TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 355, Number 4, Pages 1401–1414
S 0002-9947(02)03207-5
Article electronically published on November 20, 2002

ON ONE-DIMENSIONAL SELF-SIMILAR TILINGS AND pq-TILES

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Abstract. Let $b \geq 2$ be an integer base, $\mathcal{D} = \{0, d_1, \cdots, d_{b-1}\} \subset \mathbb{Z}$ a digit set and $T = T(b, \mathcal{D})$ the set of radix expansions. It is well known that if T has nonvoid interior, then T can tile $\mathbb{R}$ with some translation set $\mathcal{J}$ (T is called a *tile* and $\mathcal{D}$ a *tile digit set*). There are two fundamental questions studied in the literature: (i) describe the structure of $\mathcal{J}$; (ii) for a given b, characterize $\mathcal{D}$ so that T is a tile.

We show that for a given pair $(b, \mathcal{D})$, there is a unique self-replicating translation set $\mathcal{J} \subset \mathbb{Z}$, and it has period b^m for some $m \in \mathbb{N}$. This completes some earlier work of Kenyon. Our main result for (ii) is to characterize the tile digit sets for $b = pq$ when p, q are distinct primes. The only other known characterization is for $b = p^l$, due to Lagarias and Wang. The proof for the pq case depends on the techniques of Kenyon and De Bruijn on the cyclotomic polynomials, and also on an extension of the *product-form digit set* of Odlyzko.

1. Introduction

Let T be a compact subset of $\mathbb{R}$ with $T = \overline{T^\circ}$. If there is a discrete set $\mathcal{J} \subset \mathbb{R}$ such that $T + \mathcal{J} = \bigcup_{t \in \mathcal{J}} (T + t) = \mathbb{R}$ and $\{T + t\}_{t \in \mathcal{J}}$ is an essentially disjoint family (i.e., $(T + t_1)^\circ \cap (T + t_2)^\circ = \emptyset$ for any distinct t_1, t_2 in $\mathcal{J}$), then we call T a *tile* (or a *prototile*), $\mathcal{J}$ a *translation set* and $(T, \mathcal{J})$ a *(translation) tiling* of $\mathbb{R}$. If, further, there is a $\lambda \neq 0$ such that $\mathcal{J} + \lambda = \mathcal{J}$, then we say that $(T, \mathcal{J})$ is a *periodic* tiling with period λ.

Let $b \geq 2$ be an integer, and let $\mathcal{D} = \{d_0, d_1, \cdots, d_{b-1}\}$ be a subset of $\mathbb{R}$ which we call a *digit set*. The pair $(b, \mathcal{D})$ defines an iterated function system $\{\phi_i\}_{i=0}^{b-1}$:

$$\phi_i(x) = b^{-1}(x + d_i), \qquad 0 \leq i \leq b - 1.$$

These maps are contractions, and there is a unique nonempty compact set $T = T(b, \mathcal{D})$ that satisfies the set equation $T = \bigcup_{i=0}^{b-1} \phi_i(T)$ [H]. An equivalent form of the set equation is

$$bT = \bigcup_{i=0}^{b-1} (T + d_i) = T + \mathcal{D}. \tag{1.1}$$

Received by the editors February 13, 2002 and, in revised form, September 11, 2002.
2000 *Mathematics Subject Classification.* Primary 52C20, 52C22; Secondary 42B99.
The authors are partially supported by an HKRGC grant and also a direct grant from CUHK. The second author is supported by CNSF 19901025.

More explicitly, we can express elements of T as radix expansions with base b and digits $d \in \mathcal{D}$, i.e.,

$$T = T(b, \mathcal{D}) = \{\sum_{k=1}^{\infty} b^{-k} x_k : x_k \in \mathcal{D}\}.$$

We call $T = T(b, \mathcal{D})$ a *self-similar tile* and $\mathcal{D}$ a *tile digit set* if $T^o \neq \emptyset$. The condition is equivalent to $\overline{T^o} = T$; it also equivalent to the Lebesgue measure of T being positive [LW1]. The justification for calling T a tile is due to the following fundamental theorem (see, e.g., [LW1, Theorem 1.2]).

Theorem 1.1 (Tiling Theorem). *If $T = T(b, \mathcal{D})$ is a self-similar tile, then there is a discrete set $\mathcal{J} \subseteq \mathbb{R}$ such that $(T, \mathcal{J})$ is a tiling of $\mathbb{R}$.*

The theorem is also true for $\mathbb{R}^d$, and an investigation of high-dimensional self-similar tilings can be found in [LW1] and the references there. In the case that the digit set $\mathcal{D}$ is a subset of $\mathbb{Z}$ and $T^o \neq \emptyset$, then we call T an *integral* self-similar tile. The investigations in [K2] and [LW3] showed that every 1-dimensional self-similar tile is an integral self-similar tile in essence: there is a real number c such that

$$\mathcal{D} = \{d_0, \cdots, d_{b-1}\} = c\{d'_0, \cdots, d'_{b-1}\},$$

where the d'_j are all integers. If $\mathcal{D} = c\mathcal{D}'$, then $T(b, \mathcal{D}) = cT(b, \mathcal{D}')$. Therefore the study of self-similar tilings on $\mathbb{R}$ can be reduced to integral self-similar tilings. From now on, we always assume, unless otherwise specified, that

$$\mathcal{D} \subset \mathbb{Z} \quad \text{and} \quad \text{g.c.d.}(\mathcal{D}) = 1.$$

If $\mathcal{D} = \mathcal{D}' + x$, then $T(b, \mathcal{D}) = T(b, \mathcal{D}') + \sum_{i=1}^{\infty} x b^{-i}$. Hence, without loss of generality, we assume that $0 \in \mathcal{D}$, and in §5 we assume in addition that $\mathcal{D} \subset \mathbb{Z}_+$, the set of nonnegative integers.

In the theory of self-similar tiles, there are two fundamental questions that have been studied extensively:

Q1. *If $T(b, \mathcal{D})$ is a tile, what is the structure of the translation set $\mathcal{J}$?*

Q2. *For which pair $(b, \mathcal{D})$ is the set $T(b, \mathcal{D})$ a tile?*

For the first question, a basic result is due to Kenyon [K2] (see also Lagarias and Wang [LW3]):

$$\text{(1.2)} \qquad \begin{array}{l} \text{For a tile } T(b, \mathcal{D}), \text{ there is a self-replicating translation set } \mathcal{J}, \\ \text{i.e., } \mathcal{J} = b\mathcal{J} + \mathcal{D}; \text{ any such tiling set is periodic.} \end{array}$$

For higher dimensions, the periodicity of tilings is still unsettled [LW2]. For self-similar tiles with standard digit sets $\mathcal{D} \subset \mathbb{Z}$ (to be defined in the following), Gröchenig and Haas [GH] proved that the translation set in the statement can actually be taken to be $\mathbb{Z}$, and a higher-dimensional analog of this is also true [LW4].

The second question is still largely unsettled. The most basic result is due to Bandt [B]:

If $\mathcal{D}$ is a complete residue set modulo b, then $T(b, \mathcal{D})$ is a tile.

We call such a digit set $\mathcal{D}$ a *standard digit set.* In particular, when b is a prime, then such standard digit sets characterize the tile digit sets [K1, p. 262]. For non-primes the most important class is the *product-form* digit sets (see §4), first introduced by Odlyzko [O] and later on extended by Lagarias and Wang [LW2].

If $\mathcal{D}$ is a product-form digit set, then $T(b, \mathcal{D})$ is a self-similar tile.

The above two assertions are also valid for higher dimensions. In [LW2], a modification of the product-form digit set is used to characterize the tile digit sets in $\mathbb{R}$ for $b = p^l$, a prime power.

In this paper we will investigate these two fundamental questions further. First we give a complete answer to Q1 on the existence, uniqueness and periodicity of the self-replicating tiling.

Theorem 1.2. *Assume* $0 \in \mathcal{D} \subset \mathbb{Z}$ *and* $\text{g.c.d.}(\mathcal{D}) = 1$. *If* $T = T(b, \mathcal{D})$ *is a self-similar tile, then*

(i) there exists a self-replicating translation set $\mathcal{J} \subseteq \mathbb{Z}$, *and it is the unique such set contained in* $\mathbb{Z}$;

(ii) $\mathcal{J}$ *is periodic with a period* b^m *for some* m;

(iii) if $\mathcal{S} \subseteq \mathbb{Z}$ *is periodic and* $\mathcal{S} = b\mathcal{S} + \mathcal{D}$, *then* $(T, \mathcal{S})$ *is a tiling and* $\mathcal{S} = \mathcal{J}$.

One might say here concerning (i) that there may exist other self-replicating translation sets $\mathcal{J}$ not contained in $\mathbb{Z}$, see Remark 3.2. Theorem 1.2 is proved in §3 as Theorem 3.1. For the tile digit sets, we give two sufficient conditions: a condition which covers the cases not included in the product-form digit sets, and another condition which relaxes the product-form digit sets.

Theorem 1.3. *Assume* $\mathcal{D} = \{0, d_1, \cdots, d_{b-1}\} \subset \mathbb{Z}$ *and* $\text{g.c.d.}(\mathcal{D}) = 1$. *If* $d_i \not\equiv 0 \pmod{b}$ *for all* $1 \leq i \leq b-1$, *then* $T(b, \mathcal{D})$ *is a self-similar tile if and only if* $\mathcal{D}$ *is a standard digit set.*

Theorem 1.3 is proved in §4 as Theorem 4.1. The condition $d_i \not\equiv 0 \pmod{b}$ in the theorem includes those $\mathcal{D}$ that are typically non-product-form digit sets (§4). It provides a very simple way to check that a digit set is *not* a tile digit set (we guess the generic cases of tiling sets come from some sort of product-form digit sets). The proof makes use of the period b^m of $\mathcal{J}$ in Theorem 1.2.

We define the weak product-form digit set in §4 as an extension of the product-form digit set.

Theorem 1.4. *If* $\mathcal{D}$ *is a weak product-form digit set, then* $T(b, \mathcal{D})$ *is a tile.*

Theorem 1.4 is proved as Theorem 4.7. For $b = p^l$, Lagarias and Wang [LW2] gave a characterization of the tile digit sets $\mathcal{D}$. However the expression of the $\mathcal{D}$ is quite complicated. We remark that their characterization is actually related to the weak product-form digit set considered here. As was pointed out in [LW2], it will be more difficult when the base b is not a prime power. Our next result is a major step in this direction:

Theorem 1.5. *Let* $b = pq$, *where* p, q *are distinct primes, let* $\mathcal{D} = \{0, d_1, \ldots, d_{b-1}\} \subset \mathbb{Z}_+$ *and* $\text{g.c.d.}(\mathcal{D}) = 1$. *Then* $\mathcal{D}$ *is a tile digit set if and only if* $\mathcal{D}$ *is of weak product-form. Explicitly,* $\mathcal{D} \equiv \mathcal{E}_0 + b^{k-1}\mathcal{E}_1 \pmod{b^k}$ *for some* $k > 0$, *where* $\mathcal{E}_0 = \{0, \ldots, q-1\}$, $\mathcal{E}_1 = \{0, q, \ldots, q(p-1)\}$ *(or the other way around).*

Theorem 1.5 is proved as Theorem 5.1. The proof is based on Kenyon's chararacterization of tile digit sets using the roots of unity, and also De Bruijn's factorization theorem concerning the cyclotomic polynomials of order $p^\alpha q^\beta$. We are still not able to prove the theorem for a more general base b.

For the organization of the paper, in §2 we give a simple proof of the existence of a self-replicating, periodic tiling, improving the original proof in the literature ([K2]). In §3 we show that there is a unique tiling contained in $\mathbb{Z}$ which necessarily has period b^m for some m (Theorem 1.2). This is used in §4 to give a new characterization of $\mathcal{D}$ being a tile digit set. Also in §4, we introduce and discuss the notion of weak product-form digit set. In §5, we use the weak product-form digit set to give a characterization of the tile digit sets with respect to the base $b = pq$. Finally, we make some remarks and raise some open problems in §6.

2. Existence of self-replicating tiling

We first give a useful lemma on the uniqueness of a tiling set. Let $T \subset \mathbb{R}$ be a bounded prototile. We say an open interval (α, β) is a *gap* of T if $\alpha, \beta \in T$ and $(\alpha, \beta) \cap T = \emptyset$.

Lemma 2.1 (Uniqueness Lemma). *Let T be a bounded prototile of $\mathbb{R}$, and let $(T, \mathcal{J})$, $(T, \mathcal{J}')$ be two tilings. Suppose $\mathcal{P} \subseteq (\mathcal{J} \cap \mathcal{J}')$ and $\mathcal{P} + T$ contains an interval whose length is strictly greater than the length of the largest gap in T. Then $\mathcal{J} = \mathcal{J}'$.*

Proof. The main idea of the proof comes from [K2, Lemma 4]. We show that $\mathcal{P}$ has a *unique* extension as a translation set and it must equal $\mathcal{J}$. This will imply $\mathcal{J} = \mathcal{J}'$.

First we observe that $\mathcal{P}$ is contained in a translation set (say $\mathcal{J}$). Let $I \subset \mathcal{P} + T$ be the interval as in the hypothesis; furthermore we assume that $I = [\alpha, \beta]$ is a connected component of $\mathcal{P} + T$. Let v denote the rightmost end point of T and let $t_1 = \alpha - v$. Then $t_1 + T$ has α as the rightmost endpoint.

We claim that $t_1 \in \mathcal{J} \setminus \mathcal{P}$. Since T is a tile, it has nonempty interior. Let J be a maximal interval contained in T. Because α is in the boundary of $\mathcal{P} + T$, we can find a point c such that $\alpha - |J| < c < \alpha$ and $c \notin \mathcal{P} + T$. Since $\mathcal{P}$ is contained in a translation set, we can find a $t_2 \in \mathcal{J}$ such that $c \in t_2 + T$. Note that $t_2 + T^\circ$ and $\mathcal{P} + T^\circ$ are disjoint by the tiling property of $(T, \mathcal{J})$, and thus $t_2 \notin \mathcal{P}$. In the following we prove that $t_1 = t_2$.

If $t_2 < t_1$, then $t_2 + |J| \le t_1$ (otherwise $t_1 + T$ and $t_2 + T$ will overlap, following from the maximal property of J). Since $t_1 + T$ is on the left side of α, this implies the rightmost point of $t_2 + T$ is on the left side of $\alpha - |J|$, and thus c is not covered by $t_2 + T$.

Therefore we must have $t_2 > t_1$. This means the rightmost point of $t_2 + T$, say α', is on the right side of α. Moreover, $\alpha' \notin I$, because $t_2 + T$ and $\mathcal{P} + T$ are disjoint. Thus α' must be on the right side of β. Now by the gap condition of $\mathcal{P} + T$, the leftmost point of $t_2 + T$ must also be on the right side of β, which again implies $t_2 + T$ cannot cover c.

Hence $t_1 = t_2$, and the claim is proved. We can use the same argument on β, the right endpoint of I, and show that there exists a unique $s_1 \in \mathcal{J} \setminus \mathcal{P}$ such that $s_1 + T$ has β as leftmost endpoint. We let $\mathcal{P}_1 = \mathcal{P} \cup \{s_1, t_1\}$. Then $\mathcal{P}_1 \subset \mathcal{J}$, and $\mathcal{P}_1 + T$ tiles an interval I_1 properly containing I. By repeating the process inductively, we can extend $\mathcal{P}$ to a translation set of T, and it must be $\mathcal{J}$. This completes the proof. $\square$

Suppose $T(b, \mathcal{D})$ is a self-similar tile and $(T, \mathcal{J})$ is a tiling. We say that $(T, \mathcal{J})$ is a *self-replicating tiling* (and $\mathcal{J}$ a *self-replicating translation set*) if $\mathcal{J} = b\mathcal{J} + \mathcal{D}$. For a self-similar tile of $\mathbb{R}$, the translation set $\mathcal{J}$ can be chosen to be self-replicating and periodic. This fact has been used in the literature ([K2], [LW1]); however, the proof is not explicit. In Proposition 2.2 we give a proof of this for completeness. We will then give a stronger version of this result in Theorem 3.1.

We will use the following notation: $\mathcal{D}_0 = \{0\}$ and $\mathcal{D}_n = b\mathcal{D}_{n-1} + \mathcal{D}$. Then $\mathcal{D}_{n-1} \subseteq \mathcal{D}_n$ because $0 \in \mathcal{D}$; also

$$\mathcal{D}_n = b^{n-1}\mathcal{D} + \cdots + b\mathcal{D} + \mathcal{D} \qquad \text{and} \qquad b^n T = T + \mathcal{D}_n. \tag{2.1}$$

Notice that if T is a self-similar tile, then $T + \mathcal{D}_n$ is an essentially disjoint union of copies of T.

Proposition 2.2. *Assume $0 \in \mathcal{D} \subset \mathbb{R}$. If $T(b, \mathcal{D})$ is a self-similar tile, then there is a self-replicating, periodic tiling set $\mathcal{J}$ with $0 \in \mathcal{J}$.*

Proof. We first construct a special translation set $\tilde{\mathcal{J}}$. Let $\mathcal{J}_0 = \{0\}$. Since T has nonvoid interior, we can choose m_1 large enough so that $b^{m_1}T$ contains a ball $B(v_1; 2)$ with center $v_1 \in \mathcal{D}_{m_1}$ and radius 2. Let $\mathcal{J}_1 = \mathcal{D}_{m_1} - v_1$; then $\mathcal{J}_0 \subseteq \mathcal{J}_1$, and $T + \mathcal{J}_1$ covers the ball $B(0; 2)$.

Observe that for $m > m_1$, $b^m T = b^{m_1}T + b^{m_1}\mathcal{D}_{m-m_1}$. By the same argument as above we can choose $m = m_2$ large enough so that $b^{m_2}T$ contains a ball $B(v_2; 4+v_1)$ with center $v_2 \in b^{m_1}\mathcal{D}_{m_2-m_1}$. Let $\mathcal{J}_2 = D_{m_2} - v_2 - v_1$; then

$$B(0; 4) \subset b^{m_2}T - v_1 - v_2 = T + (\mathcal{D}_{m_2} - v_1 - v_2) = T + \mathcal{J}_2.$$

It follows from $\mathcal{D}_{m_2} = b^{m_1}\mathcal{D}_{m_2-m_1} + \mathcal{D}_{m_1}$ (by (2.1)) and the choice of v_2 that

$$\mathcal{J}_1 = \mathcal{D}_{m_1} - v_1 \subset \mathcal{D}_{m_2} - v_2 - v_1 = \mathcal{J}_2.$$

By repeating this process, we obtain a sequence of discrete sets $\mathcal{J}_k$ with $\mathcal{J}_k \subset \mathcal{J}_{k+1}$ such that each $T + \mathcal{J}_k$ tiles the ball $B(0; 2^k)$. Therefore if we let $\tilde{\mathcal{J}} = \bigcup_{k=0}^{\infty} \mathcal{J}_k$, then $(T, \tilde{\mathcal{J}})$ will tile the real line.

Now to construct the self-replicating translation set, we choose n large enough so that $\mathcal{D}_n = \mathcal{P}$ satisfies the "gap condition" in Lemma 2.1. Note that from the above construction, we have $\mathcal{D}_n - v \subset \tilde{\mathcal{J}}$ for some v. Let $\mathcal{J} = \tilde{\mathcal{J}} + v$; then $\mathcal{D}_n \subset \mathcal{J}$. That $(bT, b\mathcal{J})$ is a tiling implies that $(T, b\mathcal{J} + \mathcal{D})$ is a tiling. Since $0 \in \mathcal{D}$, we have $\mathcal{D}_n \subset \mathcal{D}_{n+1} = b\mathcal{D}_n + \mathcal{D} \subset b\mathcal{J} + \mathcal{D}$; hence

$$\mathcal{D}_n \subset (\mathcal{J} \cap (b\mathcal{J} + \mathcal{D})).$$

It follows from Lemma 2.1 that $\mathcal{J} = b\mathcal{J} + \mathcal{D}$, and hence $(T, \mathcal{J})$ is self-replicating.

Now we are going to show that $\mathcal{J}$ is periodic. Pick any $0 \neq d \in \mathcal{D}$. By using the self-replicating property of $\mathcal{J}$ we have

$$b^n d + \mathcal{D}_n \subseteq \mathcal{D}_{n+1} = b\mathcal{D}_n + \mathcal{D} \subseteq b\mathcal{J} + \mathcal{D} = \mathcal{J}.$$

Let $\mathcal{J}' = \mathcal{J} - b^n d$. Then $\mathcal{D}_n \subset (\mathcal{J} \cap \mathcal{J}')$ and, by Lemma 2.1, $\mathcal{J} = \mathcal{J}' = \mathcal{J} - b^n d$. Thus $b^n d$ is a period of $\mathcal{J}$. $\square$

3. Uniqueness and periodicity

We will strengthen the result of Kenyon [K2] in Proposition 2.2 as follows.

Theorem 3.1. *Let $0 \in \mathcal{D} \subset \mathbb{Z}$ and* g.c.d.$(\mathcal{D}) = 1$. *If $T = T(b, \mathcal{D})$ is a self-similar tile, then:*

(i) There exists a unique self-replicating translation set $\mathcal{J}$ with the property that $\mathcal{J} \subseteq \mathbb{Z}$. The set $\mathcal{J}$ contains 0 and is periodic with period b^m for some m.

(ii) If $\mathcal{S} \subseteq \mathbb{Z}$ is periodic and $\mathcal{S} = b\mathcal{S} + \mathcal{D}$, then $(T, \mathcal{S})$ is a tiling and $\mathcal{S} = \mathcal{J}$.

Proof. (i) The existence of a self-replicating, periodic tiling set $\mathcal{J}$ follows from Proposition 2.2. Let m be the smallest integer such that $\mathcal{D}_m + T$ satisfies the gap condition of the uniqueness lemma (Lemma 2.1). We will first prove that $\mathcal{J}$ has period b^m. Indeed, let $d \in \mathcal{D}$ and $d \neq 0$. By the proof in Proposition 2.2, we see that $b^m d$ is a period of $\mathcal{J}$. Since $d \in \mathcal{D}$ is arbitrary and g.c.d.$(\mathcal{D}) = 1$, we see that b^m is also a period of $\mathcal{J}$.

To prove the uniqueness we let $\mathcal{J}_1 \subseteq \mathbb{Z}$ be any self-replicating translation set. Pick any $z \in \mathcal{J}_1$; by the self-replicating property, $b^k z + \mathcal{D}_k \subseteq \mathcal{J}_1$ for any $k > 0$. Note that $\mathcal{D}_k \subset \mathcal{J}$. Since $\mathcal{J}$ has period b^m, we can choose $k = m$ and then $b^k z + \mathcal{D}_k \subseteq \mathcal{J} \cap \mathcal{J}_1$, by using $z \in \mathbb{Z}$. The gap condition in Lemma 2.1 is satisfied if we take $\mathcal{P} = \mathcal{D}_m$. The uniqueness lemma implies $\mathcal{J}_1 = \mathcal{J} + b^m z = \mathcal{J}$ (the second equality is by periodicity of $\mathcal{J}$).

(ii) Suppose $\mathcal{S} \subseteq \mathbb{Z}$ is periodic and $\mathcal{S} = b\mathcal{S} + \mathcal{D}$. We first claim that $(T, \mathcal{S})$ is a covering of $\mathbb{R}$. Otherwise there is an interval I which is not covered by $T + \mathcal{S}$, i.e., $(T + \mathcal{S}) \cap I = \emptyset$. Since

$$b(T + \mathcal{S}) = bT + b\mathcal{S} = T + \mathcal{D} + b\mathcal{S} = T + \mathcal{S},$$

it follows that $bI \cap (T + \mathcal{S}) = b(I \cap (T + \mathcal{S})) = \emptyset$. Repeating this argument, we can show that there is an arbitrarily large interval which has no intersection with $T + \mathcal{S}$. This contradicts the periodicity of $\mathcal{S}$, and the claim follows.

Now let $\mathcal{J}$ be the self-replicating translation set with period b^m as in (i). From the self-replicating property, we have

$$\mathcal{S} = b^m \mathcal{S} + \mathcal{D}_m \subseteq b^m \mathbb{Z} + \mathcal{D}_m.$$

Likewise, $\mathcal{J} \subseteq b^m \mathbb{Z} + \mathcal{D}_m$. Since $\mathcal{D}_m \subset \mathcal{J}$ and $\mathcal{J}$ has period b^m, it follows that $\mathcal{J} = \mathcal{D}_m + b^m \mathbb{Z}$. This implies that $\mathcal{S}$ is a subset of $\mathcal{J}$. But $T + \mathcal{S}$ is a covering of $\mathbb{R}$; hence $\mathcal{S} = \mathcal{J}$. ◻

Remark 3.2. Let $0 \in \mathcal{D} \subset \mathbb{Z}$, g.c.d.$(\mathcal{D}) = 1$, and let $T = T(b, \mathcal{D})$ be a self-similar tile. If we assume $\mathcal{J}$ is a self-replicating translation set and $0 \in \mathcal{J}$, then $\mathcal{J}$ is unique by Lemma 2.1, and $\mathcal{J}$ is a subset of $\mathbb{Z}$ by Proposition 2.2. On the other hand, if we assume that $\mathcal{J}$ is a self-replicating translation set and $\mathcal{J} \subset \mathbb{Z}$, by Theorem 3.1, $\mathcal{J}$ is unique and thus $0 \in \mathcal{J}$. If we do not assume $0 \in \mathcal{J}$ or $\mathcal{J} \subset \mathbb{Z}$, then there may exist some other self-replicating translation sets. For exmaple, $b = 3$, $\mathcal{D} = \{0, 1, 2\}$. Then both $\mathbb{Z}$ and $\mathbb{Z} + 1/2$ are self-replicating translation sets. The above argument and example were given by the referee.

The theorem gives a satisfactory answer to the fundamental question Q1 about the construction and periodicity of translation sets. Part (ii) of the theorem is also convenient to use, since for a given pair $(b, \mathcal{D})$ it is not difficult to find an $\mathcal{S} \subseteq \mathbb{Z}$ that is periodic and satisfies $\mathcal{S} = b\mathcal{S} + \mathcal{D}$. For example, if $\mathcal{D}$ is a standard digit set, it is trivial that $\mathbb{Z}$ satisfies the conditions in (ii), and hence $T(b, \mathcal{D})$ tiles $\mathbb{R}$ by $\mathbb{Z}$. This case is actually a result of Gröchenig and Haas [GH].

4. Tile digit sets

We will consider two cases for $\mathcal{D}$ to be *tile* digit sets. The first case deals with a large class of digit sets that are left out by the product-form digit set criterion, and the second case is a relaxation of the product-form digit set.

Theorem 4.1. *Let* $0 \in \mathcal{D} = \{0, d_1, \cdots, d_{b-1}\} \subset \mathbb{Z}$, g.c.d.$(\mathcal{D}) = 1$, *and suppose that* $d_j \not\equiv 0 \pmod b$ *for* $1 \le j \le b-1$. *Then* $T(b, \mathcal{D})$ *is a self-similar tile if and only if* $\mathcal{D}$ *is a standard digit set.*

Proof. The sufficiency is well known (without the special hypothesis). We need only prove the necessity by assuming that T is a tile. Let $\mathcal{J} \subseteq \mathbb{Z}$ be the self-replicating translation set with a period b^m (Theorem 3.1). We first claim that $x \in \mathcal{J}$ if and only if $bx \in \mathcal{J} \cap b\mathbb{Z}$: Indeed, the necessity follows from the self-replicating property of $\mathcal{J}$ and $0 \in \mathcal{D}$. Conversely, if $bx \in \mathcal{J} \cap b\mathbb{Z}$, then by the self-replicating property again, $bx \in b\mathcal{J} + d_j$ for some j; hence $bx = bt + d_j$. The assumption that $d_j \not\equiv 0 \pmod b$ for $j \ne 0$ forces that $d_j = 0$, and hence $bx \in b\mathcal{J}$. Therefore $x \in \mathcal{J}$, and the claim is proved.

Next we assert that $x \in \mathcal{J}$ implies $x + 1 \in \mathcal{J}$. Indeed, the claim above implies that $b^m x \in \mathcal{J}$; the periodicity implies that $b^m(x+1) = b^m x + b^m \in \mathcal{J}$. By using the claim again, we have $x + 1 \in \mathcal{J}$. Thus $\mathcal{J} = \mathbb{Z}$. The self-replicating property of $\mathcal{J}$ yields $\mathbb{Z} = b\mathbb{Z} + \mathcal{D}$; so $\mathcal{D}$ must be a complete residue set modulo b, i.e., a standard digit set. $\square$

By a translation of the digit set we have the following corollary.

Corollary 4.2. *Let* $0 \in \mathcal{D} = \{0, d_1, \cdots, d_{b-1}\} \subset \mathbb{Z}$ *and* g.c.d.$(\mathcal{D}) = 1$. *If there is a* j *such that* $d_i \not\equiv d_j \pmod b$ *for all* $i \ne j$, *then* $T(b, \mathcal{D})$ *is a self-similar tile if and only if* $\mathcal{D}$ *is a standard digit set.*

The above theorem and corollary are useful to determine digit sets that do not give tiles. For example, we see that for $b = 4$, $\mathcal{D} = \{0, 1, 7, 15\}$ is not a tile-digit set by Theorem 4.1, and $\mathcal{D} = \{0, 4, 7, 17\}$ is not a tile digit set by the corollary with $d_j = 7$.

Let $\#\mathcal{E}$ denote the cardinality of $\mathcal{E}$. We say that $\mathcal{D} \subseteq \mathbb{Z}_+$ is a *product-form* digit set with respect to b if

$$\mathcal{D} = \mathcal{E}_0 + b^{l_1}\mathcal{E}_1 + \cdots + b^{l_k}\mathcal{E}_k, \tag{4.1}$$

where $1 \le l_1 \le \cdots \le l_k$ are integers, $0 \in \mathcal{E}_i \subseteq \mathbb{Z}_+$,

$$\mathcal{E}_0 + \cdots + \mathcal{E}_k = \mathcal{E}, \qquad \#\mathcal{E} = \prod_{i=0}^{k} \#\mathcal{E}_i,$$

and $\mathcal{E}$ is a complete set modulo b such that g.c.d.$(\mathcal{E}) = 1$. $\mathcal{D}$ is called a *strict product-form digit set* if in addition $\mathcal{E} = \{0, 1, \cdots, b-1\}$.

It is easy to see from the definition that if b is a prime, then a product-form digit set is merely a complete residue set modulo b. Non-trivial cases occur whenever b is not a prime. For example, if $b = 6$, then $\mathcal{D}_1 = \{0, 1, 2, 18, 19, 20\} = \{0, 1, 2\} + 6\{0, 3\}$ is a strict product-form digit set, while $\mathcal{D}_2 = \{0, 1, 5, 18, 19, 23\} = \{0, 1, 5\} + 6\{0, 3\}$ is a product-form digit set but not a strict product-form digit set.

The concept of (strict) product-form digit set was first used by Odlyzko in [O] to study the radix expansions of real numbers with respect to the base b, and was formally named by Lagarias and Wang in [LW2] (in higher dimensions also). They proved

Theorem 4.3. *Suppose $\mathcal{D} \subseteq \mathbb{Z}_+$ is a product-form digit set with respect to b. Then $T(b, \mathcal{D})$ is a self-similar tile and*

$$\mu(T(b,\mathcal{D})) = \mu(T(b,\mathcal{E})) \prod_{i=1}^{k} (\#\mathcal{E}_i)^{l_i},$$

where $\mu(\cdot)$ is the Lebesgue measure of $\mathbb{R}$.

Also, it was pointed out in [LW2] that $\mathcal{D}$ is a strictly product-form digit set if and only if $T(b, \mathcal{D})$ is a finite union of intervals. For the two examples above, we see that $T(6, \mathcal{D}_1) = [0,1] \cup [3,4]$, but $T(6, \mathcal{D}_2)$ has infinitely many connected components; in fact for $T(6, \mathcal{D}_2)$, its structure is quite complicated and the boundary has positive Hausdorff dimension (see e.g., [HLR]).

In the proof of Theorem 4.3 in [LW2], it is shown that for the above $\mathcal{D}$ and $\mathcal{E}$ there exists $W \subseteq \mathbb{Z}$ such that

$$b^{l_k} T(b, \mathcal{E}) = T(b, \mathcal{D}) + W.$$

Since $(T(b, \mathcal{E}), \mathbb{Z})$ is a tiling of $\mathbb{R}$, it follows that $(b^{l_k} T(b, \mathcal{E}),\ b^{l_k}\mathbb{Z})$ is also a tiling of $\mathbb{R}$. Therefore $(T(b, \mathcal{D}), W + b^{l_k}\mathbb{Z})$ is a tiling of $\mathbb{R}$. We will call

$$\mathcal{J} = W + b^{l_k}\mathbb{Z} \tag{4.2}$$

a *natural translation set* of $T(b, \mathcal{D})$. The following proposition shows that such a translation set has the self-replicating property.

Proposition 4.4. *Let $\mathcal{D} \subset \mathbb{Z}_+$ be a product-form digit set with respect to b, and let $\mathcal{J} = W + b^{l_k}\mathbb{Z}$ be a natural translation set. Then $\mathcal{J} = b\mathcal{J} + \mathcal{D}$.*

Proof. It is easy to see that $\mathcal{J}$ has a period b^{l_k}; that is, $\mathcal{J} = b^{l_k} + \mathcal{J}$. From

$$b^{l_k+1} T(b, \mathcal{E}) = b^{l_k}(\mathcal{E} + T(b, \mathcal{E})) = b^{l_k}\mathcal{E} + (W + T(b, \mathcal{D}))$$

and

$$b^{l_k+1} T(b, \mathcal{E}) = b(W + T(b, \mathcal{D})) = bW + (\mathcal{D} + T(b, \mathcal{D})),$$

we see that $b^{l_k}\mathcal{E} + W = \mathcal{D} + bW$. Hence

$$b\mathcal{J} + \mathcal{D} = bW + b^{l_k+1}\mathbb{Z} + \mathcal{D} = b^{l_k}\mathcal{E} + W + b^{l_k+1}\mathbb{Z} = W + b^{l_k}\mathbb{Z} = \mathcal{J}$$

(the third equality is because $\mathcal{E}$ is a complete residue set modulo b). This completes the proof. □

There are tile digit sets that are not product-form digit sets. For example, for $b = 4$, $\mathcal{D} = \{0, 1, 8, 25\}$ is a tile digit set but it is not the product-form nor does it satisfy the criterion of Corollary 4.2. To consider sets of this type we introduce a slight extension of the product-form digit set.

Definition 4.5. A digit set $\mathcal{D} \subseteq \mathbb{Z}_+$ is called a *weak product-form digit set* if there is a product-form digit set $\mathcal{D}'$ with

$$\mathcal{D}' = \mathcal{E}_0 + b^{l_1}\mathcal{E}_1 + \cdots + b^{l_k}\mathcal{E}_k$$

(see (4.1)) such that

$$\mathcal{D} \equiv \mathcal{D}' \pmod{b^{l_k+1}}.$$

The above example, $b = 4$, $\mathcal{D} = \{0, 1, 8, 25\}$ is a weak product-form digit set, because $\mathcal{D}' \equiv \{0, 1, 8, 9\} \pmod{16}$ and $\{0, 1, 8, 9\} = \{0, 1\} + 4\{0, 2\}$ is a product-form digit set.

Lemma 4.6. *Let $\mathcal{D} = \{0, d_1, \cdots, d_{b-1}\} \subseteq \mathbb{Z}$ be a digit set. Suppose there is a set $\mathcal{J} \subseteq \mathbb{Z}$ such that (i) $\mathcal{J} \subseteq b\mathcal{J} + \mathcal{D}$; (ii) $\mathcal{J}$ has positive upper density in $\mathbb{Z}$ (i.e., $\overline{\lim}_{n\to\infty} \frac{1}{2n} \#\{t \in \mathcal{J} : |t| \leq n\} > 0$). Then $\mathcal{D}$ is a tile digit set.*

Proof. It is known that $\mathcal{D}$ is a tile digit set if and only if $\#\mathcal{D}_k = b^k$ for each k [LW2]. Hence if $\mathcal{D}$ is not a tile digit set, there exists a k such that $\#\mathcal{D}_k < b^k$. From

$$\mathcal{D}_{k^n} = (\mathcal{D}_k)_n = b^{(n-1)k}\mathcal{D}_k + \cdots + b^k\mathcal{D}_k + \mathcal{D}_k,$$

we deduce that $\#\mathcal{D}_{k^n} \leq (\#\mathcal{D}_k)^n$. Therefore,

$$\lim_{n\to\infty} \frac{\#\mathcal{D}_{k^n}}{b^{k^n}} \leq \lim_{n\to\infty} \big(\frac{\#\mathcal{D}_k}{b^k}\big)^n \leq \lim_{n\to\infty} \big(\frac{b^k - 1}{b^k}\big)^n = 0.$$

Notice that

$$\frac{\#\mathcal{D}_{n+1}}{b^{n+1}} = \frac{\#(b\mathcal{D}_n + \mathcal{D})}{b^{n+1}} \leq \frac{\#\mathcal{D}_n}{b^n} \cdot \frac{\#\mathcal{D}}{b} = \frac{\#\mathcal{D}_n}{b^n},$$

which means $\#\mathcal{D}_n/b^n$ is non-increasing on n. This yields that $\lim_{n\to\infty} \#\mathcal{D}_n/b^n = 0$.

Now if we iterate the inclusion in (i) k times, we have

$$\mathcal{J} \subseteq b^k\mathcal{J} + \mathcal{D}_k \subseteq b^k\mathbb{Z} + \mathcal{D}_k.$$

So the density of $\mathcal{J}$ is no more than the density of $b^k\mathbb{Z} + \mathcal{D}_k$, which is at most $\#\mathcal{D}_k/b^k$. We see from this that the density of $\mathcal{J}$ is 0. This is a contradiction; so $\mathcal{D}$ must be a tile digit set. □

Theorem 4.7. *If $\mathcal{D}$ is a weak product-form digit set, then $\mathcal{D}$ is a tile digit set.*

Proof. Let $\mathcal{D}'$ be the associated product-form digit set as in the definition, and let $\mathcal{J} = W + b^{l_k}\mathbb{Z}$ be the natural translation set of $T(b, \mathcal{D}')$ as in (4.2). Clearly $\mathcal{J}$ has positive density in $\mathbb{Z}$. On the other hand, b^{l_k+1} is a period of $b\mathcal{J}$. So

$$b\mathcal{J} + d = b\mathcal{J} + d + tb^{l_k+1}$$

for any $d \in \mathcal{D}$ and $t \in \mathbb{Z}$. Hence we conclude that $b\mathcal{J} + \mathcal{D} = b\mathcal{J} + \mathcal{D}'$, which equals $\mathcal{J}$ by Proposition 4.4. We see that $\mathcal{J}$ satisfies the two conditions of Lemma 4.6. Therefore $\mathcal{D}$ is a tile digit set. □

We remark that from the above proof, it is easy to see that the translation set $\mathcal{J} = W + b^{l_k}\mathbb{Z}$ for $T(b, \mathcal{D}')$ is also a translation set for $T(b, \mathcal{D})$.

5. Tile digit sets for $b = pq$

Our main theorem is to characterize the tile digit sets for $b = pq$ in terms of the weak product-form digit set.

Theorem 5.1. *Let $b = pq$, where p, q are distinct primes, let $\mathcal{D} = \{0, d_1, \cdots, d_{b-1}\} \subset \mathbb{Z}_+$, and let* g.c.d.$(\mathcal{D}) = 1$. *Then $\mathcal{D}$ is a tile digit set if and only if $\mathcal{D}$ is a weak product-form digit set, i.e., there is an integer $k \geq 1$ such that*

$$\mathcal{D} \equiv \mathcal{E}_0 + b^{k-1}\mathcal{E}_1 \pmod{b^k},$$

where $\mathcal{E}_0 = \{0, \ldots, q-1\}, \mathcal{E}_1 = \{0, q, \ldots, (p-1)q\}$ (or the other way around).

The proof depends on two major ingredients. First we let

$$\mathcal{P}_{\mathcal{D}}(x) = \sum_{d\in\mathcal{D}} x^d = 1 + x^{d_1} + \cdots + x^{d_{b-1}}. \tag{5.1}$$

A criterion due to Kenyon [K1, Theorem 15] says that

Theorem 5.2. *Let $\mathcal{D} = \{0, d_1, \cdots, d_{b-1}\} \subset \mathbb{Z}_+$. Then $\mathcal{D}$ is a tile digit set if and only if for each integer $m \neq 0$, there exists $k \geq 1$ (depending on m) such that*

$$\mathcal{P}_{\mathcal{D}}(e^{2\pi i m/b^k}) = 0.$$

In order to make use of this criterion, we will need some special properties of the cyclotomic polynomials $F_n(x)$, i.e., the minimal polynomial of the algebraic integer $e^{2\pi i/n}$. It is clear that if $n = p$ is a prime, then $F_p(x) = 1 + x + \cdots + x^{p-1}$. If $n = b^k$, then $F_{b^k}(x) = F_b(x^{b^{k-1}})$. For $d|n$, we let

$$G_{n,d}(x) = \frac{x^n - 1}{x^{n/d} - 1} = 1 + x^{n/d} + \cdots + x^{(d-1)n/d}.$$

It is known that for $n = p^\alpha q^\beta$, α, $\beta \geq 1$,

$$F_n(x) = P(x)G_{n,p}(x) + Q(x)G_{n,q}(x) \tag{5.2}$$

and $P(x), Q(x)$ are in $\mathbb{Z}[x]$, the set of all polynomials with integer coefficients [DB]. In fact, $P(x)$ and $Q(x)$ are determined from

$$x^{n/pq} - 1 = P(x)(x^{n/p} - 1) + Q(x)(x^{n/q} - 1).$$

For example, for $n = 6$, $p, q = 2, 3$, we see from $x - 1 = (x^3 - 1) - x(x^2 - 1)$ that $P(x) = 1$, $Q(x) = -x$ and

$$F_6(x) = 1 - x + x^2 = (1 + x^2 + x^4) - x(1 + x^3).$$

To prove Theorem 5.1, we need the following theorem of De Bruijn [DB, p. 374]. Let $\mathbb{Z}_+[x]$ denote the set of polynomials with nonnegative integer coefficients.

Theorem 5.3. *Let $b = p^\alpha q^\beta$, where α, $\beta \geq 0$ and p, q are distinct primes. If $f(x) \in \mathbb{Z}_+[x]$ with degree $\leq b$ and $F_b(x)|f(x)$, then there exist polynomials $P(x)$, $Q(x) \in \mathbb{Z}_+[x]$ such that*

$$f(x) = P(x)G_{b,p}(x) + Q(x)G_{b,q}(x). \tag{5.3}$$

For a polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, we let

$$\mathcal{N}(f) := f(1) = a_0 + \cdots + a_n.$$

Clearly $\mathcal{N}(f + g) = \mathcal{N}(f) + \mathcal{N}(g)$ and $\mathcal{N}(fg) = \mathcal{N}(f)\mathcal{N}(g)$. Then $\mathcal{N}(F_{p^k}) = \mathcal{N}(F_p) = p$.

Corollary 5.4. *Under the assumptions of Theorem 5.3, if $\mathcal{N}(f) = pq$, then $P(x)$ and $Q(x)$ in identity (5.3) satisfy either $P(x) \equiv 0$ or $Q(x) \equiv 0$.*

Proof. By (5.3),

$$pq = \mathcal{N}(P)p + \mathcal{N}(Q)q.$$

Since $\mathcal{N}(P) \geq 0$, $\mathcal{N}(Q) \geq 0$, it follows that either $\mathcal{N}(P) = 0$ or $\mathcal{N}(Q) = 0$. We can hence conclude that $P(x) \equiv 0$ or $Q(x) \equiv 0$ by noting that P, Q have nonnegative coefficients. □

Lemma 5.5. *Under the assumptions of Theorem 5.1, if $\mathcal{D}$ is a tile digit set, then there are integers n and $n' > 1$ such that $F_{q^n}(x)|\mathcal{P}_{\mathcal{D}}(x)$ and $F_{p^{n'}}|\mathcal{P}_{\mathcal{D}}(x)$.*

Proof. Since $\deg(F_\ell(x)) \to \infty$ as $\ell \to \infty$, we can choose a large integer $t \geq 1$ such that for any integer $s \geq 0$,

$$\deg(F_{q^t b^s}(x)) > \deg(\mathcal{P}_{\mathcal{D}}(x)). \tag{5.4}$$

Set $m = p^t$ in Theorem 5.2. There is a $k = k(m)$ such that

$$\mathcal{P}_{\mathcal{D}}(e^{2\pi i p^t/b^k}) = 0.$$

We observe that $k < t$ (for otherwise $e^{2\pi i/(q^t b^{k-t})} = e^{2\pi i p^t/b^k}$ being a root of $\mathcal{P}_{\mathcal{D}}(x)$ implies that $F_{q^t b^{k-t}}$, the minimal polynomial of $e^{2\pi i/(q^t b^{k-t})}$, divides $\mathcal{P}_{\mathcal{D}}(x)$, which contradicts (5.4)). Since $e^{2\pi i p^t/b^k} = e^{2\pi i p^{t-k}/q^k}$ and $e^{2\pi i/q^k}$ share the same minimal polynomial, if we set $n = k$, then we have $F_{q^n}(x)|\mathcal{P}_{\mathcal{D}}(x)$. The proof for $F_{p^{n'}}(x)|\mathcal{P}_{\mathcal{D}}(x)$ is the same. □

We use $P(x) \pmod{Q(x)}$ to denote the remainder of $P(x)$ when divided by $Q(x)$.

Lemma 5.6. *Let m, ℓ be any positive integers and let $P(x) \in \mathbb{Z}[x]$ be a polynomial. Then*

$$x^{m\ell}P(x)(\text{mod } (x^\ell - 1)) = P(x)(\text{mod } (x^\ell - 1)).$$

Proof. The lemma is an easy consequence of the following identity:

$$x^{m\ell}P(x) = (x^{(m-1)\ell} + \cdots + 1)(x^\ell - 1)P(x) + P(x)$$

and a comparison of the remainders after dividing by $(x^\ell - 1)$. □

Proof of Theorem 5.1. In view of Theorem 4.7, we need only prove the necessity. Let $\mathcal{D}$ be a tile digit set. By applying the criterion in Theorem 5.2 for $m = 1$, we can find a positive integer k such that $F_{b^k}(x)|\mathcal{P}_{\mathcal{D}}(x)$, and we choose k to be the smallest such number.

We observe that the remainder $\mathcal{R}(x) := \mathcal{P}_{\mathcal{D}}(x) \ (\text{mod}(x^{b^k} - 1))$ has positive coefficients. Now $F_{b^k}|\mathcal{R}(x)$, and so by applying Corollary 5.4 to $\mathcal{R}(x)$, we can write

$$\mathcal{P}_{\mathcal{D}}(x) \ (\text{mod } (x^{b^k} - 1)) = P(x)G_{b^k,p}(x) \tag{5.5}$$

with $P(x)$ having nonnegative coefficients (or the alternative form $Q(x)G_{b^k,q}(x)$). Let

$$\mathcal{E}_0 = \{s : \ x^s \text{ is a nonzero term in } P(x)\}$$

and

$$\mathcal{E}_1 = \{0, q, \cdots, (p-1)q\}.$$

Then by comparing the powers of the polynomials in (5.5) we have

$$\mathcal{D} \equiv \mathcal{E}_0 + b^{k-1}\mathcal{E}_1 \pmod{b^k}. \tag{5.6}$$

Next we observe that, by Lemma 5.5, there exists an integer n such that

$$F_{q^n}(x)|\mathcal{P}_{\mathcal{D}}(x). \tag{5.7}$$

Since $F_{q^n}(x) = 1 + x^{q^{n-1}} + \cdots x^{(q-1)q^{n-1}}$, it follows that

$$\mathcal{P}_{\mathcal{D}}(x) = Q(x)(1 + x^{q^{n-1}} + \cdots + x^{(q-1)q^{n-1}}), \tag{5.8}$$

where $Q(x) \in \mathbb{Z}_+[x]$. We claim that $n = 1$ in (5.7) by eliminating the following two cases for $n > 1$.

(*i*) $n > k$. Observe that $G_{b^k,p}(x) = 1 + x^{b^k/p} + \cdots + x^{(p-1)b^k/p}$. Using (5.5) and applying Lemma 5.6 to each term of $G_{b^k,p}(x)$, we have (since $x^{q^k} - 1 | x^{b^k} - 1$)

$$\mathcal{P}_{\mathcal{D}}(x) \pmod{(x^{q^k} - 1)} = P(x) G_{b^k,p}(x) \pmod{(x^{q^k} - 1)} = P_1(x)p,$$

where $P_1(x) = P(x) \pmod{(x^{q^k} - 1)}$. On the other hand, by (5.8) and Lemma 5.6 (we use $n > k$ here), we have

$$\mathcal{P}_{\mathcal{D}}(x) \pmod{(x^{q^k} - 1)} = Q_1(x)q,$$

where $Q_1(x) = Q(x) \pmod{(x^{q^k} - 1)}$. It follows that

$$P_1(x)p = Q_1(x)q.$$

Now $pq = \mathcal{N}(\mathcal{P}_{\mathcal{D}}) = \mathcal{N}(P_1)p$ implies $\mathcal{N}(P_1) = q$. Since $P_1(x), Q_1(x) \in \mathbb{Z}_+[x]$, we have $P_1(x) = qx^l$ for some integer $l \geq 0$. Now $0 \in \mathcal{D}$ implies $P_1(x) = q$. It follows that

$$\mathcal{E}_0 \equiv \{0\} \pmod{q^k},$$

and hence, by (5.6), q is a factor of g.c.d.$(\mathcal{D})$, a contradiction.

(*ii*) $1 < n \leq k$. Then $G_{b^k,p}(e^{2\pi i/q^n}) = p$, and so (5.5) and (5.7) imply that $F_{q^n}(x) | P(x)$. Since $P(x) \in \mathbb{Z}_+[x]$, by Theorem 5.3 we have

$$P(x) \pmod{(x^{q^n} - 1)} = P'(x)(1 + x^{q^{n-1}} + \cdots + x^{(q-1)q^{n-1}})$$

with $P'(x) \in \mathbb{Z}_+[x]$. Note that $\mathcal{N}(P(x)) = q$; hence $\mathcal{N}(P') = 1$, and thus $P'(x) = x^t$ for some $t \geq 0$. Finally, $0 \in \mathcal{D}$ implies $P'(x) = 1$. So we have

$$P(x) \pmod{(x^{q^n} - 1)} = 1 + x^{q^{n-1}} + \cdots + x^{(q-1)q^{n-1}}, \tag{5.9}$$

and thus

$$\mathcal{E}_0 \equiv \{0\} \pmod{q^{n-1}}.$$

This again implies that q is a factor of g.c.d.$(\mathcal{D})$, a contradiction.

Therefore $n = 1$, and the claim $F_q(x) | \mathcal{P}_{\mathcal{D}}(x)$ is proved. By the same argument as in (*ii*), we have $F_q(x) | P(x)$. Setting $n = 1$ in (5.9), we obtain

$$P(x) \pmod{(x^q - 1)} = 1 + x + \cdots + x^{q-1}.$$

In other words,

$$\mathcal{E}_0 \equiv \{0, 1, \cdots, q-1\} \pmod{q}.$$

Now it is easy to check that

$$\mathcal{E}_0 + \mathcal{E}_1 \equiv \{0, 1, \cdots, b-1\} \pmod{b}.$$

This together with (5.6) implies that $\mathcal{D}$ is a weak product-form digit set. $\square$

6. Remarks

We do not yet have a characterization for the more general base b. The difficulties come from Theorem 5.3 and Corollary 5.4, for which we do not have replacements. We conjecture that the weak product-form digit set can be used to characterize tile digit sets with base $b = pq \cdots r$, a product of distinct primes. However, if the b involves a product of prime powers, we might need to extend the definition further in view of thecase $b = p^\ell$ in [LW2].

For the d-dimensional case, we let A be a $d \times d$ expanding integral matrix with $|\det A| = b$ (expanding means all eigenvalues have modulus greater than 1), and let $\mathcal{D} = \{0, d_1, \cdots, d_{b-1}\} \subset \mathbb{Z}^d$ be a digit set. We can ask the same question for the attractor $T(A, \mathcal{D})$ as in Q1 and Q2, but the questions are more difficult (see [GH], [K1], [LW1]-[LW4]). For example, for Q1 we would like to know whether Theorem 1.2 is true in $\mathbb{R}^d$: Does there exist a self-replicating translation set $\mathcal{J}$ (i.e., $\mathcal{J} = A\mathcal{J} + \mathcal{D}$)? Is it unique? Will it be periodic? For Q2, unlike the one-dimensional case, the following basic question is still unanswered [LW2]:

For an expanding integral matrix A with $|\det A| = b$ a prime, is it true that $\mathcal{D}$ is a tile digit set if and only if $\mathcal{D}$ is a complete set of coset representations of $\mathbb{Z}^d/A\mathbb{Z}^d$?

For digit sets in higher dimension, a direct analog to the one-dimensional case is the *collinear* digit sets: $\mathcal{D} = \{0, t_1 v, \cdots, t_{b-1} v\}$, where $v \in \mathbb{Z}^d$ and $t_i \in \mathbb{Z}$. Even for this simple setup it is not clear that all the one-dimensional results can be extended. For instance, it is not known whether, for the case $\mathcal{D} = \{0, v, \cdots, (b-1)v\}$, the set $T(A, \mathcal{D})$ will be connected or disk-like (it is trivial in one dimension). This has been explored in [KL], [KLR], [BW], but the question is only partially settled.

Acknowlegement

The authors would like to thank the referee for many comments and suggestions that improved the paper. They also thank Professor Y. Wang for bringing up the questions in the paper, and Dr. D. J. Feng for many helpful discussions.

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