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MONOMIAL BASES FOR q-SCHUR ALGEBRAS

JIE DU AND BRIAN PARSHALL

ABSTRACT. Using the Beilinson-Lusztig-MacPherson construction of the quantized enveloping algebra of $\mathfrak{gl}_n$ and its associated monomial basis, we investigate q-Schur algebras $\mathbf{S}_q(n,r)$ as "little quantum groups". We give a presentation for $\mathbf{S}_q(n,r)$ and obtain a new basis for the integral q-Schur algebra $S_q(n,r)$, which consists of certain monomials in the original generators. Finally, when $n \geqslant r$, we interpret the Hecke algebra part of the monomial basis for $S_q(n,r)$ in terms of Kazhdan-Lusztig basis elements.

1. INTRODUCTION

Let $\mathbf{U} = \mathbf{U}(\mathfrak{g})$ be the quantized enveloping algebra over $\mathbb{Q}(v)$ associated to a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$, and let U be its Lusztig $\mathcal{Z}$-form, where $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$. When $\mathfrak{g}$ has a simply laced root system, monomial bases for the positive (resp., negative) part U^+ (resp., U^-) of U have been constructed in [19, 7.8] using the theory of Ringel-Hall algebras; see [3] for an approach that works for all finite types. For example, the monomial basis for U^+ consists of certain explicit (ordered) monomials in the standard generators for U^+. The algebra $\mathbf{U}$ also has a PBW-type basis, but monomial bases are simpler than PBW bases and, in addition, they are closely related to canonical (or crystal) bases.

Let $\mathbf{S}_q(n,r)$ be a q-Schur algebra over $\mathbb{Q}(v)$; see below for the definition of $\mathbf{S}_q(n,r)$ and the associated Hecke algebra $\mathbf{H} = \mathbf{H}(\mathfrak{S}_r)$. The q-Schur algebras were introduced by Dipper and James [5], [6] (see [16] for an earlier version in the context of quantum groups). These algebras, as well as their analogues over other fields, play an important role in the non-defining representation and cohomology theories of the finite general linear groups. It is natural to ask how to construct monomial bases for the $\mathbf{S}_q(n,r)$. Using a beautiful geometric setting for q-Schur algebras, Beilinson, Lusztig, and MacPherson [1] studied the quantized enveloping algebra $\mathbf{U} = \mathbf{U}(\mathfrak{gl}_n)$ of the reductive Lie algebra $\mathfrak{gl}_n$ as a "limit" of q-Schur algebras and described a monomial basis for $\mathbf{U}$ in terms of another basis whose elements are formal infinite sums indexed by certain $n \times n$ matrices over $\mathbb{Z}$. In particular, there is a natural surjection $\mathbf{U} \twoheadrightarrow \mathbf{S}_q(n,r)$, arising as a "truncation" map; it carries an infinite sum in $\mathbf{U}$ to a finite sum in $\mathbf{S}_q(n,r)$. In addition, the results of [1] have an integral version (i.e., over $\mathcal{Z}$) [9], and there is a corresponding surjection $U \twoheadrightarrow S_q(n,r)$. Eventually, this work leads to a quantum Weyl reciprocity [11], also valid at the integral level.

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This paper applies the approach of [1] to obtain a natural monomial basis for $\mathbf{S}_q(n,r)$. Thus, Theorems 5.4 and 6.4 present monomial bases for $\mathbf{S}_q(n,r)$ and its integral version $S_q(n,r)$, while Theorem 5.5 describes the monomial basis elements in terms of certain elements constructed in [1]. As we show elsewhere in the paper, these bases are very natural and enjoy nice properties not shared by the PBW basis; see, e.g., 4.9, 7.2, 9.4. For example, if $n \geqslant r$, then $\mathbf{H} \subset \mathbf{S}_q(n,r)$, and Theorem 9.4 shows how the monomial basis "restricts" to a monomial basis for $\mathbf{H}$ (given as monomials in the Kazhdan-Lusztig elements $C'_s \in \mathbf{H}$ [15]).

This work was initially motivated by [7] (as well as by an announcement of the results in [8] by Doty at the 2001 New Orleans AMS meeting). Their work gives an explicit presentation of $\mathbf{S}_q(n,r)$ as well as a PBW-type basis $\mathbf{S}_q(n,r)$. We were motivated to see how to cast these results in the more geometric setting of [1]. Both a presentation (in a slightly different form) and a PBW basis also can be obtained as a new application of [1]; see Theorems 5.4 and 6.6.

The table below displays three different bases for $S_q(n,r)$, indicating how they stand in relation to bases for both the integral quantum enveloping algebra $U_{\mathcal{Z}}(\mathfrak{gl}_n)$ and the integral Hecke algebra $H(\mathfrak{S}_r)$ with $r \leqslant n$.

$H(\mathfrak{S}_r)$	$S_q(n,r)$	$U_{\mathcal{Z}}(\mathfrak{gl}_n)$
$\tilde{T}_w$	$[A]$	PBW basis
C'_w	$\{A\}$	Canonical basis (for $U^+_{\mathcal{Z}}(\mathfrak{gl}_n)$)
$C'_{s_1} \cdots C'_{s_k}$	$\mathtt{m}^{(A)}$	Monomial basis

The "orbital" basis elements $[A]$ are indexed by $n \times n$ matrices over $\mathbb{N}$ whose entries sum to r. This basis is the normalized version of the usual standard basis for a centralizer algebra, whose elements are denoted by $\phi^d_{\lambda\mu}$ in [6, 1.4]. Hence, its Hecke algebra counterpart consists of the normalized basis elements $\tilde{T}_w = v^{-\ell(w)}T_w$. In $U_{\mathcal{Z}}(\mathfrak{gl}_n)$, this basis corresponds to a PBW basis by means of its connection with a canonical (or crystal) basis; see [14] for further connections. All three algebras have canonical bases indicated in the second row in the table. They arise naturally from the corresponding monomial bases.

This work provides a foundation for [12], which directly relates the geometric approach [1] to the theory of Ringel-Hall algebras for linear quivers. In particular, this leads to a new connection between the theories of Ringel-Hall algebras and q-Schur algebras.

Some notation. Throughout, $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$ is the ring of Laurent polynomials in a variable v. Write $q = v^2$ and let $^- : \mathcal{Z} \to \mathcal{Z}$ be the ring automorphism satisfying $v^i \mapsto v^{-i}$ for all i. For $m \in \mathbb{N}$, put

$$[m]^! = [1][2]\cdots[m], \quad \text{where } [i] = \frac{v^i - v^{-i}}{v - v^{-1}}.$$

We also let, for $c \in \mathbb{Z}, t \in \mathbb{N}$,

$$\begin{bmatrix} c \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{v^{c-s+1} - v^{-c+s-1}}{v^s - v^{-s}}.$$

If $|c| < t$, then $\left[{c \atop t}\right] = 0$, and $\left[{c \atop t}\right] = \frac{[c]^!}{[t]^![c-t]^!}$ for $c \geqslant t \geqslant 0$.

Let $H = H(\mathfrak{S}_r)$ be the Hecke algebra over $\mathcal{Z}$ for the symmetric group $\mathfrak{S}_r$. If $S = \{(1,2),(2,3),\cdots,(r-1,r)\}$, then H has $\mathcal{Z}$-basis T_w, $w \in \mathfrak{S}_r$, and relations

(1.0.1)
$$\begin{cases} T_sT_w = T_{sw}, \quad l(sw) = 1 + l(w), \quad s \in S, w \in W; \\ (T_s+1)(T_s-q) = 0, \quad s \in S. \end{cases}$$

If V is a free $\mathcal{Z}$-module of rank n, there is a natural right action of H on $V^{\otimes r}$ by "place" permutations. The q-Schur algebra $S_q(n,r)$ over $\mathcal{Z}$ is the centralizer ring

(1.0.2)
$$S_q(n,r) = \mathrm{End}_H(V^{\otimes r}).$$

The algebra $S_q(n,r)$ is $\mathcal{Z}$-free of rank $\binom{n^2+r-1}{r}$. For more details, see [5], [11]. Put $\mathbf{H} = \mathbb{Q}(v) \otimes H$ and $\mathbf{S}_q(n,r) = \mathbb{Q}(v) \otimes S_q(n,r)$.

2. The quantized enveloping algebra of $\mathfrak{gl}_n$

The definition below for the quantized enveloping algebra of $\mathfrak{gl}_n$ is a slightly modified version of Jimbo [16]; see [23, 3.2], [9, 1.1].

Definition 2.1. The quantized enveloping algebra of $\mathfrak{gl}_n$ is the algebra $\mathbf{U}$ over $\mathbb{Q}(v)$ generated by the elements

$$E_i, F_i \quad (1 \leqslant i \leqslant n-1), \qquad K_i, K_i^{-1} \quad (1 \leqslant i \leqslant n)$$

subject to the following relations:

(a) $K_iK_j = K_jK_i, \quad K_iK_i^{-1} = 1$;
(b) $K_iE_j = v^{\epsilon(i,j)}E_jK_i$, where $\epsilon(i,i) = 1$, $\epsilon(i+1,i) = -1$ and $\epsilon(i,j) = 0$, otherwise;
(c) $K_iF_j = v^{-\epsilon(i,j)}F_jK_i$ with $\epsilon(i,j)$ as in (b) above;
(d) $E_iE_j = E_jE_i, F_iF_j = F_jF_i$ when $|i-j| > 1$;
(e) $E_iF_j - F_jE_i = \delta_{i,j}\frac{\tilde{K}_i - \tilde{K}_i^{-1}}{v - v^{-1}}$, where $\tilde{K}_i = K_iK_{i+1}^{-1}$;
(f) $E_i^2E_j - (v+v^{-1})E_iE_jE_i + E_jE_i^2 = 0$ when $|i-j| = 1$;
(g) $F_i^2F_j - (v+v^{-1})F_iF_jF_i + F_jF_i^2 = 0$ when $|i-j| = 1$.

Relations 2.1(f),(g) are called the quantum Serre relations. The subalgebra generated by the E_i, F_i and $\tilde{K}_i$ $(1 \leqslant i \leqslant n-1)$ is the quantized enveloping algebra $\mathbf{U}_v(\mathfrak{sl}_n)$.

There is a unique $\mathbb{Q}$-algebra anti-isomorphism $\Omega : \mathbf{U} \to \mathbf{U}$ defined by

(2.1.1)
$$\Omega(E_i) = F_i, \quad \Omega(F_i) = E_i, \quad \Omega(K_j) = K_j^{-1} \quad \text{and} \quad \Omega(v) = v^{-1}.$$

Clearly, Ω extends the anti-isomorphism Ω defined in [18, 1.2(a)] for $\mathbf{U}_v(\mathfrak{sl}_n)$.

Let $\mathbf{U}^+$ (resp., $\mathbf{U}^-$, $\mathbf{U}^0$) be the subalgebra of $\mathbf{U}$ generated by the E_i (resp., F_i, K_i). There is a triangular decomposition $\mathbf{U}^+ \otimes \mathbf{U}^0 \otimes \mathbf{U}^- \xrightarrow[\sim]{\text{mult}} \mathbf{U}^+\mathbf{U}^0\mathbf{U}^- = \mathbf{U}$ which is an isomorphism of vector spaces—see below for references in the integral case. Clearly, the elements $K^{\mathbf{j}} := K_1^{j_1}\cdots K_n^{j_n}$ for all $\mathbf{j} = (j_1,\cdots,j_n) \in \mathbb{Z}^n$ form a basis for $\mathbf{U}^0$. The subalgebras $\mathbf{U}^+$ and $\mathbf{U}^-$ are both $\mathbb{N}$-graded in terms of the degrees of monomials in the E_i and F_i. For monomials M in the E_i and M' in the F_i, and an element $h \in \mathbf{U}^0$, write $\deg(MhM') = \deg(M) + \deg(M')$. Observe that deg does *not* define an algebra grading on $\mathbf{U}$: the appropriate algebra grading (which we do not use) would be given by $\deg'(MhM') = \deg(M) - \deg(M')$.

For an analogue of a Kostant $\mathbb{Z}$-form over $\mathcal{Z}$, define, for $m, t \in \mathbb{N}$ and $c \in \mathbb{Z}$,

$$E_i^{(m)} = \frac{E_i^m}{[m]^!}, \quad F_i^{(m)} = \frac{F_i^m}{[m]^!} \quad \text{and} \quad \begin{bmatrix} K_i; c \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{K_i v^{c-s+1} - K_i^{-1} v^{-c+s-1}}{v^s - v^{-s}}.$$

Following [23], let U (resp., U^+, U^-) be the $\mathcal{Z}$-subalgebra of $\mathbf{U}$ generated by all $E_i^{(m)}$, $F_i^{(m)}$, K_i and $\begin{bmatrix} K_i; 0 \\ t \end{bmatrix}$ (resp., $E_i^{(m)}$, $F_i^{(m)}$). Let U^0 be the $\mathcal{Z}$-subalgebra of $\mathbf{U}$ generated by all K_i and $\begin{bmatrix} K_i; 0 \\ t \end{bmatrix}$. Then U, U^+, U^0, U^- are $\mathcal{Z}$-forms for $\mathbf{U}, \mathbf{U}^+, \mathbf{U}^0$, $\mathbf{U}^-$, respectively, and there is a triangular decomposition $U^+ \otimes U^0 \otimes U^- \cong U^+ U^0 U^- = U$ as free $\mathcal{Z}$-modules (apply the anti-automorphism Ω to the triangular decomposition given in [23, (3.2.6)]).[1] The following is known from [18, 2.14] and [9, Lemma 2.1].

Lemma 2.2. *The algebra U^0 has a $\mathcal{Z}$-basis*

$$K_1^{\delta_1} \cdots K_n^{\delta_n} \begin{bmatrix} K_1; 0 \\ t_1 \end{bmatrix} \cdots \begin{bmatrix} K_n; 0 \\ t_n \end{bmatrix}, \quad \delta_i \in \{0, 1\}, t_i \in \mathbb{N}.$$

The formulas below will be useful; see [18, p.269], [17, 4.1(a)].

Lemma 2.3. *The following formulas hold in $\mathbf{U}$:*

(1) $\left[\begin{smallmatrix} K_i;0 \\ t \end{smallmatrix}\right] \left[\begin{smallmatrix} K_i;-t \\ s \end{smallmatrix}\right] = \left[\begin{smallmatrix} t+s \\ t \end{smallmatrix}\right] \left[\begin{smallmatrix} K_i;0 \\ t+s \end{smallmatrix}\right] \quad (t, s \geqslant 0)$;
(2) $\left[\begin{smallmatrix} K_i;c \\ t \end{smallmatrix}\right] - v^{-t} \left[\begin{smallmatrix} K_i;c+1 \\ t \end{smallmatrix}\right] = -v^{-(c+1)} K_i^{-1} \left[\begin{smallmatrix} K_i;c \\ t-1 \end{smallmatrix}\right] \quad (t \geqslant 1)$;
(3) $\left[\begin{smallmatrix} K_i;-c \\ t \end{smallmatrix}\right] = \sum_{0 \leqslant j \leqslant t} (-1)^j v^{c(t-j)} \left[\begin{smallmatrix} c+j-1 \\ j \end{smallmatrix}\right] K_i^j \left[\begin{smallmatrix} K_i;0 \\ t-j \end{smallmatrix}\right] \quad (t \geqslant 0, c \geqslant 1)$;
(4) $\left[\begin{smallmatrix} K_i;c \\ t \end{smallmatrix}\right] = \sum_{0 \leqslant j \leqslant t} v^{c(t-j)} \left[\begin{smallmatrix} c \\ j \end{smallmatrix}\right] K_i^{-j} \left[\begin{smallmatrix} K_i;0 \\ t-j \end{smallmatrix}\right] \quad (t \geqslant 0, c \geqslant 0)$;
(5) $E_i^{(m)} \left[\begin{smallmatrix} K_i;c \\ t \end{smallmatrix}\right] = \left[\begin{smallmatrix} K_i;c-m \\ t \end{smallmatrix}\right] E_i^{(m)}$ *and* $E_i^{(m)} \left[\begin{smallmatrix} K_{i+1};c \\ t \end{smallmatrix}\right] = \left[\begin{smallmatrix} K_{i+1};c+m \\ t \end{smallmatrix}\right] E_i^{(m)}$;
(6) $F_i^{(m)} \left[\begin{smallmatrix} K_i;c \\ t \end{smallmatrix}\right] = \left[\begin{smallmatrix} K_i;c+m \\ t \end{smallmatrix}\right] F_i^{(m)}$ *and* $F_i^{(m)} \left[\begin{smallmatrix} K_{i+1};c \\ t \end{smallmatrix}\right] = \left[\begin{smallmatrix} K_{i+1};c-m \\ t \end{smallmatrix}\right] F_i^{(m)}$;
(7) *For any positive integers k, l, we have*

$$E_i^{(k)} F_i^{(l)} = \sum_{t=0}^{\min(k,l)} F_i^{(l-t)} \begin{bmatrix} \tilde{K}_i; 2t-k-l \\ t \end{bmatrix} E_i^{(k-t)}.$$

Proof. The formulas (1)-(4) and (7) are proved exactly as in [17]. Finally, (5) and (6) follow from 2.1(b),(c) by induction on m. $\square$

The commutator formula 2.3(7) plus 2.3(5),(6) show that, for monomials M in the E_i and M' in the F_i, $MM' = M'M + \sum_j M_j h_j M'_j$, where M_j (resp.,M'_j) are monomials in the E_i's (resp., F_i's), $h_j \in U^0$ and

$$\deg(M_j h_j M'_j) \leqslant \deg(MM') - 2.$$

[1] Note that the Lusztig $\mathcal{Z}$-form $'U$ of $\mathbf{U}_v(\mathfrak{sl}_n)$ is generated by all $E_i^{(m)}$, $F_i^{(m)}$, $\tilde{K}_i$, and $\tilde{K}_i^{-1}$. Thus, by [9, 2.6], the $\mathcal{Z}$-form U can be generated by all $E_i^{(m)}$, $F_i^{(m)}$, $\tilde{K}_i$, $\tilde{K}_i^{-1}$, K_1 and $\begin{bmatrix} K_1; 0 \\ t \end{bmatrix}$.

3. The Beilinson-Lusztig-MacPherson construction and monomial bases

Let $\widetilde{\Xi}$ be the set of all $n \times n$ matrices over $\mathbb{Z}$ with all off-diagonal entries in $\mathbb{N}$, and let $\Xi = M_n(\mathbb{N})$ be the subset of $\widetilde{\Xi}$ consisting of matrices with entries all in $\mathbb{N}$. Let $\sigma : \Xi \to \mathbb{N}$ be the map sending a matrix to the sum of its entries. Then, for $r \in \mathbb{N}$, the inverse image $\Xi_r := \sigma^{-1}(r)$ is the set of $n \times n$ matrices in Ξ whose entries sum to r. For $1 \leqslant i, j \leqslant n$, let $E_{i,j} \in \Xi$ be the matrix $(a_{k,l})$ with $a_{k,l} = \delta_{i,k}\delta_{j,l}$.

Let U_r be the algebra over $\mathcal{Z}$ introduced in [1, 1.2].[2] It has a normalized $\mathcal{Z}$-basis $\{[A]\}_{A\in\Xi_r}$ defined in [1, 1.4]. In particular, if $\lambda \in \mathbb{N}^n$ with $D = \operatorname{diag}(\lambda) \in \Xi_r$, then (cf. [1, 1.3])

(3.0.1)

$$[D][A] = \begin{cases} [A], & \text{if } \lambda = ro(A) \\ 0, & \text{otherwise;} \end{cases} \quad \text{and} \quad [A][D] = \begin{cases} [A], & \text{if } \lambda = co(A) \\ 0, & \text{otherwise,} \end{cases}$$

where $ro(A) = \left(\sum_j a_{1,j}, \cdots, \sum_j a_{n,j}\right)$ and $co(A) = (\sum_i a_{i,1}, \cdots, \sum_i a_{i,n})$ are the sequences of row and column sums of $A = (a_{i,j})$. We put $\mathbf{U}_r = \mathbb{Q}(v) \otimes_{\mathcal{Z}} U_r$.

In [9, 1.4], the algebra U_r is shown to be naturally isomorphic to the q-Schur algebra $S_q(n,r)$ as defined in (1.0.2). In the sequel, we often call U_r and $\mathbf{U}_r$ q-Schur algebras.[3]

Let $\mathbf{K}$ be the $\mathcal{Z}$-algebra (without 1), defined in [1, §4], with basis $\{[A]\}_{A\in\widetilde{\Xi}}$, and let $\dot{\mathbf{U}} = \mathbb{Q}(v)\otimes_{\mathcal{Z}}\mathbf{K}$.[4] The multiplication $\cdot$ in $\mathbf{K}$ (and hence in $\dot{\mathbf{U}}$) is defined in [1, 4.4] by specializing v' to 1 from another algebra over $\mathbb{Q}(v)[v', v'^{-1}]$ whose multiplication is induced from the stabilization property of the multiplication of q-Schur algebras. By the definition in [1, 4.5], the relations (3.0.1), $D \in \widetilde{\Xi}$, continue to hold in $\dot{\mathbf{U}}$.

As in [1, 5.1], $\dot{\mathbf{U}}_\infty$ is the vector space of all formal (possibly infinite) $\mathbb{Q}(v)$-linear combinations $\sum_{A\in\widetilde{\Xi}} \beta_A[A]$ satisfying: for diagonal $D, D' \in \widetilde{\Xi}$, the sums

$$\sum_{A\in\widetilde{\Xi}} \beta_A[D]\cdot[A] \quad \text{and} \quad \sum_{A\in\widetilde{\Xi}} \beta_A[A]\cdot[D']$$

are finite. Defining the product of $\sum_{A\in\widetilde{\Xi}} \beta_A[A], \sum_{B\in\widetilde{\Xi}} \gamma_B[B] \in \dot{\mathbf{U}}_\infty$ to be

$$\sum_{A,B} \beta_A\gamma_B[A]\cdot[B]$$

gives $\dot{\mathbf{U}}_\infty$ an associative algebra structure, with $1 = \sum[D]$, the sum over all diagonal $D \in \widetilde{\Xi}$. Also, $\dot{\mathbf{U}}$ is naturally a subalgebra (without 1) of $\dot{\mathbf{U}}_\infty$.

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[2] The algebra U_r is denoted $\mathbf{K}_r$ in [1].

[3] The algebra U_r can be roughly described as follows: Let $G = GL_r(p^d)$ for some prime power p^d. Let $\mathcal{S}_q(n,r) = \mathbb{C}\otimes S_q(n,r)$ via the base change $\mathcal{Z} \to \mathbb{C}$, $v \mapsto p^{d/2}$. It is well known (see, e.g., [5, (2.24]) that

$$\mathcal{S}_q(n,r) \cong \operatorname{End}_G(\bigoplus \operatorname{ind}_{P_\lambda}^G \mathbb{C}); \tag{3.0.2}$$

here λ runs over all compositions λ of r into n parts, and P_λ denotes the corresponding parabolic subgroup of G. Using the geometry of relative positions of pairs of n-step filtrations on r-dimensional space, [1] defines U_r directly as a kind of "deformation" of (3.0.2).

[4] The notation here has been abused as in [1]: the basis $\{[A]\}_{A\in\Xi_r}$ for a q-Schur algebra is *not* a subset of the basis $\{[A]\}_{A\in\widetilde{\Xi}}$ for $\dot{\mathbf{U}}$. Given $A \in \Xi$, it should always be clear from the context whether $[A]$ is to be regarded as a basis element of $\mathbf{U}$ or of $\mathbf{U}_r$.

Let $\Xi^{\pm}$ be the set of all $A \in \Xi$ whose diagonal entries are zero. Given $r \in \mathbb{N}$, $r > 0$, $A \in \Xi^{\pm}$ and $\mathbf{j} = (j_1, \cdots, j_n) \in \mathbb{Z}^n$, we define

$$\begin{aligned} A(\mathbf{j}, r) &= \sum_{\substack{D \in \Xi^0 \\ \sigma(A+D)=r}} v^{\sum_i d_i j_i}[A+D] \in \mathbf{U}_r, \\ A(\mathbf{j}) = A(\mathbf{j}, \infty) &= \sum_{D \in \widetilde{\Xi}^0} v^{\sum_i d_i j_i}[A+D] \in \dot{\mathbf{U}}_\infty, \end{aligned}$$

where Ξ^0 (resp., $\widetilde{\Xi}^0$) denotes the subset of diagonal matrices in Ξ (resp., $\widetilde{\Xi}$) and $D = \operatorname{diag}(d_1, \cdots, d_n)$. If $\sigma(A) > r$, then $A(\mathbf{j}, r) = 0$. For any diagonal matrix $D \in \Xi_r$, we have from (3.0.1)

(3.0.3)
$$A(\mathbf{0}, r)[D] = \begin{cases} [A+D'], & \text{if } co(D') = ro(D) - co(A) \in \mathbb{N}^n; \\ 0, & \text{if such a } D' \in \mathbb{N}^n \text{ does not exist.} \end{cases}$$

Of course, $\mathbf{0} := (0, \cdots, 0) \in \mathbb{N}^n$. Obviously, the D' in (3.0.3) satisfying $co(D') = ro(D) - co(A)$ is unique, if it exists.

Let $\mathbf{V}$ be the subspace of $\dot{\mathbf{U}}_\infty$ spanned by

$$\mathcal{B} = \{A(\mathbf{j}) \mid A \in \Xi^{\pm}, \mathbf{j} \in \mathbb{Z}^n\}.$$

The next result is proved in [1, 5.5, 5.7].

Proposition 3.1. *(1) $\mathbf{V}$ is a subalgebra of $\dot{\mathbf{U}}_\infty$ with $\mathbb{Q}(v)$-basis $\mathcal{B}$. It is generated by $E_{h,h+1}(\mathbf{0})$, $E_{h+1,h}(\mathbf{0})$ and $0(\mathbf{j})$ for all $1 \leqslant h < n$ and $\mathbf{j} \in \mathbb{Z}^n$.*

(2) For any positive integer r, the q-Schur algebra $\mathbf{U}_r$ is generated by the elements

$$E_{h,h+1}(\mathbf{0}, r), \quad E_{h+1,h}(\mathbf{0}, r), \quad \text{and} \quad 0(\mathbf{j}, r)$$

for all $1 \leqslant h < n$ and $\mathbf{j} \in \mathbb{N}^n$.

(3) There is an algebra isomorphism $\mathbf{U} \xrightarrow{\sim} \mathbf{V}$ satisfying

$$E_h \mapsto E_{h,h+1}(\mathbf{0}), \quad K^{\mathbf{j}} \mapsto 0(\mathbf{j}), \quad F_h \mapsto E_{h+1,h}(\mathbf{0})$$

and an algebra epimorphism $\zeta_r : \mathbf{U} \twoheadrightarrow \mathbf{U}_r$ satisfying

$$E_h \mapsto E_{h,h+1}(\mathbf{0}, r), \quad K^{\mathbf{j}} \mapsto 0(\mathbf{j}, r), \quad F_h \mapsto E_{h+1,h}(\mathbf{0}, r).$$

We shall identify $\mathbf{U}$ with $\mathbf{V}$ and hence identify E_h with $E_{h,h+1}(\mathbf{0})$, etc., in the sequel. We now describe a monomial basis for $\mathbf{U}$.

Let Ξ^+ (resp., Ξ^-) be the subset of Ξ consisting of those matrices $(a_{i,j})$ with $a_{i,j} = 0$ for all $i \geqslant j$ (resp., $i \leqslant j$). For $A \in \Xi$, write $A = A^+ + A^0 + A^-$ with $A^+ \in \Xi^+$, $A^0 \in \Xi^0$, and $A^- \in \Xi^-$. We also introduce the degree function:[5]

(3.1.1)
$$\deg(A) = \sum_{1 \leqslant i,j \leqslant n} |j - i| a_{i,j}.$$

Let $A = (a_{ij}) \in \Xi$. For $i < j$, let $\sigma_{i,j}(A) = \sum_{s \leqslant i; t \geqslant j} a_{s,t}$ and $\sigma_{j,i}(A) = \sum_{s \leqslant i; t \geqslant j} a_{t,s}$. Define, following [1, 3.5], $A' \preccurlyeq A$ if and only if $\sigma_{i,j}(A') \leqslant \sigma_{i,j}(A)$ and $\sigma_{j,i}(A') \leqslant \sigma_{j,i}(A)$ for all $1 \leqslant i < j \leqslant n$. Put $A' \prec A$ if $A' \preccurlyeq A$ and, for some pair (i,j) with $i < j$, either $\sigma_{i,j}(A') < \sigma_{i,j}(A)$ or $\sigma_{j,i}(A') < \sigma_{j,i}(A)$. Since $\deg A^+ = \sum_{i=1}^{n-1} \sigma_{i,i+1}(A)$, and $\deg A^- = \sum_{i=1}^{n-1} \sigma_{i+1,i}(A)$, the lemma below holds.

[5] The degree function here differs from the Ψ function [1, p.668]. But, it plays a similar role for induction; see 4.14.

Lemma 3.2. *If $A' \preccurlyeq A$, then* $\deg(A') \leqslant \deg(A)$.

Note that $A' \prec A$ does not necessarily imply that $\deg(A') < \deg(A)$.

For $A \in \Xi^{\pm}$ and $\mathbf{j} \in \mathbb{Z}^n$, let

$$M^{(A,\mathbf{j})} = E^{(A^+)} \cdot 0(\mathbf{j}) \cdot F^{(A^-)},$$

where

$$E^{(A^+)} = \prod_{1 \leqslant i \leqslant h < j \leqslant n} E_h^{(a_{i,j})} \text{ and } F^{(A^-)} = \prod_{1 \leqslant j \leqslant h < i \leqslant n} F_h^{(a_{i,j})}.$$

The orders in which the products $E^{(A^+)}$ and $F^{(A^-)}$ are taken are defined as follows: For the jth column (reading upwards) $a_{j-1,j}, \cdots, a_{1,j}$ $(2 \leqslant j \leqslant n)$ of A^+, fix the following reduced expression for the longest word $w_{0,j}$ of $\mathfrak{S}_j$:

$$\begin{aligned} w_{0,j} &= s_{j-1}(s_{j-2}s_{j-1})(s_{j-3}s_{j-2}s_{j-1}) \cdots (s_1 s_2 \cdots s_{j-1}) \\ &= (s_{j-1} \cdots s_1)(s_{j-1} \cdots s_2) \cdots (s_{j-1}s_{j-2})s_{j-1}. \end{aligned} \tag{3.2.1}$$

Here (and later), $s_i = (i, i+1)$ for $1 \leqslant i < j$. Put[6]

$$M_j = M_j(A^+) = E_{j-1}^{(a_{j-1,j})}(E_{j-2}^{(a_{j-2,j})}E_{j-1}^{(a_{j-2,j})}) \cdots (E_1^{(a_{1,j})}E_2^{(a_{1,j})} \cdots E_{j-1}^{(a_{1,j})}).$$

Similarly, for the jth row (reading to the right) $a_{j,1}, \cdots, a_{j,j-1}$ $(2 \leqslant j \leqslant n)$ of A^-, put

$$M_j' = (F_{j-1}^{(a_{j,1})} \cdots F_2^{(a_{j,1})}F_1^{(a_{j,1})}) \cdots (F_{j-1}^{(a_{j,j-2})}F_{j-2}^{(a_{j,j-2})})F_{j-1}^{(a_{j,j-1})} = \Omega(M_j(A'))$$

(cf. (2.1.1)), where $A' = (A^-)^T$ is the transpose of A^-. Then we have $E^{(A^+)} = M_n M_{n-1} \cdots M_2$ and $F^{(A^-)} = M_2' M_3' \cdots M_n'$. Clearly, $\deg E^{(A^+)} = \deg(A^+)$ and $\deg F^{(A^-)} = \deg(A^-)$. The following result is also essentially proved in [1].

Proposition 3.3. *The set*

$$\mathcal{M} = \{M^{(A,\mathbf{j})} \mid A \in \Xi^{\pm}, \mathbf{j} \in \mathbb{Z}^n\}$$

forms a basis for $\mathbf{U}$. *For $A \in \Xi^{\pm}, \mathbf{j} \in \mathbb{Z}^n$, there exist $a \in \mathbb{Z}$, $f_{\mathbf{j}',B}, g_{\mathbf{j}'',C} \in \mathbb{Q}(v)$ such that*

$$M^{(A,\mathbf{j})} = v^a A(\mathbf{j}) + \sum_{\substack{\mathbf{j}' \in \mathbb{Z}^n, B \in \Xi^{\pm} \\ B \prec A}} f_{\mathbf{j}',B} B(\mathbf{j}'), \tag{3.3.1}$$

$$A(\mathbf{j}) = v^{-a} M^{(A,\mathbf{j})} + \sum_{\substack{\mathbf{j}'' \in \mathbb{Z}^n, C \in \Xi^{\pm} \\ C \prec A}} g_{\mathbf{j}'',C} M^{(C,\mathbf{j}'')}. \tag{3.3.2}$$

In particular, the set $\{E^{(A)}\}_{A \in \Xi^+}$ (resp., $\{F^{(B)}\}_{B \in \Xi^-}$) forms a basis for $\mathbf{U}^+$ (resp., $\mathbf{U}^-$).

---

[6] Observe that changing from the first reduced expression in (3.2.1) to the second requires only relations of the form $s_i s_j = s_j s_i$, $|i - j| > 1$. Thus, by definition 2.1, we have

$$M_j = (E_{j-1}^{(a_{j-1,j})} \cdots E_1^{(a_{1,j})})(E_{j-2}^{(a_{j-2,j})} \cdots E_2^{(a_{1,j})}) \cdots (E_{j-1}^{(a_{2,j})}E_{j-2}^{(a_{1,j})})E_{j-1}^{(a_{1,j})}.$$

This is the original definition given in [1].

Proof. For the first assertion, see [1, 5.7]. Next, [1, 5.4(c)][7] (and the discussion following it) implies (3.3.1), while (3.3.2) is obtained by solving (3.3.1) inductively. The final assertion follows from an argument along the line of that of [1, 5.5] (see [12, 4.3] for some details). □

The basis $\mathcal{M}$ is the *monomial basis* associated to the given ordering on the $w_{0,j}$ above.

Corollary 3.4. *Let $A \in \Xi^+$ and let M be any monomial in the E_i. Then the product $ME^{(A)}$ is a (finite) $\mathcal{Z}$-linear combination of $E^{(B)}$ with $B \in \Xi^+$ and $\deg(B) \leqslant \deg(M) + \deg(A)$. Thus, M itself can be written as a linear combination of $E^{(B)}$ with $B \in \Xi^+$ and $\deg(B) \leqslant \deg(M)$. A similar statement holds for the F_i.*

Proof. Using [1, 5.4(c)] again, we have

$$E^{(A)} = M^{(A,\mathbf{0})} = A(\mathbf{0}) + \sum_{\substack{\mathbf{j}\in\mathbb{Z}^n, B'\in\Xi^\pm \\ B'\prec A}} f_{\mathbf{j},B'}B'(\mathbf{j}). \tag{3.4.1}$$

The first formula given in [1, 5.3] implies that, if $A' \in \Xi^+$, then $E_h A'(\mathbf{0})$ is a linear combination of terms $B(\mathbf{0})$ with $B \in \Xi^+$. (The fact that $A' \in \Xi^+$ is essential to guarantee that the summands in [1, 5.3] of the form $B(\mathbf{j})$ with $\mathbf{j} \neq \mathbf{0}$ *all* have zero coefficient.) Since $E^{(A)}$ is a product (up to a scalar) of various $E_h = E_{h,h+1}(\mathbf{0})$, induction shows that the only $B'(\mathbf{j})$ that occur in (3.4.1) are those with $B' \in \Xi^+$ and $\mathbf{j} = \mathbf{0}$. Therefore, we obtain

$$E^{(A)} = A(\mathbf{0}) + \sum_{B'\in\Xi^+ : B'\prec A} f_{\mathbf{0},B'}B'(\mathbf{0}). \tag{3.4.2}$$

Clearly, if $f_{\mathbf{0},B'} \neq 0$, then $B' \prec A$ implies $\deg(B') \leqslant \deg(A)$, by 3.2, and $B' \in \Xi^+$ as well.

To prove the corollary, we can easily reduce to the special case when $M = E_h$ for $1 \leqslant h \leqslant n-1$. Applying [1, 5.3, p. 672] again shows that the product $E_h A(\mathbf{0})$ (resp., $E_h B'(\mathbf{0})$) is a linear combination of $B''(\mathbf{0})$ with $B'' \in \Xi^+$, $\deg(B'') \leqslant \deg(A) + 1$ (resp., $\deg(B'') \leqslant \deg(B') + 1 \leqslant \deg(A) + 1$). Every $B''(\mathbf{0})$ is a linear combination of $E^{(B)}$ with $B \in \Xi^+$, $B \preccurlyeq B''$ by (3.4.2). So $E_h E^{(A)}$ is a linear combination of $E^{(B)}$ with $B \in \Xi^+$ and $\deg(B) \leqslant \deg(B'') \leqslant \deg(A) + 1$. □

Remark 3.5. We note that the inequalities $\deg(B) \leqslant \deg(A) + \deg(M)$ in the statement of 3.4 can even be replaced by the equalities $\deg(B) = \deg(A) + \deg(M)$, using the fact that the relations defining $\mathbf{U}^+$ are homogeneous and the monomial basis given in the last assertion of 3.3 preserves the graded structure on $\mathbf{U}^+$. We thank Fu Qiang for pointing this out.

4. The algebra $\mathbf{S}_r$

Let X be another indeterminate which is independent of v. For $t \in \mathbb{N}$, put

$$[X;t]^! = (X-1)(X-v)\cdots(X-v^{t-1}),$$

with $[X;0]^! = 1$ by definition.

[7] We will make much use of this result, which holds for both $\mathbf{U}$ and for $\mathbf{U}_r$, in the sequel.

Definition 4.1. Let $\mathbf{S}_r$ be the associative algebra over $\mathbb{Q}(v)$ generated by the elements

$$\mathtt{e}_i, \mathtt{f}_i, \mathtt{k}_i \quad (1 \leqslant i \leqslant n-1),$$

subject to the relations:

(a) $\mathtt{k}_i\mathtt{k}_j = \mathtt{k}_j\mathtt{k}_i$;
(b) $[\mathtt{k}_1; t_1]^! [\mathtt{k}_2; t_2]^! \cdots [\mathtt{k}_{n-1}; t_{n-1}]^! = 0 \ \forall t_i \in \mathbb{N}$ such that $t_1 + \cdots + t_{n-1} = r+1$;
(c) $\mathtt{e}_i\mathtt{e}_j = \mathtt{e}_j\mathtt{e}_i, \mathtt{f}_i\mathtt{f}_j = \mathtt{f}_j\mathtt{f}_i \quad (|i-j| > 1)$;
(d) $\mathtt{e}_i^2\mathtt{e}_j - (v+v^{-1})\mathtt{e}_i\mathtt{e}_j\mathtt{e}_i + \mathtt{e}_j\mathtt{e}_i^2 = 0 \quad (|i-j| = 1)$;
(e) $\mathtt{f}_i^2\mathtt{f}_j - (v+v^{-1})\mathtt{f}_i\mathtt{f}_j\mathtt{f}_i + \mathtt{f}_j\mathtt{f}_i^2 = 0 \quad (|i-j| = 1)$;
(f) $\mathtt{k}_i\mathtt{e}_j = v^{\epsilon(i,j)}\mathtt{e}_j\mathtt{k}_i, \quad \mathtt{k}_i\mathtt{f}_j = v^{-\epsilon(i,j)}\mathtt{f}_j\mathtt{k}_i$ with $\epsilon(i,j)$ as in 2.1(b);
(g) $\mathtt{e}_i\mathtt{f}_j - \mathtt{f}_j\mathtt{e}_i = \delta_{i,j}\frac{\tilde{\mathtt{k}}_i - \tilde{\mathtt{k}}_i^{-1}}{v-v^{-1}}$, where $\tilde{\mathtt{k}}_i = \mathtt{k}_i\mathtt{k}_{i+1}^{-1}$, $1 \leqslant i \leqslant n-1$, with $\mathtt{k}_n = v^r\mathtt{k}_1^{-1}\cdots\mathtt{k}_{n-1}^{-1}$.

Since $[\mathtt{k}_i; t_i]^! = 0$ if $t_i = r+1$, each $\mathtt{k}_i$ is invertible and $\mathtt{k}_i^{-1}$ is a polynomial of $\mathtt{k}_i$ of degree r; so the definitions of $\tilde{\mathtt{k}}_i$ and $\mathtt{k}_n$ make sense. Also, 4.1(f) holds for $i = n$. By 2.1, there is a surjective homomorphism $\mathbf{U} \twoheadrightarrow \mathbf{S}_r$ in which $E_i \mapsto \mathtt{e}_i, F_i \mapsto \mathtt{f}_i$ and $K_j \mapsto \mathtt{k}_j$. In particular, for $A \in \Xi^+$, let $\mathtt{e}^{(A)}$ be the image of $E^{(A)}$ under this homomorphism, with a similar convention for $\mathtt{f}^{(A)}$, $A \in \Xi^-$. The relations in 2.3 hold with E_i, F_i, K_i replaced by $\mathtt{e}_i, \mathtt{f}_i, \mathtt{k}_i$, respectively. The definition implies the following result.

Lemma 4.2. *(1) There is a unique $\mathbb{Q}(v)$-algebra anti-automorphism*[8] τ *on* $\mathbf{S}_r$ *satisfying*

$$\tau(\mathtt{e}_i) = \mathtt{f}_i, \quad \tau(\mathtt{f}_i) = \mathtt{e}_i, \quad \tau(\mathtt{k}_i) = \mathtt{k}_i.$$

(2) There is a unique $\mathbb{Q}(v)$-algebra anti-automorphism γ on $\mathbf{S}_r$ satisfying

$$\gamma(\mathtt{e}_i) = \mathtt{e}_{n-i}, \quad \gamma(\mathtt{f}_i) = \mathtt{f}_{n-i}, \quad \gamma(\mathtt{k}_i) = \mathtt{k}_{n-i+1}.$$

(3) There is a unique $\mathbb{Q}$-algebra involution $^-$ on $\mathbf{S}_r$ satisfying

$$\bar{\mathtt{e}}_i = \mathtt{e}_i, \quad \bar{\mathtt{f}}_i = \mathtt{f}_i, \quad \bar{\mathtt{k}}_i = \mathtt{k}_i^{-1}, \quad \bar{v} = v^{-1}.$$

Proposition 4.3. *Let $\mathbf{t} = (t_1, \cdots, t_n) \in \mathbb{N}^n$ and put $|\mathbf{t}| = t_1 + \cdots + t_n$. Then*

$$[\mathtt{k}; \mathbf{t}] := [\mathtt{k}_1; t_1]^! [\mathtt{k}_2; t_2]^! \cdots [\mathtt{k}_n; t_n]^! = 0 \qquad \textit{whenever } |\mathbf{t}| = r+1.$$

In particular, $[\mathtt{k}_n; r+1]^! = 0$.

Proof. If $t_n = 0$, the result holds by 4.1(b). So assume $t_n \geqslant 1$. Then

$$\begin{aligned}[\mathtt{k}_n; t_n]^! &= -v^{t_n-1}\mathtt{k}_1^{-1}\cdots\mathtt{k}_{n-1}^{-1}[\mathtt{k}_n; t_n-1]^!(\mathtt{k}_1\cdots\mathtt{k}_{n-1} - v^{r-t_n+1}) \\ &= -v^{t_n-1}\mathtt{k}_1^{-1}\cdots\mathtt{k}_{n-1}^{-1}[\mathtt{k}_n; t_n-1]^! \sum_{i=1}^{n-1} v^{t_1+\cdots+t_{i-1}}(\mathtt{k}_i - v^{t_i})\mathtt{k}_{i+1}\cdots\mathtt{k}_{n-1}.\end{aligned}$$

(Observe the above sum is telescoping.) Putting $a_i = -v^{t_1+\cdots+t_{i-1}+t_{n-1}-1}\mathtt{k}_1^{-1}\cdots\mathtt{k}_i^{-1}$,

$$[\mathtt{k}; \mathbf{t}] = \sum_{i=1}^{n-1} a_i [\mathtt{k}_1; t_1]^! \cdots [\mathtt{k}_i; t_i+1]^! \cdots [\mathtt{k}_{n-1}; t_{n-1}]^! [\mathtt{k}_n; t_n-1]^! = 0,$$

[8] The composition of this isomorphism and the bar involution $^-$ below is the "little" version of the map Ω defined in (2.1.1).

by induction on t_n. $\square$

Remarks 4.4. (1) 4.3 provides a connection between the presentation 4.1 and that given in [8]. Let $\mathbf{S}_r^0$ be the commutative subalgebra of $\mathbf{S}_r$ generated by $\mathtt{k}_1, \cdots, \mathtt{k}_{n-1}$. By 4.7 below, the relations 4.1(a),(b) provide a presentation for $\mathbf{S}_r^0$. However, by 4.3 and 4.6 below, $\mathbf{S}_r^0$ can also be described differently, taking generators $\mathtt{k}_1, \cdots, \mathtt{k}_n$ satisfying the relations $\mathtt{k}_i\mathtt{k}_j = \mathtt{k}_j\mathtt{k}_i$, $[\mathtt{k}_i; r+1]^! = 0$, and $\mathtt{k}_1 \cdots \mathtt{k}_n = v^r$; see [8, Prop. 7.4]. This gives another presentation of $\mathbf{S}_r^0$, replacing $\mathtt{k}_1, \cdots, \mathtt{k}_{n-1}$ by new generators $\mathtt{k}_1, \cdots, \mathtt{k}_n$ and replacing the relations 4.1(a),(b) by those above. This presentation is studied in [8]; it has fewer relations than 4.3, but one more generator $\mathtt{k}_n$ and the relations involving $\mathtt{k}_n$.

(2) To justify the relations for $\mathbf{S}_r^0$, let $L_v(\lambda)$ be the irreducible type 1 $\mathbf{U}$-module with highest weight λ, where λ is a partition of r with at most n parts. Suppose $u_\mu \in L_v(\lambda)_\mu$, the μ-weight space of $L_v(\lambda)$. Then $\mu \in \mathbb{N}^n$, $|\mu| = r$, $\mu \trianglelefteq \lambda$ (the dominance order—see below the proof of Prop. 4.5) and $K_i u_\mu = v^{\mu_i} u_\mu$. Thus,

$$\prod_{i=1}^{n} [K_i; t_i]^! u_\mu = \prod_{i=1}^{n} [v^{\mu_i}; t_i]^! u_\mu.$$

Since $\prod_{i=1}^n [v^{\mu_i}; t_i]^! = 0$ whenever $t_1 + \cdots + t_n = r+1$, $L_v(\lambda)$ naturally becomes an $\mathbf{S}_r$-module.

By definition, for $\mathbf{t} = (t_1, \cdots, t_n) \in \mathbb{N}^n$ and $A \in \Xi$, $\mathtt{e}^{(A^+)}, \mathtt{f}^{(A^-)}$ are the images of $E^{(A^+)}, F^{(A^-)}$, respectively, under the epimorphism $\mathbf{U} \twoheadrightarrow \mathbf{S}_r$. Thus,

$$\mathtt{e}^{(A^+)} = \prod_{1 \leqslant i \leqslant h < j \leqslant n} \mathtt{e}_h^{(a_{i,j})}, \quad \mathtt{f}^{(A^-)} = \prod_{1 \leqslant j \leqslant h < i \leqslant n} \mathtt{f}_h^{(a_{i,j})},$$

$$\mathtt{k}_{\mathbf{t}} = \prod_{i=1}^{n} \begin{bmatrix} \mathtt{k}_i; 0 \\ t_i \end{bmatrix}.$$

Here the order in the products is the same as the order used for $E^{(A^+)}$ and $F^{(A^-)}$ in §3. The next result is a direct consequence of the defining relations on the $\mathtt{k}_i$ (cf. [8, 7.4(c), 7.6(a)]).

Lemma 4.5. *Let* $\mathbf{t} = (t_1, \cdots, t_n) \in \mathbb{N}^n$. *Then:*

(1) $\mathtt{k}_{\mathbf{t}} = 0$ *if* $|\mathbf{t}| > r$.

(2) *If* $|\mathbf{t}| = r$, *then* $\mathtt{k}_i \mathtt{k}_{\mathbf{t}} = v^{t_i} \mathtt{k}_{\mathbf{t}}$; *in particular,* $\begin{bmatrix} \mathtt{k}_i; c \\ t \end{bmatrix} \mathtt{k}_{\mathbf{t}} = \begin{bmatrix} t_i + c \\ t \end{bmatrix} \mathtt{k}_{\mathbf{t}}$.

Proof. To see (1), observe that

$$\begin{bmatrix} \mathtt{k}_i; 0 \\ t_i \end{bmatrix} = \prod_{s=1}^{t_i} \frac{\mathtt{k}_i v^{-s+1} - \mathtt{k}_i^{-1} v^{+s-1}}{v^s - v^{-s}} = \prod_{s=1}^{t_i} \frac{\mathtt{k}_i^{-1} v^{-s+1} (\mathtt{k}_i^2 - v^{2(s-1)})}{v^s - v^{-s}},$$

so that $[\mathtt{k}_i, t_i]^!$ is a factor of $\begin{bmatrix} \mathtt{k}_i; 0 \\ t_i \end{bmatrix}$ for every i. Thus, 4.3 implies (1). If $|\mathbf{t}| = r$, (1) implies that $(\mathtt{k}_i - v^{t_i})\mathtt{k}_{\mathbf{t}} = 0$, proving the first (and hence the last) assertion in (2). $\square$

The subalgebra $\mathbf{S}_r^0$ of $\mathbf{S}_r$ generated by the $\mathtt{k}_i$ is a quotient of $\mathbf{U}^0$. Let

$$\Lambda(n, r) = \{\mathbf{t} \mid \mathbf{t} \in \mathbb{N}^n, |\mathbf{t}| = r\}$$

be the set of compositions of r, and let $\trianglerighteq$ be the dominance partial ordering on $\Lambda(n,r)$: $\lambda \trianglerighteq \mu \iff \sum_{i=1}^j \lambda_i \geqslant \sum_{i=1}^j \mu_i, \forall j$. We have the following result (cf. [8, 7.4(b)]).

Corollary 4.6. *The algebra $\mathbf{S}_r^0$ is a commutative semisimple algebra over $\mathbb{Q}(v)$. The set $\{\mathtt{k}_\lambda\}_{\lambda\in\Lambda(n,r)}$ is a complete set of primitive orthogonal idempotents (hence a basis) for $\mathbf{S}_r^0$. In particular, the identity element $1 \in \mathbf{S}_r$ has the form $1 = \sum_{\lambda\in\Lambda(n,r)} \mathtt{k}_\lambda$.*

Proof. Let λ^+ be the partition obtained by permuting the components of $\lambda \in \Lambda(n,r)$. By 4.4(2), $\mathtt{k}_\lambda$ acts on $L_v(\lambda^+)_\lambda \neq 0$ as an identity operator. Hence, $\mathtt{k}_\lambda \neq 0$. By 4.5(2), $\mathtt{k}_\lambda\mathtt{k}_\mu = \delta_{\lambda,\mu}\mathtt{k}_\lambda$ for all $\lambda,\mu \in \Lambda(n,r)$; so the $\mathtt{k}_\lambda$ are nonzero orthogonal idempotents. The relations given in 4.1 imply that the $\#\Lambda(n,r)$ monomials $\mathtt{k}_1^{j_1}\cdots\mathtt{k}_{n-1}^{j_{n-1}}$ in the $\mathtt{k}_i$ of total degree at most r span $\mathbf{S}_r^0$. Thus, by dimension considerations, the linearly independent elements $\mathtt{k}_\lambda$, $\lambda \in \Lambda(n,r)$, must be a basis for $\mathbf{S}_r^0$. The corollary now follows. □

A dimension comparison gives rise to other bases for $\mathbf{S}_r^0$; e.g., part (1) of the corollary below follows from the proof above.

Corollary 4.7. *(1) The elements $\mathtt{k}_1^{j_1}\cdots\mathtt{k}_{n-1}^{j_{n-1}}$ ($j_i \in \mathbb{N}$, $j_1+\cdots+j_{n-1} \leqslant r$) form a basis for $\mathbf{S}_r^0$.*

(2) For any $\lambda \in \Lambda(n,r)$, let

$$\mathtt{k}'_\lambda := \begin{bmatrix}\mathtt{k}_1;0\\ \lambda_1\end{bmatrix}\begin{bmatrix}\mathtt{k}_2;0\\ \lambda_2\end{bmatrix}\cdots\begin{bmatrix}\mathtt{k}_{n-1};0\\ \lambda_{n-1}\end{bmatrix}.$$

Then the set $\{\mathtt{k}'_\lambda\}_{\lambda\in\Lambda(n,r)}$ forms a basis for $\mathbf{S}_r^0$.

Proof. We prove (2). For $\lambda,\mu \in \Lambda(n,r)$, write $\lambda \leqslant \mu \iff \lambda_i \leqslant \mu_i$, for all $i = 1,2,\cdots,n-1$. Set $\lambda < \mu$ if also $\lambda_i < \mu_i$ for some i. By 4.5, $\mathtt{k}'_\mu\mathtt{k}_\lambda \neq 0 \iff \mu \leqslant \lambda$. So 4.6 implies

$$\mathtt{k}'_\mu = \sum_{\lambda\in\Lambda(n,r),\mu\leqslant\lambda} \mathtt{k}'_\mu\mathtt{k}_\lambda = \mathtt{k}_\mu + \sum_{\lambda\in\Lambda(n,r),\mu<\lambda}\prod_{i=1}^{n-1}\begin{bmatrix}\lambda_i\\ \mu_i\end{bmatrix}\mathtt{k}_\lambda. \tag{4.7.1}$$

Now the assertion (2) follows easily. □

For $A \in \Xi$ and $1 \leqslant i \leqslant n$, define

$$\sigma_i(A) = a_{i,i} + \sum_{1\leqslant j<i}(a_{i,j}+a_{j,i}), \quad \sigma'_i(A) = a_{i,i} + \sum_{i<j\leqslant n}(a_{i,j}+a_{j,i}).$$

Then

$$\#\{(\lambda,A) \mid \lambda\in\Lambda(n,r), A\in\Xi^\pm, \lambda_i \geqslant \sigma_i(A)\,\forall i\} = \binom{r+n^2-1}{r}.$$

To see this, put $a_{i,i} = \lambda_i - \sigma_i(A)$. Then the cardinality above is the number of matrices $(a_{i,j}) \in \Xi$ such that $\sum_{i,j} a_{i,j} = r$. The identity holds if σ_i is replaced by σ'_i.

The next result was observed in [7, 4.6] and generalized in [8, 7.9]. For $1 \leqslant i \leqslant n-1$, let

$$\alpha_i = (0,\cdots,0,\underset{i}{1},-1,0,\cdots,0).$$

Lemma 4.8. *Let $\lambda \in \Lambda(n,r)$.*

(1) *If $\lambda_{i+1} \geqslant 1$, then $\mathtt{e}_i\mathtt{k}_\lambda = \mathtt{k}_{\lambda+\alpha_i}\mathtt{e}_i$.*
(2) *If $\lambda_i \geqslant 1$, then $\mathtt{f}_i\mathtt{k}_\lambda = \mathtt{k}_{\lambda-\alpha_i}\mathtt{f}_i$.*

Proof. Formula (2) results by applying the anti-automorphism τ given in 4.2(1) to (1). We prove (1). By 2.3(5), we have

$$\mathtt{e}_i\mathtt{k}_\lambda = \prod_{j\neq i,i+1}\begin{bmatrix}\mathtt{k}_j;0\\ \lambda_j\end{bmatrix}\begin{bmatrix}\mathtt{k}_i;-1\\ \lambda_i\end{bmatrix}\begin{bmatrix}\mathtt{k}_{i+1};1\\ \lambda_{i+1}\end{bmatrix}\mathtt{e}_i.$$

Multiplying on the left by $\left[\begin{smallmatrix}\mathtt{k}_i;0\\ 1\end{smallmatrix}\right]$ and applying 2.3(1),(5) and 4.5(2) gives

$$\begin{bmatrix}\lambda_i+1\\ 1\end{bmatrix}\mathtt{e}_i\mathtt{k}_\lambda = \begin{bmatrix}\lambda_i+1\\ 1\end{bmatrix}\prod_{j\neq i,i+1}\begin{bmatrix}\mathtt{k}_j;0\\ \lambda_j\end{bmatrix}\begin{bmatrix}\mathtt{k}_i;0\\ \lambda_i+1\end{bmatrix}\begin{bmatrix}\mathtt{k}_{i+1};1\\ \lambda_{i+1}\end{bmatrix}\mathtt{e}_i.$$

By 2.3(4) and 4.5, (1) follows after cancelling $\left[\begin{smallmatrix}\lambda_i+1\\ 1\end{smallmatrix}\right]$. □

Recall the sequence $ro(A)$ (resp., $co(A)$) of row (resp., column) sums of A defined in §2.

Corollary 4.9. *Let $A\in\Xi^\pm$ and $\lambda\in\Lambda(n,r)$.*

(1) *If $\lambda_i\geqslant\sigma_i(A^+)$, $1\leqslant i\leqslant n$, then $\mathtt{e}^{(A^+)}\mathtt{k}_\lambda=\mathtt{k}_{\lambda'}\mathtt{e}^{(A^+)}$, where $\lambda'=\lambda-co(A^+)+ro(A^+)$.*
(2) *If $\lambda_i\geqslant\sigma_i(A^-)$, $1\leqslant i\leqslant n$, then $\mathtt{k}_\lambda\mathtt{f}^{(A^-)}=\mathtt{f}^{(A^-)}\mathtt{k}_{\lambda''}$, where $\lambda''=\lambda+co(A^-)-ro(A^-)$.*

In these cases, we have $\lambda'\trianglerighteq\lambda$ and $\lambda''\trianglerighteq\lambda$.

Proof. If $i<j$ and $a\leqslant\lambda_l$ for all $l=i+1,\cdots,j$, by 4.8,

$$(\mathtt{e}_i^a\cdots\mathtt{e}_{j-1}^a)\mathtt{k}_\lambda=\mathtt{k}_\mu(\mathtt{e}_i^a\cdots\mathtt{e}_{j-1}^a),$$

where $\mu=\lambda+a(\alpha_i+\cdots+\alpha_{j-1})$. Now assume that $\lambda_i\geqslant\sigma_i(A^+)$ for all i. Since $\mathtt{e}^{(A^+)}$ is a product of terms $\mathtt{e}_i^{(a_{i,j})}\cdots\mathtt{e}_{j-1}^{(a_{i,j})}$, we obtain that $\mathtt{e}^{(A^+)}\mathtt{k}_\lambda=\mathtt{k}_{\lambda'}\mathtt{e}^{(A^+)}$ for $\lambda'=\lambda+\sum_{i<j}a_{i,j}(\alpha_i+\cdots+\alpha_{j-1})$, that is,

$$\begin{aligned}\lambda_1'&=\lambda_1+a_{1,2}+\cdots+a_{1,n},\\ \lambda_2'&=\lambda_2-a_{1,2}+a_{2,3}+\cdots+a_{2,n},\\ \lambda_3'&=\lambda_3-(a_{1,3}+a_{2,3})+a_{3,4}+\cdots+a_{3,n},\\ &\cdots\\ \lambda_{n-1}'&=\lambda_{n-1}-(a_{1,n-1}+\cdots+a_{n-2,n-1})+a_{n-1,n},\\ \lambda_n'&=\lambda_n-(a_{1,n}+\cdots+a_{n-1,n}),\end{aligned}\tag{4.9.1}$$

yielding the required formula. Applying τ in 4.2 to the identity in (1) gives that in (2). The last assertion follows easily from the definition. □

For part (1) of the following result, see also [8, 7.9].

Lemma 4.10. *Let $\lambda\in\Lambda(n,r)$.*

(1) *If $\lambda_{i+1}=0$ for some $1\leqslant i\leqslant n-1$, then $\mathtt{e}_i\mathtt{k}_\lambda=\mathtt{k}_\lambda\mathtt{f}_i=0$.*
(2) *More generally, if $A\in\Xi^+$ (resp., $A\in\Xi^-$) and $\lambda_i<\sigma_i(A)$ for some i, then $\mathtt{e}^{(A)}\mathtt{k}_\lambda=0$ (resp., $\mathtt{k}_\lambda\mathtt{f}^{(A)}=0$).*

Proof. It suffices to prove (1) for $\mathtt{e}_i$ and (2) for $A\in\Xi^+$; the others can be obtained by applying the anti-automorphism τ, since $\tau(\mathtt{e}^{(A)})=\mathtt{f}^{(A^T)}$ if A^T is the transpose of $A\in\Xi^+$.

To prove (1), assume $\lambda_{i+1} = 0$. By 4.1 and 4.5(2), $\mathtt{k}_{i+1}\mathtt{e}_i\mathtt{k}_\lambda = v^{-1}\mathtt{e}_i\mathtt{k}_\lambda$. Also, 4.1 and 4.6 imply that $\mathtt{e}_i\mathtt{k}_\lambda$ is a $\mathbb{Q}(v)$-linear combination of terms $\mathtt{k}_\mu\mathtt{e}_i$, $\mu \in \Lambda(n,r)$. Since $\mathtt{k}_{i+1}\mathtt{k}_\mu\mathtt{e}_i = v^{\mu_{i+1}}\mathtt{k}_\mu\mathtt{e}_i$ and $\mu_{i+1} \geqslant 0$, it follows that $\mathtt{e}_i\mathtt{k}_\lambda = 0$.

To prove (2), let i be minimal with $\lambda_i < \sigma_i(A)$. Then $i > 1$, since $\sigma_1(A) = 0$. Suppose $a_{1,i} + \cdots + a_{i'-1,i} \leqslant \lambda_i < a_{1,i} + \cdots + a_{i',i}$ for some $1 \leqslant i' \leqslant i-1$. Write $\left[{a_{i',i} \atop x}\right] \mathtt{e}^{(A)} = \mathtt{m}_1\mathtt{m}_2$, where

$$\mathtt{m}_1 = \mathtt{e}_{n-1}^{(a_{n-1,n})} \cdots \mathtt{e}_{i-1}^{(a_{i',i}-x)}, \quad \mathtt{m}_2 = \mathtt{e}_{i-1}^{(x)}\mathtt{e}_{i'-1}^{(a_{i'-1,i})}\mathtt{e}_{i'}^{(a_{i'-1,i})} \cdots \mathtt{e}_{i-1}^{(a_{i'-1,i})} \cdots \mathtt{e}_1^{(a_{1,2})}$$

and $x = \lambda_i - (a_{1,i} + \cdots + a_{i'-1,i})$. By 4.9 and 4.8, $\left[{a_{i',i} \atop x}\right] \mathtt{e}^{(A)}\mathtt{k}_\lambda = \mathtt{m}_1\mathtt{k}_\mu\mathtt{m}_2$, where $\mu = (\mu_1, \cdots, \mu_n) \in \Lambda(n,r)$ with $\mu_i = (\lambda_i - a_{1,i} - \cdots - a_{i'-1,i}) - x = 0$. Now, since $a_{i',i} - x > 0$, we have $\mathtt{m}_1\mathtt{k}_\mu = \mathtt{m}_1'\mathtt{e}_{i-1}\mathtt{k}_\mu = 0$ by part (1), and $\mathtt{e}^{(A)}\mathtt{k}_\lambda = 0$. $\square$

Let $\mathbf{S}_r^+$ (resp., $\mathbf{S}_r^-$) be the subalgebra of $\mathbf{S}_r$ generated by the $\mathtt{e}_i$ (resp., $\mathtt{f}_i$). Using PBW bases, [14, 2.5] gives a version of the following result; see also §6 below.

Corollary 4.11. *The algebra $\mathbf{S}_r^+$ (resp., $\mathbf{S}_r^-$) is spanned by the elements*

$$\{\mathtt{e}^{(A)} : A \in \Xi^+, \sigma(A) \leqslant r\} \quad (\textit{resp.}, \{\mathtt{f}^{(A)} : A \in \Xi^-, \sigma(A) \leqslant r\}).$$

Proof. If $\sigma(A^+) = \sum_i \sigma_i(A^+) > r$, then $\mathtt{e}^{(A)} = \sum_{\lambda\in\Lambda(n,r)} \mathtt{e}^{(A)}\mathtt{k}_\lambda = 0$ by 4.6 and 4.10(2). The result follows since $\mathbf{S}_r^+$ is spanned by all $\mathtt{e}^{(A)}$ with $A \in \Xi^+$, $\sigma(A) \leqslant r$, by 3.3. $\square$

For $A = (a_{i,j}) \in \Xi$, let $b_{i,j} = a_{n-j+1,n-i+1}$. So ${}^TA := (b_{i,j})$ is the matrix obtained by transposing A along its skew-diagonal. Thus, $\sigma_i'({}^TA) = \sigma_{n-i+1}(A)$. The following result is an application of the anti-automorphism γ in 4.2(2). Part (1) is a special case of [8, 7.9].

Corollary 4.12. *Let $\lambda \in \Lambda(n,r)$.*

(1) *If $\lambda_i = 0$ for some i with $1 \leqslant i \leqslant n-1$, then $\mathtt{k}_\lambda\mathtt{e}_i = \mathtt{f}_i\mathtt{k}_\lambda = 0$.*
(2) *More generally, if $A \in \Xi^+$ (resp., $A \in \Xi^-$) and $\lambda_i < \sigma_i'(A)$ for some i, then $\mathtt{k}_\lambda\gamma(\mathtt{e}^{({}^TA)}) = 0$ (resp., $\gamma(\mathtt{f}^{({}^TA)})\mathtt{k}_\lambda = 0$).*

Proof. Define $\lambda^{\mathrm{op}} = (\lambda_1^{\mathrm{op}}, \cdots, \lambda_n^{\mathrm{op}})$ by reversing the components of $\lambda = (\lambda_1, \cdots, \lambda_n)$ (i.e., $\lambda_i^{\mathrm{op}} = \lambda_{n-i+1}$). Then, $\gamma(\mathtt{k}_\lambda) = \mathtt{k}_{\lambda^{\mathrm{op}}}$, and (1) follows easily from 4.10(1). Since $\lambda_i < \sigma_i'(A)$ means $\lambda_{n-i+1}^{\mathrm{op}} < \sigma_{n-i+1}({}^TA)$, by 4.10(2), $\mathtt{e}^{({}^TA)}\mathtt{k}_{\lambda^{\mathrm{op}}} = 0 = \mathtt{k}_{\lambda^{\mathrm{op}}}\mathtt{f}^{({}^TA)}$. Now apply γ. $\square$

The elements $\gamma(\mathtt{e}^{({}^TA^+)})$ and $\gamma(\mathtt{f}^{({}^TA^-)})$ can be explicitly described as follows: for the jth row (reading to the left) $a_{j,n}, \cdots, a_{j,j+1}$ $(1 \leqslant j \leqslant n-1)$ of A^+, put

$$\mathtt{n}_j = (\mathtt{e}_j^{(a_{j,n})}\mathtt{e}_{j+1}^{(a_{j,n})} \cdots \mathtt{e}_{n-1}^{(a_{j,n})}) \cdots (\mathtt{e}_j^{(a_{j,j+2})}\mathtt{e}_{j+1}^{(a_{j,j+2})})\mathtt{e}_j^{(a_{j,j+1})}.$$

Similarly, for the j-th column (reading downwards) $a_{j+1,j}, \cdots, a_{n,j}$ of A^-, put

$$\mathtt{n}_j' = \mathtt{f}_j^{(a_{j+1,j})}(\mathtt{f}_{j+1}^{(a_{j+2,j})}\mathtt{f}_j^{(a_{j+2,j})}) \cdots (\mathtt{f}_{n-1}^{(a_{n,j})}\mathtt{f}_{n-2}^{(a_{n,j})} \cdots \mathtt{f}_j^{(a_{n,j})}).$$

Then $\gamma(\mathtt{e}^{({}^TA^+)}) = \mathtt{n}_{n-1}\mathtt{n}_{n-2} \cdots \mathtt{n}_1$ and $\gamma(\mathtt{f}^{({}^TA^-)}) = \mathtt{n}_1'\mathtt{n}_2' \cdots \mathtt{n}_{n-1}'$.

The following is the "little" version of 3.4, from which it follows.

Lemma 4.13. *Let $A = (a_{i,j}) \in \Xi^+$ and let $\mathtt{m} \in \mathbf{S}_r$ be any monomial in the $\mathtt{e}_i$. Then the product $\mathtt{m}\mathtt{e}^{(A)}$ is a linear combination of $\mathtt{e}^{(B)}$ with $B \in \Xi^+$ (and hence, of $\mathtt{e}^{(B)}\mathtt{k}_\lambda$ with $\lambda \in \Lambda(n,r)$, $B \in \Xi^+$), and $\deg(B) \leqslant \deg(\mathtt{m}) + \deg(A)$. In particular,*

$\mathtt{m}$ *itself can be written as a linear combination of* $\mathtt{e}^{(B)}\mathtt{k}_\lambda$ *with* $\lambda \in \Lambda(n,r)$, $B \in \Xi^+$ *and* $\deg(B) \leqslant \deg(\mathtt{m})$. *A similar result holds for the negative part of the algebra.*

For any $A \in \Xi^\pm$ and $\lambda \in \Lambda(n,r)$, let

$$\mathtt{m}^{(A,\lambda)} = \mathtt{e}^{(A^+)}\mathtt{k}_\lambda \mathtt{f}^{(A^-)}. \tag{4.13.1}$$

By 3.3, 4.6, and 4.10(2), $\mathbf{S}_r$ is spanned by all such $\mathtt{m}^{(A,\lambda)}$ with $\lambda \in \Lambda(n,r)$, and $A \in \Xi^\pm$ satisfying $\sigma(A^+) \leqslant r$ and $\sigma(A^-) \leqslant r$.

Theorem 4.14. *For* $A = (a_{i,j}) \in \Xi_r$, *let*

$$\mathtt{m}^{(A)} = \prod_{1\leqslant i\leqslant h<j\leqslant n} \mathtt{e}_h^{(a_{i,j})} \prod_{i=1}^{n} \begin{bmatrix} \mathtt{k}_i;0 \\ \lambda_i \end{bmatrix} \prod_{1\leqslant j\leqslant h<i\leqslant n} \mathtt{f}_h^{(a_{i,j})} = \mathtt{e}^{(A^+)}\mathtt{k}_\lambda \mathtt{f}^{(A^-)},$$

where $\lambda = \lambda(A) = (\sigma_1(A), \cdots, \sigma_n(A))$. *The set* $\mathfrak{M} = \{\mathtt{m}^{(A)}\}_{A\in\Xi_r}$ *is a spanning set for* $\mathbf{S}_r$.

Proof. Fix $B \in \Xi^\pm$ satisfying $\sigma(B^+) \leqslant r$ and $\sigma(B^-) \leqslant r$. Let $\lambda \in \Lambda(n,r)$. If $\lambda_i \geq \sigma_i(B)$ for all i, then there is a unique $A \in \Xi_r$ such that $\mathtt{m}^{(A)} = \mathtt{m}^{(B,\lambda)}$. Therefore, to prove the theorem, we must show that if $\lambda_i < \sigma_i(B)$ for some i, then $\mathtt{m}^{(B,\lambda)}$ lies in the span of $\mathfrak{M}$. We proceed by induction on $\deg(B)$; cf. (3.1.1). The result follows from 4.10 if $\deg(B) = 1$. Assume now that $\deg(B) > 1$, and suppose i is minimal with $\lambda_i < \sigma_i(B)$. Let B_i be the submatrix of B consisting of the first i rows and columns, and write $\mathtt{e}^{(B^+)} = \mathtt{m}_1\mathtt{e}^{(B_i^+)}$ and $\mathtt{f}^{(B^-)} = \mathtt{f}^{(B_i^-)}\mathtt{m}_1'$. Then

$$\mathtt{m}^{(B,\lambda)} = \mathtt{m}_1\mathtt{e}^{(B_i^+)}\mathtt{k}_\lambda\mathtt{f}^{(B_i^-)}\mathtt{m}_1'.$$

By 4.10(2), we can assume $\lambda_i \geqslant \sigma_i(B^+)$ (and so $\lambda_j \geqslant \sigma_j(B^+)$ for all $1 \leqslant j \leqslant i$ by the minimality assumption on i). Now 4.9(1) implies

$$\mathtt{m}^{(B,\lambda)} = \mathtt{m}_1(\mathtt{e}^{(B_i^+)}\mathtt{k}_\lambda)\mathtt{f}^{(B_i^-)}\mathtt{m}_1' = \mathtt{m}_1\mathtt{k}_{\lambda'}\mathtt{e}^{(B_i^+)}\mathtt{f}^{(B_i^-)}\mathtt{m}_1',$$

where $\lambda' = (\lambda_1', \cdots, \lambda_n')$ with $\lambda_i' = \lambda_i - (a_{1,i} + \cdots + a_{i-1,i}) = \lambda_i - \sigma_i(B_i^+) \geqslant 0$; cf. the last equation in (4.9.1). By remarks after 2.3,

$$\mathtt{e}^{(B_i^+)}\mathtt{f}^{(B_i^-)} = \mathtt{f}^{(B_i^-)}\mathtt{e}^{(B_i^+)} + f,$$

where f is a linear combination of monomials $\mathtt{m}_j^{\mathtt{e}}\mathtt{h}_j\mathtt{m}_j^{\mathtt{f}}$ with $\mathtt{h}_j \in \mathbf{S}_r^0$ and $\deg(\mathtt{m}_j^{\mathtt{e}}\mathtt{m}_j^{\mathtt{f}}) < \deg(B_i)$. Here, $\mathtt{m}_j^{\mathtt{e}}$ (resp., $\mathtt{m}_j^{\mathtt{f}}$) denotes a monomial in the $\mathtt{e}_i$ (resp., $\mathtt{f}_i$). Thus, $\deg(\mathtt{m}_1\mathtt{m}_j^{\mathtt{e}}\mathtt{m}_j^{\mathtt{f}}\mathtt{m}_1') < \deg(B)$. Since $\lambda_i' < \sigma_i(B_i^-)$, $\mathtt{m}_1\mathtt{k}_{\lambda'}\mathtt{f}^{(B_i^-)}\mathtt{e}^{(B_i^+)} = 0$ by 4.10. By 4.13, $\mathtt{m}_1\mathtt{m}_j^{\mathtt{e}}$ (resp., $\mathtt{m}_j^{\mathtt{f}}\mathtt{m}_1'$) is a linear combination of $\mathtt{e}^{(C)}\mathtt{k}_{\lambda''}$, $C \in \Xi^+$ (resp., $\mathtt{f}^{(C')}\mathtt{k}_{\lambda''}$, $C' \in \Xi^-$) with $\deg(C) \leqslant \deg(\mathtt{m}_1\mathtt{m}_j^{\mathtt{e}})$ (resp., $\deg(C') \leqslant \deg(\mathtt{m}_j^{\mathtt{f}}\mathtt{m}_1')$). Thus, each $\mathtt{m}_1\mathtt{k}_{\lambda'}\mathtt{m}_j^{\mathtt{e}}\mathtt{h}_j\mathtt{m}_j^{\mathtt{f}}\mathtt{m}_1'$ ($= \mathtt{m}_1\mathtt{m}_j^{\mathtt{e}}\mathtt{k}_{\lambda''}\mathtt{h}_j\mathtt{m}_j^{\mathtt{f}}\mathtt{m}_1'$) is a linear combination of $\mathtt{m}^{(B',\mu)}$ with $\deg(B') < \deg(B)$, since $\deg(\mathtt{m}_1\mathtt{m}_j^{\mathtt{e}}\mathtt{m}_j^{\mathtt{f}}\mathtt{m}_1') < \deg(B)$. By induction, $\mathtt{m}^{(B,\lambda)}$ is in the span of $\mathfrak{M}$. □

Note that all elements in $\mathfrak{M}$ are fixed under the involution $^-$ defined in 4.2(3).

5. The isomorphism between $\mathbf{S}_r$ and $\mathbf{U}_r$

Recall from 3.1(3) the surjective algebra homomorphism $\zeta_r : \mathbf{U} \twoheadrightarrow \mathbf{U}_r$. Since

$$\zeta_r(K_i) = \zeta_r(0(\mathbf{e}_i)) = 0(\mathbf{e}_i, r) = \sum_{\substack{D\in\Xi_r^0 \\ D=\mathrm{diag}(d_1,\cdots,d_n)}} v^{d_i}[D],$$

where $\mathbf{e}_i = (0, \cdots, 0, \underset{i}{1}, 0, \cdots, 0)$, and since $[D][D'] = \delta_{D,D'}[D]$, we obtain the following.

Lemma 5.1. *We have* $\zeta_r(K_1K_2\cdots K_n) = v^r 1$.

For any i, j, $1 \leqslant i \leqslant n, 0 \leqslant j \leqslant r$, let

$$\mathfrak{d}_i(j) = \sum_{D\in\Xi_r^0, d_i=j} [D];$$

so $\zeta_r(K_i) = \sum_{j=0}^r v^j \mathfrak{d}_i(j)$.

Lemma 5.2. *For any* $\mathbf{t} \in \mathbb{N}^n$, *let*

$$K(\mathbf{t}) = [K_1; t_1]^![K_2; t_2]^! \cdots [K_n; t_n]^!.$$

Then $\zeta_r(K(\mathbf{t})) = 0$ *whenever* $|\mathbf{t}| > r$.

Proof. Since $\zeta_r(K_i - v^d) = \sum_{j_i=0}^r (v^{j_i} - v^d)\mathfrak{d}_i(j_i)$, we have

$$\zeta_r([K_i; t_i]^!) = \sum_{j_i=0}^r (v^{j_i} - 1)(v^{j_i} - v) \cdots (v^{j_i} - v^{t_i-1})\mathfrak{d}_i(j_i) = \sum_{j_i=t_i}^r \prod_{l=0}^{t_i-1} (v^{j_i} - v^l)\mathfrak{d}_i(j_i).$$

Thus, if $|\mathbf{t}| > r$, the fact that $j_1 + \cdots + j_n \geqslant t_1 + \cdots + t_n > r$ implies

$$\zeta_r(K(\mathbf{t})) = \sum_{j_1\geqslant t_1,\cdots,j_n\geqslant t_n} \left(\prod_{i=1}^n \prod_{l=0}^{t_i-1} (v^{j_i} - v^l) \right) \mathfrak{d}_1(j_1)\cdots\mathfrak{d}_n(j_n) = 0,$$

since $\mathfrak{d}_1(j_1)\mathfrak{d}_2(j_2)\cdots\mathfrak{d}_n(j_n) = 0$, whenever $j_1 + j_2 + \cdots + j_n > r$. □

The following has already been obtained in [14, 2.10] using [9, (3.4.a)].

Corollary 5.3. *For any* $\mathbf{t} \in \mathbb{N}^n$, *let* $K_\mathbf{t} = \prod_{i=1}^n \left[\begin{smallmatrix} K_i;0 \\ t_i \end{smallmatrix}\right]$. *Then*

$$\zeta_r(K_\mathbf{t}) = \begin{cases} 0, & \text{if } |\mathbf{t}| > r, \\ [\mathrm{diag}(t_1, \cdots, t_n)], & \text{if } |\mathbf{t}| = r. \end{cases}$$

Proof. The case for $|\mathbf{t}| > r$ follows from 5.2. If $|\mathbf{t}| = r$, [9, (3.4.a)] implies that

$$\zeta_r \begin{bmatrix} K_i; 0 \\ t_i \end{bmatrix} = \sum_{j_i=0}^r \begin{bmatrix} j_i \\ t_i \end{bmatrix} \mathfrak{d}_i(j_i) = \sum_{j_i=t_i}^r \begin{bmatrix} j_i \\ t_i \end{bmatrix} \mathfrak{d}_i(j_i),$$

since $\left[\begin{smallmatrix} j \\ t \end{smallmatrix}\right] = 0$ for $0 \leqslant j < t$. Thus,

$$\zeta_r \left(\prod_{i=1}^n \begin{bmatrix} K_i; 0 \\ t_i \end{bmatrix} \right) = \sum_{j_1\geqslant t_1,\cdots,j_n\geqslant t_n} \prod_{i=1}^n \begin{bmatrix} j_i \\ t_i \end{bmatrix} \mathfrak{d}_1(j_1)\cdots\mathfrak{d}_n(j_n).$$

If $\mathfrak{d}_1(j_1)\cdots\mathfrak{d}_n(j_n) \neq 0$ here, then $j_i = t_i, \forall i$. Since $\mathfrak{d}_1(t_1)\mathfrak{d}_2(t_2)\cdots\mathfrak{d}_n(t_n) = [\mathrm{diag}(t_1, \cdots, t_n)]$, the result follows. □

Note that we actually have $\mathfrak{d}_1(j_1)\mathfrak{d}_2(j_2)\cdots\mathfrak{d}_{n-1}(j_{n-1}) = [\mathrm{diag}(j_1,\cdots,j_n)]$, where $j_n = r-(j_1+j_2+\cdots+j_{n-1})$. Thus, the proof above also shows that, for $\mathbf{t}\in\mathbb{N}^{n-1}$ with $|\mathbf{t}|\leqslant r$,

$$\zeta_r\left(\prod_{i=1}^{n-1}\begin{bmatrix}K_i;0\\ t_i\end{bmatrix}\right) = \sum_{j_1\geqslant t_1,\cdots,j_{n-1}\geqslant t_{n-1}}\prod_{i=1}^{n-1}\begin{bmatrix}j_i\\ t_i\end{bmatrix}\mathfrak{d}_1(j_1)\cdots\mathfrak{d}_{n-1}(j_{n-1})$$
$$= \sum_{\substack{\lambda\in\Lambda(n,r)\\ t_j\leqslant\lambda_j}}\prod_{i=1}^{n-1}\begin{bmatrix}\lambda_i\\ t_i\end{bmatrix}\zeta_r(K_\lambda),$$

yielding the $\mathbf{U}_r$ version of 4.7.1.

Theorem 5.4. *The algebra homomorphism $\zeta_r:\mathbf{U}\twoheadrightarrow\mathbf{U}_r$ induces an isomorphism $\mathbf{S}_r\cong\mathbf{U}_r$. Moreover, the set $\mathfrak{M}$ given in 4.14 forms a basis for $\mathbf{S}_r$.*

Proof. By 5.1 and 5.2, ζ_r induces a surjection $\bar\zeta_r:\mathbf{S}_r\twoheadrightarrow\mathbf{U}_r$. But 4.14 implies $\dim\mathbf{S}_r\leqslant\#\mathfrak{M}=\dim\mathbf{U}_r$; so $\bar\zeta_r$ is an isomorphism. □

We call $\mathfrak{M}$ a *monomial basis* for $\mathbf{S}_r$. Using the isomorphism $\mathbf{S}_r\cong\mathbf{U}_r$, we identify the generators $\mathtt{e}_i$, etc. with $E_{i,i+1}(\mathbf{0},r)$, etc. By 5.3,

$$\mathtt{k}_\lambda = [\mathrm{diag}(\lambda)] := [\mathrm{diag}(\lambda_1,\cdots,\lambda_n)],\quad\forall\lambda\in\Lambda(n,r). \tag{5.4.1}$$

Using [1, 5.4], we can identify the elements $(mE_{h,h+1})(\mathbf{0},r)$ with $\mathtt{e}_h^{(m)}$, and the elements $(mE_{h+1,h})(\mathbf{0},r)$ with $\mathtt{f}_h^{(m)}$.

Recall the Bruhat ordering on Ξ_r [1, 1.4]: Fix an algebraically closed field k, and let V be an r-dimensional vector space over k. As shown in [1, 1.1], the matrices $A\in\Xi_r$ correspond bijectively to $GL(V)$-orbits $\mathcal{O}_A$ of pairs of n-step filtrations of V. Then, given $A',A''\in\Xi_r$, $A'<A''$ means $\mathcal{O}_{A'}\subsetneq\overline{\mathcal{O}}_{A''}$. Here $\overline{\mathcal{O}}_{A''}$ denotes the Zariski closure of $\mathcal{O}_{A''}$. This partial ordering is independent of the algebraically closed field k.

We have the following identification of the monomial basis elements.

Theorem 5.5. *For any $A\in\Xi_r$, $\mathtt{m}^{(A)}$ is exactly the element defined in* [1, 3.9(a)]. *In particular, in $\mathbf{U}_r$ we have*

$$\mathtt{m}^{(A)} = [A] + \sum_{B\in\Xi_r,B<A} f_{B,A}[B]\quad(f_{B,A}\in\mathcal{Z}). \tag{5.5.1}$$

Proof. By 4.14 and 4.9,

$$\mathtt{m}^{(A)} = \mathtt{e}^{(A^+)}\mathtt{k}_\lambda\mathtt{f}^{(A^-)} = \mathtt{k}_{\lambda'}\mathtt{e}^{(A^+)}\mathtt{f}^{(A^-)} = \mathtt{e}^{(A^+)}\mathtt{f}^{(A^-)}\mathtt{k}_{\lambda''},$$

with $\lambda'=\lambda-co(A^+)+ro(A^+)$ and $\lambda''=\lambda+co(A^-)-ro(A^-)$. Since $\lambda=\lambda(A)=(\sigma_1(A),\cdots,\sigma_n(A))$, $\lambda'=ro(A)$ and $\lambda''=co(A)$. Recall that $\mathtt{e}^{(A^+)}$ (resp. $\mathtt{f}^{(A^-)}$) is a product of $\mathtt{e}_h^{(a_{i,j})}$ (resp. $\mathtt{f}_h^{(a_{i,j})}$) for all $1\leqslant i\leqslant h<j\leqslant n$ (resp. $1\leqslant j\leqslant h<i\leqslant n$), which are ordered as in [1, 3.9(a)] (cf. §3 and footnote 6 above). If $\nu\in\Lambda(n,r)$ and $D=\mathrm{diag}(\nu)$ are such that $\mathtt{e}_h^{(a_{i,j})}\mathtt{k}_\nu\neq 0$ then, by (3.0.3),

$$\mathtt{e}_h^{(a_{i,j})}\mathtt{k}_\nu = (a_{i,j}E_{h,h+1})(\mathbf{0},r)[D] = [a_{i,j}E_{h,h+1}+D']$$

for a unique $D'\in\Xi_r^0$ with $co(a_{i,j}E_{h,h+1}+D')=\nu$. A similar statement holds for any $\mathtt{f}_h^{(a_{i,j})}\mathtt{k}_\nu$. Thus, applying this and 4.9 (noting also that $\mathtt{k}_\nu^2=\mathtt{k}_\nu$) repeatedly from right to left beginning with $\mathtt{f}_{n-1}^{(a_{n,n-1})}\mathtt{k}_{\lambda''}$, where $\mathtt{f}_{n-1}^{(a_{n,n-1})}$ is the rightmost term of $\mathtt{f}^{(A^-)}$, we identify $\mathtt{m}^{(A)}$ with a product

$$P = \prod_{1\leqslant i\leqslant h<j\leqslant n} [D_{i,h,j} + a_{i,j}E_{h,h+1}] \times \prod_{1\leqslant j\leqslant h<i\leqslant n} [D_{i,h,j} + a_{i,j}E_{h+1,h}].$$

Here the order of the factors follows that for $\mathtt{e}^{(A^+)}$ and $\mathtt{f}^{(A^-)}$, and the diagonal matrices $D_{i,h,j} \in \Xi^0$ are inductively and uniquely determined by the conditions $co(D_{n,n-1,n-1} + a_{n,n-1}E_{n,n-1}) = \lambda'' = co(A)$ (cf. 5.4.1) and $co(X) = ro(Y)$ if $[X]$, $[Y]$ are two adjacent terms in P. However, P is identified in [1, 3.9(a)] as equaling an expression of the form given on the right-hand side of (5.5.1). $\square$

The following result will use the relation $\preccurlyeq$ on Ξ defined above 3.2. Observe that $\preccurlyeq$ does not involve diagonal entries: given $A, B \in \Xi$, put $A^\pm = A^+ + A^-$ and $B^\pm = B^+ + B^-$. Then $A \preccurlyeq B \iff A^\pm \preccurlyeq B^\pm$. In addition, [1, 3.6] states that for $A, B \in \Xi_r$, $A \leqslant B \implies A \preccurlyeq B$ and $A < B \implies A \prec B$. (For more results on these various poset structures, see [12, §5].)

Corollary 5.6. *Suppose* $\mathtt{m}^{(A,\lambda)} \neq 0$ *for some* $A \in \Xi^\pm$ *and* $\lambda \in \Lambda(n,r)$*. If there exists* $D \in \Xi^0$ *such that* $co(A + D) = \lambda + co(A^-) - ro(A^-)$*, then* $\mathtt{m}^{(A,\lambda)} = \mathtt{m}^{(A+D)}$*. Otherwise,*

$$\mathtt{m}^{(A,\lambda)} = \sum_{B\in\Xi_r, B\prec A} f_{B,A}\mathtt{m}^{(B)} \quad (f_{B,A} \in \mathbb{Q}(v)). \tag{5.6.1}$$

Proof. If $co(A+D) = \lambda + co(A^-) - ro(A^-)$ for some $D \in \Xi^0$, then $\lambda = \lambda(A+D)$, and the first assertion follows from the definition. Now suppose that no such $D \in \Xi^0$ exists. By 4.14 and 4.9, $\mathtt{m}^{(A,\lambda)} = \mathtt{e}^{(A^+)}\mathtt{f}^{(A^-)}\mathtt{k}_{\lambda''}$, with $\lambda'' = \lambda + co(A^-) - ro(A^-)$. By [1, 5.4(c)],

$$\mathtt{e}^{(A^+)}\mathtt{f}^{(A^-)} = A(\mathbf{0}, r) + \sum_{\substack{\mathbf{j}\in\mathbb{N}^n, B\in\Xi^\pm_{\leqslant r}\\ B\prec A}} f_{B,\mathbf{j},A}B(\mathbf{j}, r)$$

for $f_{B,\mathbf{j},A} \in \mathbb{Q}(v)$, where $\Xi^\pm_{\leqslant r} = \{A \in \Xi^\pm \mid \sigma(A) \leqslant r\}$. By (5.4.1), $[\mathrm{diag}(\lambda'')] = \mathtt{k}_{\lambda''}$, and so $A(\mathbf{0}, r)\mathtt{k}_{\lambda''} = 0$, since there is no $D \in \Xi^0$ satisfying $co(A+D) = \lambda + co(A^-) - ro(A^-)$. Therefore, $\mathtt{m}^{(A,\lambda)} = \sum f_{B,\mathtt{f},A}B(\mathbf{j}, r)\mathtt{k}_{\lambda''}$. But by (5.4.1) and (3.0.1) (see also (3.0.3)), $B(\mathbf{j}, r)\mathtt{k}_{\lambda''}$ is equal either to 0 or to $v^a[B + D]$ for some $a \in \mathbb{N}$ and some $D \in \Xi^0$. The corollary now follows from (5.5.1), using the fact, discussed above, that $A < B \implies A \prec B$. $\square$

Another monomial basis for $\mathbf{S}_r$ results by replacing the divided powers $\mathtt{e}_i^{(a)}$ by the ordinary powers $\mathtt{e}_i^a$. Also, if the $\mathtt{k}_\lambda$ in $\mathfrak{M}$ are replaced by the $\mathtt{k}_{\lambda'}$ defined in 4.7(2), then we obtain a new monomial basis. Monomial bases can be obtained by applying the anti-automorphism τ defined in 4.2 to the known monomial bases. However, we next show that $\mathfrak{M}$ is an integral basis.

6. Integral forms and PBW bases

Recall from §2 the various $\mathcal{Z}$-integral forms U^+, U^- and U^0. These subalgebras are all free over $\mathcal{Z}$; see 2.2 for a basis of U^0. For U^+ and U^-, the so-called PBW bases are described as follows. Given a reduced expression $w_0 = s_{i_1}s_{i_2}\cdots s_{i_\nu}$ for the longest word w_0 of $\mathfrak{S}_n$, let $\mathbf{i} = (i_1, i_2, \cdots, i_\nu)$. For any $\mathbf{c} = (c_1, \cdots, c_\nu) \in \mathbb{N}^\nu$, define

$$E_{\mathbf{i}}^{\mathbf{c}} = E_{i_1}^{(c_1)} \widetilde{T}_{i_1}(E_{i_2}^{(c_2)}) \widetilde{T}_{i_1} \widetilde{T}_{i_2}(E_{i_3}^{(c_3)}) \cdots \widetilde{T}_{i_1} \widetilde{T}_{i_2} \cdots \widetilde{T}_{i_{\nu-1}}(E_{i_\nu}^{(c_\nu)}),$$

where the $\widetilde{T}_i$ are the braid group actions on $\mathbf{U}$ [18, 1.3]. Then $\{E_{\mathbf{i}}^{\mathbf{c}}\}_{\mathbf{c}\in\mathbb{N}^\nu}$ is a $\mathcal{Z}$-basis for U^+. Let $F_{\mathbf{i}}^{\mathbf{c}} = \Omega(E_{\mathbf{i}}^{\mathbf{c}})$ (cf. [18, 1.3(d)] and (2.1.1)); so $\{F_{\mathbf{i}}^{\mathbf{c}}\}_{\mathbf{c}\in\mathbb{N}^\nu}$ is a $\mathcal{Z}$-basis for U^-.

In [19, 7.8], Lusztig established a relation between the monomial basis given in 3.3 and a PBW basis. To describe this, we choose the following reduced expression (see (3.2.1)):

$$w_0 = s_{i_1} s_{i_2} \cdots s_{i_\nu} = (s_{n-1}s_{n-2}\cdots s_1)(s_{n-1}s_{n-2}\cdots s_2)\cdots(s_{n-1}s_{n-2})s_{n-1}.$$

So $\mathbf{i} = (n-1, \cdots, 2, 1, \cdots, n-1, n-2, n-1)$. Let $\alpha_1, \cdots, \alpha_{n-1}$ be the standard list of simple roots of type A_{n-1}, and put $\beta_1 = \alpha_{n-1}$ and $\beta_k = s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k})$. Then, ordering from left to right down successive rows, we have the following listing of the positive roots:

$$\begin{aligned}\{\beta_1, \cdots, \beta_\nu\} = \{&\alpha_{n-1}, \alpha_{n-1}+\alpha_{n-2}\ , \quad \cdots,\ \alpha_{n-1}+\cdots+\alpha_1,\\ &\alpha_{n-2}, \alpha_{n-2}+\alpha_{n-3}, \cdots, \alpha_{n-2}+\cdots+\alpha_1,\\ &\cdots\\ &\alpha_2, \alpha_2+\alpha_1,\\ &\alpha_1\}.\end{aligned}$$

Write $\beta_k = \sum_{j=1}^{n-1} p_{jk}\alpha_j$, and define $\chi_{\mathbf{i}} : \mathbb{N}^\nu \to \mathbb{N}^{n-1}$ by

$$\mathbf{c} = (c_1, c_2, \cdots, c_\nu) \mapsto \chi_{\mathbf{i}}(\mathbf{c}) = \mathbf{d} = (d_1, \cdots, d_{n-1}), \quad \text{ where } d_j = \sum_{k=1}^{\nu} p_{jk}c_k.$$

Clearly, $\chi_{\mathbf{i}}(\mathbf{c}'+\mathbf{c}'') = \chi_{\mathbf{i}}(\mathbf{c}')+\chi_{\mathbf{i}}(\mathbf{c}'')$. Order the fibre $\chi_{\mathbf{i}}^{-1}(\mathbf{d})$ of $\chi_{\mathbf{i}}$ over $\mathbf{d}$ by setting

$$\mathbf{c}' <_{\dim} \mathbf{c}'' \iff \dim \mathcal{O}_{\mathbf{c}'} < \dim \mathcal{O}_{\mathbf{c}''},$$

where $\mathcal{O}_{\mathbf{c}}$ denotes the orbit of the quiver representation defined by $\mathbf{c}$ (see [19, p. 463]).

Given $\mathbf{c} \in \mathbb{N}^\nu$, define, for $1 \leqslant k \leqslant \nu$, $\mathbf{c}_k \in \mathbb{N}^\nu$ such that the entry in the k-th position is c_k, and 0 otherwise. (Thus, $\mathbf{c} = \sum_{k=1}^{\nu} \mathbf{c}_k$.) Put $\mathbf{d}^k = \chi_{\mathbf{i}}(\mathbf{c}_k)$, and define

$$E((\mathbf{c})) = E(\mathbf{d}^1)E(\mathbf{d}^2)\cdots E(\mathbf{d}^\nu), \quad \text{where } E(\mathbf{d}^k) = E_1^{(d_1^k)} E_2^{(d_2^k)} \cdots E_{n-1}^{(d_{n-1}^k)}.$$

Define a bijection $\kappa : \mathbb{N}^\nu \to \Xi^+$ by sending $\mathbf{c}$ to $A_{\mathbf{c}}^+ = (a_{i,j})$ so that the first $n-1$ components of $\mathbf{c}$ become the n-th column reading upwards, and the next $n-2$ components become the $(n-1)$-th column, and so on, i.e.,

$$c_1 = a_{n-1,n}, \cdots, c_{n-1} = a_{1,n-1}, c_n = a_{n-2,n-1}, \cdots,$$

and define $\kappa^- : \mathbb{N}^\nu \to \Xi^-$ similarly. Then we have $E^{(A_{\mathbf{c}}^+)} = E((\mathbf{c}))$. Define $F^{(A_{\mathbf{c}}^-)} = F((\mathbf{c})) := \Omega(E((\mathbf{c})))$.

The following result[9] appears in [19, 7.8(b)].

[9]This version is the result of applying the graph automorphism $E_i \mapsto E_{n-i+1}$ to Lusztig's version.

Lemma 6.1. *Let* $\mathbf{i} = (n-1, \cdots, 2, 1, \cdots, n-1, n-2, n-1)$ *and let* $\mathbf{c} \in \mathbb{N}^\nu$. *For any* $\mathbf{c}' \in \mathbb{N}^\nu$, *there exists* $h_{\mathbf{c},\mathbf{c}'} \in \mathcal{Z}$ *such that*

$$E((\mathbf{c})) = E_{\mathbf{i}}^{\mathbf{c}} + \sum_{\mathbf{c}' \in \chi_{\mathbf{i}}^{-1}(\mathbf{d}) : \mathbf{c}' <_{\dim} \mathbf{c}} h_{\mathbf{c},\mathbf{c}'} E_{\mathbf{i}}^{\mathbf{c}'}, \tag{6.1.1}$$

where $\mathbf{d} = \chi_{\mathbf{i}}(\mathbf{c})$. *A similar result holds for* $F((\mathbf{c}))$.

Remarks 6.2. (1) Using the language of quiver representations, $\mathbf{d} = \chi_{\mathbf{i}}(\mathbf{c})$ is the dimension vector of the quiver representation $V_{\mathbf{c}}$ corresponding to $\mathbf{c}$ (see [19, 4.15]). Thus, $\mathbf{c}' \in \chi_{\mathbf{i}}^{-1}(\mathbf{d})$ simply means that $V_{\mathbf{c}'}$ and $V_{\mathbf{c}}$ have the same dimension vector.

(2) By [4, 7.3(10)] and the remark [4, 7.4], (6.1.1) may be written

$$E((\mathbf{c})) = E_{\mathbf{i}}^{\mathbf{c}} + \sum_{\mathbf{c}' \in \chi_{\mathbf{i}}^{-1}(\mathbf{d}) : \mathcal{O}_{\mathbf{c}'} \underset{\neq}{\subseteq} \bar{\mathcal{O}}_{\mathbf{c}}} h_{\mathbf{c},\mathbf{c}'} E_{\mathbf{i}}^{\mathbf{c}'} = E_{\mathbf{i}}^{\mathbf{c}} + \sum_{\mathbf{c}' \in \chi_{\mathbf{i}}^{-1}(\mathbf{d}) : A_{\mathbf{c}'}^+ \prec A_{\mathbf{c}}^+} h_{\mathbf{c},\mathbf{c}'} E_{\mathbf{i}}^{\mathbf{c}'},$$

where $\bar{\mathcal{O}}_{\mathbf{c}}$ is the Zariski closure of $\mathcal{O}_{\mathbf{c}}$. Here we have used the fact, given in [2, 3.2] (see [12, 5.4] for details), that $\mathcal{O}_{\mathbf{c}'} \underset{\neq}{\subseteq} \bar{\mathcal{O}}_{\mathbf{c}}$ implies $A_{\mathbf{c}'}^+ \prec A_{\mathbf{c}}^+$.

Corollary 6.3. *The set* $\{E^{(A)}\}_{A \in \Xi^+}$ *(resp.,* $\{F^{(B)}\}_{B \in \Xi^-}$*) forms a basis for* U^+ *(resp.,* U^-*).*

By [9], $U_r = \zeta_r(U)$ is an integral $\mathcal{Z}$-form of $\mathbf{U}_r$, generated by $[D]$, $(mE_{h,h+1})(\mathbf{0}, r)$ and $(mE_{h+1,h})(\mathbf{0}, r)$ with $D \in \Xi_r^0, m \in \mathbb{N}$.

Theorem 6.4. *The* $\mathcal{Z}$*-subalgebra* U_r *is isomorphic to the subalgebra of* $\mathbf{S}_r$ *generated over* $\mathcal{Z}$ *by* $\mathtt{e}_i^{(m)}$, $\mathtt{f}_i^{(m)}$ $(m \in \mathbb{Z},\ 1 \leqslant i \leqslant n-1)$ *and* $\mathtt{k}_\lambda$ $(\lambda \in \Lambda(n,r))$. *Moreover, the set* $\mathfrak{M}$ *defined in 4.14 forms a* $\mathcal{Z}$*-basis for* U_r. *(It is called the monomial basis for* U_r.*)*

Proof. Since $\{[A]\}_{A \in \Xi_r}$ forms a $\mathcal{Z}$-basis for U_r, it follows from 5.5 that $\mathfrak{M}$ forms a $\mathcal{Z}$-basis of U_r. The first assertion follows as well. □

Recall from 4.14 that, for any $A \in \Xi_r$, $\lambda(A) = (\sigma_1(A), \cdots, \sigma_n(A))$. By 4.7(2) and its proof, we have another integral basis for U_r.

Corollary 6.5. *The set*

$$\mathfrak{M}' = \{\mathtt{e}^{(A^+)} \mathtt{k}'_{\lambda(A)} \mathtt{f}^{(A^-)} \mid A \in \Xi_r\}$$

forms a $\mathcal{Z}$*-basis for* U_r.

Let $\mathtt{e}_{\mathbf{i}}^{\mathbf{c}} = \zeta_r(E_{\mathbf{i}}^{\mathbf{c}})$ and $\mathtt{f}_{\mathbf{i}}^{\mathbf{c}} = \zeta_r(F_{\mathbf{i}}^{\mathbf{c}})$. For any $A \in \Xi_r$, let $\mathbf{c}(A^+) \in \mathbb{N}^\nu$ (resp., $\mathbf{c}(A^-) \in \mathbb{N}^\nu$) correspond to A^+ (resp., A^-) under the bijection κ (resp., κ^-) above. We now obtain the PBW-basis for U_r.

Theorem 6.6. *Let* $\mathbf{i} = (n-1, \cdots, 2, 1, \cdots, n-1, n-2, n-1)$. *Then the set*

$$\mathfrak{B}_{\mathbf{i}} = \{\mathtt{e}_{\mathbf{i}}^{\mathbf{c}(A^+)} \mathtt{k}_{\lambda(A)} \mathtt{f}_{\mathbf{i}}^{\mathbf{c}(A^-)} \mid A \in \Xi_r\}$$

forms a $\mathcal{Z}$*-basis for* U_r.

Proof. Using 6.1 and noting 4.11 and 6.3, we may write $\mathtt{e}_{\mathbf{i}}^{\mathbf{c}(A^+)} = \mathtt{e}^{(A^+)}$+lower terms and $\mathtt{f}_{\mathbf{i}}^{\mathbf{c}(A^-)} = \mathtt{f}^{(A^-)}$ + lower terms. Here the lower terms are relative to $\preccurlyeq$ by 6.2(2). By 6.4, the coefficients $f_{B,A}$ in 5.6 must lie in $\mathcal{Z}$. So (5.6.1) gives

$$\mathtt{e}_{\mathbf{i}}^{\mathbf{c}(A^+)} \mathtt{k}_{\lambda(A)} \mathtt{f}_{\mathbf{i}}^{\mathbf{c}(A^-)} = \mathtt{m}^{(A)} + \text{ lower terms (relative to } \preccurlyeq).$$

Now the assertion follows from 6.4. □

Corollary 6.7. *Maintain the notation used above. The set*

$$\mathfrak{B}'_{\mathbf{i}} = \{\mathtt{e}_{\mathbf{i}}^{\mathbf{c}(A^+)}\mathtt{k}'_{\lambda(A)}\mathtt{f}_{\mathbf{i}}^{\mathbf{c}(A^-)} \mid A \in \Xi_r\}$$

forms a $\mathcal{Z}$*-basis for* U_r*.*

7. The transfer maps $U_{n+r} \twoheadrightarrow U_r$

We define epimorphisms $U_{n+r} \twoheadrightarrow U_r$; these are the "transfer maps" in [21]. Let $\mathbf{k}_i$, $\mathbf{e}_i$ and $\mathbf{f}_i$ denote the generators for $\mathbf{U}_{n+r}$. Denote the monomial basis for $\mathbf{U}_{n+r}$ by $\{\mathbf{m}^{(A)}\}_{A\in\Xi_{n+r}}$.

Proposition 7.1. *There is a unique algebra epimorphism*

$$\psi = \psi_{n+r,r} : \mathbf{U}_{n+r} \twoheadrightarrow \mathbf{U}_r$$

satisfying $\psi(\mathbf{k}_i) = v\mathtt{k}_i$*,* $\psi(\mathbf{e}_i) = \mathtt{e}_i$ *and* $\psi(\mathbf{f}_i) = \mathtt{f}_i$*.*

Proof. It follows directly that ψ preserves the relations for $\mathbf{k}_i$, $\mathbf{e}_i$ and $\mathbf{f}_i$. □

The maps $\psi_{n+r,r}$ agree with the maps $\phi_{n+r,r}$, described (for both finite and affine cases) in [20, 9.1]. The existence of $\phi_{n+r,r}$ is proved in [21, 1.10] (cf. [10, 5.4(a)] for a dual treatment in the GL_n case). Our next result shows that $\mathfrak{M}$ shares a similar property with the canonical basis under the transfer maps. See the conjecture [20, 9.2] and a proof in [22].

Corollary 7.2. *The map* ψ *induces an epimorphism* $\psi : U_{n+r} \twoheadrightarrow U_r$*. More precisely,*

$$\psi(\mathbf{m}^{(A)}) = \begin{cases} \mathtt{m}^{(A-I_n)}, & \text{if } A - I_n \in \Xi_r, \\ 0, & \text{otherwise.} \end{cases}$$

Here I_n *denotes the* $n \times n$ *identity matrix.*

Proof. We first observe that, if $\lambda_i \geqslant 1$, then

$$\begin{bmatrix} v\mathtt{k}_i;0 \\ \lambda_i \end{bmatrix} = \frac{v\mathtt{k}_i - v^{-1}\mathtt{k}_i^{-1}}{v^{\lambda_i} - v^{-\lambda_i}} \begin{bmatrix} \mathtt{k}_i;0 \\ \lambda_i - 1 \end{bmatrix}.$$

Let $\mathbf{1} = (1, \cdots, 1) \in \mathbb{N}^n$. For any $\lambda \in \Lambda(n, n+r)$, if $\lambda - \mathbf{1} \in \Lambda(n, r)$, then $\lambda_i \geqslant 1$ for all i and, by 4.5(2),

$$\psi(\mathbf{k}_\lambda) = \prod_{i=1}^{n} \frac{v\mathtt{k}_i - v^{-1}\mathtt{k}_i^{-1}}{v^{\lambda_i} - v^{-\lambda_i}}\mathtt{k}_{\lambda-\mathbf{1}} = \mathtt{k}_{\lambda-\mathbf{1}}.$$

If $\lambda - \mathbf{1} \notin \Lambda(n, r)$, then we have clearly $\psi(\mathbf{k}_\lambda) = \mathtt{k}_\mu$ with $|\mu| = n + r - x > r$, where x is the number of i with $\lambda_i = 0$. Therefore, $\psi(\mathbf{k}_\lambda) = 0$. The rest of the proof is clear. □

8. Presentations for Borel subalgebras

For positive integers n and r, the symmetric group $\mathfrak{S}_r$ acts on the set

$$\mathcal{I} = \mathcal{I}(n,r) := \{(i_1, \cdots, i_r) \mid 1 \leqslant i_j \leqslant n\}$$

by place permutations, and then acts on $\mathcal{I} \times \mathcal{I}$ diagonally.

There is a bijection between Ξ_r and the set of all $\mathfrak{S}_r$-orbits in $\mathcal{I} \times \mathcal{I}$ defined as follows: If $A = (a_{i,j}) \in \Xi_r$ with $\lambda = ro(A)$, we let

$$\mathbf{i}_A = (\underbrace{1, \cdots, 1}_{\lambda_1 \text{ times}}, \cdots, \underbrace{n, \cdots, n}_{\lambda_n \text{ times}}) \quad \text{and} \quad \mathbf{j}_A = (\mathbf{j}_1, \cdots, \mathbf{j}_n),$$

where $\mathbf{j}_i = (\underbrace{1, \cdots, 1}_{a_{i,1} \text{ times}}, \cdots, \underbrace{n, \cdots, n}_{a_{i,n} \text{ times}})$. The map sending A to the orbit containing $(\mathbf{i}_A, \mathbf{j}_A)$ is a bijection.

We order $\mathcal{I}$ by setting $\mathbf{i} \leqslant \mathbf{j}$ if and only if $i_1 \leqslant j_1, \ldots, i_r \leqslant j_r$. Clearly, $A \in \Xi_r$ with $A^- = 0 \iff \mathbf{i}_A \leqslant \mathbf{j}_A$. It is known (see, e.g., [13, 1.3.3,5.6.1]) that the subspace $\mathbf{U}_r^{\geqslant 0}$ (resp., $\mathbf{U}_r^{\leqslant 0}$) spanned by all $[A]$ where $A \in \Xi_r$ with $A^- = 0$ (resp., $A^+ = 0$) is a subalgebra, called a Borel subalgebra. Clearly, we have the following dimension formula:

$$\dim \mathbf{U}_r^{\geqslant 0} = \dim \mathbf{U}_r^{\leqslant 0} = \binom{r + \binom{n+1}{2} - 1}{r} = \sum_{\lambda \in \Lambda(n,r)} \prod_{i=1}^{n} \binom{\lambda_i + i - 1}{i - 1}.$$

We now can state the following.

Theorem 8.1. *The subalgebra $\mathbf{U}_r^{\geqslant 0}$ is isomorphic to the algebra $\mathbf{B}$ with generators $\mathtt{e}_i, \mathtt{k}_i$, $(1 \leqslant i, j \leqslant n-1)$, subject to the following relations:*

(a) $\mathtt{k}_i \mathtt{k}_j = \mathtt{k}_j \mathtt{k}_i$;
(b) $[\mathtt{k}_1; t_1]^! [\mathtt{k}_2; t_2]^! \cdots [\mathtt{k}_{n-1}; t_{n-1}]^! = 0$, $\forall t_i \in \mathbb{N}, t_1 + \cdots + t_{n-1} = r + 1$;
(c) $\mathtt{e}_i \mathtt{e}_j = \mathtt{e}_j \mathtt{e}_i \quad (|i - j| > 1)$;
(d) $\mathtt{e}_i^2 \mathtt{e}_j - (v + v^{-1}) \mathtt{e}_i \mathtt{e}_j \mathtt{e}_i + \mathtt{e}_j \mathtt{e}_i^2 = 0$ *when* $|i - j| = 1$;
(e) $\mathtt{k}_i \mathtt{e}_j = v^{\epsilon(i,j)} \mathtt{e}_j \mathtt{k}_i$, *where* $\epsilon(i,i) = 1$, $\epsilon(i+1, i) = -1$ *and* $\epsilon(i,j) = 0$, *otherwise.*

A similar result holds for $\mathbf{U}_r^{\leqslant 0}$.

Proof. Identifying $\mathbf{U}_r$ with $\mathbf{S}_r$, it is clear that $\mathbf{U}_r^{\geqslant 0}$ is generated by the $\mathtt{e}_i$, $1 \leqslant i < n$, and the $\mathtt{k}_\lambda$, $\lambda \in \Lambda(n,r)$. Using 5.5, $\mathbf{U}_r^{\geqslant 0}$ has a basis consisting of terms $\mathtt{e}^{(A^+)} \mathtt{k}_\lambda$, with $\lambda \in \Lambda(n,r)$, and $\lambda_i \geq \sigma_i(A^+)$, $\forall i$. Temporarily denote the generators of $\mathbf{B}$ by $\mathtt{e}'_i$, $\mathtt{k}'_i$. Because 4.10(2) clearly holds for the algebra $\mathbf{B}$, it is obviously spanned by elements $\mathtt{e}'^{(A^+)} \mathtt{k}'_\lambda$ with A^+, etc., satisfying the same conditions for the basis vectors $\mathtt{e}^{(A^+)} \mathtt{k}_\lambda$ of $\mathbf{U}_r^{\geqslant 0}$. Hence, the natural algebra surjection $\mathbf{B} \twoheadrightarrow \mathbf{U}_r^{\geqslant 0}$ is an isomorphism by dimension considerations. $\square$

Let $U_r^{\geqslant 0}$ (resp., $U_r^{\leqslant 0}$) be the $\mathcal{Z}$-subalgebra of $\mathbf{U}_r$ generated by the $\mathtt{e}_i^{(m)}$ (resp., $\mathtt{f}_i^{(m)}$), $\mathtt{k}_\lambda$ with $m \in \mathbb{N}, \lambda \in \Lambda(n,r)$.

Corollary 8.2. *The set of all $\mathtt{e}^{(A)} \mathtt{k}_\lambda$ (resp., $\mathtt{k}_\lambda \mathtt{f}^{(A)}$), where $A \in \Xi^+$ (resp., $A \in \Xi^-$), $\lambda \in \Lambda(n,r)$ satisfying $\lambda_i \geqslant \sigma_i(A)$, $\forall i$, forms a $\mathcal{Z}$-basis for $U_r^{\geqslant 0}$ (resp., $U_r^{\leqslant 0}$).*

We have a further decomposition for U_r, $U_r^{\geqslant 0}$ and $U_r^{\leqslant 0}$. Let U_r^+ (resp., U_r^-, U_r^0) be the $\mathcal{Z}$-subalgebras of U_r generated by the $\mathtt{e}_i^{(m)}$ (resp., $\mathtt{f}_i^{(m)}$, $\mathtt{k}_\lambda$). Note that

$U_r^0 \otimes_{\mathcal{Z}} \mathbb{Q}(v) = \mathbf{U}_r^0 = \mathbf{S}_r^0$. A PBW basis version of the following is obtained in [14, 2.5-6].

Theorem 8.3. *The algebra U_r^+ (resp., U_r^-) is $\mathcal{Z}$-free with basis*

$$\{\mathtt{e}^{(A)} \mid A \in \Xi^+, \sigma(A) \leqslant r\} \quad (\textit{resp.}, \{\mathtt{f}^{(A)} \mid A \in \Xi^-, \sigma(A) \leqslant r\}).$$

Hence, $\dim \mathbf{S}_r^+ = \dim \mathbf{S}_r^- = \binom{N+r}{r}$, *where* $N = \frac{n(n-1)}{2}$. *Moreover, we have triangular decompositions:*

$$U_r = U_r^+ U_r^0 U_r^-, \quad U_r^{\geqslant 0} = U_r^+ U_r^0, \text{ and } U_r^{\leqslant 0} = U_r^0 U_r^-.$$

Proof. By 4.11, the two sets span U_r^+ and U_r^-, respectively. The linear independence is seen easily from 8.2 by writing $\mathtt{e}^{(A)} = \sum_{\lambda:\lambda_i \geqslant \sigma_i(A)} \mathtt{e}^{(A)}\mathtt{k}_\lambda$. □

Note that, by 4.12, the multiplication map from $U_r^+ \otimes U_r^0 \otimes U_r^-$ to U_r is no longer injective. Thus, there is no tensor product triangular decomposition in this case.

9. Identifying the monomial basis in H

In this section, assume that $n = r$ for simplicity; all results below are still valid for $r \leqslant n$. Let

$$\omega = (1, 1, \cdots, 1) \in \Lambda(n, n),$$

and let $\mathbf{H} = \mathtt{k}_\omega \mathbf{U}_n \mathtt{k}_\omega$ and $H = \mathtt{k}_\omega U_n \mathtt{k}_\omega$. By (5.4.1), $\mathtt{k}_\omega = [I_n]$, where I_n is the $n \times n$ identity matrix. It is well known (see, e.g., [6]) that H is isomorphic to the Hecke algebra defined in (1.0.1). However, in this section, we will not assume this identification, but rederive it from the monomial basis theory for q-Schur algebras. At a deeper and more interesting level, 9.4 explicitly identifies a basis $\mathfrak{M}_\omega \subset \mathfrak{M}$ of $H \subset U_n$ as a monomial basis involving certain monomials in the Kazhdan-Lusztig elements C'_s; see 9.6(2).

For $w \in \mathfrak{S}_n$, the permutation matrix $A_w \in \Xi_n$ is defined inductively by setting $A_w = A_y A_s$, where $w = ys$ with $y < w$ and $s = (i, i+1)$ for some i. Writing $\mathtt{m}^{(A_w)} = \mathtt{e}^{(A_w^+)}\mathtt{k}_\lambda \mathtt{f}^{(A_w^-)}$ as in 4.14, 4.9 implies that $\mathtt{e}^{(A_w^+)}\mathtt{k}_\lambda = \mathtt{k}_\omega \mathtt{e}^{(A_w^+)}$ and $\mathtt{k}_\lambda \mathtt{f}^{(A_w^-)} = \mathtt{f}^{(A_w^-)}\mathtt{k}_\omega$; so $\mathtt{m}^{(A_w)} \in \mathbf{H}$.

Proposition 9.1. *The algebra H is free over $\mathcal{Z}$ with basis $\mathfrak{M}_\omega = \{\mathtt{m}^{(A_w)} \mid w \in \mathfrak{S}_n\}$.*

Proof. For $A \in \Xi_n$, if $\mathtt{k}_\omega[A]\mathtt{k}_\omega \neq 0$, then $ro(A) = co(A) = \omega$. So A is necessarily a permutation matrix. Thus, $\mathfrak{M}_\omega \subseteq \mathtt{k}_\omega \mathfrak{M} \mathtt{k}_\omega$. By 5.5, if A is not a permutation matrix and $\mathtt{k}_\omega \mathtt{m}^{(A)} \mathtt{k}_\omega \neq 0$, then $\mathtt{k}_\omega \mathtt{m}^{(A)} \mathtt{k}_\omega$ is a linear combination of the elements of $\mathfrak{M}_\omega$. Therefore, $\mathfrak{M}_\omega$ forms a basis for H. □

We record the following simple commutation relations.

Lemma 9.2. *Let $\mathtt{m}$ be a monomial in the $\mathtt{f}_i$. For any $1 \leqslant i \leqslant n-2$, let $\partial_i(\mathtt{m}) = 2\deg_i(\mathtt{m}) - \deg_{i-1}(\mathtt{m}) - \deg_{i+1}(\mathtt{m})$, where $\deg_j$ denotes the degree of $\mathtt{f}_j$ in $\mathtt{m}$. Then we have the following.*

(1) *If $\partial_i(\mathtt{m}) = 1$, then $\begin{bmatrix} \tilde{\mathtt{k}}_i;0 \\ 1 \end{bmatrix} \mathtt{m}\mathtt{k}_\omega = -\mathtt{m}\mathtt{k}_\omega$.*

(2) *If $\mathtt{m} = \mathtt{m}_1 \mathtt{f}_i \mathtt{m}_2$ and $\partial_i(\mathtt{m}_2) = 0$, then $\mathtt{m}_1 \mathtt{f}_i \mathtt{e}_i \mathtt{m}_2 \mathtt{k}_\omega = \mathtt{m}_1 \mathtt{e}_i \mathtt{f}_i \mathtt{m}_2 \mathtt{k}_\omega$.*

Proof. Since $\begin{bmatrix} \tilde{\mathtt{k}}_i;0 \\ 1 \end{bmatrix} \mathtt{f}_i = \mathtt{f}_i \begin{bmatrix} \tilde{\mathtt{k}}_i;-2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} \tilde{\mathtt{k}}_i;0 \\ 1 \end{bmatrix} \mathtt{f}_j = \mathtt{f}_j \begin{bmatrix} \tilde{\mathtt{k}}_i;1 \\ 1 \end{bmatrix}$ for $j = i-1, i+1$, it follows that $\begin{bmatrix} \tilde{\mathtt{k}}_i;0 \\ 1 \end{bmatrix} \mathtt{m} = \mathtt{m} \begin{bmatrix} \tilde{\mathtt{k}}_i;-\partial_i(\mathtt{m}) \\ 1 \end{bmatrix}$. Since $\begin{bmatrix} \tilde{\mathtt{k}}_i;c \\ 1 \end{bmatrix} \mathtt{k}_\omega = c\mathtt{k}_\omega$ for $c = 0, -1$ and $\mathtt{f}_i \mathtt{e}_i = \mathtt{e}_i \mathtt{f}_i - \begin{bmatrix} \tilde{\mathtt{k}}_i;0 \\ 1 \end{bmatrix}$, the two assertions follow immediately. □

Let $s_i = (i, i+1)$ and put

$$\mathtt{C}_i = \mathtt{m}^{(A_{s_i})} = \mathtt{e}^{(A^+_{s_i})}\mathtt{k}_{\lambda(s_i)}\mathtt{f}^{(A^-_{s_i})} = \mathtt{e}_i\mathtt{k}_{\lambda(s_i)}\mathtt{f}_i = \mathtt{k}_\omega\mathtt{e}_i\mathtt{f}_i\mathtt{k}_\omega = \mathtt{k}_\omega\mathtt{f}_i\mathtt{e}_i\mathtt{k}_\omega,$$

where

$$\lambda(s_i) = \lambda(A_{s_i}) = (\sigma_1(A_{s_i}), \cdots, \sigma_n(A_{s_i})) = (1, \cdots, 1, \underset{(i)}{0}, 2, 1 \cdots, 1)$$

as defined in 4.14. For any i we have $\mathtt{k}_\omega\mathtt{e}_i = \mathtt{e}_i\mathtt{k}_{\lambda(s_i)}$ by 4.8, since $\lambda(s_i) = \omega + \alpha_i$.

Theorem 9.3. *The elements* $\mathtt{T}_i := \mathtt{C}_i - v^{-1}$, $1 \leqslant i \leqslant n-1$, *satisfy the following relations:*

(a) $(\mathtt{T}_i - v)(\mathtt{T}_i + v^{-1}) = 0$;
(b) $\mathtt{T}_i\mathtt{T}_j = \mathtt{T}_j\mathtt{T}_i$ *when* $|i-j| > 1$;
(c) $\mathtt{T}_i\mathtt{T}_{i+1}\mathtt{T}_i = \mathtt{T}_{i+1}\mathtt{T}_i\mathtt{T}_{i+1}$ *when* $1 \leqslant i \leqslant n-2$.

In particular, $\mathtt{T}_i$ *is invertible and* $\mathtt{T}_i^{-1} = \mathtt{C}_i - v$.

Proof. If $\mathtt{T}_i^{-1}$ has the required form, then (a) follows. So, to prove (a), we must show that

$$(\mathtt{C}_i - v^{-1})(\mathtt{C}_i - v) = 1.$$

This is clear, since

$$\begin{aligned}\mathtt{C}_i^2 &= (\mathtt{e}_i\mathtt{k}_{\lambda(s_i)}\mathtt{f}_i)(\mathtt{e}_i\mathtt{k}_{\lambda(s_i)}\mathtt{f}_i) = \mathtt{e}_i\mathtt{k}_{\lambda(s_i)}(\mathtt{e}_i\mathtt{f}_i - \frac{\tilde{\mathtt{k}}_i - \tilde{\mathtt{k}}_i^{-1}}{v - v^{-1}})\mathtt{k}_{\lambda(s_i)}\mathtt{f}_i \\ &= (v + v^{-1})\mathtt{e}_i\mathtt{k}_{\lambda(s_i)}\mathtt{f}_i = (v + v^{-1})\mathtt{C}_i.\end{aligned}$$

Here we use 4.5(2) and the fact that, since the i-th component of $\lambda(s_i)$ is 0, 4.12(1) implies $\mathtt{k}_{\lambda(s_i)}\mathtt{e}_i = 0$. The relation (b) follows easily from the relation $\mathtt{C}_i\mathtt{C}_j = \mathtt{C}_j\mathtt{C}_i$ whenever $|i-j| > 1$. We now prove (c). Since

$$\mathtt{T}_i\mathtt{T}_{i+1}\mathtt{T}_i = \mathtt{C}_i\mathtt{C}_{i+1}\mathtt{C}_i - \mathtt{C}_i - v^{-1}(\mathtt{C}_i\mathtt{C}_{i+1} + \mathtt{C}_{i+1}\mathtt{C}_i) + v^{-2}(\mathtt{C}_i + \mathtt{C}_{i+1}) + v^{-3},$$

and a similar formula holds with i and $i+1$ switched, it suffices to prove that

$$\mathtt{C}_i\mathtt{C}_{i+1}\mathtt{C}_i - \mathtt{C}_i = \mathtt{C}_{i+1}\mathtt{C}_i\mathtt{C}_{i+1} - \mathtt{C}_{i+1}. \tag{9.3.1}$$

For notational simplicity, put $C = \mathtt{C}_i\mathtt{C}_{i+1}\mathtt{C}_i$ and $C' = \mathtt{C}_{i+1}\mathtt{C}_i\mathtt{C}_{i+1}$. Then

$$C = \mathtt{k}_\omega(\mathtt{e}_i\mathtt{f}_i\mathtt{e}_{i+1}\mathtt{f}_{i+1}\mathtt{e}_i\mathtt{f}_i)\mathtt{k}_\omega, \quad C' = \mathtt{k}_\omega(\mathtt{e}_{i+1}\mathtt{f}_{i+1}\mathtt{e}_i\mathtt{f}_i\mathtt{e}_{i+1}\mathtt{f}_{i+1})\mathtt{k}_\omega.$$

Now, by 4.1,

$$\begin{aligned}\mathtt{e}_i\mathtt{f}_i\mathtt{e}_{i+1}\mathtt{f}_{i+1}\mathtt{e}_i\mathtt{f}_i &= \mathtt{e}_i\mathtt{e}_{i+1}(\mathtt{f}_i\mathtt{e}_i)\mathtt{f}_{i+1}\mathtt{f}_i = \mathtt{e}_i\mathtt{e}_{i+1}(\mathtt{e}_i\mathtt{f}_i - \begin{bmatrix}\tilde{\mathtt{k}}_i; 0\\ 1\end{bmatrix})\mathtt{f}_{i+1}\mathtt{f}_i \\ &= \mathtt{e}_i\mathtt{e}_{i+1}\mathtt{e}_i\mathtt{f}_i\mathtt{f}_{i+1}\mathtt{f}_i - \begin{bmatrix}\tilde{\mathtt{k}}_i; -1\\ 1\end{bmatrix}\mathtt{e}_i\mathtt{e}_{i+1}\mathtt{f}_{i+1}\mathtt{f}_i.\end{aligned}$$

By the quantum Serre relations,

$$\begin{aligned}\mathtt{e}_i\mathtt{e}_{i+1}\mathtt{e}_i\mathtt{f}_i\mathtt{f}_{i+1}\mathtt{f}_i &= (\mathtt{e}_i^{(2)}\mathtt{e}_{i+1} + \mathtt{e}_{i+1}\mathtt{e}_i^{(2)})(\mathtt{f}_i^{(2)}\mathtt{f}_{i+1} + \mathtt{f}_{i+1}\mathtt{f}_i^{(2)}) \\ &= \mathtt{e}_i^{(2)}\mathtt{e}_{i+1}(\mathtt{f}_i^{(2)}\mathtt{f}_{i+1} + \mathtt{f}_{i+1}\mathtt{f}_i^{(2)}) + \mathtt{e}_{i+1}\mathtt{e}_i^{(2)}(\mathtt{f}_i^{(2)}\mathtt{f}_{i+1} + \mathtt{f}_{i+1}\mathtt{f}_i^{(2)}).\end{aligned}$$

Since $\mathtt{k}_\omega\mathtt{e}_i^{(2)} = 0 = \mathtt{k}_\omega\mathtt{e}_{i+1}\mathtt{e}_i^{(2)}$ by 4.12(1), multiplying by $\mathtt{k}_\omega$ gives

$$C = -\mathtt{k}_\omega\begin{bmatrix}\tilde{\mathtt{k}}_i; -1\\ 1\end{bmatrix}\mathtt{e}_i\mathtt{e}_{i+1}\mathtt{f}_{i+1}\mathtt{f}_i\mathtt{k}_\omega = \mathtt{k}_\omega\mathtt{e}_i\mathtt{e}_{i+1}\mathtt{f}_{i+1}\mathtt{f}_i\mathtt{k}_\omega.$$

The calculation of C' can be done similarly by switching the subscripts i and $i+1$; so

$$\begin{aligned}\mathtt{e}_{i+1}\mathtt{e}_i\mathtt{e}_{i+1}\mathtt{f}_{i+1}\mathtt{f}_i\mathtt{f}_{i+1} &= (\mathtt{e}_{i+1}^{(2)}\mathtt{e}_i + \mathtt{e}_i\mathtt{e}_{i+1}^{(2)})(\mathtt{f}_{i+1}^{(2)}\mathtt{f}_i + \mathtt{f}_i\mathtt{f}_{i+1}^{(2)}) \\ &= \mathtt{e}_{i+1}^{(2)}\mathtt{e}_i(\mathtt{f}_{i+1}^{(2)}\mathtt{f}_i + \mathtt{f}_i\mathtt{f}_{i+1}^{(2)}) + \mathtt{e}_i\mathtt{e}_{i+1}^{(2)}\mathtt{f}_i\mathtt{f}_{i+1}^{(2)} + \mathtt{e}_i\mathtt{e}_{i+1}^{(2)}\mathtt{f}_{i+1}^{(2)}\mathtt{f}_i.\end{aligned}$$

However, in this case, only the first two summands vanish after multiplying by $\mathtt{k}_\omega$. Thus,

$$C' = \mathtt{k}_\omega(\mathtt{e}_i\mathtt{e}_{i+1}^{(2)}\mathtt{f}_{i+1}^{(2)}\mathtt{f}_i - \begin{bmatrix}\tilde{\mathtt{k}}_i;-1\\1\end{bmatrix}\mathtt{e}_{i+1}\mathtt{e}_i\mathtt{f}_i\mathtt{f}_{i+1})\mathtt{k}_\omega.$$

Now, by 2.3(7) and the last relation in 4.1, we have

$$\begin{aligned}&\mathtt{e}_i(\mathtt{e}_{i+1}^{(2)}\mathtt{f}_{i+1}^{(2)})\mathtt{f}_i \\ &\quad= \mathtt{e}_i(\mathtt{f}_{i+1}^{(2)}\mathtt{e}_{i+1}^{(2)} + \mathtt{f}_{i+1}\begin{bmatrix}\tilde{\mathtt{k}}_{i+1};-2\\1\end{bmatrix}\mathtt{e}_{i+1} + \begin{bmatrix}\tilde{\mathtt{k}}_{i+1};0\\2\end{bmatrix})\mathtt{f}_i \\ &\quad= \mathtt{e}_i\mathtt{f}_{i+1}^{(2)}\mathtt{e}_{i+1}^{(2)}\mathtt{f}_i + \begin{bmatrix}\tilde{\mathtt{k}}_{i+1};1\\2\end{bmatrix}\mathtt{e}_i\mathtt{f}_i + \begin{bmatrix}\tilde{\mathtt{k}}_{i+1};1\\1\end{bmatrix}\mathtt{e}_i(\mathtt{e}_{i+1}\mathtt{f}_{i+1} - \begin{bmatrix}\tilde{\mathtt{k}}_{i+1};0\\1\end{bmatrix})\mathtt{f}_i.\end{aligned}$$

The first two terms vanish after multiplying by $\mathtt{k}_\omega$. The last two terms equal

$$\begin{bmatrix}\tilde{\mathtt{k}}_{i+1};1\\1\end{bmatrix}\mathtt{e}_i\mathtt{e}_{i+1}\mathtt{f}_{i+1}\mathtt{f}_i - \begin{bmatrix}\tilde{\mathtt{k}}_{i+1};1\\1\end{bmatrix}^2\mathtt{e}_i\mathtt{f}_i.$$

On the other hand, we have

$$\mathtt{e}_{i+1}\mathtt{e}_i\mathtt{f}_i\mathtt{f}_{i+1} = \mathtt{e}_{i+1}(\mathtt{f}_i\mathtt{e}_i + \begin{bmatrix}\tilde{\mathtt{k}}_i;0\\1\end{bmatrix})\mathtt{f}_{i+1} = \mathtt{e}_{i+1}\mathtt{f}_i\mathtt{e}_i\mathtt{f}_{i+1} + \begin{bmatrix}\tilde{\mathtt{k}}_i;1\\1\end{bmatrix}\mathtt{e}_{i+1}\mathtt{f}_{i+1}.$$

Since $\mathtt{k}_\omega\mathtt{e}_{i+1}\mathtt{f}_i\mathtt{e}_i\mathtt{f}_{i+1}\mathtt{k}_\omega = 0$, we obtain after combining everything that

$$C' = \mathtt{k}_\omega\mathtt{e}_i\mathtt{e}_{i+1}\mathtt{f}_{i+1}\mathtt{f}_i\mathtt{k}_\omega - \mathtt{C}_i + \mathtt{C}_{i+1} = C - \mathtt{C}_i + \mathtt{C}_{i+1}.$$

This proves (9.3.1), and hence (c). □

Interestingly, the commuting relations 4.1(g) and the quantum Serre relations 4.1(d),(e) give rise to the braid relation 9.3(c). We also record the following relations on the elements $\mathtt{C}_i$ (cf. [8, (H1-3)] and [24, §2]):

(9.3.2)

$$\begin{cases}(1) \quad \mathtt{C}_i^2 = (v+v^{-1})\mathtt{C}_i; \\ (2) \quad \mathtt{C}_i\mathtt{C}_j = \mathtt{C}_j\mathtt{C}_i \text{ for } |i-j|>1; \\ (3) \quad \mathtt{C}_i\mathtt{C}_{i+1}\mathtt{C}_i - \mathtt{C}_i = \mathtt{C}_{i+1}\mathtt{C}_i\mathtt{C}_{i+1} - \mathtt{C}_{i+1} \text{ for } 1 \leqslant i \leqslant n-2.\end{cases}$$

For a permutation $y : i \mapsto y_i$ in $\mathfrak{S}_n$, let

$$I = I_y = \{i \mid i < y_i\}, \qquad J = J_y = \{j \mid j > y_j\}.$$

Since $\sum_{i=1}^n (i - y_i) = 0$, we have immediately

$$\deg(A_y^+) = \sum_{i\in I}(y_i - i) = \sum_{j\in J}(j - y_j) = \deg(A_y^-).$$

We fix an order on $I = \{i_1, \cdots, i_s\}$ $(s = \#I)$ such that $y_{i_1} > \cdots > y_{i_s}$ and an order on $J = \{j_1, \cdots, j_t\}$ $(t = \#J)$ such that $j_1 < \cdots < j_t$. For any $i \in I$ and $j \in J$ we put

$$\mathtt{e}_{[i,y_i)} = \mathtt{e}_i\mathtt{e}_{i+1}\cdots\mathtt{e}_{y_i-1}, \quad \mathtt{f}_{(j,y_j]} = \mathtt{f}_{j-1}\mathtt{f}_{j-2}\cdots\mathtt{f}_{y_j}.$$

Here for $a < b$ and $c > d$, we use the notation

$$[a, b) := \{a, a+1, \cdots, b-1\}, \qquad (c, d] := \{c-1, c-2, \cdots, d\}.$$

Then

$$\mathtt{e}^{(A_y^+)} = \mathtt{e}_{[i_1, y_{i_1})} \cdots \mathtt{e}_{[i_s, y_{i_s})}, \qquad \mathtt{f}^{(A_y^-)} = \mathtt{f}_{(j_1, y_{j_1}]} \cdots \mathtt{f}_{(j_t, y_{j_t}]}.$$

Theorem 9.4. *For any $w \in \mathfrak{S}_n$, there is a reduced expression $w = s_{i_1} \cdots s_{i_l}$ satisfying*

$$\mathtt{m}^{(A_w)} = \mathtt{C}_{i_1} \cdots \mathtt{C}_{i_l}.$$

Proof. For $i = 1, \cdots, n$, identify $\mathfrak{S}_i$ with the subgroup of $\mathfrak{S}_n$ generated by $s_1, \cdots, s_{i-1}$. We induct on the smallest integer m such that $w \in \mathfrak{S}_m$. If $m = 1$, then $w = 1$; so $\mathtt{m}^{(A_w)} = \mathtt{k}_\omega$, the identity element in H, and the result is clear. Thus, assume, for some $m > 1$, that $\mathtt{m}^{(A_z)} = \mathtt{C}_{i_1} \cdots \mathtt{C}_{i_l}$ whenever $z \in \mathfrak{S}_{m-1}$. For notational simplicity, we take $m = n$. It suffices to prove that if $z = ws_{n-1} \cdots s_i$ for some i, $i \leqslant n$ and some $w \in \mathfrak{S}_{n-1}$, then $\mathtt{m}^{(A_z)} = \mathtt{m}^{(A_w)}\mathtt{C}_{n-1} \cdots \mathtt{C}_i$.[10] In the case $i = n$, this means just that $z = w \in \mathfrak{S}_{n-1}$, and the result is true by the inductive hypothesis. So we proceed by downward induction on i.

Write $A_n = A_w$, and for $1 \leqslant i \leqslant n-1$, set

$$A_i = A_{i+1}A_{s_i} = A_n A_{s_{n-1}} \cdots A_{s_i}.$$

Since A_i is obtained from A_{i+1} by switching the ith and $(i+1)$th columns of A_i, the permutation matrix A_i has a 1 in the (n, i)-position. Also, the entries in the first $i-1$ columns of A_i and A_w agree identically. In addition, the ith, $\cdots, (n-1)$th-columns of A_w identify with the $(i+1)$th, $\cdots, n$th columns of A_i. Simply put, the matrix A_i is obtained from the matrix A_w by cyclically permuting to the right the last $n - i + 1$ columns of A_w.

Put $y = ws_{n-1} \cdots s_{i+1}$, so that $A_{i+1} = A_y$. Write $B = (b_{i,j}) = A_y$ for simplicity. We can assume that $\mathtt{m}^{(A_y)} = \mathtt{m}^{(A_w)}\mathtt{C}_{n-1} \cdots \mathtt{C}_{i+1}$. By the notation introduced before the statement of the theorem, $j_t = n$ ($t = \#J$) and $y_{j_t} = i + 1$. We must consider two cases:

Case 1. Suppose $i \neq y_a$ for all $a \in J$. Thus, the 1 appearing in column i of B does not appear in B^-. It is either on the diagonal or above the diagonal. That is, there exists $k \leqslant i$ such that $b_{k,i} = 1$.

Case 1a. If $i \notin (j_a, y_{j_a}]$ for all $a = 1, \cdots, t$, then $\mathtt{e}_i$ commutes with every factor $f_{(j, y_j]}$ appearing in $\mathtt{f}^{(B^-)}$, and hence

$$\mathtt{m}^{(B)}\mathtt{C}_i = \mathtt{k}_\omega \mathtt{e}^{(B^+)}\mathtt{f}^{(B^-)}\mathtt{e}_i\mathtt{f}_i\mathtt{k}_\omega = \mathtt{k}_\omega\mathtt{e}^{(B^+)}\mathtt{e}_i\mathtt{f}^{(B^-)}\mathtt{f}_i\mathtt{k}_\omega. \tag{9.4.1}$$

Since the 1 in the i-th column of B is not in B^-,

$$\mathtt{f}^{(B^-)}\mathtt{f}_i = \mathtt{f}^{(A_i^-)}. \tag{9.4.2}$$

If $k = i$, then $(A_i)_{i,i+1} = 1$ $(= (A_i)_{n,i})$ and

$$\mathtt{e}^{(B^+)}\mathtt{e}_i = \mathtt{e}_{[i_1, y_{i_1})} \cdots \mathtt{e}_{[i_a, y_{i_a})}\mathtt{e}_i\mathtt{e}_{[i_{a+1}, y_{i_{a+1}})} \cdots \mathtt{e}_{[i_s, y_{i_s})},$$

where $y_{i_{a+1}} < i \leqslant y_{i_a}$: if $i = y_{i_a}$, then the i-th column of B would have two 1's, which is impossible. Thus, $i < y_{i_a}$ and $\mathtt{e}^{(B^+)}\mathtt{e}_i = \mathtt{e}^{(A_i^+)}$. Now 9.4.2 gives $\mathtt{m}^{(B)}\mathtt{C}_i = \mathtt{m}^{(A_i)}$.

[10] Here we use the fact that if $w = t_1 \cdots t_u$ is a reduced expression for w, then $z = t_1 \cdots t_u s_{n-1} \cdots s_i$ is a reduced expression for z.

If $k < i$, then $i = y_a$ for some $a \in I$ and

$$\mathtt{e}^{(B^+)} = \mathtt{e}_{[i_1,y_{i_1})} \cdots \mathtt{e}_{[a,i)} \cdots \mathtt{e}_{[i_s,y_{i_s})}.$$

Since $\mathtt{e}_i$ commutes with all the factors on the right-hand side of $\mathtt{e}_{[a,i)}$ and $\mathtt{e}_{[a,i)}\mathtt{e}_i = \mathtt{e}_{[a,i+1)}$, it follows that $\mathtt{e}^{(B^+)}\mathtt{e}_i = \mathtt{e}^{(A_i^+)}$, and hence, combining 9.4.2, $\mathtt{m}^{(B)}\mathtt{C}_i = \mathtt{m}^{(A_i)}$.

Case 1b. If $i \in (j_a, y_{j_a}]$ for some $a \in J$ (perhaps more than one), then $j_a > i > y_{j_a}$. If b is the largest among those a's, the monomial $\mathtt{m} = \prod_{b<a<t} \mathtt{f}_{(j_a,y_{j_a}]}$ $(t = \#J)$ does not involve $\mathtt{f}_i$. Hence, $\mathtt{m}$ does not involve $\mathtt{f}_{i-1}$ and $\mathtt{f}_{i+1}$: if $\mathtt{f}_{i-1}$ is involved, then $j_a - 1 = i - 1$ and so $j_a = i$ for some $a > b$, but $j_b > i$, which is absurd since $j_b < j_a$; if $\mathtt{f}_{i+1}$ is involved, then $i+1 = j_{y_a}$ for some $b < a < t$ and so column $i+1$ of B has two 1's. Thus, by 4.1,

$$\begin{aligned}\prod_{b\leqslant a\leqslant t} \mathtt{f}_{(j_a,y_{j_a}]}\mathtt{e}_i\mathtt{f}_i &= \mathtt{f}_{(j_b,y_{j_b}]}\mathtt{e}_i \prod_{b<a\leqslant t} \mathtt{f}_{(j_a,y_{j_a}]}\mathtt{f}_i \\ &= \mathtt{f}_{j_b-1} \cdots (\mathtt{f}_i\mathtt{e}_i)\mathtt{f}_{i-1} \cdots \prod_{b<a\leqslant t} \mathtt{f}_{(j_a,y_{j_a}]}\mathtt{f}_i.\end{aligned}$$

Since $j_t = n$ and $y_{j_t} = i+1$, it follows that $\partial_i(\mathtt{f}_{i-1} \cdots \prod_{b<a\leqslant t} \mathtt{f}_{(j_a,y_{j_a}]}\mathtt{f}_i) = 0$. By 9.2(2), $\prod_{a\geqslant b} \mathtt{f}_{(j_a,y_{j_a}]}\mathtt{e}_i\mathtt{f}_i\mathtt{k}_\omega = \mathtt{e}_i \prod_{a\geqslant b} \mathtt{f}_{(j_a,y_{j_a}]}\mathtt{f}_i\mathtt{k}_\omega$. A similar argument, using induction and repeatedly applying 9.2, implies that $\mathtt{e}_i$ "commutes" with every $\mathtt{f}_{(j_a,y_{j_a}]}$ with $i \in (j_a, y_{j_a}]$ and $i < j_a - 1$. Finally, if $i = j_a - 1$ occurs, then for any $1 \leqslant b \leqslant a-1$ we have $i \notin (j_b, y_{j_b}]$. So, after switching $\mathtt{e}_i$ with $\mathtt{f}_{(j_a,y_{j_a}]}$, which is possible using 9.2(2) again, we may commute $\mathtt{e}_i$ with the rest of the product. This proves 9.4.1 in this case. The rest of the argument is entirely similar to the previous case. Therefore, we eventually obtain $\mathtt{m}_{(B)}\mathtt{C}_i = \mathtt{m}^{(A_i)}$.

Case 2. Suppose $i = y_j$ for some $j \in J$. Then the 1 in column i of B appears in B^-. So there exists $k \in J$, $k > i$, with $b_{k,i} = 1$. This implies that $B^+ = A_i^+$. Let $k = j_a$. For any $a < b \leqslant t$, if $i \in (j_b, y_{j_b}]$, then $i \neq y_{j_b}$. We claim that $i < j_b - 1$. Suppose the contrary: $i = j_b - 1$. Then $j_b = i+1$ and so $k = j_a < j_b = i+1$, which is absurd since $k > i$. Thus, from the argument in Case 1b, we see that

$$(\prod_{a<b\leqslant t} \mathtt{f}_{(j_b,y_{j_b}]})\mathtt{e}_i\mathtt{f}_i\mathtt{k}_\omega = \mathtt{e}_i(\prod_{a<b\leqslant t} \mathtt{f}_{(j_b,y_{j_b}]})\mathtt{f}_i\mathtt{k}_\omega.$$

Therefore, we have

$$\begin{aligned}\mathtt{f}^{(B^-)}\mathtt{e}_i\mathtt{f}_i\mathtt{k}_\omega &= \mathtt{f}_{(j_1,y_{j_1}]} \cdots \mathtt{f}_{(j_a,i]}\mathtt{e}_i \cdots \mathtt{f}_{(n,i+1]}\mathtt{f}_i\mathtt{k}_\omega \\ &= \mathtt{f}_{(j_1,y_{j_1}]} \cdots \mathtt{f}_{(j_a,i+1]}(\mathtt{e}_i\mathtt{f}_i - \begin{bmatrix}\tilde{\mathtt{k}}_i;0\\1\end{bmatrix}) \cdots \mathtt{f}_{(n,i+1]}\mathtt{f}_i\mathtt{k}_\omega \\ &= \mathtt{e}_i\mathtt{f}^{(B^-)}\mathtt{f}_i\mathtt{k}_\omega - \mathtt{f}_{(j_1,y_{j_1}]} \cdots \mathtt{f}_{(j_a,i+1]}\begin{bmatrix}\tilde{\mathtt{k}}_i;0\\1\end{bmatrix} \cdots \mathtt{f}_{(n,i+1]}\mathtt{f}_i\mathtt{k}_\omega.\end{aligned}$$

Here the first term requires a commutation between $\mathtt{f}_{(j_1,y_{j_1}]} \cdots \mathtt{f}_{(j_a,i+1]}$ and $\mathtt{e}_i$, which can be argued as above by 9.2.

We claim that $\mathtt{f}^{(B^-)}\mathtt{f}_i\mathtt{k}_\omega = 0$. Indeed, since $\mathtt{f}_i\mathtt{k}_\omega = \mathtt{k}_{\lambda(s_i)}\mathtt{f}_i$, it suffices to prove that $\mathtt{f}^{(B^-)}\mathtt{k}_{\lambda(s_i)} = 0$. We observe that the ith entry $\lambda_i^{(t)}$ of $\lambda^{(t)} := \lambda(s_i)$ is 0 and, if $\mathtt{f}_{(n,i+1]}\mathtt{k}_\lambda = \mathtt{k}_{\lambda^{(t-1)}}\mathtt{f}_{(n,i+1]}$, then, by 4.8, the ith entry $\lambda_i^{(t-1)}$ of $\lambda^{(t-1)}$ is also 0. Inductively, for any $a < b < t$, if $\mathtt{f}_{(j_b,y_{j_b}]}\mathtt{k}_{\lambda^{(b)}} = 0$, then we are done. Otherwise, we have $\mathtt{f}_{(j_b,y_{j_b}]}\mathtt{k}_{\lambda^{(b)}} = \mathtt{k}_{\lambda^{(b-1)}}\mathtt{f}_{(j_b,y_{j_b}]}$ such that $\lambda_i^{(b-1)} = 0$. This is seen as follows: if

$i \notin (j_b, y_{j_b}]$, then $i < y_{j_b}$ and $\lambda_i^{(b-1)} = \lambda_i^{(b)} = 0$; if $i \in (j_b, y_{j_b}]$, then $\mathtt{f}_{i+1}\mathtt{f}_i\mathtt{f}_{i-1}$ is a factor of $\mathtt{f}_{(j_b,y_{j_b}]}$, which guarantees $\lambda_i^{(b-1)} = 0$. In the worst case scenario, 4.12(1) implies $\mathtt{f}_{(k,i]}\mathtt{k}_{\lambda^{(a)}} = 0$, since $\lambda_i^{(a)} = 0$.

On the other hand, the argument above shows that $\prod_{a<b<t} \mathtt{f}_{(j_b,y_{j_b}]}$ has the same number of $\mathtt{f}_{i-1}$, $\mathtt{f}_i$ and $\mathtt{f}_{i+1}$; while $\mathtt{f}_{(n,i+1]}\mathtt{f}_i$ has one $\mathtt{f}_{i+1}$ and one $\mathtt{f}_i$. By 9.2(1),

$$\begin{bmatrix}\tilde{\mathtt{k}}_i;0\\1\end{bmatrix}\prod_{a<b\leqslant t}\mathtt{f}_{(j_b,y_{j_b}]}\mathtt{f}_i\mathtt{k}_\omega = -\prod_{a<b\leqslant t}\mathtt{f}_{(j_b,y_{j_b}]}\mathtt{f}_i\mathtt{k}_\omega.$$

Combining all these observations, we obtain

$$\mathtt{f}^{(B^-)}\mathtt{e}_i\mathtt{f}_i\mathtt{k}_\omega = \mathtt{f}_{(j_1,y_{j_1}]}\cdots\mathtt{f}_{(j_a,y_{j_a}+1]}\cdots\mathtt{f}_{(n,i+1]}\mathtt{f}_i\mathtt{k}_\omega = \mathtt{f}^{(A_i^-)}\mathtt{k}_\omega.$$

Since $B^+ = A_i^+$, we have proved that $\mathtt{m}^{(B)}\mathtt{C}_i = \mathtt{m}^{(A_i)}$. $\square$

Corollary 9.5. *The integral algebra H is generated over $\mathcal{Z}$ by the $\mathtt{C}_i$, and hence, by the $\mathtt{T}_i$. Therefore, H is isomorphic to the Hecke algebra $H(\mathfrak{S}_n)$* (1.0.1). *In particular, the $\mathtt{C}_i$ together with the relations* (9.3.2) *form a presentation of H.*

Proof. The first two assertions follow from 9.3 and 9.4. Thus, the generators and relations given in 9.3 form a presentation for H, easily giving the last assertion. $\square$

Remarks 9.6. (1) With this presentation, the monomial basis $\mathfrak{M}_\omega$ is a set of certain monomials in the $\mathtt{C}_i$.

(2) In the notation of [15], $\mathtt{T}_i$ corresponds to $\tilde{T}_{s_i} = v^{-1}T_{s_i}$ and $\mathtt{C}_i$ corresponds to C'_{s_i}.

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School of Mathematics, University of New South Wales, Sydney 2052, Australia
E-mail address: j.du@unsw.edu.au
URL: http://www.maths.unsw.edu.au/~jied

Department of Mathematics, University of Virginia, Charlottesville, Virginia 22904-4137
E-mail address: bjp8w@virginia.edu
URL: http://www.math.virginia.edu/~bjp8w