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ON THE CAPACITY OF SETS OF DIVERGENCE ASSOCIATED WITH THE SPHERICAL PARTIAL INTEGRAL OPERATOR

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ABSTRACT. In this article, we study the pointwise convergence of the spherical partial integral operator $S_Rf(x) = \int_{B(0,R)} \hat{f}(y)e^{2\pi ix\cdot y}dy$ when it is applied to functions with a certain amount of smoothness. In particular, for $f \in \mathcal{L}^p_\alpha(\mathbb{R}^n)$, $\frac{n-1}{2} < \alpha \leq \frac{n}{p}$, $2 \leq p < \frac{2n}{n-1}$, we prove that $S_Rf(x) \to G_\alpha * g(x)$ $C_{\alpha,p}$-quasieverywhere on $\mathbb{R}^n$, where $g \in L^p(\mathbb{R}^n)$ is such that $f = G_\alpha * g$ almost everywhere. A weaker version of this result in the range $0 < \alpha \leq \frac{n-1}{2}$ as well as some related localisation principles are also obtained. For $1 \leq p < 2 - \frac{1}{n}$ and $0 \leq \alpha < \frac{(2-p)n-1}{2p}$, we construct a function $f \in \mathcal{L}^p_\alpha(\mathbb{R}^n)$ such that $S_Rf(x)$ diverges everywhere.

INTRODUCTION AND BACKGROUND

In 1966, Carleson [8] solved the long-standing Lusin conjecture, which stated that

$$S_Nf(x) = \sum_{k-N}^{N} \hat{f}(k)e^{ikx}$$

should converge to $f(x)$ almost everywhere on $\mathbb{T}$ if $f \in L^2(\mathbb{T})$. However, it had long been known that "nice" functions had better pointwise convergence properties. The first step towards making precise this improved convergence came from Beurling [2], but the final result was proved by Salem and Zygmund [22]. They showed that if $\sum_{k=1}^{\infty} k^\beta(a_k^2 + b_k^2) < \infty$ and the Fourier series

$$\tfrac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos(kx) + b_k \sin(kx))$$

diverges on a closed set E, then E is of null $(1-\beta)$-capacity when $0 < \beta < 1$ and of null logarithmic capacity when $\beta = 1$ (information about these capacities and the results in [2], [22] can be found in [14]). In the language and context of this article, their result can be rewritten as follows:

Theorem A. *Let $0 < \alpha \leq \frac{1}{2}$ and let $f \in \mathcal{L}^2_\alpha(\mathbb{R})$. Suppose also that $g \in L^2(\mathbb{R})$ is such that $f = G_\alpha * g$ almost everywhere. Then*

$$S_Rf(x) = \int_{B(0,R)} \hat{f}(y)e^{2\pi ix\cdot y}dy \to G_\alpha * g(x)$$

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$C_{\alpha,2}$-quasieverywhere on $\mathbb{R}$. (G_α is the Bessel kernel of order α. The space $\mathcal{L}^2_\alpha(\mathbb{R})$ and its associated $C_{\alpha,2}$-capacity will be defined below.)

A counterexample due to Beurling [14, pp.47-49] shows that this is the best possible result for functions in Bessel-Sobolev spaces. In [9, pp. 50-54], Carleson extended both the result and the counterexample to some other "nice" spaces.

In this article, we will study what happens in dimensions greater than 1. More precisely, we will show that, using an idea from [5], one can easily extend Beurling's result [2, Remarque, p.9] to all dimensions:

Theorem 1. *Let $f \in \mathcal{L}^p_\alpha(\mathbb{R}^n)$, $0 < \alpha \leq \frac{n}{p}$, $2 \leq p < \frac{2n}{n-1}$, and let $g \in L^p(\mathbb{R}^n)$ be such that $f = G_\alpha * g$ almost everywhere. Then $S_R f(x) \to G_\alpha * g(x)$ $C_{\alpha-\epsilon,2}$-quasieverywhere on $\mathbb{R}^n$ for every ϵ such that $0 < \epsilon < \alpha$.*

By combining the ideas behind Theorem 1 with an estimate in [5], we will also see that it is possible to get a better result if we restrict ourselves to the localisation problem.

Theorem 2. *Let $f \in \mathcal{L}^p_\alpha(\mathbb{R}^n)$, $0 < \alpha \leq \frac{n}{p}$, $2 \leq p < \frac{2n}{n-1}$, and let $g \in L^p(\mathbb{R}^n)$ be such that $f = G_\alpha * g$ almost everywhere. Then $S_R f(x) \to G_\alpha * g(x)$ $C_{\alpha,2}$-quasieverywhere off the support of g.*

The "standard" localisation principles are all stated "off the support of f" (e.g. [5, Theorem 2.1], but the form used here is more precise for functions in $\mathcal{L}^p_\alpha$. When $\alpha \geq \frac{n-1}{2}$, the convergence in fact takes place everywhere as for the classical Riemann localisation principle.

Theorem 3. *Let $f \in \mathcal{L}^p_\alpha(\mathbb{R}^n)$, $\frac{n-1}{2} \leq \alpha \leq \frac{n}{p}$, $2 \leq p < \frac{2n}{n-1}$, and let $g \in L^p(\mathbb{R}^n)$ be such that $f = G_\alpha * g$ almost everywhere. Then $S_R f(x) \to G_\alpha * g(x)$ everywhere off the support of g.*

An example due to Il'in [13] shows that, for any $\alpha < \frac{n-1}{2}$, it is possible to find a compactly supported Lipschitz-continuous function of order α such that $f \equiv 0$ on $B(0,1)$ and $\limsup_{R\to\infty}|S_R f(0)| = \infty$. The index $\frac{n-1}{2}$ in Theorem 3 is consequently sharp when looking for an everywhere localisation principle in dimension greater than 1.

In Theorem 3, the improvement over Theorem 2 is based on some ideas similar to [6], [7]. Using the decay of G_α, it is possible to relax the condition (c) in [6, Theorem 11] to get the following key lemma:

Lemma 4. *Let $h \in L^p(\mathbb{R}^n)$ with $2 \leq p < \frac{2n}{n-1}$ and $n \geq 2$. Suppose that $f = (\phi G_\alpha) * h = \tilde{G}_\alpha * h$ on $\mathbb{R}^n$ where $\alpha \in (0, \frac{n}{p}]$ and $\phi \in C^\infty_c(\mathbb{R}^n)$ is a radial function supported in $B(0,\epsilon)$, $\epsilon > 0$, such that $0 \leq \phi(x) \leq 1$ for all x in $\mathbb{R}^n$ and $\phi \equiv 1$ on $B(0,\frac{\epsilon}{2})$. Suppose also that $\operatorname{supp}(h) \subseteq (B(0,r+\epsilon))^c$ (so that $\operatorname{supp}(f) \subseteq (B(0,r))^c$), $r \in \mathbb{R}^+$, and that μ is a finite positive Borel measure supported on $B(0,r)$ which satisfies*

$$\|\widehat{gd\mu}(R\bullet)\|_{L^2(S^{n-1})} \leq CR^{-(n-1-2\alpha)/2}\|g\|_{L^2(d\mu)} \tag{0.1}$$

for all g. Then $\|S_* f\|_{L^2(d\mu)} \leq C\|h\|_p$ where C is a constant independent of f and $S_* f(x) = \sup_{R>1}|S_R f(x)|$.

The gain over Theorem 2 obtained by using Lemma 4 is not limited to $\alpha \geq \frac{n-1}{2}$. If we add an additional restriction to closed sets of divergence, we can also get an improvement when $\frac{n-1}{4} \leq \alpha < \frac{n-1}{2}$.

Theorem 5. *Let $f \in \mathcal{L}^p_\alpha(\mathbb{R}^n)$ with $n \geq 2$, $2 \leq p < \frac{2n}{n-1}$, $\alpha > 0$ and $\alpha \in [\frac{n-1}{4}, \frac{n-1}{2})$. Suppose also that $f = G_\alpha * g$ almost everywhere, and let E be the set of divergence for $S_R f(x)$ off the support of g (i.e., $E = \{x \notin \operatorname{supp}(g) : S_R f(x)$ diverges$\}$). If $\tilde{E} \subseteq E$ is closed, then $C_{\alpha+(1/2),2}(\tilde{E}) = 0$. In particular, $S_R f(x) \to G_\alpha * g(x)$ $C_{\alpha+(1/2),2}$-quasieverywhere off the support of g if E is closed.*

In fact, we may expect Theorem 5 to hold without the additional closed set restriction. To do this we will need to follow more closely the original argument of Carbery and Soria [5, Theorem 2.1]. This aspect will be discussed in a forthcoming article.

Using the localisation provided by Theorem 3, we will be able to establish a pointwise estimate for $S_*(\phi G_\alpha)$ (where $\phi \approx \chi_{B(0,1)}$) which will enable us to give an analogue of the sharp Salem and Zygmund result (Theorem A) when $\alpha > \frac{(n-1)}{2}$.

Theorem 6. *Let $f \in \mathcal{L}^p_\alpha(\mathbb{R}^n)$, $\frac{n-1}{2} < \alpha \leq \frac{n}{p}$, $2 \leq p < \frac{2n}{n-1}$ and let $g \in L^p(\mathbb{R}^n)$ be such that $f = G_\alpha * g$ almost everywhere. Then $S_R f(x) \to G_\alpha * g(x)$ $C_{\alpha,p}$-quasieverywhere on $\mathbb{R}^n$.*

Up to now, we have considered the case where $2 \leq p < \frac{2n}{n-1}$; but what happens if $p < 2$? Until recently, it was believed that we should expect a series of results similar to the case $p \geq 2$, but T. Tao showed in the Bochner-Riesz setting that, for some α, it is possible to find a function which diverges on a set of positive measure. Just by scaling Tao's example [28, Proposition 5.1], it is very easy to verify that the same holds in the setting discussed in this article. In fact, we can do slightly better using appropriate sums of translated and scaled Tao's functions.

Theorem 7. *If $1 \leq p < 2 - \frac{1}{n}$ and $0 \leq \alpha < \frac{(2-p)n-1}{2p}$, then there is an $f \in \mathcal{L}^p_\alpha(\mathbb{R}^n)$ such that $S_R f(x)$ diverges everywhere on $\mathbb{R}^n$.*

This paper is divided into eight sections containing the proofs of all the previous results and some related discussions. The first of these sections will be used to introduce different estimates needed throughout this work and to prove Theorems 1 and 2. The three following sections will contain a discussion of our localisation principles (Lemma 4 and Theorem 3 in Section 2 and Theorem 5 in Section 3) as well as some possible generalisations (Section 4). We will then return to our main problem in Section 5 to study the pointwise convergence when $2 \leq p < \frac{2n}{n-1}$ and $\alpha > \frac{n-1}{2}$ (Theorem 6). The more difficult case $0 < \alpha \leq \frac{n-1}{2}$ will be left for Section 6, where the proof of a surprising result showing that we cannot interpolate in the "usual" way between the estimates built in the previous parts of this paper will be sketched. The situation for $p < 2$ will follow in Section 7 (Theorem 7). Finally, Section 8 will be used to try to quantify the amount of uniformity present in the previous results.

In the case corresponding to Theorems 1 and 6, we will in particular justify in this last section why the natural conjecture seems to be the following:

Conjecture 8. *Let $2 \leq p < \frac{2n}{n-1}$ and $0 < \alpha \leq \frac{n}{p}$. Suppose that $f \in \mathcal{L}^p_\alpha(\mathbb{R}^n)$ and suppose also that $g \in L^p(\mathbb{R}^n)$ is such that $f = G_\alpha * g$ almost everywhere. Then $S_R f(x) \to G_\alpha * g(x)$ $C_{\alpha,p}$-quasieverywhere on $\mathbb{R}^n$. Moreover, the convergence is taking place uniformly outside an open set of arbitrarily small $C_{\alpha,p}$-capacity.*

Remark. For $\frac{n-1}{2} < \alpha \leq \frac{n}{p}$, the first part of this conjecture is Theorem 6; what would be new here is the uniformity.

Before continuing, let us recall rapidly some definitions and results.

Definition 1. For $0 < \alpha \leq n$, the Bessel kernel of order α in $\mathbb{R}^n$, G_α, is defined to be the inverse Fourier transform of $\left(1 + |2\pi\bullet|^2\right)^{-\alpha/2}$, and the corresponding Bessel-Sobolev spaces, $\mathcal{L}^p_\alpha(\mathbb{R}^n)$, are the subspaces of $L^p(\mathbb{R}^n)$ obtained by convolving L^p with G_α.

For our purposes, we will need the following classical properties of G_α [25, pp.131-133]:

Proposition B.

(1) *$G_\alpha(x) \geq 0$ and $G_\alpha \in L^1(\mathbb{R}^n)$ for all $\alpha > 0$; moreover, $\|G_\alpha\|_1 = 1$ for these α.*
(2) *$G_\alpha * G_\beta = G_{\alpha+\beta}$ if $\alpha, \beta > 0$.*
(3) *$G_\alpha(x) = \frac{\Gamma((n-\alpha)/2)}{2^\alpha \pi^{n/2}\Gamma(\alpha/2)}|x|^{-n+\alpha} + o\big(|x|^{-n+\alpha}\big)$ near the origin for $0 < \alpha < n$.*
(4) *$G_n(x) = O\Big(\log \frac{1}{|x|}\Big)$ near the origin.*
(5) *$G_\alpha(x) = O\Big(e^{-c|x|}\Big)$ with $c > 0$ when $|x| \to \infty$.*
(6) *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $G_\alpha \notin L^q$ when $0 < \alpha \leq \frac{n}{p}$, while $\int_{(B(0,1))^c} (G_\alpha)^q(x)dx < \infty$ for all $\alpha > 0$.*

Using G_α, it is possible to define the $C_{\alpha,p}$-capacities mentioned in the different theorems above:

Definition 2. For $1 \leq p < \infty$ and $E \subseteq \mathbb{R}^n$, the p-capacity of E with respect to the kernel G_α, $C_{\alpha,p}(E)$, is defined by

$$C_{\alpha,p}(E) = \inf\left\{\int_{\mathbb{R}^n} f^p(y)dy : f \in L^p_+(\mathbb{R}^n) \text{ and } G_\alpha * f(x) \geq 1 \text{ for all } x \in E\right\}.$$

In a similar way, it is possible to build the capacities associated to other kernels, such as the $C_{K,2}$-capacities used in Theorem 4.1 (for more detail see [1]).

Remark 1. When the set E is a Borel set or more generally a Suslin set, Definition 2 is equivalent to

$$C_{\alpha,p}(E) = \left(\sup\Big\{\mu(E) : \mu \in \mathfrak{M}^+(A),\ A \subseteq E \text{ and } \|G_\alpha * d\mu\|_q \leq 1,\ \tfrac{1}{p} + \tfrac{1}{q} = 1\Big\}\right)^p.$$

Remark 2. When the set E is a compact set of $\mathbb{R}^n$, Definition 2 is equivalent to

$$C_{\alpha,p}(E) = \max\Big\{\mu(E) : \mu \in \mathfrak{M}^+(E) \text{ and } G_\alpha * (G_\alpha * \mu)^{q-1}(x) \leq 1,$$
$$\text{where } \tfrac{1}{p} + \tfrac{1}{q} = 1 \text{ for all } x \in \operatorname{supp}(\mu)\Big\}.$$

Part 6 of Proposition B implies both that a singleton has null $C_{\alpha,p}$-capacity and that $|E| = 0$ if $C_{\alpha,p}(E) = 0$ exactly when $0 < \alpha \leq \frac{n}{p}$. Thus, the $C_{\alpha,p}$-capacities are well adapted as a way to differentiate between sets of null Lebesgue measure only in the range $0 < \alpha \leq \frac{n}{p}$.

Throughout this article, $y \sim c$ will mean $\{y \in \mathbb{R} : \frac{c}{2} \leq y < 2c\}$, while C, with or without indices, will denote a constant that can change value from one appearance to the next. The only exceptions to this rule are when C is followed by either a set or a word making clear that it represents a capacity.

1. An extension of Beurling's result and a $C_{\alpha,2}$-localisation principle

It was proved in [5] that $S_R f(x) \to f(x)$ almost everywhere when $f \in \mathcal{L}^p_\alpha(\mathbb{R}^n)$, $\alpha > 0$ and $2 \le p < \frac{2n}{n-1}$. In order to prove the stronger Theorems 1 and 6, we will need to use two of Carbery and Soria's estimates that we combine in the following lemma:

Lemma C. *Let $f \in \mathcal{L}^p_\alpha(\mathbb{R}^n)$ with $\alpha > 0$ and $2 \le p < \frac{2n}{n-1}$. Then*

$$\int_{B(y,r)} |S_* f(x)|^2 dx \le C_r \|g\|_p^2,$$

where $g \in L^p(\mathbb{R}^n)$ is such that $f = G_\alpha * g$ almost everywhere, $r \in \mathbb{R}^+$, $y \in \mathbb{R}^n$ and C_r is a constant independent of y and f but depending on r.

Proof. Without loss of generality, we can, by translation and scaling, restrict ourselves to showing that $\int_{B(0,1/2)} |S_* f(x)|^2 dx \le C\|g\|_p^2$. Now, let $\phi \in C_c^\infty(\mathbb{R}^n)$ be such that $\chi_{B(0,1)} \le \phi \le \chi_{B(0,2)}$. Hence,

$$\int_{B(0,1/2)} |S_* f(x)|^2 dx \le C \int_{B(0,1/2)} \Big(|(S_* \phi f)(x)|^2 + |(S_*(1-\phi) f)(x)|^2 \Big) dx.$$

Then, the first term is bounded by $C\|g\|_p^2$ using the estimate

$$\int_{B(0,r)} |S_* f(x)|^2 dx \le C_{r,r_1} \|g\|_p^2, \tag{1.1}$$

valid when $f \in \mathcal{L}^p_\alpha(\mathbb{R}^n)$ with $2 \le p < \frac{2n}{n-1}$, $\alpha > 0$, $\operatorname{supp}(f) \subseteq B(0, r_1)$, $0 < r < r_1 < \infty$, $g \in L^p(\mathbb{R}^n)$ and $f = G_\alpha * g$ almost everywhere. The second term is bounded by $C\|f\|_p^2$ using

$$\int_{B(0,r)} |S_* f(x)|^2 dx \le C_{r,r_1} \|f\|_p^2. \tag{1.2}$$

Estimate (1.2) is valid for any $f \in L^p(\mathbb{R}^n)$, $2 \le p < \frac{2n}{n-1}$, supported in $(B(0,r_1))^c$, $0 < r < r_1 < \infty$. Both (1.1) and (1.2) follow directly from the computations in [5, Theorems 2.1 and 3.1]. $\square$

Another tool which will often be used is a simple pointwise estimate for S_*:

Basic Estimate. *If f and g are two functions in some appropriate function space, then $S_*(f * g)(x) \le \big(|f| * S_* g\big)(x)$ on $\mathbb{R}^n$.*

Proof. Fix R and then apply Fubini's theorem followed by the triangle inequality to get $|S_R(f * g)(x)| \le \big(|f| * S_* g\big)(x)$. $\square$

To discard the small part in computing the $C_{\alpha-\epsilon,2}$-capacity in Theorem 1, we will use the following lemma:

Lemma 1.1. *Suppose that f is supported outside the ball $B(0,r)$ and satisfies*

$$\int_{B(y,r_1)} |f(x)|^2 dx < C < \infty$$

for every ball $B(y, r_1) \subset \mathbb{R}^n$, with C a constant independent of $y \in \mathbb{R}^n$. Then $|G_\alpha * f(x)| < C_{r,r_1} < \infty$ for $|x| \le r_1 < r$ when $\alpha > 0$, with C_{r,r_1} a constant depending only on α, r and r_1.

Proof. To simplify the notation, fix $r_1 = 1$ and $r = 2$; the general case is similar.

Let $x \in B(0,1)$ be fixed. Then,

$$G_\alpha * f(x) = \int_{(B(0,2))^c} G_\alpha(x-y)f(y)dy = \sum_{j=2}^{\infty} \int_{D_j} G_\alpha(x-y)f(y)dy,$$

where $D_j = B(0,j+1)\backslash B(0,j)$. The first equality follows from the support property of f.

We now cover D_j with $c_n j^{n-1}$ balls $b_{i,j}$ of radius 1, where c_n is a constant depending only on the dimension n. If we replace the balls $b_{i,j}$ by $\tilde{b}_{i,j} = b_{i,j} \cap D_j$, we get $D_j = \bigcup_i \tilde{b}_{i,j}$. Hence,

$$\begin{aligned}\left|\int_{\tilde{b}_{i,j}} G_\alpha(x-y)f(y)dy\right| &\leq G_\alpha^{1/2}(j-1)\int_{\tilde{b}_{i,j}} G_\alpha^{1/2}(x-y)|f(y)|dy \\ &\leq G_\alpha^{1/2}(j-1)\left(\int_{\tilde{b}_{i,j}} G_\alpha(x-y)dy\right)^{1/2}\left(\int_{\tilde{b}_{i,j}} |f(y)|^2dy\right)^{1/2} \\ &\leq G_\alpha^{1/2}(j-1)\|G_\alpha\|_1^{1/2}C^{1/2}.\end{aligned}$$

The first inequality follows from the radial decay of G_α as $y \in \tilde{b}_{i,j} \subseteq D_j$ and $x \in B(0,1)$.

From the exponential decay of G_α away from the origin, we then have

$$\left|\sum_{j=2}^{\infty}\int_{D_j} G_\alpha(x-y)f(y)dy\right| \leq \sum_{j=2}^{\infty} G_\alpha^{1/2}(j-1)\|G_\alpha\|_1^{1/2}C^{1/2}c_n j^{n-1} = C_{r,r_1} < \infty.$$

□

Remark 1.2. A similar argument can be used if G_α is replaced by a more general kernel $\omega \in L^1(\mathbb{R}^n)$ bounded in $(B(0,r-r_1))^c$ and such that $|\omega(x)| = O(\varphi(|x|))$, where φ is a positive function satisfying $\sum_{j \text{ large}} \varphi(j)j^{n-1} < \infty$. To prove this case, we can without loss of generality replace ω by a kernel $\tilde{\omega}$ which is equal to ω where it is defined and equal to 0 otherwise. The only difference in the argument is then that the radial decay which gives a bound on the inferior border of D_j is replaced by the suprema of $|\tilde{\omega}|$ over D_j. By our hypothesis, these suprema decay fast enough to get the desired result. This more general version of Lemma 1.1 contains the fundamental idea for extending Lemma 1.6 (and all the work in this paper) to the other class of functions covered by Theorem 3.1 in [5].

We are now in a position to prove the higher-dimensional analogue of Beurling's result [2, Remarque, p. 9].

Proof of Theorem 1. Without loss of generality, we replace f by $G_\alpha * g$, since $S_R f(x) = S_R(G_\alpha * g)(x)$ for all $x \in \mathbb{R}^n$. Then, using the Basic Estimate and Proposition B, part 2, we get

$$\begin{aligned}&S_*(G_\alpha * g)(x) \\ &\quad\leq G_{\alpha-\epsilon} * \big(\chi_{B(0,2)}S_*(G_\epsilon * g)\big)(x) + G_{\alpha-\epsilon} * \big(\chi_{(B(0,2))^c}S_*(G_\epsilon * g)\big)(x) \\ &\quad= h_1(x) + h_2(x).\end{aligned}$$

This implies that

$$\begin{aligned} E &= \{x \in B(0,1) : S_*(G_\alpha * g)(x) > \lambda\} \\ &\subseteq \{x \in B(0,1) : h_1(x) > \tfrac{\lambda}{2}\} \cup \{x \in B(0,1) : h_2(x) > \tfrac{\lambda}{2}\} = E_1 \cup E_2. \end{aligned}$$

Hence, $C_{\alpha-\epsilon,2}(E) \leq C_{\alpha-\epsilon,2}(E_1) + C_{\alpha-\epsilon,2}(E_2)$ by the subadditivity of $C_{\alpha-\epsilon}$. Consequently, we will be done if we can show that $C_{\alpha-\epsilon,2}(E_j) \leq C\lambda^{-2}\|g\|_p^2$ for $\lambda > c > 0$ and $j = 1, 2$.

But, on one hand, $C_{\alpha-\epsilon,2}(E_2) = 0$ if $\lambda > 2C_n\|g\|_p$ by Lemma 1.1 combined with Lemma C. The constant $C_n\|g\|_p$ is obtained by following the proof of Lemma 1.1.

On the other hand,

$$C_{\alpha-\epsilon,2}(E_1) \leq \int_{\mathbb{R}^n} \left|2\lambda^{-1}h_1(x)\right|^2 dx \leq C\lambda^{-2}\|g\|_p^2.$$

The first inequality follows from Definition 2 while the second one follows from Lemma C. So, by translation invariance, the desired result follows. □

In fact, with exactly the same method, but with a stronger version of Lemma C implied by Carbery and Soria's estimates [5], it is possible to show the following slightly stronger result when $p = 2$:

Theorem 1.3. *Let $f \in \mathcal{L}^2_\alpha(\mathbb{R}^n)$, $0 < \alpha \leq \frac{n}{2}$. Suppose also that*

$$\int |\hat{g}(\xi)|^2 \left(1 + \log^+|\xi|\right)^2 d\xi < \infty$$

for some $g \in L^p(\mathbb{R}^n)$ such that $f = G_\alpha * g$ almost everywhere. Then $S_R f(x) \to G_\alpha * g(x)$ $C_{\alpha,2}$-quasieverywhere on $\mathbb{R}^n$.

To get most of the results in this article, we will break the action of G_α when convolved with g into two parts: one near the point, which will represent the main contribution, and one far from it, which will be negligible. The next lemma quantifies this second, nicer part.

Lemma 1.4. *Let $\alpha > 0$. Suppose that $g \in L^p(\mathbb{R}^n)$ with $2 \leq p < \frac{2n}{n-1}$ and suppose also that $\phi \in C_c^\infty(\mathbb{R}^n)$ is such that $\phi \equiv 1$ on $B(0,r)$ for some $r \in \mathbb{R}^+$. Then $S_R\big([(1-\phi)G_\alpha] * g\big)(x)$ converges everywhere on $\mathbb{R}^n$.*

Remark 1.5. This nice part could have also been controlled with the following lemma:

Lemma 1.6. *Take $0 < \alpha \leq \frac{n}{p}$ and $g \in L^p(\mathbb{R}^n)$, $2 \leq p < \frac{2n}{n-1}$. Let $\phi \in C_c^\infty(\mathbb{R}^n)$ satisfy the following conditions: ϕ is radial, $\operatorname{supp}(\phi) \subseteq B(0, 2r_1)$, where $r_1 \in \mathbb{R}^+$, $\phi \equiv 1$ on $B(0, r_1)$, and $0 \leq \phi(x) \leq 1$ for all $x \in \mathbb{R}^n$. Assume also that $\mu \in \mathfrak{M}^+(B(0, r_2))$, $r_2 \in \mathbb{R}^+$, satisfies $\mu(B(0, r_2)) \leq c < \infty$. Then*

$$\int S_*\big(((1-\phi)G_\alpha) * g\big)(x)\, d\mu(x) \leq C_{\phi,\alpha}\|g\|_p$$

where $C_{\phi,\alpha}$ is a constant depending only on ϕ, α and c. (Note that there is no relation between the values of r_1 and r_2 in this lemma.)

The proof [19, Lemma 2.3.5] of Lemma 1.6 relies on the same decay idea as Lemma 1.1 in the version of Remark 1.2. What is more interesting is that, for some spaces covered in Section 4, this is the correct way to handle the tail of the kernel when trying to extend this work. In particular, this can be done to study most of the class of functions covered by Theorem 3.1 in [5] (the kernels in these cases have the decay required by Remark 1.2, and so Lemma 1.6 can be easily modified). Extending the work presented in this paper (Theorems 1, 2, 3 and 6) then just requires minor adaptations.

Proof of Lemma 1.4. Since $G_\alpha \in C^\infty(\mathbb{R}^n\backslash\{0\})$, we have $(1-\phi)G_\alpha \in C^\infty \cap L^q(\mathbb{R}^n) \subset \mathcal{L}^q_\beta(\mathbb{R}^n)$ for all $q \in [1,\infty)$ and for all $\beta > 0$. Hence, there is an $h \in L^q(\mathbb{R}^n)$ for all $q \in [1,\infty)$ such that $(1-\phi)G_\alpha = G_\beta * h$ almost everywhere if $\beta > \frac{n}{p}$ is fixed. Now, by Young's inequality and the fact that $h \in L^1$, we have $h * g \in L^p$ (i.e., $((1-\phi)G_\alpha) * g \in \mathcal{L}^p_\beta(\mathbb{R}^n)$ with $\beta > \frac{n}{p}$). So $S_R\left([(1-\phi)G_\alpha] * g\right)(x)$ converges everywhere (uniformly) on $\mathbb{R}^n$ by Proposition 3.3 in [5]. □

Using Lemma 1.4, we can now prove our $C_{\alpha,2}$-localisation principle (Theorem 2):

Proof of Theorem 2. By translation and scaling, we can assume, without loss of generality, that $\text{supp}(g) \subseteq (B(0,2))^c$. We now restrict ourselves to showing that the statement of the theorem holds inside $B(0,1)$, i.e., that $S_Rf(x) \to G_\alpha * g(x)$ $C_{\alpha,2}$-quasieverywhere on $B(0,1)$; but the same argument can be used for any ball inside $B(0,2)$ (by defining below a function ϕ with a support small enough).

Let D represent the set of divergence inside $B(0,1)$. Then D is a Borel set. To see this, rewrite D as

$$D = \bigcup_{k=1}^{\infty}\{x \in\mathbb{R}^n : \limsup_R \text{ Re}(S_Rf(x)) - \liminf_R \text{ Re}(S_Rf(x)) + \limsup_R \text{ Im}(S_Rf(x)) - \liminf_R \text{ Im}(S_Rf(x)) > \tfrac{1}{k}\} = \bigcup_{k=1}^{\infty} D_k.$$

Then each D_k is a Borel set, because S_Rf is a continuous function, hence a Borel function, and Borel functions are closed under lim sup, lim inf and difference. Here, Re and Im denote respectively the real and imaginary part.

Now, suppose that, for every Borel measure μ such that $\text{supp}(\mu) \subseteq D$ and $\|G_\alpha * \mu\|_2 \leq 1$, we can show that $\mu(D) = 0$. Then we will have $C_{\alpha,2}(D) = 0$ by Remark 1. Using Lemma 1.4, the theorem will consequently be proven if we can show that

$$\int S_*((\phi G_\alpha) * g)(x) d\mu(x) < C\|g\|_p$$

for any positive Borel measure μ supported inside $B(0,1)$ such that $\|G_\alpha * \mu\|_2 < 1$. Here, ϕ is defined as in Lemma 1.6 with $r_1 = \frac{1}{4}$.

Now,

$$\begin{aligned}\int S_*((\phi G_\alpha) * g)(x) d\mu(x) &\leq \int (\phi G_\alpha) * S_*g(x) d\mu(x) \\ &= \int (\phi G_\alpha) * \mu(x)\chi_{B(0,3/2)}(x) S_*g(x) dx \\ &\quad \text{by } \text{supp}((\phi G_\alpha) * \mu) \subseteq B(0,3/2) \\ &\leq \|(\phi G_\alpha) * \mu\|_2 \|\chi_{B(0,3/2)} S_*g\|_2 \\ &\leq C\|g\|_p.\end{aligned}$$

The first inequality follows from the Basic Estimate, while the last one is a consequence of (1.2) and $\|G_\alpha * \mu\|_2 < 1$. □

Remark 1.7. Combining Theorem 2 with Lemma 1.4 reduces the study of $C_{\alpha,2}$ convergence to the case where both f and g are compactly supported (i.e., to the case $\tilde{f} = (\phi G_\alpha) * \tilde{g}$).

Remark 1.8. When $f \in \mathcal{L}^2_\alpha(\mathbb{R}^n)$, $0 < \alpha \leq \frac{n}{2}$, the localisation provided by Theorem 2 combined with the estimates established in [5, 15, 20, 18] implies in particular that $S_R f(x) \to G_\alpha * g(x)$ $C_{\alpha,2}$-quasieverywhere on any region where g has either some smoothness or some appropriate symmetry (radiality, for example).

2. An everywhere localisation principle

In the previous section, we built a capacitarian localisation principle (Theorem 2). Nevertheless, this result is essentially limited to functions in $\mathcal{L}^2_\alpha(\mathbb{R}^n)$, because we need at least a $C_{\alpha,p}$ localisation principle if we hope to extend Salem and Zygmund's theorem (Theorem A) to $2 < p < \frac{2n}{n-1}$ when $n \geq 2$. In this section, we will present a technique based on Fefferman's folk computation [11, p.45] to build these more precise localisation results.

Folk Lemma (Fefferman's folk computation). *Let $f \in L^p(\mathbb{R}^n)$ be compactly supported. Then*

$$S_R f(x) \approx R^{(n-1)/2}|x|^{-(n+1)/2} e^{2\pi i R|x|} \hat{f}\left(\tfrac{Rx}{|x|}\right)$$

for large values of x and R.

Our key tool in building localisation principles will be Lemma 4. But, before proving it, we want to discuss how one can slightly modify the proof of the related Theorem 11 in [6] so that this result holds not only when $p = 2$, but also when $2 < p < \frac{2n}{n-1}$. More precisely, we want to prove the following:

Lemma 2.1. *Let $n \geq 2$. Suppose that $f \in L^p(\mathbb{R}^n)$, $2 \leq p < \frac{2n}{n-1}$, is such that $\mathrm{supp}(f) \subseteq (B(0,r))^c$, $r \in \mathbb{R}^+$. Suppose also that μ is a finite positive Borel measure supported on $B(0,r)$ satisfying*

$$\|\widehat{g d\mu}(R\bullet)\|_{L^2(S^{n-1})} \leq C R^{-(n-1)/2} \|g\|_{L^2(d\mu)} \tag{2.1}$$

for all g. Then $\|S_* f\|_{L^2(d\mu)} \leq C\|f\|_p$ where C is a constant independent of f.

Proof. The case $p = 2$ was proven heuristically in [6] and formally in [7]. We will consequently allow ourselves to present a more sketchy argument for some parts of this proof. Nevertheless, what we will need for the proof of Lemma 4 and the modifications to get this result for $2 < p < \frac{2n}{n-1}$ will be clearly explained.

By dilation invariance, we can restrict ourselves to the case $r = 3$. We also suppose that $\mathrm{supp}(\mu) \subseteq B(0,1)$ to keep our notation consistent with [6], but the same argument works for any ball inside $B(0,3)$. The proof is then done in two steps. We will first study the L^2 case heuristically, using Fefferman's folk computation to obtain a weaker estimate under the lemma's hypothesis, namely,

$$\sup_{R>1} \|S_R f\|_{L^2(d\mu)} \leq C \tilde{r}^{-1} \|f\|_2 \tag{2.2}$$

for $f \in L^2(\mathbb{R}^n)$ with $\mathrm{supp}(f) \subseteq \{x : |x| \sim \tilde{r}\}$ and $\tilde{r} \in \mathbb{R}^+$. This estimate will then be used to prove another heuristic L^2 estimate implying Lemma 2.1. The rigorous versions of these two estimates are in [7].

By duality, (2.2) is equivalent to

$$\|S_R(gd\mu)\|_{L^2(\{x:|x|\sim\tilde r\})} \le C\tilde r^{-1}\|g\|_{L^2(d\mu)}. \tag{2.3}$$

Also, using Fefferman's folk computation, this can be rewritten as

$$R^{(n-1)/2}\left\| |\bullet|^{-(n+1)/2}\widehat{gd\mu}\left(\frac{R\bullet}{|\bullet|}\right)\right\|_{L^2(\{x:|x|\sim\tilde r\})} \le C\tilde r^{-1}\|g\|_{L^2(d\mu)}, \tag{2.4}$$

which is true by hypothesis (2.1).

Now, the proof of Lemma 2.1 closely follows the proof of Theorem 2.1 in [5]. After rewriting everything with the same partition of unity (i.e., let ϕ be a radial function such that $\chi_{\bar B(0,1)} \le \phi \le \chi_{\bar B(0,2)}$ and let $\psi(x) = \phi(x) - \phi(2x)$), then, $1 = \phi(0) = \phi(x) + \sum_{j=1}^\infty \left(\phi\left(\frac{x}{2^j}\right) - \phi\left(\frac{x}{2^{j-1}}\right)\right) = \phi(x) + \sum_{j=1}^\infty \psi_j(x)$, where $\psi_j(x) = \psi\left(\frac{x}{2^j}\right)$. As in Carbery and Soria's proof (we keep their notation for simplicity), we are left to show that

$$\int \sup_{t\ge1}\left|\sum_{j=1}^\infty K_t^j * (f\psi_{j+s})(x)\right|^2 d\mu(x) \le C\left(\int_{|x|\ge3}|f(x)|^p dx\right)^{2/p} = C\|f\|_p^2 \tag{2.5}$$

for $|s|<3$. Let us suppose for a moment that we have

$$\int \sup_{t\ge1}\left|K_t^j * g(x)\right|^2 d\mu(x) \le C2^{-j}\int_{|x|\sim2^j}|g(x)|^2 dx \tag{2.6}$$

for g supported in $\{x : |x|\sim 2^j\}$.

Then, by Hölder's inequality, we will have

$$\int \sup_{t\ge1}\left|K_t^j * g(x)\right|^2 d\mu(x) \le C2^{-j}2^{jn/q}\left(\int_{|x|\sim2^j}|g(x)|^p dx\right)^{2/p} \le C2^{-\epsilon j}\|g\|_p^2$$

with $\frac1q + \frac2p = 1$. But $2 \le p < \frac{2n}{n-1}$; hence $\frac nq - 1 = -\epsilon < 0$. Consequently,

$$\begin{aligned}
\int \sup_{t\ge1}\left|\sum_{j=1}^\infty K_t^j * f_{j+s}(x)\right|^2 d\mu(x) &\le \int\left(\sum_{j=1}^\infty \sup_{t\ge1}\left|K_t^j * f_{j+s}(x)\right|\right)^2 d\mu(x)\\
&\le \left(\sum_{j=1}^\infty\left(\int\sup_{t\ge1}\left|K_t^j * f_{j+s}(x)\right|^2 d\mu(x)\right)^{1/2}\right)^2\\
&\le \left(\sum_{j=1}^\infty C2^{-\epsilon j/2}\left(\int_{|x|\sim2^j}\left|f_{j+s}(x)\right|^p dx\right)^{1/p}\right)^2\\
&\le C\|f\|_p^2.
\end{aligned}$$

So, Lemma 2.1 will be proven if we can show (2.6) and formalise (2.2). But, (2.6) follows from the cases $\beta = 0, 1$ of

$$\iint_1^\infty \left|\left(\frac{d^\beta}{dt^\beta}K_t^j\right) * g(x)\right|^2 dt\,d\mu(x) \le C2^{-2j(1-\beta)}\int_{|x|\sim2^j}|g(x)|^2 dx \tag{2.7}$$

using the usual majorisation based on the fundamental theorem of calculus and the Cauchy-Schwarz inequality (the adjustment term is controlled here with (2.2)).

Finally, [6] contains a heuristic argument proving (2.7), and [7] a rigorous version of both (2.2) and (2.7). For completeness, we will now show how (2.2) can be proven rigorously using Plancherel's theorem. (2.7) is done in a similar way.

Trivially, the hypothesis (2.1) implies that

$$\int_{\{\xi:||\xi|-R|\leq 1\}} \left|\widehat{gd\mu}(\xi)\right|^2 d\xi \leq C\|g\|^2_{L^2(d\mu)}. \tag{2.8}$$

Hence, we will have finished if we can show that (2.8) implies the dual form of (2.2).

Now, let $K_R(x) = \varphi(x)K_R(x) + (1-\varphi(x))K_R(x) = K_{R,1}(x) + K_{R,2}(x)$, where $\varphi \in C_c^\infty$ with support inside $B(0,\frac{1}{2})$. Choose also a $\psi \in \mathcal{S}$ such that $\psi(x) = 1$ on $B(0,\frac{3}{2})$. Then (2.3) follows from

$$\|(1-\psi)S_R(gd\mu)\|_2 \leq C\|g\|_{L^2(d\mu)},$$

which itself follows from

$$\|(1-\psi)\left(K_{R,k} * (gd\mu)\right)\|_2 \leq C\|g\|_{L^2(d\mu)} \tag{2.9_k}$$

for $k = 1, 2$.

But, $\operatorname{supp}(gd\mu) \subseteq B(0,1)$ and $\operatorname{supp}\left(K_{R,1} * (gd\mu)\right) \subseteq B(0,\frac{3}{2})$ by our hypothesis on the support of φ. Consequently, $(1-\psi)(K_{R,1} * (gd\mu))(x) \equiv 0$, and so (2.9_1) is true. While

$$\left((1-\hat{\varphi}) * \chi_{B(0,R)}\right)(\xi) \leq \begin{cases} 1 & \text{if } ||\xi|-R| \leq 2, \\ \gamma\left(||\xi|-R|\right) & \text{if } ||\xi|-R| > 2, \end{cases}$$

because φ has been chosen so that $\hat{\varphi}\chi_{B(0,R)}$ is a smoothed version of $\chi_{B(0,R)}$ of scale 1. In the previous inequality, γ is a rapidly decreasing function (almost 0 when $||\xi|-R| > 2$). Hence, by Plancherel's theorem and (2.8), we have

$$\begin{aligned} \|(1-\psi)\left(K_{R,2} * (gd\mu)\right)\|_2 &\leq \|\left((1-\hat{\varphi}) * \chi_{B(0,R)}\right)(\xi)\widehat{gd\mu}(\xi)\|_2 \\ &\leq \int_{\{\xi:||\xi|-R|\leq 1\}} \left|\widehat{gd\mu}(\xi)\right|^2 d\xi \leq C\|g\|^2_{L^2(d\mu)}. \end{aligned}$$

□

If we add some smoothness to the function f, we can modify the argument behind Lemma 2.1 to request less decay from the Fourier transform of $gd\mu$ on S^{n-1} (Lemma 4).

Proof of Lemma 4. The proof of Lemma 4 is similar to the proof of Lemma 2.1. The only real difference is that we use the decay of $\hat{f}$ to compensate for the lack of decay of R in (2.1). We will consequently do the analogue of (2.2), and then we will limit ourselves to sketching the main modifications for the other parts of the argument.

As in Lemma 2.1, we limit ourselves to the case $r = 3$ and $\operatorname{supp}(\mu) \subseteq B(0,1)$, but all the others follow in the same way. So, we want to show that

$$\sup_{R>1} \|S_R f\|_{L^2(d\mu)} \leq C\tilde{r}^{-1}\|h\|_2 \tag{2.2'}$$

for $h \in L^2(\mathbb{R}^n)$ and $f = \tilde{G}_\alpha * h$ almost everywhere with $\operatorname{supp}(f), \operatorname{supp}(h) \subseteq \{x : |x| \sim \tilde{r}\}$ and $0 < \epsilon \ll \tilde{r}$. By duality and Fefferman's computation this is equivalent to

$$R^{(n-1)/2}\left\| |\bullet|^{-(n+1)/2}\hat{\tilde{G}}_\alpha(R)\widehat{gd\mu}\left(\tfrac{R\bullet}{|\bullet|}\right)\right\|_{L^2(\{x:|x|\sim\tilde{r}\})} \le C\tilde{r}^{-1}\|g\|_{L^2(d\mu)}, \tag{2.4'}$$

which is true by hypothesis (0.1) and $\hat{\tilde{G}}_\alpha(R) \approx R^{-\alpha}$.

Now, rather than applying the partition of unity in [5] directly to f like in Lemma 2.1, we apply it to h. The small increase in the size of the support induced by $\tilde{G}_\alpha$ to the different pieces $\tilde{G}_\alpha * (h\psi_j)$ is negligible (formally one can replace the dilations of order 2^j by dilations of order c^j with $c > 1$ large enough and keep track of the tiny perturbations). This leads us to replace (2.5) by

$$\int \sup_{t\ge 1}\left|\sum_{j=1}^{\infty} K_t^j * \left(\tilde{G}_\alpha * (h\psi_{j+s})\right)(x)\right|^2 d\mu(x) \le C\left(\int_{|x|\ge 3}|h(x)|^p dx\right)^{2/p}. \tag{2.5'}$$

As in Lemma 2.1, we will be done if we can show that

$$\int \sup_{t\ge 1}\left|K_t^j * \tilde{G}_\alpha * g(x)\right|^2 d\mu(x) \le C2^{-j}\int_{|x|\sim 2^j}|g(x)|^2 dx \tag{2.6'}$$

for g supported in $\left\{x : |x| \sim 2^j\right\}$. But, this follows as in the previous proof from the fundamental theorem of calculus if we can show that

$$\iint_1^{\infty}\left|\left(\tfrac{d^\beta}{dt^\beta}K_t^j\right) * \tilde{G}_\alpha * g(x)\right|^2 dt d\mu(x) \le C2^{-2j(1-\beta)}\int_{|x|\sim 2^j}|g(x)|^2 dx. \tag{2.7'}$$

Finally, (2.7′) is obtained easily (like (2.4′) compared to (2.4)) by using $\hat{\tilde{G}}_\alpha(R) \approx R^{-\alpha}$ in the proof of (2.7) in [6], [7]. □

When $\alpha \in [\frac{n-1}{2}, \frac{n}{p}]$, the hypothesis (0.1) in Lemma 4 can be removed because it is already contained in our conditions on μ. In this case, Lemma 4 should read:

Lemma 2.2. *Let $h \in L^p(\mathbb{R}^n)$ with $2 \le p < \frac{2n}{n-1}$ and $n \ge 2$. Suppose that $f = (\phi G_\alpha) * h = \tilde{G}_\alpha * h$ on $\mathbb{R}^n$, with ϕ defined as in Lemma 4 and $\alpha \in [\frac{n-1}{2}, \frac{n}{p}]$. Suppose also that* $\operatorname{supp}(h) \subseteq (B(0, r+\epsilon))^c$ *(so* $\operatorname{supp}(f) \subseteq (B(0,r))^c$*), $r \in \mathbb{R}^+$, and that μ is a finite positive Borel measure supported on $B(0,r)$. Then $\|S_* f\|_{L^2(d\mu)} \le C\|h\|_p$, where C is a constant independent of f.*

Proof. If $R \ge 1$, from the finiteness of μ, we have

$$\begin{aligned}\|\widehat{gd\mu}(R\bullet)\|_{L^2(S^{n-1})} &\le C\|\widehat{gd\mu}\|_\infty \\ &\le C\|gd\mu\|_1 \\ &\le C\|\mu\|^{1/2}\|g\|_{L^2(d\mu)} \\ &\le CR^{-(n-1-2\alpha)/2}\|g\|_{L^2(d\mu)} \quad \text{by the hypothesis on } \mu \text{ and } \alpha.\end{aligned}$$

Lemma 2.2 then follows from Lemma 4. □

Remark 2.3. It is easy to verify that the hypothesis $\operatorname{supp}(h) \subseteq (B(0, r+\epsilon))^c$ is not necessary in Lemma 2.2 or in Lemma 4 ($\operatorname{supp}(f) \subseteq (B(0,r))^c$ is enough). Nevertheless, we prefer to keep it here, since it is the form of these results that we will use in this article.

By combining Lemmas 1.4 and 2.2, we are immediately led to our everywhere localisation principle (Theorem 3).

Proof of Theorem 3. Without loss of generality, we can suppose that $f = G_\alpha * g$ everywhere, because $S_R f(x) = S_R(G_\alpha * g)(x)$. We can also suppose without loss of generality that g is supported in $(B(0,3))^c$. So we must show that $S_R f(x) \to G_\alpha * g(x)$ everywhere on any ball contained in $B(0,3)$. For simplicity, we limit ourselves to proving this for $B(0,1)$, but the argument also applies for any other ball.

Now, let ϕ be as in Lemma 1.6 with $r_1 = \frac{\epsilon}{2}$ where ϵ is a small positive number to be chosen later. Then $S_R([(1-\phi)G_\alpha] * g)(x)$ converges everywhere on $\mathbb{R}^n$ by Lemma 1.4. Hence we can limit ourselves to proving that $S_R((\phi G_\alpha) * g)(x)$ converges everywhere on $B(0,1)$, and we can use Remark 1 to do this, since the set of divergence is a Borel set (see the proof of Theorem 2).

Let E be this set of divergence inside $B(0,1)$. We will now compute $C_{(n+\delta)/p,p}(E)$ for a $\delta > 0$. Let μ be a positive Borel measure supported inside $B(0,1)$ which satisfies $\|G_{(n+\delta)/p} * \mu\|_q \le 1$ where $\frac{1}{p} + \frac{1}{q} = 1$. It is easy to verify that $(\mu(E))^p \le C_{(n+\delta)/p,p}(E) \le C_{(n+\delta)/p,p}(B(0,1)) < \infty$. Hence, $\mu(E) = 0$ by Lemma 2.2 provided that $\epsilon < 2$. So, this implies that $C_{(n+\delta)/p,p}(E) = 0$ by Remark 1. But, by Proposition B, part 6, and by Proposition 2.6.1 in [1], we have that the only set $A \subseteq \mathbb{R}^n$ such that $C_{(n+\delta)/p,p}(A) = 0$ is the empty set (as $(n+\delta)/p \cdot p > n$). □

Remark 2.4. Using the $C_{(n+\delta)/p,p}$-capacity is perfectly legitimate, since all the propositions in [1, Chapter 2] remain true for this capacity, but it is really unorthodox, since this is not a "meaningful" way to measure small sets. To avoid this, a limiting argument can also be used.

Remark 2.5. Theorem 3 (in the form "off the support of f") can also be proven with an argument similar to the classical proof of Riemann's localisation principle.

Remark 2.6. It does not seem to have been observed in the one-dimensional case that everywhere convergence takes place not only in the complement of the support of f, but also outside the support of g.

3. Improved localisation

In this section, we will see how we can adapt what we did in the previous section to also improve on Theorem 2 when $0 < \alpha < \frac{n-1}{2}$. To do this we will need an energy estimate for compactly supported finite Borel measures.

Lemma 3.1. *Let μ be a finite positive Borel measure on $\mathbb{R}^n$ with compact support. Then, there is a constant C, independent of μ (but depending on the size of its support), such that*

$$\int_{S^{n-1}} |\hat{\mu}(r\xi)|^2 d\sigma(\xi) \le C r^{-\gamma} \iint G_{2\alpha}(x-y) d\mu(y) d\mu(x)$$

where $\gamma = n-1-2\alpha$ if $\alpha \in (0, \frac{n-1}{4}]$, $\gamma = \frac{n-1}{2}$ if $\alpha \in (\frac{n-1}{4}, \frac{n+1}{4}]$ and $\gamma = n-2\alpha$ if $\alpha \in (\frac{n+1}{4}, \frac{n}{2})$.

Proof. In [17], [24], the same result is proven with $G_{2\alpha}$ replaced by $I_{2\alpha}$. Using $\chi_{B(0,1)} I_{2\alpha} \le C G_{2\alpha}$ (by Proposition B, part 3), it is easy to verify that Lemma 3.1 is just a rewriting of the results of Mattila and Sjölin. □

By combining Lemma 3.1 and Lemma 4, we can now easily get Theorem 5.

Proof of Theorem 5. Without loss of generality, we can suppose that $f = G_\alpha * g$ everywhere, because $S_R f(x) = S_R(G_\alpha * g)(x)$. By translation and dilation invariance, we can also suppose without loss of generality that g is supported in $(B(0,3))^c$. Hence, Theorem 5 will be proven if we can show that $C_{\alpha+(1/2),2}(\tilde{E}) = 0$ for any closed $\tilde{E} \subseteq E \subseteq B(0,3)$. Let $F \subseteq B(0,1)$ be such a set $\tilde{E}$. (A similar proof can be done if $B(0,1)$ is replaced by any other ball inside $B(0,3)$.)

Now, let ϕ be as in Lemma 1.6 with $r_1 = \frac{\epsilon}{2}$ where ϵ is a small positive number to be chosen later. Then $S_R([(1-\phi)G_\alpha] * g)(x)$ converges everywhere on $\mathbb{R}^n$ by Lemma 1.4; hence we can limit ourselves to studying $S_R((\phi G_\alpha) * g)(x)$.

Using the compactness of F, we can compute its $C_{\alpha+(1/2),2}$-capacity using Remark 2. Thus we will be done if we can show that $\mu(F) = 0$ for any $\mu \in \mathfrak{M}^+(F)$ satisfying $\sup_{x \in \operatorname{supp}(\mu)} G_{2\alpha+1} * \mu(x) \leq 1$. By our choice of μ, we have $\mu(F) \leq C_{\alpha+(1/2)}(F) \leq C_{\alpha+(1/2)}(B(0,1)) < \infty$ and, by Proposition 2.6.2 in [1], we also have $\sup_{x \in \mathbb{R}^n} G_{2\alpha+1} * \mu(x) \leq C < \infty$.

Now, by Lemma 3.1,

$$\int_{S^{n-1}} |\hat{\mu}(r\xi)|^2 d\sigma(\xi) \leq C r^{-(n-1-2\alpha)} \iint G_{2\alpha+1}(x-y) d\mu(y) d\mu(x)$$

when $\alpha \in [\frac{n-1}{4}, \frac{n-1}{2})$. Hence,

$$\begin{aligned}
\|\widehat{gd\mu}(r\bullet)\|_{L^2(S^{n-1})} &\leq C r^{-(n-1-2\alpha)/2} \left(\iint G_{2\alpha+1}(x-y) g(y) g(x) d\mu(y) d\mu(x) \right)^{1/2} \\
&\leq C r^{-(n-1-2\alpha)/2} \left(\iint G_{2\alpha+1}(x-y) |g(x)|^2 d\mu(y) d\mu(x) \right)^{1/2} \\
&\leq C r^{-(n-1-2\alpha)/2} \|g\|_{L^2(d\mu)} \left(\sup_{x \in \mathbb{R}^n} G_{2\alpha+1} * \mu(x) \right)^{1/2} \\
&\leq C r^{-(n-1-2\alpha)/2} \|g\|_{L^2(d\mu)}.
\end{aligned}$$

The second inequality follows from $2|ab| \leq a^2 + b^2$ applied to g. Thus,

$$\|S_*((\phi G_\alpha) * h)\|_{L^2(d\mu)} \leq C \|h\|_p$$

by Lemma 4, provided that $\epsilon < 2$. Consequently, $\mu(F) = 0$. $\square$

Remark 3.2. The improvement by $\frac{1}{2}$ in Theorem 5 is not really surprising. Carbery and Soria already realised implicitly in [CS1] that any function in $L^p(\mathbb{R}^n)$ with $n \geq 2$ and $2 \leq p < \frac{2n}{n-1}$ has essentially half a derivative outside its support.

Remark 3.3. In Lemma 3.1, we can take $\gamma = n - 1 - 2\alpha$ for all $\alpha \in (0, \frac{n}{2}]$. Under an additional restriction to closed sets of divergence, Theorem 2 can consequently be proven with the same technique as Theorem 5.

It seems reasonable to believe that the approach to localisation used in this section and in Section 2 can also be used to prove a $C_{\alpha,p}$ localisation principle restricted to closed sets of divergence for functions in $\mathcal{L}^p_\alpha(\mathbb{R}^n)$ with $2 \leq p < \frac{2n}{n-1}$ and $\alpha > 0$. In fact, such a result is already contained in Theorem 5 when $n \geq 2$, $2 \leq p < \frac{2n}{n-1}$, $\alpha > 0$ and $\frac{n-3}{4} < \alpha < \frac{n-1}{2}$ (this follows from Proposition 5.5.1 in [1]).

Even better, it is not perhaps too much to hope to obtain from this approach a $C_{\alpha,p}$ localisation principle without any restriction. If we had an estimate like

$$\|\hat{\mu}(r\bullet)\|^2_{L^2(S^{n-1})} \leq Cr^{-(n-1-2\alpha)}\mu(\mathbb{R}^n) \tag{3.1}$$

for any finite and boundedly supported positive Borel measure μ satisfying the energy condition $\|G_\alpha * \mu\|_q \leq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, this will be the case. But, (3.1) is a plausible estimate since it is essentially expressing that, on average, the Fourier transform of a "nice" measure behaves like an absolutely continuous measure does pointwise. In any case, the key idea to obtaining such a localisation principle appears to require a better understanding of $\|\hat{\mu}(r\bullet)\|^2_{L^2(S^{n-1})}$ when $\|G_\alpha * \mu\|_q \leq 1$.

4. A general capacitarian localisation principle

In Remarks 1.2 and 1.5, we made some comments on extending the work done in this article for $\mathcal{L}^p_\alpha$ to some related nice spaces. In this section we will give a taste of how one can start to do this. Nevertheless, we do not intend to push this digression too far, as our real goal is to study the class $\mathcal{L}^p_\alpha$. We will consequently limit ourselves to a simple case:

Theorem 4.1. *Let $K \in L^1(\mathbb{R}^n)$ be a radially decreasing convolution kernel which is lower semi-continuous on $\mathbb{R}^n$. Suppose also that K satisfies $\int_{\mathbb{R}^n} |K(x)|^2 dx = \infty$ and $\int_{(B(0,1))^c} |K(x)|^2 dx < \infty$. Then $S_R f(x) \to 0$ $C_{K,2}$-quasieverywhere off the support of f if f is a function with bounded support in the class*

$$\Big\{ f \in L^2(\mathbb{R}^n) : \int |\hat{f}(\xi)|^2 |\hat{K}(\xi)|^{-2} d\xi < \infty \Big\}.$$

The conditions on K are there to ensure meaningful $C_{K,2}$-capacities (see [1, Definition 2.3.3 and Proposition 2.6.1]).

To prove this localisation principle (Theorem 4.1), we can use the technique developed by Carbery and Soria in [5]. The equation (2.2) in their proof is then reduced to only one term in this case. Alternatively, one can use the following theorem of Sjölin [23]:

Theorem D. *Let $\phi \in C^1(\mathbb{R}^n \backslash \{0\})$ be positive and homogenous of degree 1. Suppose also that $\phi(0) = 0$. For $D_R = \{x \in \mathbb{R}^n : \phi(R^{-1}x) < 1\}$, define the kernel $\tilde{K}_R(x) = \int_{D_R} e^{2\pi i x\cdot\xi} d\xi$ and then, with $\varphi \in \mathcal{S}$ (Schwartz class) such that $\varphi(0) = 0$, define the operator $T_R = (\varphi \tilde{K}_R) * f$ for $f \in L^2(\mathbb{R}^n)$ and $R > 0$. If $T_* f(x) = \sup_{R\geq 1} |T_R f(x)|$ is its associated maximal operator, then $\|T_* f\|_2 \leq C\|f\|_2$, with C a constant independent of f.*

Theorem D is more general than what we truly need for Theorem 4.1 ($\phi(x) = |x|$ will be enough in this case). Consequently, we are able not only to extend the class of functions, but also the class of operators. More precisely, we have

Theorem 4.2. *Let $\tilde{S}_R(f) = \tilde{K}_R * f$ where $\tilde{K}_R$ is defined as in Theorem D and let $K \in L^1(\mathbb{R}^n)$ be a radially decreasing convolution kernel which is lower semi-continuous on $\mathbb{R}^n$. Suppose also that K satisfies*

$$\int_{\mathbb{R}^n} |K(x)|^2 dx = \infty \text{ and } \int_{(B(0,1))^c} |K(x)|^2 dx < \infty.$$

Then $\tilde{S}_R f(x) \to 0$ $C_{K,2}$-quasieverywhere off the support of f if f is a function with bounded support in the class

$$\Big\{ f \in L^2(\mathbb{R}^n) : \int |\hat{f}(\xi)|^2 |\hat{K}(\xi)|^{-2} d\xi < \infty \Big\}.$$

To keep the notation simple, we will limit ourselves to proving Theorem 4.1 in the case where $f \in \mathcal{L}^2_\alpha(\mathbb{R}^n)$, but the same argument also works for the other class of functions covered by Theorem 4.1 as well as for the more general setting of Theorem 4.2.

Proof. We can suppose without loss of generality that $\operatorname{supp}(f) \subseteq \{x \in \mathbb{R}^n : a < |x| < b\}$. Now let choose a function $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\varphi \equiv 1$ on $\{x \in \mathbb{R}^n : a - r < |x| < b + r\}$ and $\varphi \equiv 0$ near the origin. Here, $0 < r < a < b < \infty$. For $|x| < r$, the support restriction on f implies that

$$S_R f(x) = K_R * f(x) = \int \varphi(x-y) K_R(x-y) f(y) dy = T_R f(x). \tag{4.1}$$

Let $g \in L^2(\mathbb{R}^n)$ be such that $f = K * g$ almost everywhere. Define $D_\lambda = \{x \in B(0,r) : S_* f(x) > \lambda\}$, $\dot{D}_\lambda = \{x \in B(0,r) : K * T_* g(x) > \lambda\}$ and $\ddot{D}_\lambda = \{x \in \mathbb{R}^n : K * T_* g(x) > \lambda\}$. The Basic Estimate and (4.1) then imply that $D_\lambda \subseteq \dot{D}_\lambda \subseteq \ddot{D}_\lambda$. Hence,

$$C_{\alpha,2}(D_\lambda) \leq C_{\alpha,2}(\ddot{D}_\lambda) \leq C\lambda^{-2} \|T_* g\|_2 \leq C\lambda^{-2}\|g\|_2.$$

The second inequality follows from Definition 2, while the third is a consequence of Theorem D. Using translation and scaling, this implies the desired result. ☐

Remark 4.3. Using this theorem and the idea seen in Section 5, it is easy to show for $\frac{n-1}{2} < \alpha < \frac{n}{2}$ that $S_R f(x) \to I_\alpha * g(x)$ $\dot{C}_{\alpha,2}$-quasieverywhere on $\mathbb{R}^n$ when f is in the subset of the homogeneous Sobolev space

$$\dot{\mathcal{L}}^2_\alpha(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : f = I_\alpha * g \text{ almost everywhere for some } g \in L^2(\mathbb{R}^n)\}$$

where g is compactly supported. The $\dot{C}_{\alpha,2}$-capacity is the capacity associated with the Riesz kernel I_α defined as the inverse Fourier transform of $|2\pi \bullet|^{-\alpha}$ (see [1]).

Even if this section just intended to be a "timid" first step in the process of extending the Carleson capacitarian result [9, p.50], it is worth mentioning that, for the homogeneous Sobolev spaces, the study done in this article can be used to control the central part of the kernel I_α. The real problem in this case is the lack of decay, which forces $(1-\phi)I_\alpha * g$ to not converge better than $\dot{C}_{\alpha,p}$ off the support of g.

5. Pointwise convergence when $2 \leq p < \frac{2n}{n-1}$ and $\alpha > \frac{n-1}{2}$

In $\mathbb{R}$, Salem and Zygmund essentially deduced the sharp pointwise result (Theorem A) from a pointwise estimate for $S_* K$ where K is roughly the kernel of the $(1-\beta)$-capacity. As we will see in this section, it is possible to extend Theorem A to all dimensions when $\frac{n-1}{2} < \alpha \leq \frac{n}{2}$. Using Theorem 3, this extension also easily follows from a pointwise estimate for $S_*(\phi G_\alpha)$. To prove this main estimate, we will need the following properties[1] of the Bessel functions (J_k):

[1]Lemma 5.1 is certainly well known, but to save time we give its proof rather than try to find a reference in the extensive literature on Bessel functions.

Lemma 5.1. *If k is a positive integer or half of a positive integer, then*

(1) $x^{-1/2}J_k(x)$ *is bounded on* $\mathbb{R}^+$, *and*
(2) $\left|\int_0^y x^{-1/2}J_k(x)dx\right| \leq C_k$ *for all* $y \in \mathbb{R}^+$, *with* C_k *a constant independent of* y, *but depending on* k.

Proof. Estimate 1 follows by splitting $x^{-1/2}J_k(x)$ into a part near and a part away from 0. The part near 0 is easily controlled by Lommel's recurrence formula,

$$J_m(x) = \tfrac{2(m-1)}{x}J_{m-1}(x) - J_{m-2}(x), \tag{5.1}$$

combined with $\lim_{x\to 0^+} J_0(x) = 1$ and $\lim_{x\to 0^+} J_k(x) = 0$ when $k > 0$ [21, p.48]. The part away from 0 is controlled using the boundedness of J_k.

For 2, we use $J_k(x) = \sqrt{\frac{2}{\pi x}}\cos(x - \frac{\pi(2k+1)}{4}) + E_k(x)$ when $x > C_k > 0$ [20]. Here, C_k is a constant depending only on k, and $|E_k(x)| \leq \tilde{c}_k x^{-3/2}$. By 1, we can suppose that $y \geq C_k$; hence

$$\left|\int_{C_k}^y x^{-1/2}J_k(x)dx\right| \leq \left|\int_{C_k}^y x^{-1}\cos(x - \tfrac{\pi(2k+1)}{4})dx\right| + \tilde{c}_k\int_{C_k}^y x^{-2}dx$$

where the first integral behaves like the harmonic series. □

Remark 5.2. Under the stronger condition $y \geq \tilde{C} > 0$, it is also possible to show in a similar way that $\left|\int_{\tilde{C}}^y x^{-1/2}J_0(x)dx\right| \leq C$, with C a constant independent of y, but depending on $\tilde{C}$.

Using Lemma 5.1, we can now prove the crucial part of the pointwise estimate for $S_*(\phi G_\alpha)$:

Lemma 5.3. *Let* $N, y, s \in \mathbb{R}^+$. *If k is a positive integer, then*

$$\left|\int_0^y N(sr)^{1/2}(s^2 - r^2)^{-1}I_{k,N,r,s}dr\right| \leq C_k$$

where $I_{k,N,r,s} = sJ_{(k-2)/2}(2\pi Nr)J_{k/2}(2\pi Ns) - rJ_{k/2}(2\pi Nr)J_{(k-2)/2}(2\pi Ns)$ *and* C_k *is a constant independent of* y, s *and* N.

Proof. This proof is done in two steps. We will first prove a transference technique which reduces all the cases to either $k = 1, 2, 3$ or 4 modulo a simple term which will be easily controlled, and then we will prove the four main cases.

The transference is obtained by two consecutive applications of (5.1) on $I_{k,N,r,s}$:

$$\begin{aligned} I_{k,N,r,s} &= -sJ_{(k-2)/2}(2\pi Nr)J_{(k-4)/2}(2\pi Ns) + rJ_{(k-4)/2}(2\pi Nr)J_{(k-2)/2}(2\pi Ns) \\ &= I_{k-4,N,r,s} + \tfrac{(k-4)(r^2-s^2)}{2\pi Nrs}J_{(k-4)/2}(2\pi Nr)J_{(k-4)/2}(2\pi Ns). \end{aligned}$$

Hence, if we can deal with $\int_0^y (rs)^{-1/2}J_{(k-4)/2}(2\pi Nr)J_{(k-4)/2}(2\pi Ns)dr$, the case k will follow from the case $k - 4$ as desired.

But, by Lemma 5.1,

$$\begin{aligned} &\left|\int_0^y (rs)^{-1/2}J_{(k-4)/2}(2\pi Nr)J_{(k-4)/2}(2\pi Ns)\right| \\ &\qquad = |(Ns)^{-1/2}J_{(k-4)/2}(2\pi Ns)|\left|\int_0^{Ny} r^{-1/2}J_{(k-4)/2}(2\pi r)dr\right| \leq C. \end{aligned}$$

Now, the cases $k=1$ and $k=3$ are similar; so we will only do $k=3$. We first replace $J_{1/2}(x)$ and $J_{3/2}(x)$ by their respective values:

$$\sqrt{\tfrac{2}{\pi x}}\sin(x) \quad \text{and} \quad \sqrt{\tfrac{2}{\pi x}}(x^{-1}\sin(x)-\cos(x)).$$

After simplifications, the expression to evaluate is then

$$C\int_0^y (s^2-r^2)^{-1}(r\cos(2\pi Nr)\sin(2\pi Ns)-s\sin(2\pi Nr)\cos(2\pi Ns))dr.$$

If we now replace $\sin(a)\cos(b)$ by $\frac{1}{2}(\sin(a+b)+\sin(a-b))$ and group the terms in $\sin(2\pi N(s+r))$ and the terms in $\sin(2\pi N(s-r))$ separately, we obtain

$$C\int_0^y (s+r)^{-1}\sin(2\pi N(s+r))dr - C\int_0^y (s-r)^{-1}\sin(2\pi N(s-r))dr.$$

But, it is well known that $\left|\int_0^y x^{-1}\sin(x)dx\right| \leq C$ with a constant C independent of $y \in \mathbb{R}^+$. Hence, the desired result follows when $k=3$.

When $k=2$ and $k=4$, the technique is also essentially similar; so we will only do $k=2$. After a change of variable, the expression to evaluate is

$$\left|\int_0^{Ny}(xz)^{1/2}(z^2-x^2)^{-1}(zJ_0(2\pi x)J_1(2\pi z)-xJ_1(2\pi x)J_0(2\pi z))dx\right| \leq C$$

where $z=Ns$ to simplify the notation. Let $f(x,z)$ denote the integrand in the last integral and let $\tilde{C}=\max\{C_0,C_1\}$ where the C_j are the constants for J_0 and J_1 respectively in the asymptotic expansion seen in the proof of Lemma 5.1. The proof is now done by splitting f into four parts: 1) $x,z>\tilde{C}$, 2) $x>2\tilde{C}$ and $0\leq z\leq\tilde{C}$, 3) $0\leq x\leq\tilde{C}$ and $z>2\tilde{C}$, 4) the remainder of $\mathbb{R}^+\times\mathbb{R}^+$. For the first two parts, we will use the oscillation (Lemma 5.1, part 2), while for the other two we will show that f is bounded.

In region 1), supposing that $Ny>\tilde{C}$, we have

$$\begin{aligned}
&\left|\int_{\tilde{C}}^{Ny} f(x,z)dx\right| \\
&\qquad \approx C\left|\int_{\tilde{C}}^{Ny}(z^2-x^2)^{-1}\big(z\cos(2\pi x-\tfrac{\pi}{4})\cos(2\pi z-\tfrac{3\pi}{4}) \right.\\
&\qquad\qquad\qquad\qquad \left. - x\cos(2\pi x-\tfrac{3\pi}{4})\cos(2\pi z-\tfrac{\pi}{4})\big)dx\right| \\
&\qquad\quad \text{by the asymptotic formula in the proof of Lemma 5.1} \\
&\qquad = C\left|\int_{\tilde{C}}^{Ny}(z^2-x^2)^{-1}(z-x)\big(\cos(2\pi(x+z)-\pi)+\cos(2\pi(x-z)-\tfrac{\pi}{2})\big)dx\right| \\
&\qquad\quad \text{since } \cos(a)\cos(b)=\tfrac{1}{2}(\cos(a+b)+\cos(a-b)) \\
&\qquad = C\left|\int_{\tilde{C}}^{Ny}(x+z)^{-1}\big(\cos(2\pi(x+z))+\sin(2\pi(x-z))\big)dx\right| \leq C.
\end{aligned}$$

The last line follows as for the harmonic series, since $x+z>2\tilde{C}$.

In region 2), supposing that $Ny > 2\tilde{C}$,

$$\begin{aligned}\left|\int_{2\tilde{C}}^{Ny} f(x,z)dx\right| &\leq \int_{2\tilde{C}}^{Ny} \left|x^{1/2}z^{3/2}(z^2-x^2)^{-1}J_0(2\pi x)J_1(2\pi z)\right|dx \\ &\quad + \left|\int_{2\tilde{C}}^{Ny} x^{3/2}z^{1/2}(z^2-x^2)^{-1}J_1(2\pi x)J_0(2\pi z)dx\right| \\ &\leq C\int_{2\tilde{C}}^{Ny} x^{1/2}\left|z^2-x^2\right|^{-1}dx + C\left|\int_{2\tilde{C}}^{Ny} x^{3/2}(z^2-x^2)^{-1}J_1(2\pi x)dx\right| \\ &\quad \text{by } z \leq \tilde{C} \text{ and the boundedness of } J_0 \text{ and } J_1 \\ &\leq C\int_{2\tilde{C}}^{Ny} x^{-3/2}dx + C\left|\int_{2\tilde{C}}^{Ny} x^{3/2}(z^2-x^2)^{-1}J_1(2\pi x)dx\right| \leq C.\end{aligned}$$

In the last line, the first inequality is a consequence of $x^2 - z^2 \geq x^2 - \frac{1}{4}x^2 = \frac{3}{4}x^2$ since $x > 2\tilde{C} \geq 2z \geq 0$. The first integral obtained from this is bounded, since $x \geq 2\tilde{C}$, while the second is controlled as in Lemma 5.1, part 2, because $x > 2\tilde{C} \geq 2z \geq 0$ (for $k = 4$, this is replaced by the remark following Lemma 5.1).

Now, in region 3),

$$\begin{aligned}|f(x,z)| &\leq |xz|^{1/2}\left|z^2-x^2\right|^{-1}\left(|zJ_0(2\pi x)J_1(2\pi z)| + |xJ_1(2\pi x)J_0(2\pi z)|\right) \\ &\leq C|z|^{1/2}(z^2-x^2)^{-1}(|z|+1) \\ &\quad \text{by } x \leq \tilde{C} \text{ and the boundedness of } J_0 \text{ and } J_1 \\ &\leq C|z|^{-3/2}(|z|+1) \leq C.\end{aligned}$$

In the last line the first inequality follows from $z^2 - x^2 \geq z^2 - \frac{1}{4}z^2 = \frac{3}{4}z^2$, since $z > 2\tilde{C} \geq 2x \geq 0$. Hence, region 3) is under control as we integrate over $0 \leq x \leq \min\{Ny, \tilde{C}\}$.

Finally, when x and z are in region 4), we define a modified function f:

$$\tilde{f}(x,z) = \begin{cases} f(x,z) & \text{if } x \neq z, \\ \frac{1}{2}x\left(J_0^2(x) + J_1^2(x)\right) & \text{if } x = z. \end{cases}$$

Then, $\tilde{f}$ is continuous on the compact set where $0 \leq x, z \leq \tilde{C}$ or ($0 \leq x \leq \tilde{C}$ and $\tilde{C} \leq z \leq 2\tilde{C}$) or ($0 \leq z \leq \tilde{C}$ and $\tilde{C} \leq z \leq 2\tilde{C}$). Hence, $\tilde{f}$ is bounded on this set. Since $f = \tilde{f}$ almost everywhere, we can replace f by $\tilde{f}$ in the integral to evaluate, and so the desired result follows in region 4) as well. In a similar way one can use

$$\tilde{f}(x,z) = \begin{cases} f(x,z) & \text{if } x \neq z, \\ \frac{1}{2}\left(2J_0(x)J_1(x) - xJ_0^2(x) - xJ_1^2(x)\right) & \text{if } x = z, \end{cases}$$

for $k = 4$, after rewriting f using one application of (5.1). □

From Lemma 5.3 and some estimates in [20], it is a simple exercise to get the pointwise result alluded to above:

Lemma 5.4. *Let ϕ be a radial bump function associated with $B(0, r_1) \subset \mathbb{R}^n$ defined as in Lemma 1.6. If $\frac{n-1}{2} < \alpha \leq \frac{n}{2}$, then $S_*(\phi G_\alpha)(x) \leq C_{\alpha,r_1}(1 + G_\alpha(x))$, with C_{α,r_1} a constant independent of x.*

Proof. If f is a radial function on $\mathbb{R}^n$, then it is well known that

$$\hat{f}(t) = 2\pi \int_0^\infty |rt|^{-(n-2)/2} J_{(n-2)/2}(2\pi tr) f(r) r^{n-1} dr.$$

Here, $f(r)$ means $f(|x|)$ with $|x| = r$ (the same remark applies to $\hat{f}(t)$ and in the lines below). Hence,

$$S_R(\phi G_\alpha)(s) = CRs^{-(n-2)/2} \int_0^\infty (s^2 - r^2)^{-1} r^{n/2} (\phi G_\alpha)(r) I_{n,R,r,s} dr,$$

using the symmetry as in [20]. In the previous integral, $I_{n,R,r,s}$ is defined as in Lemma 5.3.

To evaluate this integral, we break it into two parts, $(0, \frac{s}{2})$ and $(\frac{s}{2}, r_1)$. For the first part, the estimates in [20] give the desired result as

$$s^{-(n+1)/2} \int_0^{s/2} r^{(n-1)/2} (\phi G_\alpha)(r) dr \leq C_\alpha (1 + G_\alpha(s)),$$

while the other part is controlled using Lemma 5.3 and integration by parts (on $r^{(n-1)/2}(\phi G_\alpha)(r)$ and $R(rs)^{1/2}(s^2 - r^2)^{-1} I_{n,R,r,s} dr$). □

Remark 5.5. When $\frac{n-1}{2} < \alpha \leq \frac{n}{2}$, $\phi G_\alpha \in L^q(\mathbb{R}^n)$ for a q in $(\frac{2n}{n+1}, \frac{2n}{n-1})$ and, so, $S_R(\phi G_\alpha)(x) \to \phi G_\alpha(x)$ almost everywhere by [15], [20]. Thus, Lemma 5.4 is a stronger version of these result for ϕG_α, but, when $0 < \alpha \leq \frac{n-1}{2}$, there is no such q. In fact, $S_* \phi G_\alpha(x) = \infty$ for these α. Consequently, a completely different approach will be needed to extend Theorem A to these values.

Using Theorem 3 and Lemma 5.4, we can now prove Theorem 6.

Proof of Theorem 6. By the localisation principle (Theorem 3), we can suppose without loss of generality that $\operatorname{supp}(g) \subseteq B(0,2)$, and we must show that $S_R f(x) = S_R(G_\alpha * g)(x) \to G_\alpha * g(x)$ $C_{\alpha,p}$-quasieverywhere on any ball contained inside $B(0,2)$. We will do this for $B(0,1)$, but the argument is similar for any other ball.

Let μ be a Borel measure supported on $B(0,1)$ which satisfies the conditions in Remark 1. By Lemma 1.4, we can limit ourselves to considering ϕG_α only, rather than G_α, for ϕ a radial bump function supported inside $B(0,1)$. Hence, as in Theorem 2, we will be done if we can show that $\int S_*((\phi G_\alpha) * g)(x) d\mu(x) \leq C\|g\|_p$.

But, this easily follows from Lemma 5.4:

$$\begin{aligned}
&\int S_*((\phi G_\alpha) * g)(x) d\mu(x) \\
&\qquad \leq \int \big(|g| * S_*(\phi G_\alpha)\big)(x) d\mu(x) && \text{by Basic Estimate} \\
&\qquad = \int \big(|g| * (\chi_{B(0,3)} S_*(\phi G_\alpha))\big)(x) d\mu(x) && \text{from } \operatorname{supp}(\mu) \text{ and } \operatorname{supp}(g) \\
&\qquad \leq \|g\|_p \|(\chi_{B(0,3)} S_*(\phi G_\alpha)) * \mu\|_q && \text{with } \tfrac{1}{p} + \tfrac{1}{q} = 1 \\
&\qquad \leq C\|g\|_p \|(C\chi_{B(0,3)}(1 + G_\alpha)) * \mu\|_q && \text{by Lemma 5.4} \\
&\qquad \leq C\|g\|_p \|G_\alpha * \mu\|_q \leq C\|g\|_p && \text{by } \operatorname{supp}(\mu).
\end{aligned}$$

The last inequality follows from the properties of μ, which imply that $\|G_\alpha * \mu\|_q \leq 1$ and $\mu(B(0,1)) \leq \big(C_{\alpha,p}(B(0,1))\big)^{1/p} < \infty$. □

Remark 5.6. This argument can also be used to show that $S_Rf(x) \to G_\alpha * g(x)$ $C_{\alpha,p}$-quasieverywhere on $\mathbb{R}^n$ when $f \in \mathcal{L}^p_\alpha(\mathbb{R}^n)$ with $\frac{n-1}{2} < \alpha \leq \frac{n}{p}$ and $\frac{2n}{n+1} < p < 2$, provided that a $C_{\alpha,p}$-localisation principle is proven in this case.

6. Pointwise convergence when $2 \leq p < \frac{2n}{n-1}$ and $0 < \alpha \leq \frac{n-1}{2}$

We discussed in the previous sections the situation when $p = 2$ or when $\alpha > \frac{n-1}{2}$, but what happens if $2 < p < \frac{2n}{n-1}$ and $0 < \alpha \leq \frac{n-1}{2}$? For the moment, the only answer is Theorem 1, but we should hope for a $C_{\alpha,p}$-result.

By returning for a moment to Theorem 1, it is easily observed that, if we had dealt with the tail of $G_{\alpha-\epsilon}$ using Lemma 1.6, we could have proven this result by showing that $\|S_*f\|_{L^1(d\mu)} \leq C\|g\|_p$ for any boundedly supported positive Borel measure μ satisfying $\|G_{\alpha-\epsilon} * \mu\|_2 \leq 1$. So, one might think that by interpolating between this estimate and the estimate established to prove Theorem 6, we would be able to show the following:

Hypothetical Result 6.1. *Let $f \in \mathcal{L}^p_\alpha(\mathbb{R}^n)$ with $2 \leq p < \frac{2n}{n-1}$ and $\frac{(n-1)(p-2)n}{2p} < \alpha \leq \frac{n}{p}$. Suppose also that $g \in L^p(\mathbb{R}^n)$ is such that $f = G_\alpha * g$ almost everywhere. Then $S_Rf(x) \to G_\alpha * g(x)$ $C_{\alpha-\epsilon,p}$-quasieverywhere on $\mathbb{R}^n$ for every ϵ such that $0 < \epsilon < \alpha$.*

Moreover, one might hope that this could be achieved with an argument similar to the classical interpolation built up by Stein for the Bochner-Riesz operator [27, pp. 279-281]. Unfortunately, this does not work. If it were possible to do so, then we would also be able to interpolate in a similar fashion between the estimates behind Theorems 2 and 3. But, this easily leads to a contradiction with Il'in's result [13], since we would then be able to prove everywhere localisation for α below $\frac{n-1}{2}$.

We do not know yet how to get the sharp $C_{\alpha,p}$-result when $2 < p < \frac{2n}{n-1}$ and $0 < \alpha \leq \frac{n-1}{2}$, but clearly it is impossible to use interpolation in any way close to Stein's classical interpolation.

7. Pointwise convergence when $p < 2$

When $p < 2$ and $n \geq 2$, the combination of Stein's maximal principle and Fefferman's counterexample for the ball multiplier implies that there is an $f \in L^p(\mathbb{R}^n)$ such that $S_Rf(x)$ does not converge almost everywhere on $\mathbb{R}^n$. Nevertheless, it was believed that, under an additional smoothness condition, the situation would be different. In particular, Carbery and Soria showed in [5] that $S_Rf(x)$ converges almost everywhere when $f \in \mathcal{L}^p_\alpha(\mathbb{R}^n)$ with $n \geq 2$, $1 < p \leq 2$, $\alpha > 0$ and $\frac{1}{p} < \frac{\alpha}{n-1} + \frac{1}{2}$, as well as when $f \in \mathcal{L}^1_{(n-1)/2}(\mathbb{R}^n)$. Despite these positive results, nothing can be done in general. By rescaling a recent counterexample of T. Tao [28] for the Bochner-Riesz problem, we immediately get a counterexample in our setting when p and α are small enough:

Theorem 7.1. *If $1 \leq p < 2 - \frac{1}{n}$ and $0 \leq \alpha < \frac{(2-p)n-1}{2p}$, then there is an $f \in \mathcal{L}^p_\alpha(\mathbb{R}^n)$ such that $S_Rf(x)$ does not converge almost everywhere on $\mathbb{R}^n$.*

In fact, we can construct a function which diverges everywhere (Theorem 7) by studying Tao's example more carefully. We want to point out that these results (Theorems 7 and 7.1) go against the intuition provided by the lacunary case [5, Theorem 7] when $n=2$, $\frac{4}{3+2\alpha}<p\le 2$ and $0<\alpha<\frac{3-2p}{2p}$, as well as when $n\ge 3$, $\frac{2n}{n+1+2\alpha}<p\le 2$ and $\frac{n-1}{2(n+1)}<\alpha<\frac{(2-p)n-1}{2p}$.

So what can we do? The only positive capacitarian results [19, Theorems 4.3.4 and 4.3.5] that we were able to obtain for $p<2$ follow easily from the estimates of Carbery and Soria [5, Theorems 5, 6 and 7]. Moreover, of these results, the only nice one is in the lacunary case when $n=2$, where it was possible to show that $S_{R_k}f(x)\to G_\alpha * g(x)$ $C_{\alpha-\epsilon_1,p-\epsilon_2}$-quasieverywhere on $\mathbb{R}^n$ for all $\epsilon_1,\epsilon_2>0$ small enough, when $f\in\mathcal{L}^p_\alpha(\mathbb{R}^2)$ where $\frac{2n}{n+1}<p<2$, $0<\alpha\le\frac{n-1}{2}$ and $\{R_k\}_{k=1}^\infty$ is a lacunary sequence.

Because of Theorems 7 and 7.1, it is unclear to us what should be the correct analogue of Salem and Zygmund's result (Theorem A) when $p<2$. Even when $\frac{n-1}{2}<\alpha\le\frac{n}{p}$, it is not completely evident that we can truly hope for a $C_{\alpha,p}$-result.

If this is the case for some $\frac{2n}{n+1}<p_0<2$ and $\frac{n-1}{2}<\alpha_0\le\frac{n}{p_0}$ (i.e., if there are a p_0 and an α_0 for which we can find an $f\in\mathcal{L}^{p_0}_{\alpha_0}(\mathbb{R}^n)$ such that $S_Rf(x)$ does not converge C_{α_0,p_0}-quasieverywhere), then Il'in's example [13] can be extended to $\mathcal{L}^{p_0}_{\alpha_0}(\mathbb{R}^n)$. More precisely, there will then be an $f\in\mathcal{L}^{p_0}_{\alpha_0}(\mathbb{R}^n)$ such that $f\equiv 0$ on $B(0,1)$ and $\limsup_{R\to\infty}|S_Rf(0)|=\infty$. This is an immediate consequence of the fact that we only lack a $C_{\alpha,p}$-localisation principle to get a $C_{\alpha,p}$-theorem when $\frac{2n}{n+1}<p<2$ and $\frac{n-1}{2}<\alpha\le\frac{n}{p}$ (see Remark 5.6). Nevertheless, extending Il'in's example to some $\mathcal{L}^{p_0}_{\alpha_0}(\mathbb{R}^n)$ will not rule out a C_{α_0,p_0}-result, since the reverse of the previous implication can be false.

Proof of Theorem 7. Let $\tilde f_R(y)=e^{2\pi iRy_n}\psi(R\underline{y},R^{1/2}y_n)$, where ψ is a nonnegative bump function with a small support and $R\gg 1$. Then, by stationary phase asymptotics, it can be shown that $S_t(G_\alpha*\tilde f_R)(x)\approx R^{-(n/2)-\alpha}$ when $0<x_n\sim|x|\sim 1$ and $t=\frac{|x|}{x_n}R$ (for the details, see [28, Proposition 5.1], where $K(x)$ should be replaced by $\tilde K_t^\alpha(x)=K_t*G_\alpha(x)\approx t^{(n-1-2\alpha)/2}|x|^{-(n+1)/2}e^{2\pi it|x|}$). $\tilde f_R$ is the rescaled Tao's function which satisfies Theorem 7.1.

If we define $f_{R,z}(y)=\tilde f_R(y-z)$, then, from the above, $\|f_{R,z}\|_p=CR^{(1-2n)/(2p)}$ and $S_t(G_\alpha*f_{R,z})(x)\approx R^{-(n/2)-\alpha}$ when $0<x_n-z_n\sim|x-z|\sim 1$ and $t=\frac{|x-z|}{x_n-z_n}R$.

Now, let $\{\alpha_k\}_{k=1}^\infty$ be an increasing sequence of positive numbers satisfying both $\alpha_k k^{(1-2n)/(2p)}\searrow 0$ and $\alpha_k k^{-(n/2)-\alpha}\nearrow\infty$ when $0\le\alpha<\frac{(2-p)n-1}{2p}$ (e.g., $\alpha_k=k^{(2n-1-\epsilon)/(2p)}$ with $\epsilon>0$ small enough). Then, it is possible to choose a sequence $\{k_j\}_{j=1}^\infty$ such that $\frac{k_{j+1}}{k_j}>2$ for all $j\in\mathbb{N}$, $\alpha_{k_j}k_j^{-(n/2)-\alpha}\nearrow\infty$ and $\sum_{j=1}^\infty\alpha_{k_j}k_j^{(1-2n)/(2p)}<\infty$.

Finally, let $\{f_j\}_{j=1}^\infty$ be a sequence of functions $f_{R,z}$ such that $R=k_j$, $z\in I=\{y=(y_1,y_2,\dots,y_n):2y_j\in\mathbb{Z}\ for\ j=1,2,\dots,n\}$ and every $z_0\in I$ is assigned to an infinity of different f_j (i.e., $\#\{f_j:f_j=f_{k_j,z_0}\}=\infty$ for all $z_0\in I$). Hence,

$$\|f\|_p\le\sum_{j=1}^\infty\alpha_{k_j}\|f_j\|_p=\sum_{j=1}^\infty\alpha_{k_j}k_j^{(1-2n)/(2p)}<\infty$$

for $f=\sum_{j=1}^\infty\alpha_{k_j}f_j$ (so $G_\alpha*f\in\mathcal{L}^p_\alpha(\mathbb{R}^n)$). We also know that $S_t(G_\alpha*f_j)(x)\approx k_j^{-(n/2)-\alpha}$ and $\alpha_{k_j}k_j^{-(n/2)-\alpha}\nearrow\infty$ when $0<x_n-z_n\sim|x-z|\sim 1$ and $t=\frac{|x-z|}{x_n-z_n}k_j$. But, by taking the union of all $A_z=\{y\in\mathbb{R}^n:0<x_n-z_n\sim|x-z|\sim 1\}$ such that z has integer or half integer coordinates (i.e., $z\in I$), we can cover all of $\mathbb{R}^n$.

Consequently, we will be done if we can show for each $j \in \mathbb{N}$ that $S_t(G_\alpha * f)(x)$ is essentially $S_t\left(G_\alpha * f_j\right)(x)$ when $t \sim k_j$.

But, this follows from standard estimates on oscillatory integrals in a way similar to the counterexample in [6]. The main difference between this case and [6] is that here we want some estimates valid for all $\mathbb{R}^n$, and not only for a compact set small enough. When $\frac{M}{2} > T > 1$, this difference is not important, and we get that $|S_T f_M(x)| \leq C_l \frac{T^n}{M^{n+(1/2)} M^l}$ on $\mathbb{R}^n$ for all $l > 0$. For $2M < T$, the loss of "smallness" forces the weaker estimate $|S_T f_M(x)| \leq C_l \frac{T^n M^l}{M^{n+(1/2)} T^l}$ for all $l > 0$, but the analysis in this case also stays similar to what was done in [6]. This is enough, since we want to use this estimate for T essentially fixed and $\frac{M}{T} < \frac{1}{2}$.

N.B.: In fact, one can get an additional factor of $T^{(n+1)/2+\alpha}$ in the denominator of the previous estimates by restricting the analysis to $|x| > C > 0$. □

Remark 7.2. To get the last two estimates in the proof of Theorem 7, follow the argument of Lemma 1 in [6] and use the fact that

$$\begin{aligned}|\nabla\Phi(y)|^2 = &\left|M^{1/2} - \frac{T(x_n - M^{-1/2}y_n)}{M^{1/2}|x-(M^{-1}\underline{y}, M^{-1/2}y_n)|^{1/2}}\right|^2 \\ &+ \frac{T^2}{M^2|x-(M^{-1}\underline{y}, M^{-1/2}y_n)|}\sum_{j=2}^{n}|x_j - M^{-1}y_j|^2\end{aligned}$$

is bigger than CM and $C\frac{T^2}{M^2}$ respectively when $\frac{M}{2} > T > 1$ and when $2M < T$. (For $2M < T$, we split into $|x_n - M^{-1/2}y_n| \gg |\underline{x} - M^{-1}\underline{y}|$, $|x_n - M^{-1/2}y_n| \approx |\underline{x} - M^{-1}\underline{y}|$ and $|x_n - M^{-1/2}y_n| \ll |\underline{x} - M^{-1}\underline{y}|$ to get the lower bounds $C\frac{T^2}{M}$, $C\frac{T^2}{M^2}$ and $C\frac{T^2}{M^2}$, which are all bigger than $C\frac{T^2}{M^2}$.)

8. On a question of uniformity

It is a well-known fact that some uniformity is associated to the convergence of $S_R f(x)$ when $f \in \mathcal{L}^2_\alpha(\mathbb{R}^n)$ with $\alpha > \frac{n}{2}$ [5]. In fact, one can easily show the following:

Theorem E. *Let $f \in \mathcal{L}^2_\alpha(\mathbb{R}^n)$, $\alpha > \frac{n}{2}$, and let $g \in L^2(\mathbb{R}^n)$ be such that $f = G_\alpha * g$ almost everywhere. Then $S_R f(x) \to G_\alpha * g(x)$ uniformly on $\mathbb{R}^n$.*

With Theorems 1 and 6 in mind, it is then natural to ask how "far from uniform" is the convergence when $0 < \alpha \leq \frac{n}{2}$? Using Theorem B′ in [4], we can easily answer this question in the lacunary case:

Theorem 8.1. *Let $\{R_k\}_{k=1}^\infty$ be a lacunary sequence and let $f \in \mathcal{L}^2_\alpha(\mathbb{R}^n)$, $0 < \alpha \leq \frac{n}{2}$. Suppose also that $g \in L^2(\mathbb{R}^n)$ is such that $f = G_\alpha * g$ almost everywhere. Then $S_{R_k} f(x) \to (G_\alpha * g)(x)$ $C_{\alpha,2}$-quasieverywhere on $\mathbb{R}^n$. Moreover, the convergence takes place uniformly outside an open set of arbitrarily small $C_{\alpha,2}$-capacity.*

Proof. For $\delta > 0$, let

$$\begin{aligned}\Omega_\delta f(x) = &\sup_{\substack{\delta<r\\ r\in\{R_k\}_{k=1}^\infty}} \mathrm{Re}(K_r * f)(x) - \inf_{\substack{\delta<r\\ r\in\{R_k\}_{k=1}^\infty}} \mathrm{Re}(K_r * f)(x) \\ &+ \sup_{\substack{\delta<r\\ r\in\{R_k\}_{k=1}^\infty}} \mathrm{Im}(K_r * f)(x) - \inf_{\substack{\delta<r\\ r\in\{=R_k\}_{k=1}^\infty}} \mathrm{Im}(K_r * f)(x),\end{aligned}$$

where K_r is the Dirichlet kernel (i.e., the kernel of the disc multiplier).

Now, fix $\epsilon > 0$ and choose a $\tilde{g} \in \mathcal{S}$ such that $\|g-\tilde{g}\|_2 < \epsilon$. If $\tilde{f} = G_\alpha * \tilde{g}$, then $S_{R_k}\tilde{f}(x) \to G_\alpha * \tilde{g}(x) = \tilde{f}(x)$ for all $x \in \mathbb{R}^n$ by Theorem E. Moreover, by the uniformity in Theorem E, it is possible to find a $\delta \in \{R_k\}_{k=1}^\infty$ large enough to have $\Omega_\delta \tilde{f}(x) \le \epsilon$ for all $x \in \mathbb{R}^n$. Hence, with $S_*^l f(x) = \{\sup|S_{R_k}f(x)| : k \in \mathbb{N}\}$,

$$\Omega_\delta f(x) \le \Omega_\delta(f-\tilde{f})(x) + \Omega_\delta \tilde{f}(x) \le 4S_*^l(f-\tilde{f})(x) + \epsilon.$$

Let us suppose that $\epsilon < \frac{\lambda}{2}$. Then, by the previous line,

$$O_{\delta,\lambda} = \{x \in \mathbb{R}^n : \Omega_\delta f(x) > \lambda\} \subset \{x \in \mathbb{R}^n : S_*^l(f-\tilde{f})(x) > \tfrac{\lambda}{8}\}.$$

Consequently,

$$\begin{aligned} C_{\alpha,2}(O_{\delta,\lambda}) &\le C_{\alpha,2}(\{x \in \mathbb{R}^n : S_*^l(f-\tilde{f})(x) > \tfrac{\lambda}{8}\}) \\ &\le \left(\tfrac{8}{\lambda}\right)^2 \|S_*^l(g-\tilde{g})\|_2^2 \le \tfrac{C}{\lambda^2}\|g-\tilde{g}\|_2^2 \le \tfrac{C\epsilon^2}{\lambda^2}. \end{aligned} \tag{8.1}$$

The second inequality follows from Definition 2 while the third one is a consequence of Theorem B′ in [4], which says that $\|S_*^l f\|_2 \le C\|f\|_2$ if $f \in L^2(\mathbb{R}^n)$.

Now, for $k \in \mathbb{N}$ fixed, let $\epsilon = 4^{-k}$ and $\lambda = 2^{-k}$, and let δ_k be the associated δ. By the subadditivity of $C_{\alpha,2}$ and (8.1),

$$C_{\alpha,2}(F_m) \le \sum_{k=m}^\infty C_{\alpha,2}(O_{\delta_k,2^{-k}}) \le C\sum_{k=m}^\infty 2^{-2k},$$

where $F_m = \bigcup_{k=m}^\infty O_{\delta_k,2^{-k}}$. If $m \to \infty$ the last sum tends to 0; thus the same happens to $C_{\alpha,2}(\bigcap_{m=1}^\infty F_m)$.

So, $\Omega_{\delta_m} f(x) \le 2^{-k}$ if $x \notin F_m$ for $\delta_m \ge \delta_k$ for all $k \ge m$, because $F_m^c = \bigcap_{k=m}^\infty\{x \in \mathbb{R}^n : \Omega_{\delta_k} f(x) \le 2^{-k}\}$ and $\Omega_{\delta_m} f(x) \le \Omega_{\delta_k} f(x)$ for such δ_k. Thus, $S_{R_k} f(x) \to (G_\alpha * g)(x)$ if $x \notin \bigcap_{m=1}^\infty F_m$, and the convergence takes place uniformly outside any F_m. □

Remark 8.2. The estimate in [4] is strong enough to give the first part of Theorem 8.1 when $2 < p < \frac{2n}{n-1}$, but not its second part concerning the uniformity. This is because the weak-type capacitarian inequality implied by the result in [4] is local, rather than being valid for the whole of $\mathbb{R}^n$ when $p \ne 2$.

In a way similar to Theorem 8.1, one can use the estimates in [15], [20], [18] to show analogous results when $f \in \mathcal{L}_\alpha^p(\mathbb{R}^n)$, $\frac{2n}{n+1} < p < \frac{2n}{n-1}$, is radial or satisfies some symmetry conditions. It is thus natural to expect that the same will be true in the general $\mathcal{L}_\alpha^2(\mathbb{R}^n)$ case (Conjecture 8), but some global estimates, rather than our local ones, built in this article, will be needed to prove this.

When $p \ne 2$, the situation is less clear. Using Hunt's estimates [12] in their version on $\mathbb{R}$ [16], one directly gets the following:

Theorem 8.3. *Let $0 < \alpha \le \frac{1}{p}$ and let $1 < p < \infty$. Suppose that $f \in \mathcal{L}_\alpha^p(\mathbb{R})$, and suppose also that $g \in L^p(\mathbb{R})$ is such that $f = G_\alpha * g$ almost everywhere. Then $S_R f(x) \to G_\alpha * g(x)$ $C_{\alpha,p}$-quasieverywhere on $\mathbb{R}$. Moreover, the convergence takes place uniformly outside an open set of arbitrarily small $C_{\alpha,p}$-capacity.*

But, in higher dimensions, Fefferman's well-known counterexample for the ball multiplier shows that we cannot have any L^p boundedness results similar to those in [12]. Moreover, when $2 < p < \frac{2n}{n-1}$, one of the main ideas used to get Theorems 1 and 6 is that these functions are locally in L^2. So, it is not clear if one can truly hope to get some uniformity on "all" $\mathbb{R}^n$ when $n \geq 2$ and $p \neq 2$. Nevertheless, we expect this to be true. One of our reasons for being optimistic is that a partial result in this direction can be obtained when $\frac{n-1}{2} < \alpha \leq \frac{n}{p}$.

Theorem 8.4. *Let $f \in \mathcal{L}^p_\alpha(\mathbb{R}^n)$ with $1 < p < \frac{2n}{n-1}$ and $\frac{n-1}{2} < \alpha \leq \frac{n}{p}$. Suppose also that $g \in L^p(\mathbb{R}^n)$ is such that $f = G_\alpha * g$ almost everywhere. Then $S_R f(x) \to G_\alpha * g(x)$ $C_{\gamma,q}$-quasieverywhere. Moreover, the convergence takes place uniformly outside an open set of arbitrarily small $C_{\gamma,q}$-capacity. Here, $\gamma = \alpha - (\frac{n-1}{2} + \epsilon)$, $q = \frac{2np}{2n-np-2\epsilon p+p}$ and $0 < \epsilon < \alpha - \frac{n-1}{2}$.*

Proof. By Young's inequality, the Basic Estimate and Corollary 3 in [15],

$$\|S_*(G_{(n-1)/2+\epsilon} * g)\|_a \leq \|g\|_p \|G_{(n-1)/2+\epsilon}\|_b \leq C\|g\|_p \|S_*(G_{(n-1)/2+\epsilon})\|_b,$$

where $\frac{1}{a} = \frac{1}{p} + \frac{1}{b} - 1$ and $a, b \geq 1$.

Now, we want to choose a as big as possible (because this will imply a better pointwise convergence of $S_R f(x)$); so we need to pick the biggest b in $(\frac{2n}{n+1}, \frac{2n}{n-1})$ for which $G_{(n-1)/2+\epsilon} \in L^b(\mathbb{R}^n)$. Hence, using Proposition B, part 6, $b = \frac{2n}{n+1-2\epsilon}$ and, consequently, $a = \frac{2np}{2n-np-2\epsilon p+p}$.

Using the technique seen in the proof of Theorem 1, the previous estimate then implies that $S_R f(x) \to G_\alpha * g(x)$ $C_{\alpha-((n-1)/2+\epsilon),2np/(2n-np-2\epsilon p+p)}$-quasieverywhere on $\mathbb{R}^n$. Moreover, with an argument similar to Theorem 8.1, it can be shown that the convergence takes place uniformly outside an open set of arbitrarily small $C_{\alpha-((n-1)/2+\epsilon),2np/(2n-np-2\epsilon p+p)}$-capacity. □

Remark 8.5. The new elements in Theorem 8.4 are the uniformity and the region $1 < p < 2$. The quasieverywhere result is weaker than Theorem 6 when $2 \leq p < \frac{2n}{n-1}$.

Remark 8.6. Theorem 8.4 is true for all ϵ in $(0, \alpha - \frac{n-1}{2})$, but the best result takes place when $\epsilon \to 0$.

Remark 8.7. The idea behind Theorem 8.4 is truly one-dimensional, and we therefore cannot expect to get the full Conjecture 8 from it. Nevertheless, in the limiting case (i.e., $\alpha = \frac{n}{p}$), the value obtained is quite good. In fact, Theorem 8.4 is more precise than Theorem 1 in this case, because $\gamma q = n > \frac{n}{k}$ (but it is still slightly weaker than the sharp Theorem 6). Unfortunately, this changes rapidly as α tends towards $\frac{n-1}{2}$.

It is not completely clear yet how one could prove Conjecture 8 even when $\frac{n-1}{2} < \alpha \leq \frac{n}{p}$, but, with [4] in mind, an interesting possibility is certainly to try to use some "smooth" approximations of the ball multiplier, $m_R = \chi_{B(0,R)}$. This should at least help to extend Theorem 8.4 below the critical index $\frac{n-1}{2}$. Another possibility to obtain a partial result (e.g., uniformity, except on an open set of arbitrarily small Lebesgue measure when $\frac{n(n-1)(p-2)}{2p} < \alpha \leq \frac{n-1}{2}$ and $2 \leq p < \frac{2n}{n-1}$) would be to show that the estimate in Lemma C is global when $p = 2$. Combining such an estimate with Theorem 8.4 would then imply the desired partial result using Stein's complex interpolation theorem [27, p. 205].

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