbb{C}^q$ be such that for each $a \in \mathbb{C}^p$ the fiber $M_{(a, \cdot)}$ is pluripolar. Let $C \subset \mathbb{C}^p \times \mathbb{C}^q$ be such that the set $\{ z \in \mathbb{C}^p : C(z, \cdot) \text{ is not pluripolar} \}$ is not pluripolar (e.g. $C = C' \times C''$, where $C' \subset \mathbb{C}^p$, $C'' \subset \mathbb{C}^q$ are nonpluripolar). Then $C \setminus M$ is nonpluripolar.

(c) Let $M \subset \mathbb{C}^p \times \mathbb{C}^q$ be such that for each $a \in \mathbb{C}^p$ the fiber $M_{(a, \cdot)}$ is pluripolar. Let $A \subset \mathbb{C}^p$ be locally pluriregular. Let $C := \{(a, b') \in A \times \mathbb{C}^{q-1} : M_{(a, b') \cdot} \text{ is polar} \}$. Then $C$ is locally pluriregular.

**Proof.** (a) Let $v \in \mathcal{P}SH(\mathbb{C}^{p+q})$, $v \neq -\infty$, be such that $S \subset v^{-1}(-\infty)$. Define

$$u(z) := \sup \{ v(z, w) : w \in \mathbb{C}^q \}, \quad z \in \mathbb{C}^p.$$

Then $A \subset u^{-1}(\infty)$. Moreover, $u \in \mathcal{P}SH(\mathbb{C}^p)$ and $u \neq -\infty$.

(b) Suppose that $C \setminus M$ is pluripolar. Then, by (a), there exists a pluripolar set $A \subset \mathbb{C}^p$ such that the fiber $(C \setminus M)_{(a, \cdot)}$ is pluripolar, $a \in \mathbb{C}^p \setminus A$. Consequently, the fiber $C_{(a, \cdot)}$ is pluripolar, $a \in \mathbb{C}^p \setminus A$, a contradiction.

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2 Here and in the sequel, to simplify notation we write $P_1 \times \cdots \times P_k \setminus Q$ instead of $(P_1 \times \cdots \times P_k) \setminus Q$.

3 We like to thank Professor Evgeni Chirka for explaining to us some details of the proof of Theorem 1 in [4].
(c) Fix a point \((a_0, b_0') \in C\) and a neighborhood \(U := \Delta_{(a_0, b_0')}(r)\). We have to show that \(h^*_{\mathcal{C} \cap U}(a_0, b_0') = 0\). First we show that

\[
(*) \quad h^*_{\mathcal{C} \cap U}(a_0, b_0') \leq h^*_{(A \cap \Delta_{a_0}(r)) \times \Delta_{b_0'}(r), U}(a_0, b_0').
\]

Indeed, let \(u \in \mathcal{PSH}(U)\) be such that \(u \leq 1\) and \(u \leq 0\) on \(C \cap U\). Then for any \(a \in A \cap \Delta_{a_0}(r)\) the function \(u(a, \cdot)\) is plurisubharmonic on \(\Delta_{b_0'}(r)\), and \(u(a, \cdot) \leq 0\) on the set

\[
(C \cap U)_{(a, \cdot)} = \{b' \in \Delta_{b_0'}(r) : (M_{(a, \cdot)})(b') \text{ is polar}\}.
\]

By (a) (applied to the set \( M_{(a, \cdot)}\)), the set \(\Delta_{b_0'}(r) \setminus (C \cap U)_{(a, \cdot)}\) is pluripolar. Hence \(u(a, \cdot) \leq 0\) on \(\Delta_{b_0'}(r)\). Consequently, \(u \leq 0\) on \((A \cap \Delta_{a_0}(r)) \times \Delta_{b_0'}(r)\), which implies that \(h_{\mathcal{C} \cap U, U} \leq h_{(A \cap \Delta_{a_0}(r)) \times \Delta_{b_0'}(r), U}\), and finally, \(h^*_{\mathcal{C} \cap U, U}(a_0, b_0') \leq h^*_{(A \cap \Delta_{a_0}(r)) \times \Delta_{b_0'}(r), U}(a_0, b_0')\).

Now, by virtue of the product property of the relative extremal function (cf. [11]), using (*) and the fact that \(\{a\}_{(a, \cdot)}\) is pluripolar, we get

\[
h^*_{\mathcal{C} \cap U, U}(a_0, b_0') \leq h^*_{(A \cap \Delta_{a_0}(r)) \times \Delta_{b_0'}(r), U}(a_0, b_0') = \max \left\{ h^*_{(A \cap \Delta_{a_0}(r)) \times \Delta_{b_0'}(r), U}(a_0, b_0'), h^*_{(A \cap \Delta_{a_0}(r)) \times \Delta_{b_0'}(r), U}(a_0, b_0') \right\} = h^*_{(A \cap \Delta_{a_0}(r)) \times \Delta_{b_0'}(r), U}(a_0, b_0') = 0.
\]

\[\square\]

**Lemma 9.** Let \(D_j, A_j, j = 1, \ldots, N\), and \(X\) be as in the Main Theorem. Let

\[
S_j \subset A_1 \times \cdots \times A_{j-1} \times A_{j+1} \times \cdots \times A_N
\]

be pluripolar, \(j = 1, \ldots, N\). Put

\[
T := T(A_1, \ldots, A_N; D_1, \ldots, D_N; S_1, \ldots, S_N).
\]

Then any function \(f \in \mathcal{O}_s(T) \cap \mathcal{C}(T)\) extends holomorphically to \(\tilde{X}\).

If \(N = 2\), then the result is true for any function \(f \in \mathcal{O}_s(T)\) (see the proof). In the case where \(N \geq 3\) we do not know whether the result is true for arbitrary \(f \in \mathcal{O}_s(T)\).

**Proof.** We apply induction on \(N\). The case \(N = 2\) follows from Theorem 3 and the fact that \(\tilde{X} = \tilde{T}\) (recall that if \(N = 2\), then \(T\) is a 2-fold cross). Moreover, if \(N = 2\), then the result is true for any \(f \in \mathcal{O}_s(T)\).

Assume that the result is true for \(N - 1 \geq 2\). Take an \(f \in \mathcal{O}_s(T) \cap \mathcal{C}(T)\). Let \(Q\) denote the set of all \(z_N \in A_N\) for which there exists a \(j \in \{1, \ldots, N - 1\}\) such that the fiber \((S_j)_{(z_N)}\) is not pluripolar. Then, by Lemma (b), \(Q\) is pluripolar. Take a \(z_N \in A_N \setminus Q\) and define

\[
T_{z_N} := T(A_1, \ldots, A_{N-1}; D_1, \ldots, D_{N-1}; S_1)_{(z_N)}, \ldots, (S_{N-1})_{(z_N)}).
\]

Then \(f(\cdot, z_N) \in \mathcal{O}_s(T_{z_N}) \cap \mathcal{C}(T_{z_N})\). By the inductive assumption, the function \(f(\cdot, z_N)\) extends to an \(f_{z_N}^\ast \in \mathcal{O}(\tilde{Y})\), where \(Y = X(A_1, \ldots, A_{N-1}; D_1, \ldots, D_{N-1})\).

Let \(A' := A_1 \times \cdots \times A_{N-1}\). Consider the 2-fold cross

\[
Z := T(A', A_N; \tilde{Y}, D_N; S_N, Q) = ((A' \setminus S_N) \times D_N) \cup (\tilde{Y} \times (A_N \setminus Q)).
\]

\[\square\]
Let \( g : Z \to \mathbb{C} \) be given by the formulae
\[
\begin{align*}
g(z', z_N) & := f(z', z_N), (z', z_N) \in (A' \setminus S_N) \times D_N, \\
g(z', z_N) & := \hat{f}(z'), (z', z_N) \in \hat{Y} \times (A_N \setminus Q).
\end{align*}
\]
Observe that \( g \) is well-defined.

Indeed, let \((z', z_N) \in ((A' \setminus S_N) \times D_N) \cap (\hat{Y} \times (A_N \setminus Q))\). If \( z' \in T_{z_N} \), then obviously \( \hat{f}(z') = f(z', z_N) \). If \( z' \notin T_{z_N} \), then
\[
z'_j \in P_{z_N} := \bigcap_{j=1}^{N-1} \{ (w', w_j, w'') \in (A_1 \times \cdots \times A_{j-1}) \times A_j \times (A_{j+1} \times \cdots \times A_{N-1}) : (w', w'') \in (S_j, -z_N) \};
\]
\( P_{z_N} \) is pluripolar. Take a sequence \((z''_j) \in (S_N \cup P_{z_N}) \) with \( z'' \to z' \). Then \( z'' \in T_{z_N} \).

Moreover, \( g \in \mathcal{O}_s(Z) \). Put \( V := \mathbb{X}(A', A_N; \hat{Y}, D_N) \supset Z \). Since the result is true for \( N = 2 \) (without the continuity), we get a holomorphic extension of \( g \) to \( \hat{V} \). It remains to observe that \( \hat{V} = \hat{X} \); cf. [8], the proof of Step 3.

Lemma 10. Let \( D \subset \mathbb{C}^p \), \( G \subset \mathbb{C}^q \) be pseudoconvex domains, let \( A \subset D \), \( B \subset G \) be locally pluriregular, and let \( M \subset U \) be a relatively closed subset of an open neighborhood \( U \) of the cross \( X := \mathbb{X}(A, B; D, G) \). Let \( A' \subset A \), \( B' \subset B \) be such that \( A \setminus A' \), \( B \setminus B' \) are pluripolar and for any \((a, b) \in A' \times B' \) the fibers \( M_{(a, b)} \) are pluripolar. Let \((D_j)_{j \in \mathbb{N}} \), \((G_j)_{j \in \mathbb{N}} \) be sequences of pseudoconvex domains, \( D_j \subset D \), \( G_j \subset G \), with \( D_j \not\subset D \), \( G_j \not\subset G \), such that \((A_j := A' \cap D_j) \neq \emptyset \), \((B_j := B' \cap G_j) \neq \emptyset \), \( j \in \mathbb{N} \). We assume that for each \( j \in \mathbb{N} \), \( a \in A_j \), and \( b \in B_j \), there exist:
- polydiscs \( \Delta_a(r_{a,j}) \subset D_j \), \( \Delta_b(s_{b,j}) \subset G_j \) and
- relatively closed pluripolar sets \( S_{a,j} \subset \Delta_a(r_{a,j}) \times G_j \), \( S^{b,j} \subset D_j \times \Delta_b(s_{b,j}) \) such that:
  1. \((\Delta_a(r_{a,j}) \times G_j) \cup (D_j \times \Delta_b(s_{b,j})) \subset U \cap \hat{X} \),
  2. \((A' \cap \Delta_a(r_{a,j}) \times G_j) \) \( S_{a,j} \subset M \), \((D_j \times (B' \cap \Delta_b(s_{b,j}) \)) \) \( S^{b,j} \subset M \),
  3. for any \( f \in \mathcal{O}_s(X \setminus M) \) there exist functions \( f_{a,j} \in \mathcal{O}(\Delta_a(r_{a,j}) \times G_j \setminus S_{a,j}) \), \( f^{b,j} \in \mathcal{O}(D_j \times \Delta_b(s_{b,j}) \setminus S^{b,j}) \) with
    \[
    \begin{align*}
f_{a,j} &= f & \text{ on } (A' \cap \Delta_a(r_{a,j}) \times G_j \setminus M), \\
f^{b,j} &= f & \text{ on } (D_j \times (B' \cap \Delta_b(s_{b,j}) \setminus M),
    \end{align*}
    \]

- \( S_{a,j} \) is singular with respect to the family \( \{ f_{a,j} : f \in \mathcal{O}_s(X \setminus M) \} \), while \( S^{b,j} \) is singular with respect to the family \( \{ f^{b,j} : f \in \mathcal{O}_s(X \setminus M) \} \).

Then there exists a relatively closed pluripolar set \( \hat{M} \subset \hat{X} \) such that:
- \( \hat{M} \cap X' \subset M \), where \( X' := \mathbb{X}(A', B'; D, G) \),
- for any \( f \in \mathcal{O}_s(X \setminus M) \) there exists exactly one \( \hat{f} \in \mathcal{O}(\hat{X} \setminus \hat{M}) \) with \( \hat{f} = f \) on \( X' \setminus M \), and
- the set \( \hat{M} \) is singular with respect to the family \( \{ \hat{f} : f \in \mathcal{O}_s(X \setminus M) \} \).

---

[8] Here is the only place where the continuity of \( f \) is used.
Proof. Fix a $j \in \mathbb{N}$. Put
\[
\tilde{U}_j := \bigcup_{a \in A_j', b \in B_j'} (\Delta_a(r_{a,j}) \times G_j) \cup (D_j \times \Delta_b(s_{b,j})),
\]
\[
X_j := ((A \cap D_j) \times G_j) \cup (D_j \times (B \cap G_j)),
\]
\[
X_j' := (A'_j \times G_j) \cup (D_j \times B'_j).
\]
Note that $X_j' \subset \tilde{U}_j$. Take an $f \in O_s(X \setminus M)$. We want to glue the sets $(S_{a,j})_{a \in A_j'}$, $(S^{b,j})_{b \in B_j'}$ and the functions $(f_{a,j})_{a \in A_j'}$, $(f^{b,j})_{b \in B_j'}$ to obtain a global holomorphic function $f_j := \bigcup_{a \in A_j', b \in B_j'} f_{a,j} \cup f^{b,j}$ on $\tilde{U}_j \setminus S_j$ where $S_j := \bigcup_{a \in A_j', b \in B_j'} S_{a,j} \cup S^{b,j}$.

Let $a \in A_j'$, $b \in B_j'$. Observe that
\[
f_{a,j} = f \quad \text{on } (A' \cap \Delta_a(r_{a,j})) \times G_j \setminus M,
\]
\[
f^{b,j} = f \quad \text{on } D_j \times (B' \cap \Delta_b(s_{b,j})) \setminus M.
\]
Thus $f_{a,j} = f^{b,j}$ on the non-pluripolar set $(A' \cap \Delta_a(r_{a,j})) \times (B' \cap \Delta_b(s_{b,j})) \setminus M$ (cf. Lemma 8(b)).

Using the minimality of $S_{a,j}$ and $S^{b,j}$, we conclude that
\[
S_{a,j} \cap (\Delta_a(r_{a,j}) \times \Delta_b(s_{b,j})) = S^{b,j} \cap (\Delta_a(r_{a,j}) \times \Delta_b(s_{b,j})).
\]

Now let $a', a'' \in A_j'$ be such that $C := \Delta_{a'}(r_{a',j}) \cap \Delta_{a''}(r_{a'',j}) \neq \emptyset$. Fix a $b \in B_j'$. We know that $f_{a',j} = f^{b,j} = f_{a'',j}$ on $C \times \Delta_b(r_{b,j}) \setminus (S_{a',j} \cup S^{b,j} \cup S_{a'',j})$. Hence, by the identity principle, we conclude that $f_{a',j} = f_{a'',j}$ on $C \times G_j \setminus (S_{a',j} \cup S_{a'',j})$ and, moreover,
\[
S_{a',j} \cap (C \times G_j) = S_{a'',j} \cap (C \times G_j).
\]

The same argument works for $b', b'' \in B' \cap G_j$.

Let $U_j$ be the connected component of $\tilde{U}_j \cap X_j'$ with $X_j' \subset U_j$. We have constructed a relatively closed pluripolar set $S_j \subset U_j$ such that:

- $S_j \cap X_j' \subset M$, and
- for any $f \in O_s(X \setminus M)$ there exists (exactly one) $f_j \in O(U_j \setminus S_j)$ with $f_j = f$ on $X_j' \setminus M$.

Recall that $X_j' \subset U_j \subset X_j'$. Hence the envelope of holomorphy $\hat{U}_j$ coincides with $\hat{X}_j$ (cf. [2], the proof of Step 4).

Applying the Chirka theorem (Theorem [1]), we find a relatively closed pluripolar set $\hat{M}_j \subset \hat{X}_j$ such that:

- $\hat{M}_j \cap U_j \subset S_j$,
- for any $f \in O_s(X \setminus M)$ there exists (exactly one) function $\hat{f}_j \in O(\hat{X}_j \setminus \hat{M}_j)$ with $\hat{f}_j = f_j$ on $U_j \setminus S_j$ (in particular, $\hat{f}_j = f$ on $X_j' \setminus M$), and
- the set $\hat{M}_j$ is singular with respect to the family $\{\hat{f}_j : f \in O_s(X \setminus M)\}$.

Since $A \setminus A', B \setminus B'$ are pluripolar, we get
\[
\hat{X}_j = \{(z, w) \in D_j \times G_j : h^*_A \cap D_j, D_j(z) + h^*_B \cap G_j, G_j(w) < 1\}
\]
\[
= \{(z, w) \in D_j \times G_j : h^*_A \cap D_j, D_j(z) + h^*_B \cap G_j, G_j(w) < 1\} = \hat{X}_j.
\]
So, in fact, \( \hat{f}_j \in \mathcal{O}(\hat{X}_j \setminus \hat{M}_j) \). Observe that \( \bigcup_{j=1}^{\infty} X_j = X \), \( \hat{X}_j \subset \hat{X}_{j+1} \), and \( \bigcup_{j=1}^{\infty} \hat{X}_j = \hat{X} \). Using again the minimality of the \( \hat{M}_j \)'s (and gluing the \( \hat{f}_j \)'s), we get a relatively closed pluripolar set \( \hat{M} \subset \hat{X} \) which satisfies all the required conditions.

**Lemma 11.** Let \( A \subset E^{n-1} \) be locally pluriregular, let \( G \subset \mathbb{C} \) be a domain with \( E \in G \), let \( X := X(A, E; E^{n-1}, G) \), and let \( U \subset E^{n-1} \times G \) be an open neighborhood of \( X \). Let \( M \subset U \) be a relatively closed set such that \( M \cap E^n = \emptyset \) and for any \( a \in A \) the fiber \( M(a, \cdot) \) is polar. Then there exists a relatively closed pluripolar set \( \hat{M} \subset \hat{X} \) such that:

- \( \hat{M} \cap X \subset M \),
- for any \( f \in \mathcal{O}_s(X \setminus M) \) there exists exactly one \( \hat{f} \in \mathcal{O}(\hat{X} \setminus \hat{M}) \) with \( \hat{f} = f \) on \( X \setminus M \), and
- the set \( \hat{M} \) is singular with respect to the family \( \{ \hat{f} : f \in \mathcal{O}_s(X \setminus M) \} \).

Notice that the above result is a special case of our Main Theorem with \( N = 2 \), \( n_1 = n - 1 \), \( D_1 = E^{n-1} \), \( A_1 = A \), \( n_2 = 1 \), \( D_2 = G \), \( A_2 = E \), \( \Sigma_1 = \Sigma_2 = \emptyset \).

**Proof.** By Lemma 10 it suffices to show that for any \( a_0 \in A \) and for any domain \( G' \in G \) with \( E \in G' \) there exist \( r > 0 \) and a relatively closed pluripolar set \( S \subset \Delta_{a_0}(r) \times G' \subset U \) such that:

- \( S \cap X \subset M \), and
- any function from \( \mathcal{O}_s(X \setminus M) \) extends holomorphically to \( \Delta_{a_0}(r) \times G' \setminus S \).

Fix \( a_0 \) and \( G' \). For \( b \in G \), let \( \rho = \rho_b > 0 \) be such that \( \Delta_b(\rho) \subset G \) and \( M(a_0, \cdot) \cap \partial \Delta_b(\rho) = \emptyset \) (cf. [2], Th. 7.3.9). Take \( \rho^- = \rho_b > 0 \), \( \rho^+ = \rho_b^+ > 0 \) such that \( \rho^- < \rho < \rho^+ \), \( \Delta_b(\rho^+) \subset G \), and \( M(a_0, \cdot) \cap \overline{P} = \emptyset \), where

\[ P = P_b := \{ w \in \mathbb{C} : \rho^- < |w| < \rho^+ \}. \]

Let \( \gamma : [0, 1] \rightarrow G \setminus M(a_0, \cdot) \) be a curve such that \( \gamma(0) = 0 \) and \( \gamma(1) \in \partial \Delta_b(\rho) \). There exists an \( \varepsilon = \varepsilon_b > 0 \) such that

\[ \Delta_{a_0}(\varepsilon) \times ((\gamma([0, 1]) + \Delta_b(\varepsilon)) \cup P) \subset U \setminus M. \]

Put \( V = V_b := E \cup (\gamma([0, 1]) + \Delta_b(\varepsilon)) \cup P \) and consider the cross

\[ Y = Y_b := X(A \setminus \Delta_{a_0}(\varepsilon), E; \Delta_{a_0}(\varepsilon), V). \]

Then \( f \in \mathcal{O}_s(Y) \) for any \( f \in \mathcal{O}_s(X \setminus M) \). Consequently, by Theorem 3 any function from \( \mathcal{O}_s(X \setminus M) \) extends holomorphically to \( \hat{Y} \supset \{ a_0 \} \times V \). Shrinking \( \varepsilon \) and \( V \), we may assume that any function \( f \in \mathcal{O}_s(X \setminus M) \) extends to a function \( \tilde{f} = \tilde{f}_b \in \mathcal{O}(\Delta_{a_0}(\varepsilon) \times W) \), where

\[ W = W_b := \Delta_b(1 - \varepsilon) \cup (\gamma([0, 1]) + \Delta_b(\varepsilon)) \cup P. \]

In particular, \( \tilde{f} \) is holomorphic in \( \Delta_{a_0}(\varepsilon) \times P \), and therefore may be represented by the Hartogs–Laurent series

\[ \tilde{f}(z, w) = \sum_{k=0}^{\infty} \tilde{f}_k(z)(w - b)^k + \sum_{k=1}^{\infty} \tilde{f}_{-k}(z)(w - b)^{-k} =: \tilde{f}^+(z, w) + \tilde{f}^-(z, w), \]

\((z, w) \in \Delta_{a_0}(\varepsilon) \times P \),

where \( \tilde{f}^+ \in \mathcal{O}(\Delta_{a_0}(\varepsilon) \times \Delta_b(\rho^+)) \) and \( \tilde{f}^- \in \mathcal{O}(\Delta_{a_0}(\varepsilon) \times (\mathbb{C} \setminus \overline{\Delta_b(\rho^-)})) \). Recall that for any \( a \in A \cap \Delta_{a_0}(\varepsilon) \) the function \( \tilde{f}(a, \cdot) \) extends holomorphically to \( G \setminus M(a, \cdot) \).
Consequently, for any \( a \in A \cap \Delta_{a_0}(\varepsilon) \) the function \( \tilde{f}(a, \cdot) \) extends holomorphically to \( C \setminus (M_{a, \varepsilon} \cap \overline{\Delta_b}(\rho^-)) \). Now, by Theorem 7 there exists a relatively closed pluripolar set \( S = S_b \subset \Delta_{a_0}(\varepsilon) \times \overline{\Delta_b}(\rho^-) \) such that:

- \( S \cap ((A \cap \Delta_{a_0}(\varepsilon)) \times \overline{\Delta_b}(\rho^-)) \subset M, \) and

- any function \( \tilde{f}^{-} \) extends holomorphically to a function \( \tilde{f}^{-} \in \mathcal{O}(\Delta_{a_0}(\varepsilon) \times C \setminus S) \).

Since \( \tilde{f} = \tilde{f}^{+} + \tilde{f}^{-} \), the function \( \tilde{f} \) extends holomorphically to a function \( \tilde{f} = \tilde{f}_b \in \mathcal{O}(\Delta_{a_0}(\varepsilon) \times \Delta_b(\rho^+ \setminus S)) \). We may assume that the set \( S \) is singular with respect to the family \( \{ \tilde{f} : \tilde{f} \in \mathcal{O}_s(X \setminus M) \} \).

Using the identity principle and the minimality of the \( S_b \)'s, one can easily show that for \( b', b'' \in G \), if \( B := \Delta_{b'}(\rho^+_{b'}) \cap \Delta_{b''}(\rho^+_{b''}) \neq \emptyset \), then

\[
S_{b'} \cap (\Delta_{a_0}(\eta) \times B) = S_{b''} \cap (\Delta_{a_0}(\eta) \times B), \quad \tilde{f}_{b'} = \tilde{f}_{b''} \text{ on } \Delta_{a_0}(\eta) \times B,
\]

where \( \eta := \min\{\varepsilon_{b'\varepsilon}, \varepsilon_{b''}\} \). Thus the functions \( \tilde{f}_{b'}, \tilde{f}_{b''} \) and sets \( S_{b'}, S_{b''} \) may be glued together.

Now, select \( b_1, \ldots, b_k \in G \) so that \( G' \subset \bigcup_{j=1}^{k} \Delta_{b_j}(\rho^+_{b_j}) \). Put

\[
r := \min\{\varepsilon_{b_j} : j = 1, \ldots, k\}.
\]

Then \( S := (\Delta_{a_0}(r) \times G') \cap \bigcup_{j=1}^{k} S_{b_j} \) gives the required relatively closed pluripolar subset of \( \Delta_{a_0}(r) \times G' \) such that \( S \subset X \subset M \) and for any \( f \in \mathcal{O}_s(X \setminus M) \), the function \( \tilde{f} := \bigcup_{j=1}^{k} \tilde{f}_{b_j} \) extends holomorphically \( f \) to \( \Delta_{a_0}(r) \times G' \setminus S \). \( \square \)

**Lemma 12.** Let \( A \subset E^p \) be locally pluriregular, let \( R > 1 \), let

\[
X := \mathcal{X}(A, E^p; E^p, \Delta_0^q(R)),
\]

and let \( U \subset E^p \times \Delta_0^q(R) \) be an open neighborhood of \( X \). Let \( M \subset U \) be a relatively closed set such that \( M \cap E^{p+q} = \emptyset \) and for any \( a \in A \) the fiber \( M_{a, \varepsilon} \) is pluripolar. Then there exists a relatively closed pluripolar set \( \widehat{M} \subset \bar{X} \) such that:

- \( \widehat{M} \cap X \subset M, \)

- for any \( f \in \mathcal{O}_s(X \setminus M) \) there exists exactly one \( \tilde{f} \in \mathcal{O}(\bar{X} \setminus \widehat{M}) \) with \( \tilde{f} = f \) on \( X \setminus M, \)

- the set \( \widehat{M} \) is singular with respect to the family \( \{ \tilde{f} : f \in \mathcal{O}_s(X \setminus M) \}. \)

Notice that the above result is a special case of our Main Theorem with \( N = 2, \) \( n_1 = p, \) \( D_1 = E^p, A_1 = A, \) \( n_2 = q, \) \( D_2 = \Delta_0^q(R), A_2 = E^q, \) \( \Sigma_1 = \Sigma_2 = \emptyset. \)

**Proof.** The case \( q = 1 \) follows from Lemma 11. Thus assume that \( q \geq 2 \). By Lemma 10 it suffices to show that for any \( a_0 \in A \) and for any \( R' \in (1, R) \) there exist \( r = r_{R'} > 0 \) and a relatively closed pluripolar set \( S = S_{R'} \subset \Delta_{a_0}(r) \times \Delta_0^q(R') \subset U \) such that:

- \( S \cap X \subset M, \)

- any function from \( \mathcal{O}_s(X \setminus M) \) extends holomorphically to \( \Delta_{a_0}(r) \times \Delta_0^q(R') \setminus S. \)

Let \( a_0 \in A \) and let \( R_0' \) be the supremum of all \( R' \in (0, R) \) such that \( r_{R'} \) and \( S_{R'} \) exist. Note that \( 1 \leq R_0' \leq R. \) It suffices to show that \( R_0' = R. \)

Suppose that \( R_0' < R. \) Fix \( R_0'' < R' < R \) and choose \( R' \in (0, R_0'') \) such that \( R_0'' < R' < R \). Let \( r := r_{R'}, \) \( S := S_{R'}. \)

Write \( w = (w', w_0) \in C_0 = C^q \setminus \mathbb{C}. \) Let \( C \) denote the set of all \( (a, b') \in (A \cap \Delta_{a_0}(r)) \times \Delta_0^q(R') \) such that the fiber \( (M \cup S)_{(a, b'), \varepsilon} \) is polar. By Lemma 5(a, c),
C is pluriregular. Now, by Lemma [11] applied to the cross
\[ Y_q := \mathcal{X}(C, \Delta_0(R'); \Delta_{a_0}(r) \times \Delta_0^{q-1}(R'), \Delta_0(R)) \]
and the set \( M_q := M \cup S \), we conclude that there exists a closed pluripolar set \( S_q \subset \tilde{Y}_q \) such that \( S_q \cap Y_q \subset M_q \) and any function \( f \in \mathcal{O}_s(X \setminus M) \) extends holomorphically to \( \tilde{Y}_q \setminus S_q \). Using the product property of the relative extremal function (cf. [11]), we get
\[ \tilde{Y}_q = \{(z, w', w_q) \in \Delta_{a_0}(r) \times \Delta_0^{q-1}(R') \times \Delta_0(R) : \]
\[ h^*_C,\Delta_{a_0}(r) \times \Delta_0^{q-1}(R') \langle z, w' \rangle + h^*_0(R'),\Delta_0(R)(w_q) < 1 \}
\[ \{(z, w', w_q) \in \Delta_{a_0}(r) \times \Delta_0^{q-1}(R') \times \Delta_0(R) : \]
\[ h^*_0(R),\Delta_0(R)(w_q) < 1 \}
\[ = \{(z, w', w_q) \in \Delta_{a_0}(r) \times \Delta_0^{q-1}(R') \times \Delta_0(R) : \]
\[ h^*_0(R),\Delta_0(R)(w_q) < 1 \}
\[ = \{(z, w', w_q) \in \Delta_{a_0}(r) \times \Delta_0^{q-1}(R') \times \Delta_0(R) : \]
\[ h^*_0(R),\Delta_0(R)(w_q) < 1 \}
\[ = \{(z, w', w_q) \in \Delta_{a_0}(r) \times \Delta_0^{q-1}(R') \times \Delta_0(R) : \]
\[ h^*_0(R),\Delta_0(R)(w_q) < 1 \}.

Since \( R'' < R \), we find an \( r_q \in (0, R] \) such that any function \( f \in \mathcal{O}_s(X \setminus M) \) extends holomorphically to a function \( \tilde{f}_q \) on \( \Delta_{a_0}(r_q) \times \Delta_0^{q-1}(R') \times \Delta_0(R'') \setminus S_q \). We may assume that \( S_q \) is singular with respect to the family \( \{ f : f \in \mathcal{O}_s(X \setminus M) \} \).

Repeating the above argument for the coordinates \( w_\nu, \nu = 1, \ldots, q-1 \), and gluing the obtained sets, we find an \( r_0 \in (0, R] \) and a relatively closed pluripolar set \( S_0 := \bigcup_{i=1}^q S_j \) such that any function \( f \in \mathcal{O}_s(X \setminus M) \) extends holomorphically to a function \( \tilde{f}_0 := \bigcup_{i=1}^q \tilde{f}_i \) holomorphic in \( \Delta_{a_0}(r_0) \times \Omega \setminus S_0 \), where
\[ \Omega := \bigcup_{\nu=1}^q \Delta_0^{q-1}(R') \times \Delta_0(R'') \times \Delta_0^{q-1}(R'). \]

Let \( \hat{\Omega} \) denote the envelope of holomorphy of \( \Omega \). Applying the Chirka theorem (Theorem [1]), we find a relatively closed pluripolar subset \( \hat{S}_0 \) of \( \Delta_{a_0}(r_0) \times \hat{\Omega} \) such that any function \( f \in \mathcal{O}_s(X \setminus M) \) extends to a function \( \hat{f} \) holomorphic on \( \Delta_{a_0}(r_0) \times \hat{\Omega} \setminus \hat{S}_0 \).

Let \( R''' := R'' R'' \). Observe that \( \Delta_0(R''') \subset \hat{\Omega} \). Recall that \( R''' > R'' \). We may assume that \( \hat{M} \) is singular with respect to the family \( \{ f : f \in \mathcal{O}_s(X \setminus M) \} \). To get a contradiction it suffices to show that \( \hat{M} \cap X \subset M \). We argue as in the proof of Lemma [11].

Take \((a, b) \in (A \cap \Delta_{a_0}(r_0)) \times \Delta_0^q(R''') \setminus M \). Since \( M_{(a, \cdot)} \) is pluripolar, there exists a curve \( \gamma : [0, 1] \rightarrow \Delta_0(R''') \setminus M_{(a, \cdot)} \) such that \( \gamma(0) = 0, \gamma(1) = b \). Take an \( \varepsilon > 0 \) so small that
\[ \Delta_a(\varepsilon) \times (\gamma([0, 1]) + \Delta_0^q(\varepsilon)) \subset \Delta_{a_0}(r) \times \Delta_0^q(R''') \setminus M. \]

Put \( V_0 := E_y \cup (\gamma([0, 1]) + \Delta_0^q(\varepsilon)) \) and consider the cross
\[ Y := \mathcal{X}(A \cap \Delta_a(\varepsilon), E^y; \Delta_a(\varepsilon), V_0). \]

Then \( f \in \mathcal{O}_s(Y) \) for any \( f \in \mathcal{O}_s(X \setminus M) \). Consequently, by Theorem [3] \( \hat{Y} \subset \Delta_{a_0}(r) \times \Delta_0(R''') \setminus \hat{M} \), which implies that \( \hat{M}_{(a, \cdot)} \cap \Delta_0^q(R''') \subset M_{(a, \cdot)}. \)

\[ \square \]
3. Proof of the Main Theorem for $N = 2$

To simplify notation, put $p := n_1$, $D := D_1$, $A := A_1$, $A' := A \setminus \Sigma_2$, $q := n_2$, $G := D_2$, $B := A_2$, $B' := B \setminus \Sigma_1$.

It suffices to verify the assumptions of Lemma 10. Let $(D_j)_{j=1}^\infty$, $(G_j)_{j=1}^\infty$ be approximation sequences: $D_j \Subset D_{j+1} \Subset D$, $G_j \Subset G_{j+1} \Subset G$, $D_j \not\supset D$, $G_j \not\supset G$, $A' \cap D_j \not= \emptyset$, and $B' \cap G_j \not= \emptyset$, $j \in \mathbb{N}$.

Fix $j \in \mathbb{N}$, $a \in A' \cap D_j$, and let $\Omega_j$ be the set of all $b \in G_{j+1}$ such that there exist a polydisc $\Delta\pars{a,b} \subset D_j \times G_{j+1}$ and a relatively closed pluripolar set $S_b \subset \Delta\pars{a,b}$ such that:

1. $S_b \cap ((A' \cap \Delta\pars{a}) \times \Delta_b) \subset M$,
2. any function $f \in \mathcal{O}_a (X \setminus M)$ extends to a function $\tilde{f}_b \in \mathcal{O}(\Delta\pars{a,b}) \setminus S_b$ with $\tilde{f}_b = f$ on $(A' \cap \Delta\pars{a}) \times \Delta_b \setminus M$, and
3. $S_b$ is singular with respect to the family $\{ \tilde{f}_b : f \in \mathcal{O}_a (X \setminus M) \}$.

It is clear that $\Omega_j$ is open. Observe that $\Omega_j \neq \emptyset$. Indeed, since $B \cap G_j \setminus M(a,.), \neq \emptyset$, we find a point $b \in B \cap G_j \setminus M(a,.).$ Therefore there is a polydisc $\Delta\pars{a,b}(r) \subset D_j \times G_j \setminus M$. Put

$$Y := X(A \cap \Delta_a), B \cap \Delta_b; \Delta_a, \Delta_b(r).$$

By Theorem 13 we find an $r_0 \in (0, r)$ such that any function $f \in \mathcal{O}_a (X \setminus M)$ extends to $\tilde{f}_b \in \mathcal{O}(\Delta\pars{a,b})$ with $\tilde{f}_b = f$ on $\Delta\pars{a,b} \cap Y \supset (A \cap \Delta_a(r)) \times \Delta_b(r)$. Consequently, $b \in \Omega_j$.

Moreover, $\Omega_j$ is relatively closed in $G_{j+1}$. Indeed, let $c$ be an accumulation point of $\Omega_j$ in $G_{j+1}$ and let $\Delta_c (3R) \subset G_{j+1}$. Take a point $b \in \Omega_j \cap \Delta_c(R) \setminus M(a,.)$ and let $r \in (0, r_0]$, $r < 2R$, be such that $\Delta\pars{a,b}(r) \cap M = \emptyset$. Observe that $\tilde{f}_b \in \mathcal{O}(\Delta\pars{a,b})$ and $\tilde{f}_b (z, \cdot) = f (z, \cdot) \in \mathcal{O}(\Delta_b (2R)) \setminus M(z, -)$ for any $z \in A' \cap \Delta_a$. Hence, by Lemma 12 (with $R' := R$), there exists a relatively closed pluripolar set $S \subset \Delta_a \times \Delta_b$ with $\rho' \in (0, r)$ such that any $f$ has an extension $\tilde{f}_b \in \mathcal{O}(\Delta\pars{a,b}) \setminus S$. Take an $r_c > 0$ so small that $\Delta\pars{a,c}(r_c) \subset \Delta\pars{a,b} \times \Delta_b(R)$, and put $S_c := S \cap \Delta\pars{a,c}(r_c)$.

Thus $\Omega_j = G_{j+1}$. There exists a finite set $T \subset \Omega_j$ such that

$$\overline{T} \subset \bigcup_{b \in T} \Delta_b \left( r_b \right).$$

Define $r_{a,j} := \min \{ r_b : b \in T \}$. Take $b', b'' \in T$ with $C := \Delta_{b'} \setminus \Delta_{b''} \neq \emptyset$. Then $\tilde{f}_{b'} = f = \tilde{f}_{b''}$ on $(A' \cap \Delta_{a,j}(r_{a,j})) \times (\Delta_{b'} \setminus \Delta_{b''} \setminus C) \setminus M$. Consequently, $\tilde{f}_{b'} = \tilde{f}_{b''}$ on $\Delta_a \times \Delta_{b'} \setminus \Delta_{b''} \setminus C$. In particular, using the minimality of the sets $S_{b'}$ and $S_{b''}$, we conclude that they coincide on $\Delta_{a,j} \times C$ and that the functions $f_{b'}$ and $f_{b''}$ glue together. Thus we get a relatively closed pluripolar set $S_{a,j} \subset \Delta_{a} \times G_j$ such that $S_{a,j} \cap (A' \cap \Delta_{a,j}) \times G_j \subset M$ and any function $f \in \mathcal{O}_a (X \setminus M)$ extends holomorphically to an $f_{a,j} \in \mathcal{O}(\Delta_{a,j} \times G_j \setminus S_{a,j})$ with $f_{a,j} = f$ on $(A' \cap \Delta_{a,j}) \times G_j \setminus M$.

Changing the roles of $z$ and $w$, we get $S_{b,j}^{b,j}$ and $f_{b,j}^{b,j}$, $b \in B' \cap G_j$. □

The above proof of the Main Theorem for $N = 2$ shows that the following generalization of Lemma 12 is true.
Theorem 13. Let $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$ be pseudoconvex domains, let $A \subset D$ be locally pluriregular, let $B \subset G$ be open and nonempty, and let $M \subset U$ be a relatively closed subset of an open neighborhood $U$ of the cross $X := \mathbb{X}(A; B; D, G)$ such that $M \cap (D \times B) = \emptyset$ and for any $a \in A$ the fiber $M(a, \cdot)$ is pluripolar. Then there exists a relatively closed pluripolar set $\hat{M} \subset \hat{X}$ such that:

- $\hat{M} \cap X \subset M$,
- for any $f \in \mathcal{O}_s(X \setminus M)$ there exists exactly one $\hat{f} \in \mathcal{O}(\hat{X} \setminus \hat{M})$ with $\hat{f} = f$ on $X \setminus M$, and
- the set $\hat{M}$ is singular with respect to the family $\{\hat{f} : f \in \mathcal{O}_s(X \setminus M)\}$.

Observe that if $G = \mathbb{C}^q$, then $\hat{X} = D \times \mathbb{C}^q$. Consequently, Theorem 13 also generalizes Theorem 7.

Proof. We apply Lemma 10 (as in the proof of the Main Theorem for $N = 2$). The functions $f_{a,j}$ are constructed exactly as in that proof (with $A' = A$). The functions $f^{b,j}$ are simply given as $f^{b,j} := f|_{D_j \times \Delta_b(s_{b,j})}$ with $\Delta_b(s_{b,j}) \subset B \cap D_j$ ($S^{b,j} := \emptyset$). \hfill $\square$

4. Proof of the Main Theorem

First observe that, by Lemma 3(b), the set $X' \setminus M$ is not pluripolar. Consequently, the function $\hat{f}$ is uniquely determined.

We proceed by induction on $N$. The case $N = 2$ is proved.

Let $D_{j,k} \neq D_j, D_{j,k} \in D_{j,k+1} \setminus D_j$, where the $D_{j,k}$ are pseudoconvex domains with $A_{j,k} := A_j \cap D_{j,k} \neq \emptyset$, $j = 1, \ldots, N$. Put

$$X_k := \mathbb{X}(A_{1,k}, \ldots, A_{N,k}; D_{1,k}, \ldots, D_{N,k}) \subset X,$$

$$\Sigma_j := (A_{1,k} \times \cdots \times A_{j-1,k} \times A_{j+1,k} \times \cdots \times A_{N,k}) \cap \Sigma_j, \quad j = 1, \ldots, N,$$

$$X'_k := \mathbb{T}(A_{1,k}, \ldots, A_{N,k}; D_{1,k}, \ldots, D_{N,k}; \Sigma_1, \ldots, \Sigma_N) \subset X_k.$$

It suffices to show that for each $k \in \mathbb{N}$ the following condition (*) holds:

\begin{itemize}
  \item There exist a domain $U_k$, $X'_k \subset U_k \subset \hat{X}_k$, and a relatively closed pluripolar set $M_k \subset U_k$, such that:
    \begin{itemize}
      \item $M_k \cap X'_k \subset M$,
      \item for any $f \in \mathcal{O}_s(X \setminus M)$ there exists an $\tilde{f}_k \in \mathcal{O}(U_k \setminus M_k)$ with $\tilde{f}_k = f$ on $X'_k \setminus M$.
    \end{itemize}
\end{itemize}

Indeed, fix a $k \in \mathbb{N}$ and observe that, by Lemma 9, $\hat{X}_k$ is the envelope of holomorphy of $U_k$. Hence, by virtue of the Chirka theorem (Theorem 9), there exists a relatively closed pluripolar set $\hat{M}_k$ of $\hat{X}_k$, $\hat{M}_k \cap U_k \subset M_k$, such that $\hat{X}_k \setminus \hat{M}_k$ is the envelope of holomorphy of $U_k \setminus M_k$. In particular, for each $f \in \mathcal{O}_s(X \setminus M)$ there exists an $\tilde{f}_k \in \mathcal{O}(\hat{X}_k \setminus \hat{M}_k)$ with $\tilde{f}_k|_{U_k \setminus M_k} = \tilde{f}_k$. We may assume that $\hat{M}_k$ is singular with respect to the family $\{\hat{f}_k : f \in \mathcal{O}_s(X \setminus M)\}$.

In particular, $\hat{M}_{k+1} \cap \hat{X}_k = \hat{M}_k$. Consequently:

\begin{itemize}
  \item $\tilde{M} := \bigcup_{k=1}^\infty \hat{M}_k$ is a relatively closed pluripolar subset of $\hat{X}$ with $\tilde{M} \cap X' \subset M$,
  \item for each $f \in \mathcal{O}_s(X \setminus M)$, the function $\tilde{f} := \bigcup_{k=1}^\infty \tilde{f}_k$ is holomorphic on $\hat{X} \setminus \tilde{M}$ with $\tilde{f}$ is on $X' \setminus M$, and
  \item $\tilde{M}$ is singular with respect to the family $\{\hat{f} : f \in \mathcal{O}_s(X \setminus M)\}$.
\end{itemize}

It remains to prove (*). Fix a $k \in \mathbb{N}$. For any $a = (a_1, \ldots, a_N) \in A_{1,k} \times \cdots \times A_{N,k} \setminus M$
Now, we apply Theorem 13 to the 2-fold cross $Z_{i,j} = \{z \in \mathbb{C}^2 \mid \phi(z) \in \mathbb{D}^2 \cap \mathbb{R}^2 \}$ and we assume that $O$.

Observe that any function from $(0,\tau)$ to $(0,\tau)$ is so small that $\Delta_a(\rho) \subset \hat{Y}_a$.

If $N \geq 4$, then we additionally define $(N-2)$-fold crosses

$$Y_{k,\mu,\nu} := \mathbb{X}(A_1, \ldots, A_{\mu-1}, A_{\mu+1}, \ldots, A_{\nu-1}, A_{\nu+1}, \ldots, A_{N,k}) : D_{1,k}, \ldots, D_{\mu-1,k}, D_{\mu+1,k}, \ldots, D_{\nu-1,k}, D_{\nu+1,k}, \ldots, D_{N,k})$$

and we assume that $\rho$ is so small that

$$\Delta(a_1,\ldots,a_{\mu-1},a_{\mu+1},\ldots,a_{\nu-1},a_{\nu+1},\ldots,a_N)(\rho) \subset \hat{Y}_{k,\mu,\nu}, \quad 1 \leq \mu < \nu \leq N.$$ For $j \in \{1,\ldots,N\}$, define the 2-fold crosses

$$Z'_{k,a,j} := \left\{ (z',z_j,z'') \in (A_1 \cap \Delta_{a_1}(\rho)) \times \cdots \times (A_{j-1} \cap \Delta_{a_{j-1}}(\rho)) \times D_{j,k+1} \times ((A_{j+1} \cap \Delta_{a_{j+1}}(\rho)) \times \cdots \times (A_N \cap \Delta_{a_N}(\rho)) : (z',z'') \notin \Sigma_j \right\} \cup \Delta_a(\rho),$$

$$Z_{k,a,j} := \left( (A_1 \cap \Delta_{a_1}(\rho)) \times \cdots \times (A_{j-1} \cap \Delta_{a_{j-1}}(\rho)) \times D_{j,k+1} \times (A_{j+1} \cap \Delta_{a_{j+1}}(\rho)) \times \cdots \times (A_N \cap \Delta_{a_N}(\rho)) \right) \cup \Delta_a(\rho).$$

Now, we apply Theorem 13 to the 2-fold cross $Z'_{k,a,j}$ and the set $M$. We find a relatively closed pluripolar set $S_{k,a,j} \subset \hat{Z}'_{k,a,j} = \hat{Z}_{k,a,j}$ such that:

- $S_{k,a,j} \cap Z'_{k,a,j} \subset M$,
- for any function $f \in \mathcal{O}_s(X \setminus M)$ there exists an $\hat{f}_{k,a,j} \in \mathcal{O}(\hat{Z}_{k,a,j} \setminus S_{k,a,j})$ such that $\hat{f}_{k,a,j} = f$ on $Z'_{k,a,j} \setminus M$, and
- $S_{k,a,j}$ is singular with respect to the space $\{ \hat{f}_{k,a,j} : f \in \mathcal{O}_s(X \setminus M) \}$.

Observe that $(a_1,\ldots,a_{j-1}) \times D_{j,k} \times \{ (a_{j+1},\ldots,a_N) \} \subset \hat{Z}_{k,a,j}$. Consequently, we find $r = r_k(a) \in (0,\rho]$ such that

$$V_{k,a,j} := \Delta(a_1,\ldots,a_{j-1})(r) \times D_{j,k} \times \Delta(a_{j+1},\ldots,a_N)(r) \subset \hat{Z}_{k,a,j}, \quad j = 1,\ldots,N.$$ Let

$$V_k := \bigcup_{a \in A_1 \times \cdots \times A_N,k} V_{k,a,j}. \quad j \in \{1,\ldots,N\}$$

Note that $X_k' \subset V_k$. Let $U_k$ be the connected component of $V_k \cap \hat{X}_k$ that contains $X_k$.

It remains to glue the sets $S_{k,a,j}$ and functions $\hat{f}_{k,a,j}$. Then

$$S_k := \bigcup_{a \in A_1 \times \cdots \times A_N,k} S_{k,a,j} \cup U_k, \quad \hat{f}_k := \bigcup_{a \in A_1 \times \cdots \times A_N,k} \hat{f}_{k,a,j}|_{V_{k,a,j} \cap U_k \setminus S_k}$$

will satisfy (*).

To check that the gluing process is possible, let $a,b \in A_1 \times \cdots \times A_N,k \setminus M$, $i,j \in \{1,\ldots,N\}$ be such that $V_{k,a,i} \cap V_{k,b,j} = \emptyset$. We have the following two cases:
Let $w = (w', w'') \in \mathbb{C}^{n_1 + \cdots + n_{N-1}} \times \mathbb{C}^{n_{N-1} + n_N}$. Observe that

\[ V_{k,a,N-1} \cap V_{k,b,N} = \left( \Delta_a(r_k(a)) \cap \Delta_b(r_k(b)) \right) \times \Delta_{b,N-1}(r_k(b)) \times \Delta_{a_N}(r_k(a)). \]

We consider the following three subcases:

- $N = 2$ (cf. the proof of Lemma 10): Then $V_{k,a,1} \cap V_{k,b,2} = \Delta_{b_1}(r_k(b)) \times \Delta_{a_2}(r_k(a))$.

We know that $\hat{f}_{k,a,1} = \hat{f}_{k,b,2}$ on the non-pluripolar set

\[ (A_1 \cap \Delta_{b_1}(r_k(b)) \setminus \Sigma_2) \times (A_2 \cap \Delta_{a_2}(r_k(a)) \setminus \Sigma_1) \setminus M; \]

cf. Lemma 8(b). Hence, by the identity principle, $\hat{f}_{k,a,1} = \hat{f}_{k,b,2}$ on $V_{k,a,1} \cap V_{k,b,2} \setminus (S_{k,a,1} \cup S_{k,b,2})$. Consequently, the sets $S_{k,a,1}$, $S_{k,b,2}$ and the functions $\hat{f}_{k,a,1}$, $\hat{f}_{k,b,2}$ glue together.

- $N = 3$: Then $V_{k,a,2} \cap V_{k,b,3} = (\Delta_{a_1}(r_k(a)) \cap \Delta_{b_1}(r_k(b)) \times \Delta_{a_2}(r_k(b)) \times \Delta_{b_3}(r_k(a))$.

Let

\[ C'' := (A_2 \cap \Delta_{a_2}(r_k(b)) \times (A_3 \cap \Delta_{a_3}(r_k(a))) \setminus \Sigma_1. \]

Recall that for any $c'' \in C''$ the fiber $M_{(\cdot,c'')}$ is pluripolar. We have $\hat{f}_{k,a,2}(\cdot,c'') = \hat{f}_{k,a,2}(\cdot,c'') \times \Delta_{a_1}(r_k(a)) \cap \Delta_{b_1}(r_k(b)) \setminus M_{(\cdot,c')}$. 

Now, let $C''$ denote the set of all $c' \in \Delta_{a_1}(r_k(a)) \cap \Delta_{b_1}(r_k(b))$ such that the fiber $(S_{k,a,2} \cup S_{k,b,3})(c',\cdot)$ is pluripolar. Recall that the complement of $C''$ is pluripolar (Lemma 8(a)). If $c' \in C''$, then $\hat{f}_{k,a,2}(c',\cdot) = \hat{f}_{k,b,3}(c',\cdot)$ on $C'' \setminus (S_{k,a,2} \cup S_{k,b,3})(c',\cdot)$. Consequently, by the identity principle, $\hat{f}_{k,a,2}(c',\cdot) = \hat{f}_{k,b,3}(c',\cdot)$ on $\Delta_{b_2}(r_k(b)) \times \Delta_{a_3}(r_k(a)) \setminus (S_{k,a,2} \cup S_{k,b,3})(c',\cdot)$, $c' \in C''$. Finally, $\hat{f}_{k,a,2} = \hat{f}_{k,b,3}$ on $V_{k,a,2} \cap V_{k,b,3} \setminus (S_{k,a,2} \cup S_{k,b,3})$. Consequently, the sets $S_{k,a,2}$, $S_{k,b,3}$ and the functions $\hat{f}_{k,a,2}$, $\hat{f}_{k,b,3}$ glue together.

If $N \in \{2,3\}$, then we jump directly to (b), and we conclude that the Main Theorem is true for $N \in \{2,3\}$.

- $N \geq 4$: Here is the only place where the induction over $N$ is used. We assume that the Main Theorem is true for $N - 1 \geq 3$.

Let

\[ C'' := \{ c'' \in (A_{N-1} \cap \Delta_{b,N-1}(r_k(b)) \times (A_N \cap \Delta_{a_N}(r_k(a)) : (\Sigma_s)(\cdot,c'') \text{ is pluripolar, } s = 1, \ldots, N - 2) \}; \]

note that, by Lemma 8(a), $C''$ is not pluripolar. For any $c'' \in C''$ the function $f_{c''} := f(\cdot,c'')$ is separately holomorphic on $Y_{k,N-1,N} \setminus M_{(\cdot,c'')}$. Moreover, the set $M_{(\cdot,c'')}$ satisfies all the assumptions of the Main Theorem. Indeed,

\[ \Sigma_s(A_{1,k}, \ldots, A_{N-2,k}; M_{(\cdot,c'')}) = \left( \Sigma_s(A_{1,k}, \ldots, A_{N,k}; M) \right)_{(\cdot,c'')} \subset (\Sigma_s)(\cdot,c''), \]

\[ s = 1, \ldots, N - 2. \]

By the inductive assumption, the function $f_{c''}$ extends to a function

\[ \hat{f}_{c''} \in \mathcal{O}(\hat{Y}_{k,N-1,N} \setminus \hat{M}(c'')), \]

where $\hat{M}(c'')$ is a relatively closed pluripolar subset of $\hat{Y}_{k,N-1,N}$ such that $\hat{M}(c'') \cap Y'_{k,N-1,N} \subset M_{(\cdot,c'')}$. Recall that

\[ \Delta_a(r_k(a)) \cup \Delta_b(r_k(b)) \subset \hat{Y}_{k,N-1,N}. \]
Since $\tilde{f}_{k,a,N-1}(\cdot, c') = f_{c'}$ on $\Delta_{\varphi}(r_k(a)) \cap Y'_{k,N-1,N} \setminus M(\cdot, c')$ and $\tilde{f}_{k,b,N}(\cdot, c'') = f_{c''}$ on $\Delta_{\psi}(r_k(b)) \cap Y'_{k,N-1,N} \setminus M(\cdot, c'')$, we conclude that $\tilde{f}_{k,a,N-1}(\cdot, c') = \tilde{f}_{k,b,N}(\cdot, c'')$ on $\Delta_{\varphi}(r_k(a)) \cap \Delta_{\psi}(r_k(b)) \setminus M(\cdot, c'').$

Let $c' \in \Delta_{\varphi}(r_k(a)) \cap \Delta_{\psi}(r_k(b))$ be such that the fiber $(S_{k,a,N-1} \cup S_{k,b,N})(c', c')$ is pluripolar. Then $\tilde{f}_{k,a,N-1}(c', \cdot) = \tilde{f}_{k,b,N}(c', \cdot)$ on $C'' \setminus (S_{k,a,N-1} \cup S_{k,b,N})(c', c')$. Consequently, by the identity principle, $\tilde{f}_{k,a,N-1}(c', \cdot) = \tilde{f}_{k,b,N}(c', \cdot)$ on $(\Delta_{\varphi}(r_k(a)) \times \Delta_{\psi}(r_k(b))) \setminus (S_{k,a,N-1} \cup S_{k,b,N})$ and, finally, $\tilde{f}_{k,a,N-1} = \tilde{f}_{k,b,N}$ on $(V_{k,a,N-1} \cap V_{k,b,N}) \setminus (S_{k,a,N-1} \cup S_{k,b,N})$. Consequently, the sets $S_{k,a,N-1}$, $S_{k,b,N}$ and the functions $\tilde{f}_{k,a,N-1}$, $\tilde{f}_{k,b,N}$ glue together.

(b) $i = j$: We may assume that $i = j = N$. Observe that

$$V_{k,a,N} \cap V_{k,b,N} = \left(\Delta_{(a_1, ..., a_{N-1})}(r_k(a)) \cap \Delta_{(b_1, ..., b_{N-1})}(r_k(b))\right) \times D_{N,k}.$$ 

By (a) we know that

$$\tilde{f}_{k,a,N} = \tilde{f}_{k,a,N-1} \quad \text{on} \quad V_{k,a,N} \cap V_{k,a,N-1} \setminus (S_{k,a,N} \cup S_{k,a,N-1}),$$

$$\tilde{f}_{k,a,N-1} = \tilde{f}_{k,b,N} \quad \text{on} \quad V_{k,a,N-1} \cap V_{k,b,N} \setminus (S_{k,a,N-1} \cup S_{k,b,N}).$$

Hence (we write $w = (w', w_N) \in \mathbb{C}^{n_1+\cdots+n_{N-1}} \times \mathbb{C}^n$)

$$\tilde{f}_{k,a,N} = \tilde{f}_{k,b,N}$$

on

$$V_{k,a,N} \cap V_{k,a,N-1} \cap V_{k,b,N} \setminus (S_{k,a,N-1} \cup S_{k,a,N} \cup S_{k,b,N})$$

$$= \left(\Delta_{\varphi}(r_k(a)) \cap \Delta_{\psi}(r_k(b))\right) \times \Delta_{\psi}(r_k(a)) \setminus (S_{k,a,N-1} \cup S_{k,a,N} \cup S_{k,b,N}),$$

and finally, by the identity principle,

$$\tilde{f}_{k,a,N} = \tilde{f}_{k,b,N} \quad \text{on} \quad V_{k,a,N} \cap V_{k,b,N} \setminus (S_{k,a,N} \cup S_{k,b,N}).$$

Consequently, the sets $S_{k,a,N}$, $S_{k,b,N}$ and the functions $\tilde{f}_{k,a,N}$, $\tilde{f}_{k,b,N}$ glue together.

The proof of the Main Theorem is completed. \qed

References


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