TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 355, Number 5, Pages 1729–1772
S 0002-9947(03)03202-1
Article electronically published on January 8, 2003

# EXTENDER-BASED RADIN FORCING

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Abstract. We define extender sequences, generalizing measure sequences of Radin forcing.

Using the extender sequences, we show how to combine the Gitik-Magidor forcing for adding many Prikry sequences with Radin forcing.

We show that this forcing satisfies a Prikry-like condition, destroys no cardinals, and has a kind of properness.

Depending on the large cardinals we start with, this forcing can blow the power of a cardinal together with changing its cofinality to a prescribed value. It can even blow the power of a cardinal while keeping it regular or measurable.

## 1. Introduction

We give some background on previous work relating directly to the present work. The first forcing which changed the cofinality of a cardinal without changing the cardinal structure was Prikry forcing [6]. In this forcing a measurable cardinal, $\kappa$, was 'invested' in order to get $\mathrm{cf}(\kappa) = \omega$ without collapsing any cardinal. Developing that idea, Magidor [2] used a coherent sequence of measures of length $\lambda < \kappa$ in order to get $\mathrm{cf}(\kappa) = \lambda$ without collapsing any cardinals. In [7] Radin, introducing the notion of measure sequence, showed that it is useful to continue the coherent sequence to $\lambda > \kappa$. For example, $\kappa$ remains regular when $\lambda = \kappa^+$. In general, the longer the measure sequence, the more resemblance there is between $\kappa$ in the generic extension and the ground model.

As is well known, and unlike regular cardinals, blowing the power of a singular cardinal is not an easy task. A natural approach to try was to blow the power of a cardinal while it was regular and after that make it singular by one of the above methods. A crucial idea of Gitik and Magidor [3] was to combine the power set blowing and the cofinality change in one forcing. They introduced a forcing notion which added many Prikry sequences at once and still collapsed no cardinals. The 'investment' they needed for this was an extender of length which is the size of the power they wanted. Building on the idea of Gitik and Magidor, Segal [8] implemented the idea of adding many sequences to Magidor forcing. So by investing

Received by the editors October 19, 1998.

2000 *Mathematics Subject Classification.* Primary 03E35, 03E55.

*Key words and phrases.* Forcing, Radin forcing, extender, extender-based forcing, generalized continuum hypothesis, singular cardinal hypothesis.

This work is a part of research which, hopefully, will become the author's Ph.D. thesis. It was done at Tel-Aviv University under the supervision of M. Gitik. The author thanks M. Gitik for his help with this work, with other works and just in general.

We thank Asaf Sharon for reading and pointing out some problems in a preliminary version of this work.

a coherent sequence of extenders of length $\lambda < \kappa$ she was able to get a singular cardinal of cofinality $\lambda$ together with power as large as the length of the extenders in question. Our work also builds on the idea of Gitik and Magidor. However, we implement the idea of adding many sequences to Radin forcing. So we introduce the notion of extender sequence and show that it makes sense to deal with quite long extender sequences. As in Radin forcing, for long enough sequences we are left with $\kappa$ which is regular and even measurable. The power size will be the length of the extenders we start with.

The structure of this work is as follows. In section 2 we define extender sequences. In section 3 we define $P_{\bar{E}}$, the forcing notion which is the purpose of this work. In section 4 we define Radin forcing as a means to prove lemmas we need later. The definition is not the usual one. The definition we give here is more in the spirit of the forcing we defined in section 3. In section 5 we show the chain-conditions satisfied by $P_{\bar{E}}$ and how 'locally' it resembles Radin forcing. We also show here that there are many new subsets in the generic extension. In section 6 we investigate the structure of dense open subsets of $P_{\bar{E}}$. We show that they satisfy a strong homogeneity property. In section 7 we prove Prikry's condition for $P_{\bar{E}}$. The proof is a simple corollary of the strong homogeneity of dense open subsets. In section 8 we show that $P_{\bar{E}}$ satisfies a kind of properness. In section 9 we combine the machinery developed so far in order to show that no cardinals are collapsed. In section 10 we show how the length of the extender sequences affects the properties of $\kappa$. Section 11 summarizes what the forcing $P_{\bar{E}}$ does. In section 12 we have a result concerning $P_{\bar{E}}$ when $\mathrm{l}(\bar{E}) = 1$. We show that there is, in $V$, a generic filter over an elementary submodel in an $\omega$-iterate of $V$. We were not able to prove something equivalent (or weaker) for the general case. Section 13 contains a list of missing or unknown points to check. The last point in this list is in preparation.

Our notation is standard. We assume fluency with forcing and extenders. Some basic properties of Radin forcing are taken for granted.

## 2. Extender Sequences

2.1. **Constructing from elementary embedding.** Suppose we have an elementary embedding $j{:}V \to M \supset V_\lambda$, $\mathrm{crit}(j) = \kappa$. The value of $\lambda$ will be determined later, according to the different applications we have.

Construct from $j$ a nice extender such as in [3]:

$$E(0) = \langle \langle E_\alpha(0) \mid \alpha \in \mathcal{A} \rangle, \langle \pi_{\beta,\alpha} \mid \beta \geq_{\mathcal{A}} \alpha \ \alpha, \beta \in \mathcal{A} \rangle \rangle.$$

We recall the properties of this extender:

(1) $\mathcal{A} \subseteq |V_\lambda| \setminus \kappa$,
(2) $|\mathcal{A}| = |V_\lambda|$,
(3) $\langle \mathcal{A}, \leq_{\mathcal{A}} \rangle$ is a $\kappa^+$-directed partial order,
(4) $\forall \alpha, \beta \in \mathcal{A}\ \beta \geq_{\mathcal{A}} \alpha \implies \pi_{\beta,\alpha}{:}V_\kappa \to V_\kappa$,
(5) $\kappa \in \mathcal{A}$,
(6) $\forall \alpha \in \mathcal{A}\ \kappa \leq_{\mathcal{A}} \alpha$. We write $\pi_{\alpha,0}$ instead of $\pi_{\alpha,\kappa}$,
(7) $\forall \alpha, \beta \in \mathcal{A}\ \forall \nu < \kappa\ \nu^0 = \pi_{\alpha,0}(\nu) = \pi_{\beta,0}(\nu)$,
(8) $\forall \alpha, \beta \in \mathcal{A}\ \beta \geq_{\mathcal{A}} \alpha \implies \forall \nu < \kappa\ \pi_{\beta,0}(\nu) = \pi_{\alpha,0}(\pi_{\beta,\alpha}(\nu))$,
(9) $\forall \alpha, \beta, \gamma \in \mathcal{A}\ \gamma \geq_{\mathcal{A}} \beta \geq_{\mathcal{A}} \alpha \implies$

$$\exists A \in E_\gamma(0) \forall \nu \in A \pi_{\gamma,\alpha}(\nu) = \pi_{\beta,\alpha}(\pi_{\gamma,\beta}(\nu)).$$

If, for example, we need $|E(0)| = \kappa^{+3}$, then, under GCH, we require $\lambda = \kappa + 3$. A typical large set in this extender concentrates on singletons.

If $j$ is not sufficiently closed, then $E(0) \notin M$ and the construction stops. We set

$$\forall \alpha \in \mathcal{A}\ \bar{E}_\alpha = \langle \alpha, E(0) \rangle.$$

We say that $\bar{E}_\alpha$ is an extender sequence of length 1 ($\mathrm{l}(\bar{E}_\alpha) = 1$).

If, on the other hand, $E(0) \in M$, we can construct for each $\alpha \in \operatorname{dom} E(0)$ the following ultrafilter:

$$A \in E_{\langle \alpha, E(0) \rangle}(1) \iff \langle \alpha, E(0) \rangle \in j(A).$$

Such an $A$ concentrates on elements of the form $\langle \xi, e(0) \rangle$ where $e(0)$ is an extender on $\xi^0$ and $\xi \in \operatorname{dom} e(0)$. Note that $e(0)$ concentrates on singletons below $\xi^0$. If, for example, $|E(0)| = \kappa^{+3}$, then on a large set we have $|e(0)| = (\xi^0)^{+3}$.

We define $\pi_{\langle \beta, E(0) \rangle, \langle \alpha, E(0) \rangle}$ as

$$\pi_{\langle \beta, E(0) \rangle, \langle \alpha, E(0) \rangle}(\langle \xi, e(0) \rangle) = \langle \pi_{\beta,\alpha}(\xi), e(0) \rangle.$$

From this definition we get

$$j(\pi_{\langle \beta, E(0) \rangle, \langle \alpha, E(0) \rangle})(\langle \beta, E(0) \rangle) = \langle \alpha, E(0) \rangle.$$

Hence we have here an extender

$$E(1) = \langle \langle E_{\langle \alpha, E(0) \rangle}(1) \mid \alpha \in \mathcal{A} \rangle, \langle \pi_{\langle \beta, E(0) \rangle, \langle \alpha, E(0) \rangle} \mid \beta \geq_{\mathcal{A}} \alpha\ \alpha, \beta \in \mathcal{A} \rangle \rangle.$$

Note that the difference between $\pi_{\beta,\alpha}$ and $\pi_{\langle \beta, E(0) \rangle, \langle \alpha, E(0) \rangle}$ is quite superficial. We can define $\pi_{\langle \beta, E(0) \rangle, \langle \alpha, E(0) \rangle}$ in a uniform way for both extenders. Just project the first element of the argument using $\pi_{\beta,\alpha}$.

If $\langle E(0), E(1) \rangle \notin M$, then the construction stops. In this case we set

$$\forall \alpha \in \mathcal{A}\ \bar{E}_\alpha = \langle \alpha, E(0), E(1) \rangle.$$

We say that $\bar{E}_\alpha$ is an extender sequence of length 2 ($\mathrm{l}(\bar{E}_\alpha) = 2$).

If $\langle E(0), E(1) \rangle \in M$, then we construct the extender $E(2)$ in the same way as we constructed $E(1)$ from $E(0)$.

The above special case being worked out, we continue with the general case. Assume we have constructed

$$\langle E(\tau') \mid \tau' < \tau \rangle.$$

If $\langle E(\tau') \mid \tau' < \tau \rangle \notin M$, then the construction stops here. We set

$$\forall \alpha \in \mathcal{A}\ \bar{E}_\alpha = \langle \alpha, E(\tau') \mid \tau' < \tau \rangle,$$

and we say that $\bar{E}_\alpha$ is an extender sequence of length $\tau$ ($\mathrm{l}(\bar{E}_\alpha) = \tau$).

If, on the other hand, $\langle E(\tau') \mid \tau' < \tau \rangle \in M$, then we construct

$$A \in E_{\langle \alpha, E(0), \ldots, E(\tau'), \ldots \mid \tau' < \tau \rangle}(\tau) \iff \langle \alpha, E(0), \ldots, E(\tau'), \ldots \mid \tau' < \tau \rangle \in j(A).$$

Defining $\pi_{\langle \beta, E(0), \ldots, E(\tau'), \ldots \mid \tau' < \tau \rangle, \langle \alpha, E(0), \ldots, E(\tau'), \ldots, \mid \tau' < \tau \rangle}$ using the first coordinate as before gives the needed projection.

We are quite casual in writing the indices of the projections and ultrafilters. By this we mean that we sometimes write $\pi_{\beta,\alpha}$ when we should have written $\pi_{\langle \beta, E(0), \ldots, E(\tau'), \ldots \mid \tau' < \tau \rangle, \langle \alpha, E(0), \ldots, E(\tau'), \ldots, \mid \tau' < \tau \rangle}$ and $E_\alpha(\tau)$ when we should have written $E_{\langle \alpha, E(0), \ldots, E(\tau'), \ldots, \mid \tau' < \tau \rangle}(\tau)$.

With this abuse of notation the projection we just defined satisfies

$$j(\pi_{\beta,\alpha})(\langle \beta, E(0), \ldots, E(\tau'), \ldots \mid \tau' < \tau \rangle) = \langle \alpha, E(0), \ldots, E(\tau'), \ldots \mid \tau' < \tau \rangle,$$

and we have the extender

$$E(\tau) = \langle\langle E_\alpha(\tau) \mid \alpha \in \mathcal{A}\rangle, \langle \pi_{\beta,\alpha} \mid \beta \geq_{\mathcal{A}} \alpha \ \alpha, \beta \in \mathcal{A}\rangle\rangle.$$

We let the construction run until it stops due to the extender sequence not being in $M$.

**Definition 2.1.** We call $\bar{\mu}$ an *extender sequence* if there is an elementary embedding $j{:}V \to M$ such that $\bar{\nu}$ is an extender sequence generated as above and $\bar{\mu} = \bar{\nu}{\restriction}\tau$ for $\tau \leq \mathrm{l}(\bar{\nu})$. $\kappa(\bar{\mu})$ is the ordinal at the beginning of the sequence (i.e., $\kappa(\bar{E}_\alpha) = \alpha$), and $\kappa^0(\bar{\mu})$ is $(\kappa(\bar{\mu}))^0$ (i.e., $\kappa^0(\bar{E}_\alpha) = \kappa$).

That is, we do not have to construct the extender sequence until it is not in $M$. We can stop anywhere on the way.

The generalization of the measure on the $\alpha$ coordinate in Gitik-Magidor forcing [3] is $\bar{E}_\alpha$.

**Definition 2.2.** A sequence of extender sequences $\langle \bar{\mu}_1, \ldots, \bar{\mu}_n\rangle$ is called ${}^0$-*increasing* if $\kappa^0(\bar{\mu}_1) < \cdots < \kappa^0(\bar{\mu}_n)$.

**Definition 2.3.** Let $\langle \bar{\mu}_1, \ldots, \bar{\mu}_n\rangle$ be ${}^0$-increasing. An extender sequence $\bar{\mu}$ is called *permitted to* $\langle \bar{\mu}_1, \ldots, \bar{\mu}_n\rangle$ if $\kappa(\bar{\mu}_n) < \kappa^0(\bar{\mu})$.

**Definition 2.4.** We say $A \in \bar{E}_\alpha$ if $\forall \xi < \mathrm{l}(\bar{E}_\alpha),\ A \in E_\alpha(\xi)$.

**Definition 2.5.** $\bar{E} = \langle \bar{E}_\alpha \mid \alpha \in \mathcal{A}\rangle$ is an *extender sequence system* if there is an elementary embedding $j{:}V \to M$ such that all $\bar{E}_\alpha$ are extender sequences generated from $j$ as prescribed above and $\forall \alpha, \beta \in \mathcal{A}, \mathrm{l}(\bar{E}_\alpha) = \mathrm{l}(\bar{E}_\beta)$. This common length is called the *length* of the system, $\mathrm{l}(\bar{E})$. We write $\bar{E}(\bar{\mu})$ for the extender sequence system to which $\bar{\mu}$ belongs (i.e., $\bar{E}(\bar{E}_\alpha) = \bar{E}$).

We point out that there is a $\kappa^+$-directed partial order on $\bar{E}$ inherited from $\mathcal{A}$. That is, $\bar{E}_\beta \geq_{\bar{\mathrm{E}}} \bar{E}_\alpha \iff \beta \geq_{\mathcal{A}} \alpha$. Of course, this implies that there is $\min \bar{E}$, namely $\bar{E}_\kappa$. From now on we use only the order $\geq_{\bar{\mathrm{E}}}$, even for $\mathcal{A}$, and we write $\operatorname{dom} \bar{E}$ for $\mathcal{A}$.

### 2.2. $\bar{E}_\gamma$-tree.

**Definition 2.6.** A tree $T$ is an $\bar{E}_\alpha$-*tree* if its elements are of the form

$$\langle\langle \bar{\mu}_1, \ldots, \bar{\mu}_n\rangle, S\rangle$$

where

(1) if we set $\operatorname{dom} T = \{\langle \bar{\mu}_1, \ldots, \bar{\mu}_n\rangle \mid \langle\langle \bar{\mu}_1, \ldots, \bar{\mu}_n\rangle, S\rangle \in T\}$, then the function

$$\langle \bar{\mu}_1, \ldots, \bar{\mu}_n\rangle \mapsto \langle\langle \bar{\mu}_1, \ldots, \bar{\mu}_n\rangle, S\rangle$$

from $\operatorname{dom} T$ to $T$ is 1-1 and onto,

(2) $t \in \mathrm{Lev}_n(\operatorname{dom} T) \implies |t| = n+1$,

(3) $\langle \bar{\mu}_1, \ldots, \bar{\mu}_n\rangle$ is a ${}^0$-increasing sequence,

(4) $\mathrm{Lev}_0(\operatorname{dom} T) \in \bar{E}_\alpha$, and for each $t \in \operatorname{dom} T$, $\mathrm{Suc}_{\operatorname{dom} T}(t) \in \bar{E}_\alpha$,

(5) $S$ is a $\bar{\mu}_n$-tree. When $\mathrm{l}(\bar{\mu}_n) = 0$ we set $S = \emptyset$.

Note that this clause is recursive.

*Note* 2.7. While we call $T$ a tree, formally speaking, it is a function with domain a tree (in the usual sense) of finite sequences.

*Note* 2.8. Later on, we abuse notation and use $T$ instead of $\operatorname{dom} T$. That is, we write $\operatorname{Suc}_T(t)$ instead of $\operatorname{Suc}_{\operatorname{dom} T}(t)$.

**Definition 2.9.** Assume $T$ is a $\bar{E}_\alpha$-tree and $t \in T$. Then:

(1) $T_t = \{\langle s, S\rangle \mid \langle t^\frown s, S\rangle \in T\}$.
(2) $T(\bar{\mu})$ is the tree, $S$, satisfying $\langle \bar{\mu}, S\rangle \in \operatorname{Lev}_0(T)$.
(3) $T_t(\bar{\mu})$ is the tree, $S$, satisfying $\langle \bar{\mu}, S\rangle \in \operatorname{Suc}_T(t)$.

**Definition 2.10.** Let $T$, $S$ be $\bar{E}_\alpha$-trees, where $\mathrm{l}(\bar{E}) = 1$. We say that $T \leq S$ if

(1) $\operatorname{Lev}_0(T) \subseteq \operatorname{Lev}_0(S)$, and
(2) $\forall t \in T\, \operatorname{Suc}_T(t) \subseteq \operatorname{Suc}_S(t)$.

**Definition 2.11.** Let $T$, $S$ be $\bar{E}_\alpha$-trees. We say that $T \leq S$ if

(1) $\operatorname{Lev}_0(T) \subseteq \operatorname{Lev}_0(S)$,
(2) $\forall t \in T\, \operatorname{Suc}_T(t) \subseteq \operatorname{Suc}_S(t)$,
(3) $\forall \langle \bar{\mu}\rangle \in \operatorname{Lev}_0(T)\ T(\bar{\mu}) \leq S(\bar{\mu})$, and
(4) $\forall t \in T\ \forall \langle \bar{\mu}\rangle \in T_t\ T_t(\bar{\mu}) \leq S_t(\bar{\mu})$.

Note that the last 2 conditions are recursive.

**Definition 2.12.** Let $S$ be $\bar{E}_\alpha$-tree and $\beta > \alpha$. Define $T = \pi^{-1}_{\beta,\alpha}(S)$ by

(1) $\operatorname{dom} T = \pi^{-1}_{\beta,\alpha}(\operatorname{dom} S)$,
(2) $T_{\langle \bar{\mu}_1,\ldots,\bar{\mu}_{n-1}\rangle}(\bar{\mu}_n) = \pi^{-1}_{\bar{\mu}_n,\pi_{\beta,\alpha}(\bar{\mu}_n)} S_{\langle \pi_{\beta,\alpha}(\bar{\mu}_1),\ldots,\pi_{\beta,\alpha}(\bar{\mu}_{n-1})\rangle}(\pi_{\beta,\alpha}(\bar{\mu}_n))$.

**Definition 2.13.** Let $T$, $S$ be $\bar{E}_\beta$, $\bar{E}_\alpha$-trees respectively, where $\beta \geq_{\bar{\mathrm{E}}} \alpha$. We say that $T \leq S$ if

(1) $T \leq \pi^{-1}_{\beta,\alpha}(S)$.

**Definition 2.14.** Assume we have $A_{\langle \bar{\nu}, R\rangle}$, where $\bar{\nu}$ is an extender sequence such that each element in $A_{\langle \bar{\nu}, R\rangle}$ is of the form $\langle \bar{\mu}, S\rangle$ where $\bar{\mu}$ is an extender sequence and $S$ is a tree (in this work $S$ is always a $\bar{\mu}$-tree). We define $\triangle^0_{\langle \bar{\nu}, R\rangle} A_{\langle \bar{\nu}, R\rangle}$ as

$$\langle \bar{\mu}, S\rangle \in \underset{\langle \bar{\nu}, R\rangle}{\triangle^0} A_{\langle \bar{\nu}, R\rangle} \iff \forall \langle \bar{\nu}, R\rangle\, \kappa(\bar{\nu}) < \kappa^0(\bar{\mu}) \to \langle \bar{\mu}, S_{\langle \bar{\nu}, R\rangle}\rangle \in A_{\langle \bar{\nu}, R\rangle}.$$

## 3. $P_{\bar{E}}$-Forcing

The definition of the forcing is done by induction on $\mathrm{l}(\bar{E})$. Originally we started by directly giving the general case. At the suggestion of the referee we have added concrete definitions also for the cases $\mathrm{l}(\bar{E}) = 1$, $\mathrm{l}(\bar{E}) = 2$ and several figures we hope will enhance the intuition standing behind the definitions.

**Definition 3.1.** Assume $\mathrm{l}(\bar{E}) = 0$. A condition in $P^*_{\bar{E}}$ is of the form

$$\left\{\langle \langle \kappa\rangle, p^{\langle \kappa\rangle}\rangle\right\},$$

where $p^{\langle \kappa\rangle} \in V_\kappa$ is an extender sequence. Call it $p^0$. ($p^{\langle \kappa\rangle} = \emptyset$ is allowed.)

When $\mathrm{l}(\bar{E}) = 0$, the partial order $\leq^*$ degenerates into $=$.

**Definition 3.2.** Assume $\mathrm{l}(\bar{E}) = 0$. Let $p, q \in P^*_{\bar{E}}$. We say that $p$ is a *Prikry extension* of $q$ ($p \leq^* q$ or $p \leq^0 q$) if $p^0 = q^0$.

**Definition 3.3.** Assume $\mathrm{l}(\bar{E}) = 0$. We set $\langle P_{\bar{E}}, \leq\rangle = \langle P^*_{\bar{E}}, \leq^*\rangle$.

Hence, when $\mathrm{l}(\bar{E}) = 0$, $P_{\bar{E}}$ is not really a forcing. We use it more as a place-holder in order not to convolute later proofs with two cases everywhere.

**Definition 3.4.** Assume $\mathrm{l}(\bar{E}) = 1$. A condition in $P^*_{\bar{E}}$ is of the form

$$\{\langle\langle\alpha, E(0)\rangle, p^{\langle\alpha,E(0)\rangle}\rangle \mid \langle\alpha, E(0)\rangle \in g\} \cup \{T\}$$

where

(1) $g \subseteq \bar{E}$, $|g| \leq \kappa$,
(2) $\min \bar{E} = \langle\kappa, E(0)\rangle \in g$ and $g$ has a maximal element,
(3) $p^{\langle\kappa,E(0)\rangle} \in V_{\kappa^0(\bar{E})}$ is an extender sequence which we call $p^0$ (we allow $p^0 = \emptyset$),
(4) $\forall\langle\alpha, E(0)\rangle \in g \setminus \{\langle\kappa, E(0)\rangle\}$ $p^{\langle\alpha,E(0)\rangle} \in [V_{\kappa^0(\bar{E})}]^{<\omega}$ is a ${}^0$-increasing sequence (we allow $p^{\langle\alpha,E(0)\rangle} = \emptyset$),
(5) $T$ is a $\max g$-tree such that for all $t \in T$, and if $p^{\max g} {}^\frown t$ is ${}^0$-increasing, $\langle\bar{\nu}\rangle \in T$, then $\bar{\nu} = \langle\xi\rangle$ for some $\xi < \kappa$ (meaning that $\bar{\nu}$ is an extender sequence of length 0),
(6) for all $\langle\alpha, E(0)\rangle \in g$, $p^0$ is not permitted to $p^{\langle\alpha,E(0)\rangle}$,
(7) $\forall\langle\bar{\nu}\rangle \in T$ $|\{\langle\alpha, E(0)\rangle \in g \mid \bar{\nu}$ is permitted to $p^{\langle\alpha,E(0)\rangle}\}| \leq \kappa^0(\bar{\nu})$, and
(8) $\forall\langle\bar{\nu}\rangle \in T$, if $\bar{\nu}$ is permitted to $p^{\langle\beta,E(0)\rangle}, p^{\langle\alpha,E(0)\rangle}$, then

$$\pi_{\max g,\langle\beta,E(0)\rangle}(\bar{\nu}) \neq \pi_{\max g,\langle\alpha,E(0)\rangle}(\bar{\nu}).$$

We write $\mathrm{mc}(p)$, $p^{\mathrm{mc}}$, $T^p$, $\bar{E}(p)$, $\operatorname{supp} p$ for $\max g$, $p^{\max g}$, $T$, $\bar{E}$, $g$ respectively.

**Definition 3.5.** Assume $\mathrm{l}(\bar{E}) = 1$. Let $p, q \in P^*_{\bar{E}}$. We say that $p$ is a *Prikry extension* of $q$ ($p \leq^* q$ or $p \leq^0 q$) if

(1) $\operatorname{supp} p \supseteq \operatorname{supp} q$,
(2) $\forall\langle\alpha, E(0)\rangle \in \operatorname{supp} q$ $p^{\langle\alpha,E(0)\rangle} = q^{\langle\alpha,E(0)\rangle}$,
(3) $T^p \leq \pi^{-1}_{\mathrm{mc}(p),\mathrm{mc}(q)} T^q$,
(4) $\forall\langle\alpha, E(0)\rangle \in \operatorname{supp} q$ $\forall\langle\bar{\nu}\rangle \in T^p$

$$\max \kappa(p^{\langle\alpha,E(0)\rangle}) < \kappa^0(\bar{\nu}) \implies \pi_{\mathrm{mc}(p),\alpha}(\bar{\nu}) = \pi_{\mathrm{mc}(q),\alpha}(\pi_{\mathrm{mc}(p),\mathrm{mc}(q)}(\bar{\nu})).$$

**Definition 3.6.** Assume $\mathrm{l}(\bar{E}) = 1$. A condition in $P_{\bar{E}}$ is of the form

$$p_n {}^\frown \cdots {}^\frown p_0$$

where

- $p_0 \in P^*_{\bar{E}}$,
- $p_1 \in P^*_{\bar{\mu}_1}$,
- $\vdots$
- $p_n \in P^*_{\bar{\mu}_n}$,

where $\bar{E}, \bar{\mu}_1, \ldots, \bar{\mu}_n$ are extender sequence systems such that $\mathrm{l}(\bar{\mu}_1) = 0, \ldots, \mathrm{l}(\bar{\mu}_n) = 0$ satisfying

$$\forall i \leq n-1 \ \kappa(\bar{\mu}_{i+1}) < \kappa^0(\bar{\mu}_i) \text{ where } \bar{\mu}_0 = \bar{E}.$$

**Definition 3.7.** Assume $\mathrm{l}(\bar{E}) = 1$. Let $p, q \in P_{\bar{E}}$. We say that $p$ is a *Prikry extension* of $q$ ($p \leq^* q$ or $p \leq^0 q$) if $p, q$ are of the form

$$p = p_n {}^\frown \cdots {}^\frown p_0,$$
$$q = q_n {}^\frown \cdots {}^\frown q_0,$$

and

- $p_0, q_0 \in P^*_{\bar{E}},\ p_0 \leq^* q_0$,
- $p_1, q_1 \in P^*_{\bar{\mu}_1},\ p_1 \leq^* q_1$,
- $\vdots$
- $p_n, q_n \in P^*_{\bar{\mu}_n},\ p_n \leq^* q_n$.

$p_{0\langle\bar{\nu}\rangle}$, defined now, is the basic non-direct extension in $P_{\bar{E}}$ of the condition $p_0$, which adds the extender sequence $\bar{\nu} \in T^{p_0}$ to the finite sequence.

**Definition 3.8.** Assume $\mathrm{l}(\bar{E}) = 1$. Let $p \in P^*_{\bar{E}}$ and $\langle\bar{\nu}\rangle \in T^p$. In this case $\bar{\nu} = \langle\xi\rangle$ for some $\xi < \kappa$. We define $(p)_{\langle\bar{\nu}\rangle}$ to be $p'_1 {}^\frown p'_0$ where

(1) $\operatorname{supp} p'_0 = \operatorname{supp} p$,
(2) $\forall\langle\alpha, E(0)\rangle \in \operatorname{supp} p'_0$,

$$p_0'^{\langle\alpha,E(0)\rangle} = \begin{cases} \langle\pi_{\mathrm{mc}(p),\alpha}(\xi)\rangle, & \max\kappa(p^{\langle\alpha,E(0)\rangle}) < \kappa^0(\xi),\ \alpha = \kappa, \\ p^{\langle\alpha,E(0)\rangle} {}^\frown \langle\pi_{\mathrm{mc}(p),\alpha}(\xi)\rangle, & \max\kappa(p^{\langle\alpha,E(0)\rangle}) < \kappa^0(\xi),\ \alpha \neq \kappa, \\ p^{\langle\alpha,E(0)\rangle}, & \text{otherwise}, \end{cases}$$

(3) $T^{p'_0} = T^p_{\langle\bar{\nu}\rangle}$,
(4) $\operatorname{supp} p'_1 = \{\langle\pi_{\mathrm{mc}(p),\kappa}(\xi)\rangle\}$,
(5) $p_1'^{\langle\pi_{\mathrm{mc}(p),\kappa}(\xi)\rangle} = p^{\langle\kappa,E(0)\rangle}$. (Note that $p'_1 \in P^*_{\langle\pi_{\mathrm{mc}(p),\kappa}\rangle}$.)

**Definition 3.9.** Assume $\mathrm{l}(\bar{E}) = 1$. Let $p, q \in P_{\bar{E}}$. We say that $p$ is a 1-*point extension* of $q$ ($p \leq^1 q$) if $p, q$ are of the form

$$p = p_{n+1} {}^\frown p_n {}^\frown \cdots {}^\frown p_0,$$
$$q = q_n {}^\frown \cdots {}^\frown q_0,$$

and

- $p_{i+1}, q_i \in P^*_{\bar{\mu}_i},\ p_{i+1} \leq^* q_i$ for $i = 1, \ldots, n$,
- there is $\langle\bar{\nu}\rangle \in T^{q_0}$ such that $p_1 {}^\frown p_0 \leq^* (q_0)_{\langle\bar{\nu}\rangle}$.

**Definition 3.10.** Assume $\mathrm{l}(\bar{E}) = 1$. Let $p, q \in P_{\bar{E}}$. We say that $p$ is an $n$-*point extension* of $q$ ($p \leq^n q$) if there are $p^n, \ldots, p^0$ such that

$$p = p^n \leq^1 \cdots \leq^1 p^0 = q.$$

**Definition 3.11.** Assume $\mathrm{l}(\bar{E}) = 1$. Let $p, q \in P_{\bar{E}}$. We say that $p$ is an *extension* of $q$ ($p \leq q$) if there is an $n$ such that $p \leq^n q$.

It is quite clear that when $\mathrm{l}(\bar{E}) = 1$, $\langle P_{\bar{E}}, \leq\rangle$ is the Gitik-Magidor forcing from [3] with the following slight changes:

(1) The support elements are $\langle\alpha, E(0)\rangle$ instead of $\alpha$.
(2) $p^{\langle\kappa,E(0)\rangle}$ has at most one element and not a finite sequence.

We give some figures in order to enhance the intuition behind these definitions. In Figure 1 we show a typical condition $p_0 \in P^*_{\bar{E}}$. In Figure 2 we show a 1-point extension of $p_0$ using $\langle\langle\nu\rangle\rangle \in T^{p_0}$, assuming $\xi^0_{1,2} \geq \nu^0$. In Figure 3 we show a 1-point extension of $p_{0\langle\bar{\nu}\rangle}$ using $\langle\langle\mu\rangle\rangle \in T^{p_0}_{\langle\bar{\nu}\rangle}$, again assuming $\xi_{2,1} \geq \mu^0$.

If $G$ is $P_{\bar{E}}$-generic and we set $C^\alpha = \bigcup\{p^{\langle\alpha,E(0)\rangle} \mid p \in G,\ \langle\alpha, E(0)\rangle \in \operatorname{supp} p\}$, then $C^\alpha$ is an $\omega$-sequence unbounded in $\kappa$.

We continue with the definition of $P_{\bar{E}}$ for $\mathrm{l}(\bar{E}) = 2$.

$$\begin{array}{cccc} & \langle\xi_{1,2}\rangle & & \\ & \langle\xi_{1,1}\rangle & \langle\xi_{2,1}\rangle & \\ \langle\xi_{0,0}\rangle & \langle\xi_{1,0}\rangle & \langle\xi_{2,0}\rangle & T^{p_0} \\ \hline \langle\kappa, E(0)\rangle & \langle\alpha_1, E(0)\rangle & \langle\alpha_2, E(0)\rangle & \end{array}$$

FIGURE 1. An example of $p_0 \in P^*_{\bar{E}}$, $\mathrm{l}(\bar{E}) = 1$

$$\frac{\langle\xi_{0,0}\rangle}{\langle\nu^0\rangle}$$

$$\begin{array}{cccc} & \langle\xi_{1,2}\rangle & \langle\nu\rangle & \\ & \langle\xi_{1,1}\rangle & \langle\xi_{2,1}\rangle & \\ \langle\nu^0\rangle & \langle\xi_{1,0}\rangle & \langle\xi_{2,0}\rangle & T^{p_0}_{\langle\langle\nu\rangle\rangle} \\ \hline \langle\kappa, E(0)\rangle & \langle\alpha_1, E(0)\rangle & \langle\alpha_2, E(0)\rangle & \end{array}$$

FIGURE 2. $p_{0\langle\langle\nu\rangle\rangle}$, a 1-point extension of $p_0$ using $\langle\nu\rangle$

$$\frac{\langle\xi_{0,0}\rangle}{\langle\nu^0\rangle}$$

$$\frac{\langle\nu^0\rangle}{\langle\mu^0\rangle}$$

$$\begin{array}{cccc} & & \langle\mu\rangle & \\ & \langle\xi_{1,2}\rangle & \langle\nu\rangle & \\ & \langle\xi_{1,1}\rangle & \langle\xi_{2,1}\rangle & \\ \langle\mu^0\rangle & \langle\xi_{1,0}\rangle & \langle\xi_{2,0}\rangle & T^{p_0}_{\langle\langle\nu\rangle,\langle\mu\rangle\rangle} \\ \hline \langle\kappa, E(0)\rangle & \langle\alpha_1, E(0)\rangle & \langle\alpha_2, E(0)\rangle & \end{array}$$

FIGURE 3. $p_{0\langle\langle\nu\rangle,\langle\mu\rangle\rangle}$, a 1-point extension of $p_{0\langle\langle\nu\rangle\rangle}$ using $\langle\mu\rangle$

**Definition 3.12.** Assume $\mathrm{l}(\bar{E}) = 2$. A condition in $P^*_{\bar{E}}$ is of the form

$$\{\langle\langle\alpha, E(0), E(1)\rangle, p^{\langle\alpha,E(0),E(1)\rangle}\rangle \mid \langle\alpha, E(0), E(1)\rangle \in g\} \cup \{T\}$$

where

(1) $g \subseteq \bar{E}$, $|g| \leq \kappa$,
(2) $\min \bar{E} = \langle\kappa, E(0), E(1)\rangle \in g$ and $g$ has a maximal element,
(3) $p^{\langle\kappa,E(0),E(1)\rangle} \in V_{\kappa^0(\bar{E})}$ is an extender sequence that we call $p^0$ (we allow $p^0 = \emptyset$),
(4) $\forall\langle\alpha, E(0), E(1)\rangle \in g \setminus \{\langle\kappa, E(0), E(1)\rangle\}$ $p^{\langle\alpha,E(0),E(1)\rangle} \in [V_{\kappa^0(\bar{E})}]^{<\omega}$ is a $^0$-increasing sequence (we allow $p^{\langle\alpha,E(0),E(1)\rangle} = \emptyset$),
(5) $T$ is a $\max g$-tree such that for all $t \in T$, $p^{\max g} \frown t$ is $^0$-increasing, and if $\langle\bar{\nu}\rangle \in T$, then either $\bar{\nu} = \langle\xi\rangle$ or $\bar{\nu} = \langle\xi, e(0)\rangle$, where $e(0)$ is extender and $\xi \in \mathrm{dom}\, e(0)$,
(6) for all $\langle\alpha, E(0), E(1)\rangle \in g$, $p^0$ is not permitted to $p^{\langle\alpha,E(0),E(1)\rangle}$,
(7) $\forall\langle\bar{\nu}\rangle \in T$ $|\{\langle\alpha, E(0), E(1)\rangle \in g \mid \bar{\nu}$ is permitted to $p^{\langle\alpha,E(0),E(1)\rangle}\}| \leq \kappa^0(\bar{\nu})$,

(8) $\forall\langle\bar{\nu}\rangle \in T$, if $\bar{\nu}$ is permitted to $p^{\langle\beta,E(0),E(1)\rangle}, p^{\langle\alpha,E(0),E(1)\rangle}$, then

$$\pi_{\max g,\langle\beta,E(0),E(1)\rangle}(\bar{\nu}) \neq \pi_{\max g,\langle\alpha,E(0),E(1)\rangle}(\bar{\nu}).$$

We write $\mathrm{mc}(p)$, $p^{\mathrm{mc}}$, $T^p$, $\bar{E}(p)$, $\operatorname{supp} p$ for $\max g$, $p^{\max g}$, $T$, $\bar{E}$, $g$ respectively.

**Definition 3.13.** Assume $\mathrm{l}(\bar{E}) = 2$. Let $p, q \in P^*_{\bar{E}}$. We say that $p$ is a *Prikry extension* of $q$ ($p \leq^* q$ or $p \leq^0 q$) if

(1) $\operatorname{supp} p \supseteq \operatorname{supp} q$,
(2) $\forall\langle\alpha, E(0), E(1)\rangle \in \operatorname{supp} q\ p^{\langle\alpha,E(0),E(1)\rangle} = q^{\langle\alpha,E(0),E(1)\rangle}$,
(3) $T^p \leq \pi^{-1}_{\mathrm{mc}(p),\mathrm{mc}(q)} T^q$,
(4) $\forall\langle\alpha, E(0), E(1)\rangle \in \operatorname{supp} q\ \forall\langle\bar{\nu}\rangle \in T^p$

$$\max \kappa(p^{\langle\alpha,E(0),E(1)\rangle}) < \kappa^0(\bar{\nu}) \implies \pi_{\mathrm{mc}(p),\alpha}(\bar{\nu}) = \pi_{\mathrm{mc}(q),\alpha}(\pi_{\mathrm{mc}(p),\mathrm{mc}(q)}(\bar{\nu})).$$

**Definition 3.14.** Assume $\mathrm{l}(\bar{E}) = 2$. A condition in $P_{\bar{E}}$ is of the form

$$p_n {}^\frown \cdots {}^\frown p_0$$

where

- $p_0 \in P^*_{\bar{E}}$,
- $p_1 \in P^*_{\bar{\mu}_1}$,
- $\vdots$
- $p_n \in P^*_{\bar{\mu}_n}$,

where $\bar{E}, \bar{\mu}_1, \ldots, \bar{\mu}_n$ are extender sequence systems such that $\mathrm{l}(\bar{\mu}_1) < 2, \ldots, \mathrm{l}(\bar{\mu}_n) < 2$ satisfying

$$\forall i \leq n-1\ \kappa(\bar{\mu}_{i+1}) < \kappa^0(\bar{\mu}_i) \text{ where } \bar{\mu}_0 = \bar{E}.$$

**Definition 3.15.** Assume $\mathrm{l}(\bar{E}) = 2$. Let $p, q \in P_{\bar{E}}$. We say that $p$ is a *Prikry extension* of $q$ ($p \leq^* q$ or $p \leq^0 q$) if $p, q$ are of the form

$$p = p_n {}^\frown \cdots {}^\frown p_0,$$
$$q = q_n {}^\frown \cdots {}^\frown q_0,$$

and

- $p_0, q_0 \in P^*_{\bar{E}},\ p_0 \leq^* q_0$,
- $p_1, q_1 \in P^*_{\bar{\mu}_1},\ p_1 \leq^* q_1$,
- $\vdots$
- $p_n, q_n \in P^*_{\bar{\mu}_n},\ p_n \leq^* q_n$.

$p_{0\langle\bar{\nu}\rangle}$, defined now, is the basic non-direct extension in $P_{\bar{E}}$ of the condition $p_0$, which adds the extender sequence $\bar{\nu} \in T^{p_0}$ to the finite sequence. If $\bar{\nu} = \langle\xi, e(0)\rangle$, then a condition $p'_1 \in P_{e(0)}$ is added.

**Definition 3.16.** Assume $\mathrm{l}(\bar{E}) = 2$. Let $p \in P^*_{\bar{E}}$ and $\langle\bar{\nu}\rangle \in T^p$. We define $p_{\langle\bar{\nu}\rangle}$ to be $p'_1 {}^\frown p'_0$, where

(1) $\operatorname{supp} p'_0 = \operatorname{supp} p$,

(2) $\forall \langle \alpha, E(0), E(1)\rangle \in \operatorname{supp} p'_0 \; p_0'^{\langle \alpha, E(0), E(1)\rangle} =$

$$\begin{cases} \langle \pi_{\mathrm{mc}(p),\alpha}(\xi), e(0)\rangle, & \max \kappa(p^{\langle \alpha, E(0), E(1)\rangle}) < \kappa^0(\xi),\ \bar{\nu} = \langle \xi, e(0)\rangle, \\ p^{\langle \alpha, E(0), E(1)\rangle} \frown \langle \pi_{\mathrm{mc}(p),\alpha}(\xi), e(0)\rangle, & \max \kappa(p^{\langle \alpha, E(0), E(1)\rangle}) < \kappa^0(\xi),\ \bar{\nu} = \langle \xi\rangle, \\ & \alpha \neq \kappa, \\ \langle \pi_{\mathrm{mc}(p),\alpha}(\xi)\rangle, & \max \kappa(p^{\langle \alpha, E(0), E(1)\rangle}) < \kappa^0(\xi),\ \bar{\nu} = \langle \xi\rangle, \\ & \alpha = \kappa, \\ p^{\langle \alpha, E(0), E(1)\rangle}, & \text{otherwise}, \end{cases}$$

(3) $T^{p'_0} = T^p_{\langle \bar{\nu}\rangle}$,

(4) if $\bar{\nu} = \langle \xi\rangle$, then

(4.1) $\operatorname{supp} p'_1 = \{\langle \pi_{\mathrm{mc}(p),\kappa}(\xi)\rangle\}$,

(4.2) $p_1'^{\langle \pi_{\mathrm{mc}(p),\kappa}(\xi)\rangle} = p^{\langle \kappa, E(0), E(1)\rangle}$,

(4.3) $T^{p'_1} = \emptyset$,

(5) if $\bar{\nu} = \langle \xi, e(0)\rangle$, then

(5.1) $\operatorname{supp} p'_1 = \{\langle \pi_{\mathrm{mc}(p),\alpha}(\xi), e(0)\rangle \mid \langle \alpha, E(0), E(1)\rangle \in \operatorname{supp} p,$ $\max \kappa(p^{\langle \alpha, E(0), E(1)\rangle}) < \kappa^0(\xi)\}$,

(5.2) $\forall \langle \pi_{\mathrm{mc}(p),\alpha}(\xi), e(0)\rangle \in \operatorname{supp} p'_1 \; p_1'^{\langle \pi_{\mathrm{mc}(p),\alpha}(\xi), e(0)\rangle} = p^{\langle \alpha, E(0), E(1)\rangle}$,

(5.3) $T^{p'_1} = T^p(\langle \xi, e(0)\rangle)$ (note that $p'_1 \in P^*_{e(0)}$).

**Definition 3.17.** Assume $\mathrm{l}(\bar{E}) = 2$. Let $p, q \in P_{\bar{E}}$. We say that $p$ is a 1-*point extension* of $q$ ($p \leq^1 q$) if $p, q$ are of the form

$$p = p_{n+1} \frown p_n \frown \cdots \frown p_0,$$
$$q = q_n \frown \cdots \frown q_0,$$

and there is $0 \leq k \leq n$ such that

- $p_i, q_i \in P^*_{\bar{\mu}_i}$, $p_i \leq^* q_i$ for $i = 0, \ldots, k-1$,
- $p_{i+1}, q_i \in P^*_{\bar{\mu}_i}$, $p_{i+1} \leq^* q_i$ for $i = k+1, \ldots, n$,
- there is $\langle \bar{\nu}\rangle \in T^{q_k}$ such that $p_{k+1} \frown p_k \leq^* (q_k)_{\langle \bar{\nu}\rangle}$.

**Definition 3.18.** Assume $\mathrm{l}(\bar{E}) = 2$. Let $p, q \in P_{\bar{E}}$. We say that $p$ is an $n$-*point extension* of $q$ ($p \leq^n q$) if there are $p^n, \ldots, p^0$ such that

$$p = p^n \leq^1 \cdots \leq^1 p^0 = q.$$

**Definition 3.19.** Assume $\mathrm{l}(\bar{E}) = 2$. Let $p, q \in P_{\bar{E}}$. We say that $p$ is an *extension* of $q$ ($p \leq q$) if there is an $n$ such that $p \leq^n q$.

When $\mathrm{l}(\bar{E}) = 2$, the forcing $P_{\bar{E}}$ is similar to a special case of the forcing defined in [8]. Again we give some figures in order to enhance the intuition behind these definitions. In Figure 4 we show a typical condition $p_0 \in P^*_{\bar{E}}$. In Figure 5 we show a 1-point extension of $p_0$ using $\langle\langle \nu\rangle\rangle \in T^{p_0}$, assuming $\xi^0_{1,2} \geq \nu^0$. In Figure 6 we show a 1-point extension of $p_{0\langle \bar{\nu}\rangle}$ using $\langle\langle \mu\rangle\rangle \in T^{p_0}_{\langle \bar{\nu}\rangle}$, again assuming $\xi_{2,1} \geq \mu^0$.

Let $G$ be $P_{\bar{E}}$-generic, and let $C^\alpha = \bigcup\{p^{\langle \alpha, E(0), E(1)\rangle} \mid p \in G,\ \langle \alpha, E(0), E(1)\rangle \in \operatorname{supp} p\}$. Then $C^\alpha$ is an $\omega^2$-sequence unbounded in $\kappa$.

Set also $C^\mu = \bigcup\{p^{\langle \mu, e(0)\rangle} \mid p \in G,\ \langle \mu, e(0)\rangle \in \operatorname{supp} p\}$. Then $C^\mu$ is an $\omega$-sequence unbounded in $\nu^0$.

We now leave the special cases and define the forcing notion for arbitrary $\mathrm{l}(\bar{E})$.

$$\begin{array}{cccc}
 & \xi_{1,2} & & \\
 & \xi_{1,1} & \xi_{2,1} & \\
\xi_{0,0} & \xi_{1,0} & \xi_{2,0} & T^{p_0} \\
\hline
\langle \kappa, E(0), E(1)\rangle & \langle \alpha_1, E(0), E(1)\rangle & \langle \alpha_2, E(0), E(1)\rangle &
\end{array}$$

FIGURE 4. An example of $p_0 \in P^*_{\bar{E}}$, $\mathrm{l}(\bar{E}) = 2$

$$\begin{array}{c}
\xi_{0,0} \\
\hline
\langle \nu^0 \rangle
\end{array}$$

$$\begin{array}{cccc}
 & \xi_{1,2} & \nu & \\
 & \xi_{1,1} & \xi_{2,1} & \\
\nu^0 & \xi_{1,0} & \xi_{2,0} & T^{p_0}_{\langle \nu \rangle} \\
\hline
\langle \kappa, E(0), E(1)\rangle & \langle \alpha_1, E(0), E(1)\rangle & \langle \alpha_2, E(0), E(1)\rangle &
\end{array}$$

FIGURE 5. $p_{0\langle \bar{\nu} \rangle}$, a 1-point extension of $p_0$ using $\langle \nu \rangle$

$$\begin{array}{c}
\xi_{0,0} \\
\hline
\langle \nu^0 \rangle
\end{array}$$

$$\begin{array}{ccc}
 & \nu & \\
 & \xi_{2,1} & \\
\nu^0 & \xi_{2,0} & T^{p_0}_{\langle \nu \rangle}(\langle \mu, e(0)\rangle) \\
\hline
\langle \mu^0, e(0)\rangle & \langle \mu, e(0)\rangle &
\end{array}$$

$$\begin{array}{cccc}
 & \xi_{1,2} & & \\
 & \xi_{1,1} & & \\
\langle \mu^0, e(0)\rangle & \xi_{1,0} & \langle \mu, e(0))\rangle & T^{p_0}_{\langle \langle \nu \rangle, \langle \mu, e(0)\rangle\rangle} \\
\hline
\langle \kappa, E(0), E(1)\rangle & \langle \alpha_1, E(0), E(1)\rangle & \langle \alpha_2, E(0), E(1)\rangle &
\end{array}$$

FIGURE 6. $p_{0\langle \nu, \langle \mu, e(0)\rangle\rangle}$, a 1-point extension of $p_{0\langle \nu \rangle}$ using $\langle \mu, e(0)\rangle$

**Definition 3.20.** A condition in $P^*_{\bar{E}}$ is of the form

$$\{\langle \bar{\gamma}, p^{\bar{\gamma}} \rangle \mid \bar{\gamma} \in g\} \cup \{T\}$$

where

(1) $g \subseteq \bar{E}$, $|g| \leq \kappa$,
(2) $\min \bar{E} = \bar{E}_\kappa \in g$ and $g$ has a maximal element,
(3) $p^{\min \bar{E}} \in V_{\kappa^0(\bar{E})}$ is an extender sequence, call it $p^0$ (we allow $p^0 = \emptyset$),
(4) $\forall \bar{\gamma} \in g \setminus \{\min \bar{E}\}$ $p^{\bar{\gamma}} \in [V_{\kappa^0(\bar{E})}]^{<\omega}$ is a $^0$-increasing sequence (we allow $p^{\bar{\gamma}} = \emptyset$),
(5) $T$ is a $\max g$-tree such that for all $t \in T$, $p^{\max g} \frown t$ is $^0$-increasing,
(6) for all $\bar{\gamma} \in g$, $p^0$ is not permitted to $p^{\bar{\gamma}}$,
(7) $\forall \langle \bar{\nu} \rangle \in T$ $|\{\bar{\gamma} \in g \mid \bar{\nu} \text{ is permitted to } p^{\bar{\gamma}}\}| \leq \kappa^0(\bar{\nu})$,
(8) $\forall \langle \bar{\nu} \rangle \in T$ if $\bar{\nu}$ is permitted to $p^{\bar{\beta}}, p^{\bar{\gamma}}$, then $\pi_{\max g, \bar{\beta}}(\bar{\nu}) \neq \pi_{\max g, \bar{\gamma}}(\bar{\nu})$.

We write $\mathrm{mc}(p)$, $p^{\mathrm{mc}}$, $T^p$, $\bar{E}(p)$, $\operatorname{supp} p$ for $\max g$, $p^{\max g}$, $T$, $\bar{E}$, $g$ respectively.

**Definition 3.21.** Let $p, q \in P^*_{\bar{E}}$. We say that $p$ is a *Prikry extension* of $q$ ($p \leq^* q$ or $p \leq^0 q$) if

(1) $\operatorname{supp} p \supseteq \operatorname{supp} q$,
(2) $\forall \gamma \in \operatorname{supp} q \; p^\gamma = q^\gamma$,
(3) $T^p \leq \pi^{-1}_{\mathrm{mc}(p),\mathrm{mc}(q)} T^q$,
(4) $\forall \bar{\gamma} \in \operatorname{supp} q \; \forall \langle \bar{\nu} \rangle \in T^p$

$$\max \kappa(p^{\bar{\gamma}}) < \kappa^0(\bar{\nu}) \implies \pi_{\mathrm{mc}(p),\bar{\gamma}}(\bar{\nu}) = \pi_{\mathrm{mc}(q),\bar{\gamma}}(\pi_{\mathrm{mc}(p),\mathrm{mc}(q)}(\bar{\nu})).$$

Note that if $p, q \in P_{\bar{E}}$ satisfy all details of the above definition except 4, then by just shrinking $T^p$ we can get 4 also. The proof is the same as the one given in [3].

In many cases in the sequel we use $\pi^{-1}_{\bar{\beta},\bar{\alpha}} T$. Whenever we do so we automatically assume the above shrinkage, which of course depends on the conditions used.

**Definition 3.22.** A condition in $P_{\bar{E}}$ is of the form

$$p_n \frown \cdots \frown p_0$$

where

- $p_0 \in P^*_{\bar{E}}$,
- $p_1 \in P^*_{\bar{\mu}_1}$,
- $\vdots$
- $p_n \in P^*_{\bar{\mu}_n}$,

where $\bar{E}, \bar{\mu}_1, \ldots, \bar{\mu}_n$ are extender sequence systems such that $\mathrm{l}(\bar{\mu}_1) < \mathrm{l}(\bar{E})$, $\ldots$, $\mathrm{l}(\bar{\mu}_n) < \mathrm{l}(\bar{E})$ satisfying

$$\forall i \leq n-1 \; \kappa(\bar{\mu}_{i+1}) < \kappa^0(\bar{\mu}_i) \text{ where } \bar{\mu}_0 = \bar{E}.$$

**Definition 3.23.** Let $p, q \in P_{\bar{E}}$. We say that $p$ is a *Prikry extension* of $q$ ($p \leq^* q$ or $p \leq^0 q$) if $p, q$ are of the form

$$p = p_n \frown \cdots \frown p_0,$$
$$q = q_n \frown \cdots \frown q_0,$$

and

- $p_0, q_0 \in P^*_{\bar{E}}, \; p_0 \leq^* q_0$,
- $p_1, q_1 \in P^*_{\bar{\mu}_1}, \; p_1 \leq^* q_1$,
- $\vdots$
- $p_n, q_n \in P^*_{\bar{\mu}_n}, \; p_n \leq^* q_n$.

$p_{0\langle \bar{\nu} \rangle}$, defined now, is the basic non-direct extension in $P_{\bar{E}}$ of the condition $p_0$, which adds the extender sequence $\bar{\nu} \in T^{p_0}$ to the finite sequence. A condition $p'_1 \in P_{\bar{E}(\bar{\nu})}$ is added.

**Definition 3.24.** Let $p \in P^*_{\bar{E}}$ and $\langle \bar{\nu} \rangle \in T^p$. We define $(p)_{\langle \bar{\nu} \rangle}$ to be $p'_1 \frown p'_0$ where

(1) $\operatorname{supp} p'_0 = \operatorname{supp} p$,
(2) $\forall \bar{\gamma} \in \operatorname{supp} p'_0 \; p'^{\bar{\gamma}}_0 = \begin{cases} \pi_{\mathrm{mc}(p),\bar{\gamma}}(\bar{\nu}), & \max \kappa(p^{\bar{\gamma}}) < \kappa^0(\bar{\nu}),\ \mathrm{l}(\bar{\nu}) > 0, \\ p^{\bar{\gamma}} \frown \pi_{\mathrm{mc}(p),\bar{\gamma}}(\bar{\nu}), & \begin{array}{l} \max \kappa(p^{\bar{\gamma}}) < \kappa^0(\bar{\nu}),\ \mathrm{l}(\bar{\nu}) = 0, \\ \bar{\gamma} \neq \bar{E}_\kappa, \end{array} \\ \pi_{\mathrm{mc}(p),\bar{\gamma}}(\bar{\nu}), & \begin{array}{l} \max \kappa(p^{\bar{\gamma}}) < \kappa^0(\bar{\nu}),\ \mathrm{l}(\bar{\nu}) = 0, \\ \bar{\gamma} = \bar{E}_\kappa. \end{array} \\ p^{\bar{\gamma}}, & \text{otherwise}, \end{cases}$

(3) $T^{p_0'} = T^p_{\langle\bar{\nu}\rangle}$,
(4) if $\mathrm{l}(\bar{\nu}) = 0$, then
  (4.1) $\operatorname{supp} p_1' = \{\pi_{\mathrm{mc}(p),\bar{E}_\kappa}(\bar{\nu})\}$,
  (4.2) $p_1'^{\pi_{\mathrm{mc}(p),\bar{E}_\kappa}(\bar{\nu})} = p^{\bar{E}_\kappa}$,
  (4.3) $T^{p_1'} = \emptyset$,
(5) if $\mathrm{l}(\bar{\nu}) > 0$, then
  (5.1) $\operatorname{supp} p_1' = \{\pi_{\mathrm{mc}(p),\bar{\gamma}}(\bar{\nu}) \mid \bar{\gamma} \in \operatorname{supp} p,\ \max \kappa(p^{\bar{\gamma}}) < \kappa^0(\bar{\nu})\}$,
  (5.2) $\forall \pi_{\mathrm{mc}(p),\bar{\gamma}}(\bar{\nu}) \in \operatorname{supp} p_1'\ p_1'^{\pi_{\mathrm{mc}(p),\bar{\gamma}}(\bar{\nu})} = p^{\bar{\gamma}}$,
  (5.3) $T^{p_1'} = T^p(\bar{\nu})$ (note that $p_1' \in P^*_{\bar{\epsilon}}$ for $\bar{\epsilon}$ associated with $\bar{\nu}$).

**Definition 3.25.** Let $p, q \in P_{\bar{E}}$. We say that $p$ is a 1-*point extension* of $q$ ($p \leq^1 q$) if $p, q$ are of the form

$$p = p_{n+1} {}^\frown p_n {}^\frown \cdots {}^\frown p_0,$$
$$q = q_n {}^\frown \cdots {}^\frown q_0,$$

and there is $0 \leq k \leq n$ such that

- $p_i, q_i \in P^*_{\bar{\mu}_i}$, $p_i \leq^* q_i$ for $i = 0, \ldots, k-1$,
- $p_{i+1}, q_i \in P^*_{\bar{\mu}_i}$, $p_{i+1} \leq^* q_i$ for $i = k+1, \ldots, n$,
- there is $\langle\bar{\nu}\rangle \in T^{q_k}$ such that $p_{k+1} {}^\frown p_k \leq^* (q_k)_{\langle\bar{\nu}\rangle}$.

**Definition 3.26.** Let $p, q \in P_{\bar{E}}$. We say that $p$ is an *n-point extension* of $q$ ($p \leq^n q$) if there are $p^n, \ldots, p^0$ such that

$$p = p^n \leq^1 \cdots \leq^1 p^0 = q.$$

**Definition 3.27.** Let $p, q \in P_{\bar{E}}$. We say that $p$ is an *extension* of $q$ ($p \leq q$) if there is an $n$ such that $p \leq^n q$.

Later on by $P_{\bar{E}}$ we mean $\langle P_{\bar{E}}, \leq\rangle$.

| | | | |
|---|---|---|---|
| | $\bar{\xi}_{1,2}$ | | |
| | $\bar{\xi}_{1,1}$ | $\bar{\xi}_{2,1}$ | |
| $\bar{\xi}_{0,0}$ | $\bar{\xi}_{1,0}$ | $\bar{\xi}_{2,0}$ | $T^{p_0}$ |
| $\bar{E}_\kappa$ | $\bar{E}_{\alpha_1}$ | $\bar{E}_{\alpha_2}$ | |

FIGURE 7. An example of $p_0 \in P^*_{\bar{E}}$

*Note* 3.28. When $\mathrm{l}(\bar{E}) = 1$ the forcing $P_{\bar{E}}$ is the Gitik-Magidor forcing from section 1 of [3]. When $\mathrm{l}(\bar{E}) < \kappa$ the forcing $P_{\bar{E}}$ is similar to the forcing defined in [8].

In several places we want to prevent enlargement of the support of a condition. This makes all the conditions which are stronger than some condition but with the same support resemble Radin forcing. The following definition catches the meaning of not enlarging the support. The 'resemblance' we look for is 5.3.

**Definition 3.29.** Let $p, q \in P^*_{\bar{E}}$. We say that $p \leq^*_R q$ if

(1) $p \leq^* q$, and
(2) $\operatorname{supp} p = \operatorname{supp} q$.

$$\frac{\bar{\xi}_{0,0}}{\langle\nu^0\rangle}$$

$$\begin{array}{cccc}
 & \bar{\xi}_{1,2} & \langle\nu\rangle & \\
 & \bar{\xi}_{1,1} & \bar{\xi}_{2,1} & \\
\langle\nu^0\rangle & \bar{\xi}_{1,0} & \bar{\xi}_{2,0} & T^{p_0}_{\langle\langle\nu\rangle\rangle} \\
\hline
\bar{E}_\kappa & \bar{E}_{\alpha_1} & \bar{E}_{\alpha_2} &
\end{array}$$

FIGURE 8. $p_{0\langle\bar{\nu}\rangle}$, a 1-point extension of $p_0$ using $\langle\nu\rangle$

$$\frac{\bar{\xi}_{0,0}}{\langle\nu^0\rangle}$$

$$\begin{array}{ccc}
 & \langle\nu\rangle & \\
 & \bar{\xi}_{2,1} & \\
\langle\nu^0\rangle & \bar{\xi}_{2,0} & T^{p_0}_{\langle\langle\nu\rangle\rangle}(\bar{\mu}) \\
\hline
\bar{\mu}^0 & \bar{\mu} &
\end{array}$$

$$\begin{array}{cccc}
 & \bar{\xi}_{1,2} & & \\
 & \bar{\xi}_{1,1} & & \\
\bar{\mu}^0 & \bar{\xi}_{1,0} & \bar{\mu} & T^{p_0}_{\langle\langle\nu\rangle,\bar{\mu}\rangle} \\
\hline
\bar{E}_\kappa & \bar{E}_{\alpha_1} & \bar{E}_{\alpha_2} &
\end{array}$$

FIGURE 9. $p_{0\langle\langle\nu\rangle,\bar{\mu}\rangle}$, a 1-point extension of $p_{0\langle\nu\rangle}$ using $\bar{\mu}$

**Definition 3.30.** Let $p, q \in P_{\bar{E}}$. We say that $p \leq^1_R q$ if

(1) $p \leq^1 q$, and
(2) in the definition of $\leq^1$, we can replace $\leq^*$ by $\leq^*_R$.

**Definition 3.31.** Let $p, q \in P_{\bar{E}}$. We say that $p \leq^n_R q$ if there are $p^n, \ldots, p^0$ such that

$$p = p^n \leq^1_R \cdots \leq^1_R p^0 = q.$$

**Definition 3.32.** Let $p, q \in P_{\bar{E}}$. We say that $p \leq_R q$ if there is an $n$ such that $p \leq^n_R q$.

*Note* 3.33. The above definitions imply that if $p \leq q$, then there is an $r$ such that $p \leq^* r \leq_R q$.

*Note* 3.34. When $\mathrm{l}(\bar{E}) = 1$ and we force with $\langle P_{\bar{E}}, \leq_R\rangle$ below some $p$ with maximal coordinate $\alpha$, we are forcing just the tree Prikry forcing for the measure $E_\alpha(0)$.

**Definition 3.35.** Let $\bar{\epsilon}$ be an extender sequence such that $\kappa^0(\bar{\epsilon}) < \kappa^0(\bar{E})$. Then

$$P_{\bar{E}}/P_{\bar{\epsilon}} = \{p \mid \exists q \in P_{\bar{\epsilon}},\ q^\frown p \in P_{\bar{E}}\}.$$

## 4. Radin Forcing

The main aims of this section are 4.8, 4.10. Since the simplest way we found to formulate them was with Radin forcing [7], [5], [10], we took the opportunity to

depart from the usual formulation in order to work in the spirit of the extender-based forcing we defined in section 3.

The main point is that possible extensions of a condition are stored in an $\bar{E}_\alpha$-tree and not in a set. The $\alpha$ is fixed; so practically we deal here with a measure sequence and not an extender sequence.

**Definition 4.1.** A condition in $R_\alpha$ is of the form

$$\langle\langle\bar{\mu}_n, s^n\rangle, S^n, \ldots, \langle\bar{\mu}_1, s^1\rangle, S^1, \langle\bar{\mu}_0, s^0\rangle, S^0\rangle,$$

where

(1) $\bar{\mu}_0, \ldots, \bar{\mu}_n$ are extender sequences,
(2) $\forall i \leq n-1\ \kappa(\bar{\mu}_{i+1}) < \kappa^0(\bar{\mu}_i)$,
(3) $\bar{\mu}_0 = \bar{E}_\alpha$,
(4) $\forall i \leq n\ S^i$ is a $\bar{\mu}_i$-tree,
(5) $\forall i \leq n\ s^i \in V_{\kappa^0(\bar{\mu}_i)}$ is an extender sequence.

**Definition 4.2.** Let $p, q \in R_\alpha$. We say that $p$ is a *Prikry extension* of $q$ ($p \leq^* q$ or $p \leq^0 q$) if $p, q$ are of the form

$$p = \langle\langle\bar{\mu}_n, s^n\rangle, S^n, \ldots, \langle\bar{\mu}_1, s^1\rangle, S^1, \langle\bar{E}_\alpha, s^0\rangle, S^0\rangle,$$
$$q = \langle\langle\bar{\mu}_n, t^n\rangle, T^n, \ldots, \langle\bar{\mu}_1, t^1\rangle, T^1, \langle\bar{E}_\alpha, t^0\rangle, T^0\rangle,$$

and

- $\forall i \leq n\ S^i \leq T^i$,
- $\forall i \leq n\ s^i = t^i$.

**Definition 4.3.** Let $p = \langle\langle\bar{\mu}_n, s^n\rangle, S^n, \ldots, \langle\bar{\mu}_1, s^1\rangle, S^1, \langle\bar{\mu}_0, s^0\rangle, S^0\rangle$ where $\bar{\mu}_0 = \bar{E}_\alpha$. Let $\langle\bar{\nu}\rangle \in S^i$. We define $(p)_{\langle\bar{\nu}\rangle}$ to be

$$\begin{aligned}(p)_{\langle\bar{\nu}\rangle} = \langle\langle\bar{\mu}_n, s^n\rangle, S^n, \ldots, \langle\bar{\mu}_{i+1}, s^{i+1}\rangle, S^{i+1},\\ \langle\bar{\nu}, s^i\rangle, S^i(\bar{\nu}), \langle\bar{\mu}_i, \bar{\nu}\rangle, S^i_{\langle\bar{\nu}\rangle},\\ \langle\bar{\mu}_{i-1}, s^{i-1}\rangle, S^{i-1}, \ldots, \langle\bar{\mu}_0, s^0\rangle, S^0\rangle.\end{aligned}$$

Note the degenerate case in this definition when $\mathrm{l}(\bar{\nu}) = 0$. In this case $S^i(\bar{\nu}) = \emptyset$.

**Definition 4.4.** Let $p, q \in R_\alpha$ where

$$q = \langle\langle\bar{\mu}_n, s^n\rangle, S^n, \ldots, \langle\bar{\mu}_1, s^1\rangle, S^1, \langle\bar{\mu}_0, s^0\rangle, S^0\rangle.$$

We say that $p$ is a 1-*point extension* of $q$ ($p \leq^1 q$) if there is $\langle\bar{\nu}\rangle \in S^i$ such that $p \leq^* (q)_{\langle\bar{\nu}\rangle}$.

**Definition 4.5.** Let $p, q \in R_\alpha$. We say that $p$ is an $n$-*point extension* of $q$ ($p \leq^n q$) if there are $p^n, \ldots, p^0$ such that

$$p = p^n \leq^1 \ldots \leq^1 p^0 = q.$$

**Definition 4.6.** Let $p, q \in R_\alpha$. We say that $p$ is an *extension* of $q$ ($p \leq q$) if there is an $n$ such that $p \leq^n q$.

Lemma 4.8 is needed in the proof of Theorem 7.1. Very loosely speaking, Lemma 4.8 means that if "something" happens on a measure-one set for one of the measures, that "something" is happening on a measure-one set for all the measures.

Lemma 4.8 is proved by induction, and Lemma 4.7 is the first case of the induction.

**Lemma 4.7.** *Suppose* $\mathrm{l}(\bar{E}_\alpha) = 2$, $i < 2$, *and there is a tree* $T$ *(*not *an* $\bar{E}_\alpha$ *tree) such that* $\mathrm{Lev}_0(T) \in E_\alpha(i)$, *and* $\forall \langle \bar{\nu} \rangle \in T$ $T_{\langle \bar{\nu} \rangle}$ *is an* $\bar{E}_\alpha$*-tree. Then there is an* $\bar{E}_\alpha$*-tree,* $T^*$, *satisfying*

(1) $\forall \langle \bar{\nu} \rangle \in T \cap T^*$ $T^*_{\langle \bar{\nu} \rangle} \leq T_{\langle \bar{\nu} \rangle}$, $T^*(\bar{\nu}) \leq T(\bar{\nu})$, *and*

(2) *if* $p \leq \big\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^* \big\rangle$, *then there is* $\langle \bar{\mu} \rangle \in T^* \cap T$ *such that*

$$p \parallel \big( \big\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^* \big\rangle \big)_{\langle \bar{\mu} \rangle}.$$

*Proof.* There are two cases to deal with:

- $A_0 = \mathrm{Lev}_0(T) \in E_\alpha(0)$: If $A_0 \in E_\alpha(1)$, we set $T^* = T$, and the proof is finished. So suppose $A_0 \notin E_\alpha(1)$. We would like to build $A_1 \in E_\alpha(1)$. Set

  $$\forall \langle \bar{\mu}_0 \rangle \in T \ A_{\langle \bar{\mu}_0 \rangle, 1} = \{ \langle \bar{\mu}_1, S \rangle \in T_{\langle \bar{\mu}_0 \rangle} \mid A_0 \cap \kappa^0(\bar{\mu}_1) \in \bar{\mu}_1(0)_{\kappa(\bar{\mu}_1)} \}.$$

  Since $A_0 \in E_\alpha(0)$ and $\mathrm{Suc}_T(\langle \bar{\mu}_0 \rangle) \in E_\alpha(1)$, we get that $A_{\langle \bar{\mu}_0 \rangle, 1} \in E_\alpha(1)$. Let

  $$A_1 = \underset{\langle \bar{\mu}_0 \rangle \in T}{\triangle^0} A_{\langle \bar{\mu}_0 \rangle, 1}.$$

  We can now construct $T^*$:

  $$\begin{aligned} &\mathrm{Lev}_0(T^*) = A_0 \cup A_1, \\ &\forall \bar{\mu}_0 \in A_0 \ T^*_{\langle \bar{\mu}_0 \rangle} = T_{\langle \bar{\mu}_0 \rangle}, \\ &\forall \bar{\mu}_1 \in A_1 \ T^*_{\langle \bar{\mu}_1 \rangle} = \bigcap_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle \in T} T_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle}. \end{aligned}$$

- $A_1 = \mathrm{Lev}_0(T) \in E_\alpha(1)$: If $A_1 \in E_\alpha(0)$, we set $T^* = T$ and finish the proof. So assume $A_1 \notin E_\alpha(0)$. We would like to build $A_0 \in E_\alpha(0)$. Set

  $$\begin{aligned} S &= j(T)\big(\langle \alpha, E(0) \rangle\big), \\ A_0 &= \mathrm{Lev}_0(S) \setminus A_1. \end{aligned}$$

  We construct $T^*$:

  $$\begin{aligned} &\mathrm{Lev}_0(T^*) = A_0 \cup A_1, \\ &\forall \bar{\mu}_1 \in A_1 \ T^*_{\langle \bar{\mu}_1 \rangle} = T_{\langle \bar{\mu}_1 \rangle}. \end{aligned}$$

  We are left with the construction of $T^*_{\langle \bar{\mu}_0 \rangle}$ for $\bar{\mu}_0 \in A_0$. For all $\bar{\mu} \in A_0$, set

  $$\begin{aligned} &A_{\langle \bar{\mu}_0 \rangle, 0} = \mathrm{Suc}_S\big(\langle \bar{\mu}_0 \rangle\big), \\ &A_{\langle \bar{\mu}_0 \rangle, 1} = \{ \langle \bar{\mu}_1, T(\bar{\mu}_1)_{\langle \bar{\mu}_0 \rangle} \rangle \mid \langle \bar{\mu}_1 \rangle \in T, \ \langle \bar{\mu}_0 \rangle \in T(\bar{\mu}_1) \}, \\ &\mathrm{Suc}_{T^*}\big(\langle \bar{\mu}_0 \rangle\big) = A_{\langle \bar{\mu}_0 \rangle, 0} \cup A_{\langle \bar{\mu}_0 \rangle, 1}, \\ &\forall \bar{\mu}_1 \in A_{\langle \bar{\mu}_0 \rangle, 1} \ T^*_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle} = T_{\langle \bar{\mu}_1 \rangle}. \end{aligned}$$

  We continue one more level and hope this will convince the reader we indeed can complete $T^*$. We are left with the construction of $T^*_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle}$ for $\langle \bar{\mu}_0, \bar{\mu}_1 \rangle \in A_0 \times A_{\langle \bar{\mu}_0 \rangle, 0}$. For all $\langle \bar{\mu}_0, \bar{\mu}_1 \rangle \in A_0 \times A_{\langle \bar{\mu}_0 \rangle, 0}$, set

  $$\begin{aligned} &A_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle, 0} = \mathrm{Suc}_S\big(\langle \bar{\mu}_0, \bar{\mu}_1 \rangle\big), \\ &A_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle, 1} = \{ \langle \bar{\mu}_2, T(\bar{\mu}_2)_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle} \rangle \mid \langle \bar{\mu}_2 \rangle \in T, \langle \bar{\mu}_0, \bar{\mu}_1 \rangle \in T(\bar{\mu}_2) \}, \\ &\mathrm{Suc}_{T^*}\big(\langle \bar{\mu}_0, \bar{\mu}_1 \rangle\big) = A_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle, 0} \cup A_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle, 1}, \\ &\forall \bar{\mu}_2 \in A_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle, 1} \ T^*_{\langle \bar{\mu}_0, \bar{\mu}_1, \bar{\mu}_2 \rangle} = T_{\langle \bar{\mu}_2 \rangle}. \end{aligned}$$

We are left with the construction of $T^*_{\langle\bar\mu_0,\bar\mu_1,\bar\mu_2\rangle}$ for $\langle\bar\mu_0,\bar\mu_1,\bar\mu_2\rangle \in A_0 \times A_{\langle\bar\mu_0\rangle,0} \times A_{\langle\bar\mu_0,\bar\mu_1\rangle,0}$, and we hope that by now the continuation is clear. $\square$

**Lemma 4.8.** *Let $\xi_0 < \mathrm{l}(\bar E_\alpha)$, and let $T$ be a tree such that $\mathrm{Lev}_0(T) \in E_\alpha(\xi_0)$ and $\forall\langle\bar\mu\rangle \in T\ T_{\langle\bar\mu\rangle}$ is an $\bar E_\alpha$-tree. Then there is an $\bar E_\alpha$-tree, $T^*$, satisfying*

(1) *$\forall\langle\bar\nu\rangle \in T\cap T^*\ T^*_{\langle\bar\nu\rangle} \le T_{\langle\bar\nu\rangle}$, $T^*(\bar\nu) \le T(\bar\nu)$, and*
(2) *if $p \le \big\langle\langle\bar E_\alpha,\langle\rangle\rangle, T^*\big\rangle$, then there is $\langle\bar\mu\rangle \in T^*\cap T$ such that*

$$p \parallel \big(\big\langle\langle\bar E_\alpha,\langle\rangle\rangle, T^*\big\rangle\big)_{\langle\bar\mu\rangle}.$$

*Proof.* Our induction hypothesis is that this lemma is true for $\bar\mu$'s with $\mathrm{l}(\bar\mu) < \mathrm{l}(\bar E_\alpha)$. The previous lemma is the case for $\mathrm{l}(\bar\mu) = 2$.

Let $S = j(T)(\bar E_\alpha\restriction\xi_0)$. The tree $S$ is an $\bar E_\alpha\restriction\xi_0$-tree. We extend it step by step to a full $\bar E_\alpha$-tree as required.

Let

$$\begin{aligned} A_{\xi_0} &= \mathrm{Lev}_0(T),\\ A_{<\xi_0} &= \mathrm{Lev}_0(S)\setminus A_{\xi_0}. \end{aligned}$$

For $\xi_0 < \xi < \mathrm{l}(\bar E_\alpha)$, do the following:

$$\begin{aligned} N_\xi &= \mathrm{Ult}(V, E_\alpha(\xi)),\\ k_\xi([h]_{E_\alpha(\xi)}) &= j(h)(\bar E_\alpha\restriction\xi), \end{aligned}$$

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d] & M \\ N_\xi = \mathrm{Ult}(V, E_\alpha(\xi)) \arrow[ur, "k_\xi"'] & \end{tikzcd}$$

Since

$$\langle\alpha, E(0),\dots,E(\tau),\dots \mid \tau<\xi\rangle = k_\xi([id]_{E_\alpha(\xi)}) \in \mathrm{ran}(k_\xi),$$

there is in $N_\xi$ a preimage for it:

$$\langle\alpha', E'(0),\dots,E'(\tau'),\dots \mid \tau'<\xi'\rangle.$$

Since $A_{\xi_0} \in E_\alpha(\xi_0)$, $\xi_0 < \xi$, we have $\tau' < \xi'$ such that $A_{\xi_0} \in E'_{\alpha'}(\tau')$ (where $\alpha' = [\kappa(id)]_{E_\alpha(\xi)}$). Taking a function $h_\xi$ such that $[h_\xi]_{E_\alpha(\xi)} = \tau'$, we get

$$\{\bar\mu \mid A_{\xi_0}\cap\kappa^0(\bar\mu) \in \bar\mu(h_\xi(\bar\mu))_{\kappa(\bar\mu)}\} \in E_\alpha(\xi).$$

For each $\bar\mu_0 \in A_{\xi_0}$ we set

$$\begin{aligned} A_{\xi,\langle\bar\mu_0,T(\bar\mu_0)\rangle} &= \{\langle\bar\mu_1,R\rangle \in T_{\langle\bar\mu_0\rangle} \mid A_{\xi_0}\cap\kappa^0(\bar\mu_1) \in \bar\mu_1(h_\xi(\bar\mu_1))_{\kappa(\bar\mu)}\},\\ A'_\xi &= \mathop{\triangle^0}_{\langle\bar\mu_0,T(\bar\mu_0)\rangle} A_{\xi,\langle\bar\mu_0,T(\bar\mu_0)\rangle}. \end{aligned}$$

For any tree $R$ which appears in a pair $\langle\bar\mu_1,R\rangle \in A'_\xi$ we can invoke our lemma by induction and generate $R^*$ which is a $\bar\mu_1$-tree. Now define $A_\xi$ as

$$\langle\bar\mu_1,R^*\rangle \in A_\xi \iff \langle\bar\mu_1,R\rangle \in A'_\xi.$$

When we have $\{A_\xi \mid \xi_0 < \xi < \mathrm{l}(\bar{E}_\alpha)\}$ we set

$$\begin{aligned}
&A_{>\xi_0} = \bigcup_{\xi_0<\xi<\mathrm{l}(\bar{E}_\alpha)} A_\xi \setminus (A_{<\xi_0} \cup A_{\xi_0}),\\
&\mathrm{Lev}_0(T^*) = A_{<\xi_0} \cup A_{\xi_0} \cup A_{>\xi_0},\\
&\forall \langle \bar{\mu}_0 \rangle \in A_{\xi_0}\ T^*_{\langle \bar{\mu}_0 \rangle} = T_{\langle \bar{\mu}_0 \rangle},\\
&\forall \langle \bar{\mu}_1 \rangle \in A_{>\xi_0}\ T^*_{\langle \bar{\mu}_1 \rangle} = \bigcap T_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle}.
\end{aligned}$$

We are left to define $T_{\langle \bar{\mu}_0 \rangle}$ for $\langle \bar{\mu}_0 \rangle \in A_{<\xi_0}$. For each $\bar{\mu}_0 \in A_{<\xi_0}$, set

$$\begin{aligned}
&A_{\langle \bar{\mu}_0 \rangle, \xi_0} = \{\langle \bar{\mu}_1, R_{\langle \bar{\mu}_0 \rangle} \rangle \mid \langle \bar{\mu}_1, R \rangle \in A_{\xi_0},\ \langle \bar{\mu}_0 \rangle \in R\},\\
&A_{\langle \bar{\mu}_0 \rangle, <\xi_0} = \mathrm{Suc}_S(\langle \bar{\mu}_0 \rangle) \setminus A_{\langle \bar{\mu}_0 \rangle, \xi_0}.
\end{aligned}$$

For each $\bar{\mu}_1 \in A_{\xi_0}$ we set

$$\begin{aligned}
&A_{\langle \bar{\mu}_0 \rangle, \xi, \langle \bar{\mu}_1, T(\bar{\mu}_1)_{\langle \bar{\mu}_0 \rangle} \rangle} = \{\langle \bar{\mu}_2, R \rangle \in T_{\langle \bar{\mu}_1 \rangle} \mid\ A_{\xi_0} \cap \kappa^0(\bar{\mu}_2) \in \bar{\mu}_2(h_\xi(\bar{\mu}_2))_{\kappa(\bar{\mu}_2)}\},\\
&A'_{\langle \bar{\mu}_0 \rangle, \xi} = \underset{\langle \bar{\mu}_1, T(\bar{\mu}_1)_{\langle \bar{\mu}_0 \rangle} \rangle}{\triangle^0} A_{\langle \bar{\mu}_0 \rangle, \xi, \langle \bar{\mu}_1, T(\bar{\mu}_1)_{\langle \bar{\mu}_0 \rangle} \rangle}.
\end{aligned}$$

We define

$$\langle \bar{\mu}_0, R^* \rangle \in A_{\langle \bar{\mu}_0 \rangle, \xi} \iff \langle \bar{\mu}_0, R \rangle \in A'_{\langle \bar{\mu}_0 \rangle, \xi}$$

where $R^*$ is generated from $R$ using the current lemma by induction. Now we set

$$\begin{aligned}
&A_{\langle \bar{\mu}_0 \rangle, >\xi_0} = \bigcup_{\xi_0<\xi<\mathrm{l}(\bar{E}_\alpha)} A_{\langle \bar{\mu}_0 \rangle, \xi} \setminus (A_{\langle \bar{\mu}_0 \rangle, <\xi_0} \cup A_{\langle \bar{\mu}_0 \rangle, \xi_0}),\\
&\mathrm{Suc}_{T^*}(\langle \bar{\mu}_0 \rangle) = A_{\langle \bar{\mu}_0 \rangle, <\xi_0} \cup A_{\langle \bar{\mu}_0 \rangle, \xi_0} \cup A_{\langle \bar{\mu}_0 \rangle, >\xi_0},\\
&\forall \langle \bar{\mu}_1 \rangle \in A_{\bar{\mu}_0, \xi_0}\ T^*_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle} = T_{\langle \bar{\mu}_1 \rangle},\\
&\forall \langle \bar{\mu}_2 \rangle \in A_{\bar{\mu}_0, >\xi_0}\ T^*_{\langle \bar{\mu}_0, \bar{\mu}_2 \rangle} = \bigcap T_{\langle \bar{\mu}_1, \bar{\mu}_2 \rangle}.
\end{aligned}$$

This leaves us with the definition of $T_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle}$ for $\langle \bar{\mu}_0, \bar{\mu}_1 \rangle \in A_{<\xi_0} \times A_{\langle \bar{\mu}_0 \rangle, <\xi_0}$, which is done exactly as in this step. □

Lemma 4.10 is needed in the proof of Theorem 6.1. Loosely speaking, it says that if "something" happens on all extensions which are taken from $\mathrm{dom}\, T$, then that "something" happens on all extensions from $T$.

Lemma 4.10 is proved by induction where Lemma 4.9 is the first case.

**Lemma 4.9.** *Assume* $\mathrm{l}(\bar{E}_\alpha) = 2$, *and let* $T$ *be an* $\bar{E}_\alpha$*-tree. Then there is* $T^* \leq T$ *such that, if*

$$p \leq \big\langle \langle \bar{E}_\alpha, \langle\rangle \rangle, T^* \big\rangle,$$

*then there is* $\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in T$ *such that*

$$p \leq^* \big(\big\langle \langle \bar{E}_\alpha, \langle\rangle \rangle, T \big\rangle\big)_{\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle}.$$

*Proof.* As is usual in this section, the proof is done level by level. Let us set $T^1 = T$. It is trivially true that if

$$p \leq^1 \big\langle \langle \bar{E}_\alpha, \langle\rangle \rangle, T^1 \big\rangle,$$

then there is $\langle \bar{\nu}_1 \rangle \in T^1 = T$ such that

$$p \leq^* \big(\big\langle \langle \bar{E}_\alpha, \langle\rangle \rangle, T^1 \big\rangle\big)_{\langle \bar{\nu}_1 \rangle}.$$

We continue to the second level. Let us set

$$\begin{aligned}
&A_{\langle\bar\nu_1,S^1\rangle} = \mathrm{Suc}_{T^1}(\langle\bar\nu_1\rangle),\\
&A_{\langle\rangle} = \underset{\langle\bar\nu_1,S^1\rangle}{\triangle^0} A_{\langle\bar\nu_1,S^1\rangle},\\
&B_{\langle\rangle} = \{\langle\bar\nu_2\rangle \mid \mathrm{l}(\bar\nu_2)=0 \text{ or } \mathrm{Lev}_0(T^1)\cap\kappa^0(\bar\nu_2)\in\bar\nu_2\},\\
&\mathrm{Lev}_0(T^{(0)}) = A_{\langle\rangle}\cap B_{\langle\rangle},\\
&T^{(0)}_{\langle\bar\nu_2\rangle} = \bigcap_{\langle\bar\nu_1\rangle\in T^{(0)}(\bar\nu_2)} T^1_{\langle\bar\nu_1,\bar\nu_2\rangle},\\
&T^2 = T^1\cap T^{(0)}.
\end{aligned}$$

Let us assume that

$$p \le^2 \big\langle\langle\bar E_\alpha,\langle\rangle\rangle, T^2\big\rangle.$$

There are two cases to consider here:

(1) $p \le^2 \big(\big\langle\langle\bar E_\alpha,\langle\rangle\rangle, T^2\big\rangle\big)_{\langle\bar\nu_1,\bar\nu_2\rangle}$ where $\langle\bar\nu_1,\bar\nu_2\rangle\in T^2$: At once we have $\langle\bar\nu_1,\bar\nu_2\rangle\in T^1\le T$.

(2) $p \le^2 \big(\big\langle\langle\bar E_\alpha,\langle\rangle\rangle, T^2\big\rangle\big)_{\langle\bar\nu_2,\bar\nu_1\rangle}$ where $\langle\bar\nu_2\rangle\in T^2$, $\langle\bar\nu_1\rangle\in T^2(\bar\nu_2)$: By construction, $\forall\langle\bar\mu\rangle\in T^2(\bar\nu_2)$ $\langle\bar\nu_2\rangle\in T^1_{\langle\bar\mu\rangle}$. Since $\langle\bar\nu_1\rangle\in T^2(\bar\nu_2)$, we get $\langle\bar\nu_1,\bar\nu_2\rangle\in T^1\le T$.

We show how to continue to the third level. Let us set

$$\begin{aligned}
&A_{\langle\bar\nu_1,S^1,\bar\nu_2,S^2\rangle} = \mathrm{Suc}_{T^2}(\langle\bar\nu_1,\bar\nu_2\rangle),\\
&A_{\langle\bar\nu_1,S^1\rangle} = \underset{\langle\bar\nu_2,S^2\rangle}{\triangle^0} A_{\langle\bar\nu_1,S^1,\bar\nu_2,S^2\rangle},\\
&B_{\langle\bar\nu_1,S^1\rangle} = \{\langle\bar\nu_3\rangle \mid \mathrm{l}(\bar\nu_3)=0 \text{ or } \mathrm{Suc}_{T^2}(\langle\bar\nu_1,S^1\rangle)\cap\kappa^0(\bar\nu_3)\in\bar\nu_3\},\\
&\mathrm{Lev}_0(T^{(0)}) = \mathrm{Lev}_0(T^2),\\
&\mathrm{Suc}_{T^{(0)}}(\langle\bar\nu_1\rangle) = A_{\langle\bar\nu_1,S^1\rangle}\cap B_{\langle\bar\nu_1,S^1\rangle},\\
&T^{(0)}_{\langle\bar\nu_1,\bar\nu_3\rangle} = \bigcap_{\langle\bar\nu_2\rangle\in T^{(0)}_{\langle\bar\nu_1\rangle}(\bar\nu_3)} T^2_{\langle\bar\nu_1,\bar\nu_2,\bar\nu_3\rangle},\\
&A_{\langle\rangle} = \underset{\langle\bar\nu_1,S^1\rangle}{\triangle^0} A_{\langle\bar\nu_1,S^1\rangle},\\
&B_{\langle\rangle} = \{\langle\bar\nu_3\rangle \mid \mathrm{l}(\bar\nu_3)=0 \text{ or } \mathrm{Lev}_0(T^2)\cap\kappa^0(\bar\nu_3)\in\bar\nu_3\},\\
&\mathrm{Lev}_0(T^{(1)}) = A_{\langle\rangle}\cap B_{\langle\rangle},\\
&\mathrm{Suc}_{T^{(1)}}(\langle\bar\nu_3\rangle) = \bigcap_{\langle\bar\nu_1,\bar\nu_2\rangle\in T^{(0)}(\bar\nu_3)} T^2_{\langle\bar\nu_1,\bar\nu_2,\bar\nu_3\rangle},\\
&T^3 = T^2\cap T^{(0)}\cap T^{(1)}.
\end{aligned}$$

Let us assume that

$$p \le^3 \big\langle\langle\bar E_\alpha,\langle\rangle\rangle, T^3\big\rangle.$$

There are three cases to consider here:

(1) $p \le^* \big(\big\langle\langle\bar E_\alpha,\langle\rangle\rangle, T^3\big\rangle\big)_{\langle\bar\nu_1,\bar\nu_2,\bar\nu_3\rangle}$ where $\langle\bar\nu_1,\bar\nu_2,\bar\nu_3\rangle\in T^3$: At once we get $\langle\bar\nu_1,\bar\nu_2,\bar\nu_3\rangle\in T$.

(2) $p \leq^* \left(\langle\langle \bar{E}_\alpha, \langle\rangle\rangle, T^3\rangle\right)_{\langle\bar{\nu}_1,\bar{\nu}_3,\bar{\nu}_2\rangle}$ where $\langle\bar{\nu}_1,\bar{\nu}_3\rangle \in T^3$, $\langle\bar{\nu}_2\rangle \in T^3_{\langle\bar{\nu}_1\rangle}(\bar{\nu}_3)$: In this case $\forall\langle\bar{\mu}\rangle \in T^3_{\langle\bar{\nu}_1\rangle}(\bar{\nu}_3)$ $\langle\bar{\nu}_3\rangle \in T^3_{\langle\bar{\nu}_1,\bar{\mu}\rangle}$. Since $\langle\bar{\nu}_2\rangle \in T^3_{\langle\bar{\nu}_1\rangle}(\bar{\nu}_3)$, we get $\langle\bar{\nu}_3\rangle \in T_{\langle\bar{\nu}_1,\bar{\nu}_2\rangle}$.

(3) $p \leq^* \left(\langle\langle \bar{E}_\alpha, \langle\rangle\rangle, T^3\rangle\right)_{\langle\bar{\nu}_3,\bar{\nu}_1,\bar{\nu}_2\rangle}$ where $\langle\bar{\nu}_3\rangle \in T^3$, $\langle\bar{\nu}_1,\bar{\nu}_2\rangle \in T^3(\bar{\nu}_3)$: Then $\forall\langle\bar{\mu}_1,\bar{\mu}_2\rangle \in T^3(\bar{\nu}_3)$ $\langle\bar{\nu}_3\rangle \in T^3_{\langle\bar{\mu}_1,\bar{\mu}_2\rangle}$; hence $\langle\bar{\nu}_3\rangle \in T_{\langle\bar{\nu}_1,\bar{\nu}_2\rangle}$.

In this way we continue to all levels. □

**Lemma 4.10.** *Let $T$ be an $\bar{E}_\alpha$ tree. Then there is $T^* \leq T$ such that if*

$$p \leq \left\langle\langle \bar{E}_\alpha, \langle\rangle\rangle, T^*\right\rangle,$$

*then there is $\langle\bar{\nu}_1,\ldots,\bar{\nu}_n\rangle \in T$ such that*

$$p \leq^* \left(\langle\langle \bar{E}_\alpha, \langle\rangle\rangle, T\rangle\right)_{\langle\bar{\nu}_1,\ldots,\bar{\nu}_n\rangle}.$$

*Proof.* The proof is by induction on $\mathrm{l}(\bar{E}_\alpha)$. The first case was done in Lemma 4.9. The proof is almost the same. We just make sure to invoke the induction hypothesis while repeating the construction.

Construction of $T^1$ and $T^2$ is exactly as in Lemma 4.9. We show the construction at the 3rd level.

Let us set

$$\begin{aligned}
&A_{\langle\bar{\nu}_1,S^1,\bar{\nu}_2,S^2\rangle} = \mathrm{Suc}_{T^2}(\langle\bar{\nu}_1,\bar{\nu}_2\rangle),\\
&A_{\langle\bar{\nu}_1,S^1\rangle} = \underset{\langle\bar{\nu}_2,S^2\rangle}{\triangle^0} A_{\langle\bar{\nu}_1,S^1,\bar{\nu}_2,S^2\rangle},\\
&B_{\langle\bar{\nu}_1,S^1\rangle} = \{\langle\bar{\nu}_3\rangle \mid \mathrm{l}(\bar{\nu}_3) = 0 \text{ or } \mathrm{Suc}_{T^2}(\langle\bar{\nu}_1,S^1\rangle) \cap \kappa^0(\bar{\nu}_3) \in \bar{\nu}_3\},\\
&\mathrm{Lev}_0(T^{(0)}) = \mathrm{Lev}_0(T^2),\\
&\mathrm{Suc}_{T^{(0)}}(\langle\bar{\nu}_1\rangle) = A_{\langle\bar{\nu}_1,S^1\rangle} \cap B_{\langle\bar{\nu}_1,S^1\rangle},\\
&T^{(0)}_{\langle\bar{\nu}_1,\bar{\nu}_3\rangle} = \bigcap_{\langle\bar{\nu}_2\rangle \in T^{(0)}_{\langle\bar{\nu}_1\rangle}(\bar{\nu}_3)} T^2_{\langle\bar{\nu}_1,\bar{\nu}_2,\bar{\nu}_3\rangle},\\
&A'_{\langle\rangle} = \underset{\langle\bar{\nu}_1,S^1\rangle}{\triangle^0} A_{\langle\bar{\nu}_1,S^1\rangle},\\
&A_{\langle\rangle} = \{\langle\bar{\nu}_3,S^3\rangle \mid \langle\bar{\nu}_3,S^{3\prime}\rangle \in A'_{\langle\rangle} \text{ and } S \text{ is generated from } S^{3\prime} \text{ by induction}\},\\
&B_{\langle\rangle} = \{\langle\bar{\nu}_3\rangle \mid l(\bar{\nu}_3) = 0 \text{ or } \mathrm{Lev}_0(T^2) \cap \kappa^0(\bar{\nu}_3) \in \bar{\nu}_3\},\\
&\mathrm{Lev}_0(T^{(1)}) = A_{\langle\rangle} \cap B_{\langle\rangle},\\
&\mathrm{Suc}_{T^{(1)}}(\langle\bar{\nu}_3\rangle) = \bigcap_{\langle\bar{\nu}_1,\bar{\nu}_2\rangle \in T^{(0)}(\bar{\nu}_3)} T^2_{\langle\bar{\nu}_1,\bar{\nu}_2,\bar{\nu}_3\rangle},\\
&T^3 = T^2 \cap T^{(0)} \cap T^{(1)}.
\end{aligned}$$

Let us assume that

$$p_2 \frown p_1 \frown p_0 = p \leq^3 \left\langle\langle \bar{E}_\alpha, \langle\rangle\rangle, T^3\right\rangle.$$

There are three cases to consider here:

(1) $p \leq^* \left(\langle\langle \bar{E}_\alpha, \langle\rangle\rangle, T^3\rangle\right)_{\langle\bar{\nu}_1,\bar{\nu}_2,\bar{\nu}_3\rangle}$ where $\langle\bar{\nu}_1,\bar{\nu}_2,\bar{\nu}_3\rangle \in T^3$: At once we get $\langle\bar{\nu}_1,\bar{\nu}_2,\bar{\nu}_3\rangle \in T$.

(2) $p \leq^* \left(\left\langle\langle \bar{E}_\alpha, \langle\rangle\rangle, T^3\right\rangle\right)_{\langle\bar{\nu}_1,\bar{\nu}_3,\bar{\nu}_2\rangle}$ where $\langle\bar{\nu}_1,\bar{\nu}_3\rangle \in T^3$, $\langle\bar{\nu}_2\rangle \in T^3_{\langle\bar{\nu}_1\rangle}(\bar{\nu}_3)$: In this case, $\forall\langle\bar{\mu}\rangle \in T^3_{\langle\bar{\nu}_1\rangle}(\bar{\nu}_3)$ $\langle\bar{\nu}_3\rangle \in T^3_{\langle\bar{\nu}_1,\bar{\mu}\rangle}$. Since $\langle\bar{\nu}_2\rangle \in T^3_{\langle\bar{\nu}_1\rangle}(\bar{\nu}_3)$, we get $\langle\bar{\nu}_3\rangle \in T_{\langle\bar{\nu}_1,\bar{\nu}_2\rangle}$.

(3) $p \leq^2 \left(\left\langle\langle \bar{E}_\alpha, \langle\rangle\rangle, T^3\right\rangle\right)_{\langle\bar{\nu}_3\rangle}$ where $\langle\bar{\nu}_3\rangle \in T^3$ and $p_2 \frown p_1 \leq^2 \left\langle\langle\bar{\nu}_3, \langle\rangle\rangle, T^3(\bar{\nu}_3)\right\rangle$. By induction there is $\langle\bar{\nu}_1,\bar{\nu}_2\rangle \in T(\bar{\nu}_3)$ such that

$$p_2 \frown p_1 \leq^* \left(\left\langle\langle\bar{\nu}_3, \langle\rangle\rangle, T^2(\bar{\nu}_3)\right\rangle\right)_{\langle\bar{\nu}_1,\bar{\nu}_2\rangle}.$$

By construction, $\forall\langle\bar{\mu}_1,\bar{\mu}_2\rangle \in T^2(\bar{\nu}_3)$ $\langle\bar{\nu}_3\rangle \in T^2_{\langle\bar{\mu}_1,\bar{\mu}_2\rangle}$; hence $\langle\bar{\nu}_3\rangle \in T_{\langle\bar{\nu}_1,\bar{\nu}_2\rangle}$.

In this way we continue to all levels. □

## 5. Basic Properties of $P_{\bar{E}}$

**Claim 5.1.** *$P_{\bar{E}}$ satisfies $\kappa^{++}$-c.c.*

*Proof.* The usual $\Delta$-lemma argument on the support will do. □

**Claim 5.2.** *Let $p \in P_{\bar{E}}$, $P^* = \{q \leq_R p \mid p \in P_{\bar{E}}\}$. Then*

(1) *$\langle P^*, \leq_R\rangle$ satisfies $\kappa^+$-c.c., and*
(2) *$\langle P^*, \leq_R\rangle$ is sub-forcing of $\langle P_{\bar{E}}/p, \leq\rangle$.*

*Proof.* Showing $\kappa^+$-c.c. is trivial.

Showing that $P^*$ is sub-forcing of $P_{\bar{E}}/p$ amounts to showing that any maximal anti-chain of $P^*$ is also a maximal anti-chain of $P_{\bar{E}}/p$.

Let $A$ be a maximal anti-chain of $P^*$. Let $q \in P_{\bar{E}}/p$. Then $q \leq p$, and there is $r' \in P^*$ such that $q \leq^* r' \leq_R p$. Assume that $r' = r'_n \frown \cdots \frown r'_0$. Then also $q = q_n \frown \cdots \frown q_0$. Let $r_i$ be $r'_i$ with $T^{r'_i}$ replaced by $T^{r'_i} \cap \pi_{\mathrm{mc}(q_i),\mathrm{mc}(r'_i)}(T^{q_i})$ and $r = r_n \frown \cdots \frown r_0$. Since $r \in P^*$ and $A$ is a maximal anti-chain, there is an $a \in A$ such that $a \parallel r$. Take $s \leq_R a, r$. Considering how we constructed $r$ from $r'$, we must have $t \leq^* s$ such that $t \leq q$. Hence $q \parallel a$. So we get that $A$ is a maximal anti-chain of $P_{\bar{E}}/p$. □

**Claim 5.3.** *Let $p \in P_{\bar{E}}$, $P^* = \{q \leq_R p \mid p \in P_{\bar{E}}\}$. Then there is $r \in R_{\mathrm{mc}(p)}$ such that $P^* \simeq R_{\mathrm{mc}(p)}/r$.*

*Proof.* For simplicity, assume that $p = p_0$. Then we set $r = \langle\langle \mathrm{mc}(p_0), p_0^{\mathrm{mc}}\rangle, T^p\rangle$.

We give the isomorphism: The image of $q \in P^*$ is $s \in \langle R_{\mathrm{mc}(p)}/r, \leq\rangle$ such that

(1) $q \leq^* s$, and
(2) $T^{s_i} = T^{q_i}$ where $s = s_n \frown \cdots \frown s_0$, $q = q_n \frown \cdots \frown q_0$. □

Let $G$ be $P_{\bar{E}}$-generic.

**Definition 5.4.** $\bar{E}_G$ is the enumeration of $\{\bar{E}(p_k) \mid p_n \frown \cdots \frown p_0 \in G\}$ ordered increasingly by $\kappa^0(\bar{E}(p_k))$.

**Definition 5.5.** Let $\zeta < \mathrm{otp}(\bar{E}_G)$. Then

(1) $G{\restriction}\zeta = \{p_n \frown \cdots \frown p_k \mid p_n \frown \cdots \frown p_k \frown \cdots \frown p_0 \in G, \bar{E}(p_k) = \bar{E}_G(\zeta)\}$,
(2) $G \setminus \zeta = \{p_{k-1} \frown \cdots \frown p_0 \mid p_n \frown \cdots \frown p_k \frown \cdots \frown p_0 \in G, \bar{E}(p_k) = \bar{E}_G(\zeta)\}$.

**Definition 5.6.**

$$M^\alpha_G = \bigcup\{p^{\bar{E}_\alpha} \mid p \in G,\ \bar{E}_\alpha \in \mathrm{supp}\, p\}, \qquad ,$$
$$C^\alpha_G = \{\kappa(\bar{\mu}) \mid \bar{\mu} \in M^\alpha_G\}.$$

**Proposition 5.7.** (1) *$C^\kappa_G$ is a club in $\kappa$.*
(2) *$C^\alpha_G$ is unbounded in $\kappa$.*
(3) *$\alpha \neq \beta \implies C^\alpha_G \neq C^\beta_G$.*

*Proof.* The first two claims are immediate, since these are sequences generated by Radin forcing.

The last is by density and noticing that for $p \in P_{\bar{E}}$ when $\bar{\nu} \in T^p$ is permitted for $p^{\bar{E}_\alpha}, p^{\bar{E}_\beta}$ we required $\pi_{\mathrm{mc}(p),\bar{E}_\alpha}(\bar{\nu}) \neq \pi_{\mathrm{mc}(p),\bar{E}_\beta}(\bar{\nu})$. □

## 6. Homogeneity in Dense Open Subsets

Our aim in this section is to prove the following

**Theorem 6.1.** *Let $D \subseteq P_{\bar{E}}$ be dense open and $p = p_k \frown \cdots \frown p_0 \in P_{\bar{E}}$. Then there is $p^* \leq^* p$ such that*

$$\exists S^k\ \exists n_k \forall \langle \bar{\nu}_{k,1}, \ldots, \bar{\nu}_{k,n_k}\rangle \in S^k \ \ldots \exists S^0\ \exists n_0 \forall \langle \bar{\nu}_{0,1}, \ldots, \bar{\nu}_{0,n_0}\rangle \in S^0$$
$$(p^*_k)_{\langle \bar{\nu}_{k,1},\ldots,\bar{\nu}_{k,n_k}\rangle} \frown \cdots \frown (p^*_0)_{\langle \bar{\nu}_{0,1},\ldots,\bar{\nu}_{0,n_0}\rangle} \in D$$

*where*

(1) *$S^i \subseteq T^{p^*_i} \restriction [V_\kappa]^{n_i}$, and*
(2) *$\forall l < n_i\, \forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_l\rangle \in S^i\, \exists \xi\ \ \mathrm{Suc}_{S^i}(\langle \bar{\nu}_1, \ldots, \bar{\nu}_l\rangle) \in E_{\mathrm{mc}(p^*_i)}(\xi)$.*

The proof is done by a series of lemmas.

**Definition 6.2.** Let $p \in P^*_{\bar{E}}$. Let $s$ be a function such that $\mathrm{dom}\, s \subseteq \bar{E}$ and for all $\bar{\alpha}, \bar{\beta} \in \mathrm{dom}\, s$, $\bar{\alpha} \neq \bar{\beta}$,

(1) $s(\bar{\alpha})$ is an extender sequence,
(2) $\mathrm{l}(s(\bar{\alpha})) = \mathrm{l}(s(\bar{\beta}))$,
(3) $\kappa^0(s(\bar{\alpha})) = \kappa^0(s(\bar{\beta}))$,
(4) $s(\bar{\alpha}) \neq s(\bar{\beta})$.

We define $(p)_{\langle s\rangle}$ to be $p'_1 \frown p'_0$ where

(1) $\mathrm{supp}\, p'_0 = \mathrm{supp}\, p$,
(2) $\forall \bar{\alpha} \in \mathrm{supp}\, p'_0$

$$p'^{\bar{\alpha}}_0 = \begin{cases} s(\bar{\alpha}), & \bar{\alpha} \in \mathrm{dom}\, s,\ \max \kappa(p^{\bar{\alpha}}) < \kappa^0(s(\bar{\alpha})),\ \mathrm{l}(s(\bar{\alpha})) > 0, \\ p^{\bar{\alpha}}_0 \frown s(\bar{\alpha}), & \bar{\alpha} \in \mathrm{dom}\, s,\ \max \kappa(p^{\bar{\alpha}}) < \kappa^0(s(\bar{\alpha})),\ \mathrm{l}(s(\bar{\alpha})) = 0, \\ p^{\bar{\alpha}}, & \text{otherwise}, \end{cases}$$

(3) if $s(\mathrm{mc}(p)) \in T^p$, then $T^{p'_0} = T^p_{\langle s(\mathrm{mc}(p))\rangle}$; otherwise we leave $T^{p'_0}$ undefined,
(4) if $\forall \bar{\alpha} \in \mathrm{dom}\, s\ \mathrm{l}(s(\bar{\alpha})) = 0$, then $p'_1 = \emptyset$,
(5) if $\forall \bar{\alpha} \in \mathrm{dom}\, s\ \ \mathrm{l}(s(\bar{\alpha})) > 0$, then
  (5.1) $\mathrm{supp}\, p'_1 = \{s(\bar{\alpha}) \mid \bar{\alpha} \in \mathrm{supp}\, p \cap \mathrm{dom}\, s,\ \max \kappa(p^{\bar{\alpha}}) < \kappa^0(s(\bar{\alpha}))\}$,
  (5.2) $\forall \bar{\alpha} \in \mathrm{supp}\, p'_0\ \forall s(\bar{\alpha}) \in \mathrm{supp}\, p'_1\ p'^{s(\bar{\alpha})}_1 = p^{\bar{\alpha}}$,
  (5.3) if $s(\mathrm{mc}(p)) \in T^p$, then $T^{p'_1} = T^p(s(\mathrm{mc}(p)))$; otherwise we leave $T^{p'_1}$ undefined.

**Definition 6.3.** Let $p \in P^*_{\bar{E}}$. Let $s$ be a function with $\mathrm{dom}\, s = 1, \ldots, n$ such that for all $i$, $s(i)$ satisfies Definition 6.2. Then we define $(p)_{\langle s\rangle}$ as $p^n$ where $p^n$ is defined

by induction as follows:

$$p^0 = p,$$
$$p^{i+1} = p^i_i \frown \cdots \frown p^i_1 \frown (p^i_0)_{\langle s(i+1)\rangle}.$$

We note the following: If $\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in T^p$ and for all $1 \leq i \leq n$ we set

$$s(i) = \{\langle \bar{\alpha}, \pi_{\mathrm{mc}(p),\bar{\alpha}}(\bar{\nu}_i)\rangle \mid \bar{\alpha} \in \operatorname{supp} p\},$$

then

$$(p)_{\langle \bar{\nu}_1,\ldots,\bar{\nu}_n\rangle} = (p)_{\langle s\rangle}.$$

We use this operation also in cases where $p$ is not strictly a condition. That is, if $p \cup \{T\} \in P_{\bar{E}}$, we also use $(p)_{\langle s\rangle}$. In this case we ignore the trees in the definition.

This definition is used in the proof of the homogeneity, because we do not know beforehand what a legitimate extension is. By checking all possible $\mu$'s we check all legitimate conditions which *might* be extensions.

**Claim 6.4.** *Let $D$ be dense open in $P_{\bar{E}}/P_{\bar{\epsilon}}$, $p = p_0 \in P_{\bar{E}}/P_{\bar{\epsilon}}$, $0 < n < \omega$. Then there is $p^* \leq^* p$ such that one and only one of the following is true:*

(1) *There is $S \subseteq T^{p^*} \restriction [V_\kappa]^n$ such that*
  (1.1) $\forall k < n\, \exists \xi < \mathrm{l}(\bar{E})\ \mathrm{Suc}_S(\langle \bar{\nu}_1, \ldots, \bar{\nu}_k\rangle) \in E_{\mathrm{mc}(p^*)}(\xi)$,
  (1.2) $\forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_n\rangle \in S\ (p^*)_{\langle \bar{\nu}_1,\ldots,\bar{\nu}_n\rangle} \in D$.

(2) $\forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_n\rangle \in T^{p^*} \forall q \leq^* (p^*)_{\langle \bar{\nu}_1,\ldots,\bar{\nu}_n\rangle}\ q \notin D$.

*Proof.* We give the proof for $n = 1$. Adapting the proof for higher $n$'s requires that whenever we enumerate singletons we should enumerate $n$-tuples and when we use $j$ we should use $j_n$.

We start an induction on $\xi$ in which we build

$$\langle \bar{\alpha}^\xi, u^\xi \mid \xi < \kappa\rangle.$$

We start by setting

$$u^0 = p_0 \setminus \{T^{p_0}\},$$
$$\bar{\alpha}^0 = \mathrm{mc}(p_0),$$
$$T^0 = T^{p_0} \restriction \pi^{-1}_{\bar{\alpha}^0,0}\{\bar{\nu} \mid \kappa^0(\bar{\nu}) \text{ is inaccessible}\},$$

and taking an increasing enumeration

$$\{\kappa^0(\bar{\nu}) \mid \langle \bar{\nu}\rangle \in T^0\} = \langle \tau_\xi \mid \xi < \kappa\rangle.$$

Assume that we have constructed

$$\langle \bar{\alpha}^\xi, u^\xi \mid \xi < \xi_0\rangle.$$

We have two cases. If $\xi_0$ is limit, choose $\bar{\alpha}^{\xi_0} >_{\bar{\mathrm{E}}} \bar{\alpha}^\xi$ for all $\xi < \xi_0$ and set

$$u^{\xi_0} = \bigcup_{\xi<\xi_0} u^\xi \cup \left\{\langle \bar{\alpha}^{\xi_0}, t\rangle\right\} \text{ where } \kappa^0(t) = \tau_{\xi_0}.$$

If $\xi_0 = \xi + 1$, for each $\bar{\nu}$ such that $\kappa^0(\bar{\nu}) = \tau_\xi$ we set

$$S(\bar{\nu}) = \Big(\prod_{\substack{\bar{\alpha}\in\operatorname{supp} u^\xi \\ \kappa((u^\xi)^{\bar{\alpha}})<\tau_\xi}} \{\bar{\mu} \mid \kappa^0(\bar{\mu}) = \kappa^0(\bar{\nu})\}\Big) \times \{\langle \bar{\nu}\rangle\}.$$

Let

$$S = \bigcup_{\kappa^0(\bar{\nu})=\tau_\xi} S(\bar{\nu})$$

and fix an enumeration of $S$,

$$S = \langle s^{\xi_0,\rho} \mid \rho < \tau_{\xi_0} \rangle.$$

There are fewer than $\tau_{\xi_0}$ elements in $S$. We use $\tau_{\xi_0}$, because this is the maximum size $S$ can have without "killing" the induction.

We do induction on $\rho$ which builds

$$\langle \bar{\alpha}^{\xi_0,\rho}, u_0^{\xi_0,\rho}, T_0^{\xi_0,\rho}, u_1^{\xi_0,\rho}, T_1^{\xi_0,\rho} \mid \rho < \tau_{\xi_0} \rangle,$$

from which we build $\langle \bar{\alpha}^{\xi_0}, u^{\xi_0} \rangle$. Set

$$\bar{\alpha}^{\xi_0,0} = \bar{\alpha}^{\xi},$$
$$u_0^{\xi_0,0} = u_0^{\xi}.$$

Assume we have constructed $\langle \bar{\alpha}^{\xi_0,\rho}, u_0^{\xi_0,\rho}, T_0^{\xi_0,\rho} \mid \rho < \rho_0 \rangle$.

We have two cases.

If $\rho_0$ is limit, set

$$\forall \rho < \rho_0 \ \bar{\alpha}^{\xi_0,\rho_0} >_{\bar{\mathrm{E}}} \bar{\alpha}^{\xi_0,\rho},$$
$$u^{\xi_0,\rho_0} = \bigcup_{\rho<\rho_0} u^{\xi_0,\rho} \cup \left\{ \langle \bar{\alpha}^{\xi_0,\rho_0}, t \rangle \right\} \text{ where } \kappa^0(t) = \tau_\xi.$$

We set $T_0^{\xi_0,\rho_0}$ and $T_1^{\xi_0,\rho_0}$ equal to anything we like, because we do not use them later.

If $\rho_0 = \rho + 1$, let $\langle \bar{\nu} \rangle = s^{\xi_0,\rho}(2)$. Set

$$u'' = u_1'' \frown u_0'' = (u_0^{\xi_0,\rho})_{\langle s^{\xi_0,\rho} \rangle},$$
$$T_0'' = \pi^{-1}_{\bar{\alpha}^{\xi},\bar{\alpha}^0}(T^0_{\langle \bar{\nu} \rangle}),$$
$$T_1'' = \pi^{-1}_{\mathrm{mc}(u_1''),\bar{\nu}}(T^0(\bar{\nu})).$$

If there are

$$q_1' \leq^* u_1'' \cup \{T_1''\},$$
$$q_0' \leq^* u_0'' \cup \{T_0''\}$$

such that

$$q_1' \frown q_0' \in D,$$

then set

$$\bar{\alpha}^{\xi_0,\rho_0} = \mathrm{mc}(q_0'),$$
$$u_0^{\xi_0,\rho_0} = u^{\xi_0,\rho} \cup (q_0' \setminus (u_0'' \cup \left\{ T^{q_0'} \right\})),$$
$$T_0^{\xi_0,\rho_0} = T^{q_0'},$$
$$u_1^{\xi_0,\rho_0} = q_1' \setminus (u_1'' \cup \left\{ T^{q_1'} \right\}),$$
$$\bar{\alpha}_1^{\xi_0,\rho_0} = \mathrm{mc}(q_1'),$$
$$T_1^{\xi_0,\rho_0} = T^{q_1'};$$

otherwise set

$$\begin{aligned}
\bar{\alpha}^{\xi_0,\rho_0} &= \bar{\alpha}^{\xi_0,\rho},\\
u_0^{\xi_0,\rho_0} &= u^{\xi_0,\rho},\\
T_0^{\xi_0,\rho_0} &= T_0'',\\
u_1^{\xi_0,\rho_0} &= \emptyset,\\
\bar{\alpha}_1^{\xi_0,\rho_0} &= \mathrm{mc}(u''),\\
T_1^{\xi_0,\rho_0} &= T_1''.
\end{aligned}$$

When the induction on $\rho$ terminates we have $\langle \bar{\alpha}^{\xi_0,\rho}, u_0^{\xi_0,\rho}, T_0^{\xi_0,\rho}, u_1^{\xi_0,\rho}, T_1^{\xi_0,\rho} \mid \rho < \tau_{\xi_0}\rangle$. We continue with the induction on $\xi$. We set

$$\begin{aligned}
&\forall \rho < \tau_{\xi_0}\ \bar{\alpha}^{\xi_0} >_{\bar{\mathrm{E}}} \bar{\alpha}^{\xi_0,\rho},\\
&u_0^{\xi_0} = \bigcup_{\rho<\tau_{\xi_0}} u_0^{\xi_0,\rho} \cup \left\{\langle \bar{\alpha}^{\xi_0}, t\rangle\right\} \text{ where } \max \kappa^0(t) = \tau_\xi.
\end{aligned}$$

When the induction on $\xi$ terminates we have $\langle \bar{\alpha}^{\xi}, u_0^{\xi} \mid \xi < \kappa\rangle$. Let

$$\begin{aligned}
&\forall \xi < \kappa\ \bar{\alpha}^{*\prime} >_{\bar{\mathrm{E}}} \bar{\alpha}^{\xi},\\
&p_0^{*\prime} = \bigcup_{\xi<\kappa} u_0^{\xi} \cup \{\langle \bar{\alpha}^{*\prime}, t\rangle\} \text{ where } \max \kappa^0(t) = \max p_0^0.
\end{aligned}$$

We set

$$\mathrm{Lev}_0(T^{p_0^{*\prime}}) = \pi^{-1}_{\bar{\alpha}^{*\prime},\bar{\alpha}^0}\, \mathrm{Lev}_0(T^0).$$

Let us consider $\langle \bar{\nu}\rangle \in \mathrm{Lev}_0(T^{p_0^{*\prime}})$. There is $\xi$ such that $\kappa^0(\bar{\nu}) = \tau_\xi$. We set

$$\begin{aligned}
s(1) &= \{\langle \bar{\alpha}, \pi_{\bar{\alpha}^{*\prime},\bar{\alpha}}(\bar{\nu})\rangle \mid \bar{\alpha} \in \mathrm{supp}\, p_0^*\},\\
s(2) &= \left\{\langle \pi_{\bar{\alpha}^{*\prime},\bar{\alpha}^0}(\bar{\nu})\rangle\right\}.
\end{aligned}$$

Let $\xi_0 = \xi + 1$. By our construction there is $\rho$ such that

$$(u_0^{\xi_0,\rho_0})_{\langle s\rangle} = (u_0^{\xi_0,\rho_0})_{\langle s^{\xi_0,\rho}\rangle}$$

where $\rho_0 = \rho + 1$. We set

$$\begin{aligned}
&T^{p_0^{*\prime}}_{\langle\bar{\nu}\rangle} = \pi^{-1}_{\bar{\alpha}^{*\prime},\bar{\alpha}^{\xi_0,\rho_0}}(T^{\xi_0,\rho_0}) \cap \pi^{-1}_{\bar{\alpha}^{*\prime},\bar{\alpha}^0}(T^0_{\langle \pi_{\bar{\alpha}^*,\bar{\alpha}^0}(\bar{\nu})\rangle}),\\
&T^{p_0^{*\prime}}(\bar{\nu}) = T_1^{\xi_0,\rho_0},\\
&\bar{\alpha}_1'(\bar{\nu}) = \bar{\alpha}_1^{\xi_0,\rho_0},\\
&p_1'(\bar{\nu}) = u_1^{\xi_0,\rho_0}.
\end{aligned}$$

Note that $T^{p_0^{*\prime}}$ is not legal as a tree in a condition, because if $p_1'(\bar{\nu}) \neq \emptyset$, then $T^{p_0^{*\prime}}(\bar{\nu})$ might be defined on too high a coordinate. We abuse the notation, because wherever $T^{p_0^{*\prime}}(\bar{\nu})$ appears, so does $p_1'(\bar{\nu})$.

Let us show that $p_0^{*\prime}$ approximates the $p^*$ we are looking for. So let $\langle\bar{\nu}\rangle \in T^{p_0^{*\prime}}$ and assume

$$q_1' \leq^* p_1'(\bar{\nu}) \cup ((p_0^{*\prime})_{\langle\bar{\nu}\rangle})_1, \tag{6.4.1a}$$

$$q_0' \leq^* ((p_0^{*\prime})_{\langle\bar{\nu}\rangle})_0, \tag{6.4.1b}$$

$$q_1' \frown q_0' \in D. \tag{6.4.1c}$$

Let $\xi$ be such that $\kappa^0(\bar{\nu}) = \tau_\xi$. Set

$$s(1) = \{\langle\bar{\alpha}, \pi_{\bar{\alpha}^{*\prime},\bar{\alpha}}(\bar{\nu})\rangle \mid \bar{\alpha} \in \operatorname{supp} p_0^*\},$$
$$s(2) = \left\{\langle\pi_{\bar{\alpha}^{*\prime},\bar{\alpha}^0}(\bar{\nu})\rangle\right\},$$

where $\xi_0 = \xi + 1$, $\rho_0 = \rho + 1$. By our construction there is a $\rho$ such that

$$(u^{\xi_0,\rho_0})_{\langle s^{\xi_0,\rho}\rangle} = (u^{\xi_0,\rho_0})_{\langle s\rangle}.$$

Let us set

$$r = \left((u^{\xi_0,\rho_0}_{\langle s^{\xi_0,\rho}\rangle})_1 \cup \left\{T_1^{\xi_0,\rho_0}\right\}\right) \frown \left((u^{\xi_0,\rho_0}_{\langle s^{\xi_0,\rho}\rangle})_0 \cup \left\{T_0^{\xi_0,\rho_0}\right\}\right).$$

By construction we have

$$\left(p_1'(\bar{\nu}) \cup ((p_0^{*\prime})_{\langle\bar{\nu}\rangle})_1\right) \frown ((p_0^{*\prime})_{\langle\bar{\nu}\rangle})_0 \leq^* r.$$

So what we have is

$$D \ni q_1' \frown q_0' \leq^* r.$$

This is a positive answer to the question in the induction. Hence

$$r \in D,$$

which gives us, by openness of $D$, that

$$\left(p_1'(\bar{\nu}) \cup ((p_0^{*\prime})_{\langle\bar{\nu}\rangle})_1\right) \frown ((p_0^{*\prime})_{\langle\bar{\nu}\rangle})_0 \in D. \tag{6.4.2}$$

Having proved this approximation property of $p_0^{*\prime}$, let us consider the set

$$B = \{\langle\bar{\nu}\rangle \in T^{p_0^{*\prime}} \mid \exists q \leq^* \left(p_1'(\bar{\nu}) \cup ((p_0^{*\prime})_{\langle\bar{\nu}\rangle})_1\right) \frown ((p_0^{*\prime})_{\langle\bar{\nu}\rangle})_0 \; q \in D\}.$$

Let $\bar{\alpha}^{*\prime} = \mathrm{mc}(p_0^{*\prime})$. There are two cases to be considered.

(1) $\exists\zeta < \mathrm{l}(\bar{E})\ B \in E_{\bar{\alpha}^{*\prime}}(\zeta)$.

Let us set

$$p_1^\zeta = j(p_1')(\bar{E}_{\bar{\alpha}^{*\prime}} \restriction \zeta),$$
$$\bar{\beta}^{\zeta\prime} = \mathrm{mc}(p_1^\zeta),$$
$$A^\zeta = \{\langle\bar{\nu}\rangle \mid \left((p_1^\zeta)_{\langle\bar{\nu}\rangle}\right)_1 = p_1'(\pi_{\bar{\beta}^{\zeta\prime},\bar{\alpha}^{*\prime}}(\bar{\nu}))\}.$$

Clearly

$$A^\zeta \in E_{\bar{\beta}^{\zeta\prime}}(\zeta).$$

Let $\bar{\beta}^\zeta >_{\bar{\mathrm{E}}} \bar{\beta}^{\zeta\prime}, \bar{\alpha}^{*\prime}$. Set $T^\zeta = \pi^{-1}_{\bar{\beta}^\zeta,\bar{\alpha}^{*\prime}}(T^{p_0^{*\prime}})$. Since $T^{p_0^{*\prime}}$ is not legal as a tree, we give the necessary modification making $T^\zeta$ legal:

$$\forall\bar{\nu} \in \pi^{-1}_{\bar{\beta}^\zeta,\bar{\beta}^{\zeta\prime}}(A^\zeta)\ T^\zeta(\bar{\nu}) = \pi^{-1}_{\bar{\nu},\bar{\alpha}_1'(\pi_{\bar{\beta}^\zeta,\bar{\alpha}^{*\prime}}(\bar{\nu}))} T^{p_0^{*\prime}}(\pi_{\bar{\beta}^\zeta,\bar{\alpha}^{*\prime}}(\bar{\nu})),$$
$$\forall\bar{\nu} \notin \pi^{-1}_{\bar{\beta}^\zeta,\bar{\beta}^{\zeta\prime}}(A^\zeta)\ T^\zeta(\bar{\nu}) = \pi^{-1}_{\bar{\nu},\bar{\alpha}_1'(\pi_{\bar{\beta}^\zeta,\bar{\alpha}^{*\prime}}(\bar{\nu}))} T^{p_0}(\pi_{\bar{\beta}^\zeta,\bar{\alpha}^{*\prime}}(\bar{\nu})),$$
$$p^\zeta = p_1^\zeta \cup p_0^{*\prime} \cup \left\{\langle\beta^\zeta, t\rangle\right\} \cup \left\{T^\zeta\right\} \text{ where } \max t^0 = p_0^0.$$

The nice property of $p^\zeta$ is that when $\langle\bar{\nu}\rangle \in T^\zeta \restriction (\pi_{\bar{\beta}^\zeta,\bar{\beta}^{\zeta\prime}})^{-1}(A^\zeta)$ we get

$$(p^\zeta)_{\langle\bar{\nu}\rangle} \leq^* \left(p_1'(\pi_{\bar{\beta}^\zeta,\bar{\alpha}^{*\prime}}(\bar{\nu})) \cup ((p_0^{*\prime})_{\langle\pi_{\bar{\beta}^\zeta,\bar{\alpha}^{*\prime}}(\bar{\nu})\rangle})_1\right) \frown ((p_0^{*\prime})_{\langle\pi_{\bar{\beta}^\zeta,\bar{\alpha}^{*\prime}}(\bar{\nu})\rangle})_0.$$

We set

$$p^* = p^\zeta,$$
$$A = \mathrm{Lev}_0(T^\zeta) \cap (\pi_{\bar{\beta}^\zeta,\bar{\beta}^{\zeta\prime}})^{-1} A^\zeta,$$

and show that the claim is satisfied. Assume that

$$\langle\bar{\nu}\rangle \in A,$$
$$q_1' \leq^* ((p^*)_{\langle\bar{\nu}\rangle})_1,$$
$$q_0' \leq^* ((p^*)_{\langle\bar{\nu}\rangle})_0,$$
$$q_1' \frown q_0' \in D.$$

Note that

$$((p^*)_{\langle\bar{\nu}\rangle})_1 \leq^* p_1'(\pi_{\bar{\beta}^\zeta,\bar{\alpha}^{*\prime}}) \cup ((p_0^{*\prime})_{\langle\pi_{\bar{\beta}^\zeta,\bar{\alpha}^{*\prime}}(\bar{\nu})\rangle})_1,$$
$$((p^*)_{\langle\bar{\nu}\rangle})_0 \leq^* ((p_0^{*\prime})_{\langle\pi_{\bar{\beta}^\zeta,\bar{\alpha}^{*\prime}}(\bar{\nu})\rangle})_0.$$

Hence, we know that

$$q_1' \leq^* p_1'(\pi_{\bar{\beta}^\zeta,\bar{\alpha}^{*\prime}}(\bar{\nu})) \cup ((p_0^{*\prime})_{\langle\pi_{\bar{\beta}^\zeta,\bar{\alpha}^{*\prime}}(\bar{\nu})\rangle})_1,$$
$$q_0' \leq^* ((p_0^{*\prime})_{\langle\pi_{\bar{\beta}^\zeta,\bar{\alpha}^{*\prime}}(\bar{\nu})\rangle})_0,$$
$$q_1' \frown q_0' \in D.$$

This is the assumption (6.4.1). So from (6.4.2) we know that

$$\left(p_1'(\pi_{\bar{\beta}^\zeta,\bar{\alpha}^{*\prime}}(\bar{\nu})) \cup ((p_0^{*\prime})_{\langle\pi_{\bar{\beta}^\zeta,\bar{\alpha}^{*\prime}}(\bar{\nu})\rangle})_1\right) \frown ((p_0^{*\prime})_{\langle\pi_{\bar{\beta}^\zeta,\bar{\alpha}^{*\prime}}(\bar{\nu})\rangle})_0 \in D.$$

Hence, by openness of $D$,

$$(p^*)_{\langle\bar{\nu}\rangle} \in D.$$

(2) $\forall\zeta < \mathrm{l}(\bar{E})\ B \notin E_{\bar{\alpha}^{*\prime}}(\zeta)$, which is the same as saying that

$$\{\langle\bar{\nu}\rangle \in T^{p_0^{*\prime}} \mid \forall q \leq^* \left(p_1'(\bar{\nu}) \cup ((p_0^{*\prime})_{\langle\bar{\nu}\rangle})_1\right) \frown ((p_0^{*\prime})_{\langle\bar{\nu}\rangle})_0\ q \notin D\} \in \bar{E}_{\bar{\alpha}^{*\prime}}.$$

In fact, from the construction we can see that

$$\{\langle\bar{\nu}\rangle \in T^{p_0^{*\prime}} \mid p_1'(\bar{\nu}) = \emptyset\} \in \bar{E}_{\bar{\alpha}^{*\prime}}.$$

So we really have

$$A = \{\langle\bar{\nu}\rangle \in T^{p_0^{*\prime}} \mid \forall q \leq^* (p_0^{*\prime})_{\langle\bar{\nu}\rangle}\ q \notin D\} \in \bar{E}_{\bar{\alpha}^{*\prime}},$$

and the completion is quite easy now; we set

$$T^{p^*} = T^{p^{*\prime}} \restriction A,$$
$$p^* = p_0^{*\prime} \cup \left\{T^{p^*}\right\}.$$

□

**Claim 6.5.** *Let $D$ be dense open in $P_{\bar{E}}/P_{\bar{\epsilon}}$, $p = p_0 \in P_{\bar{E}}/P_{\bar{\epsilon}}$. Then there is $p^* \leq^* p$ such that one and only one of the following is true:*

(1) *There are $n < \omega$, $S \subseteq T^{p^*} \restriction [V_\kappa]^n$ such that*

(1.1) $\forall k < n\, \exists \xi < \mathrm{l}(\bar{E})\ \mathrm{Suc}_S(\langle \bar{\nu}_1, \dots, \bar{\nu}_k \rangle) \in E_{\mathrm{mc}(p^*)}(\xi)$,
(1.2) $\forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in S\ (p^*)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle} \in D$.

(2) $\forall n < \omega\ \forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in T^{p^*} \forall q \leq^* (p^*)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle}\ q \notin D$.

*Proof.* Let $p^0 = p$.

Generate $p^{n+1} \leq^* p^n$ by invoking Claim 6.4 for $n+1$ levels.

Take $\forall n < \omega\ p^* \leq p^n$. □

**Claim 6.6.** *Let $D$ be dense open in $P_{\bar{E}}/P_{\bar{\epsilon}}$, $p = p_0 \in P_{\bar{E}}/P_{\bar{\epsilon}}$. Then there are $n < \omega$, $p^* \leq^* p$, $S \subseteq T^{p^*} \restriction [V_\kappa]^n$ such that*

(1) $\forall k < n\, \exists \xi < \mathrm{l}(\bar{E})\ \mathrm{Suc}_S(\langle \bar{\nu}_1, \dots, \bar{\nu}_k \rangle) \in E_{\mathrm{mc}(p^*)}(\xi)$,
(2) $\forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in S\ (p^*)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle} \in D$.

*Proof.* Towards a contradiction, let us assume that the conclusion is false. That means that for all $p^* \leq^* p$, for all $n < \omega$, for all $S \subseteq T^{p^*} \restriction [V_\kappa]^n$ such that

$$\forall k < n\, \forall \langle \bar{\nu}_1, \dots, \bar{\nu}_k \rangle \in S\, \exists \xi < \mathrm{l}(\bar{E})\ \mathrm{Suc}_S(\langle \bar{\nu}_1, \dots, \bar{\nu}_k \rangle) \in E_{\mathrm{mc}(p^*)}(\xi)$$

we have

$$\exists \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in S\ (p^*)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle} \notin D.$$

We construct a $\leq^*$-decreasing sequence as follows: We set $p^0 = p$. We construct $p^{n+1}$ from $p^n$ using Claim 6.4 for $n+1$ levels. Due to our assumption we get

$$\forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in T^{p^n}\ \forall q \leq^* (p^n)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle}\ q \notin D.$$

Choosing $p^{*\prime}$ such that $\forall n < \omega\ p^{*\prime} \leq^* p^n$, we get

$$\forall n < \omega\, \forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in T^{p_0^{*\prime}}\, \forall q \leq^* (p^{*\prime})_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle}\ q \notin D.$$

Construct a tree $T$ from $T^{p^{*\prime}}$ using Lemma 4.10. Let us call $p^*$ the condition $p^{*\prime}$ with $T$ substituted for $T^{p^{*\prime}}$. Now if we have

$$q \leq p^*,$$

then there is $\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in T^{p^{*\prime}}$ such that

$$q \leq^* (p^{*\prime})_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle}.$$

Hence

$$q \notin D.$$

However, $D$ is dense, a contradiction. □

**Claim 6.7.** *Let $D$ be dense open in $P_{\bar{E}}$, and let $p = p_1 \frown p_0 \in P_{\bar{E}}$. Then there is $p^* \leq^* p$ such that*

$$\exists S^1\ \exists n_1 \forall \langle \bar{\nu}_{1,1}, \dots, \bar{\nu}_{1,n_1} \rangle \in S^1\ \dots \exists S^0\ \exists n_0 \forall \langle \bar{\nu}_{0,1}, \dots, \bar{\nu}_{0,n_0} \rangle \in S^0$$
$$(p_1^*)_{\langle \bar{\nu}_{1,1}, \dots, \bar{\nu}_{1,n_1} \rangle} \frown \dots \frown (p_0^*)_{\langle \bar{\nu}_{0,1}, \dots, \bar{\nu}_{0,n_0} \rangle} \in D$$

*where*

(1) $S^i \subseteq T^{p_i^*} \restriction [V_\kappa]^{n_i}$,
(2) $\forall l < n_i\, \forall \langle \bar{\nu}_1, \dots, \bar{\nu}_l \rangle \in S^i\, \exists \xi\ \mathrm{Suc}_{S^i}(\langle \bar{\nu}_1, \dots, \bar{\nu}_l \rangle) \in E_{\mathrm{mc}(p_i^*)}(\xi)$.

*Proof.* Let $\bar{\epsilon}$ be such that $p_1 \in P_{\bar{\epsilon}}$. We prove that there are $n < \omega$, $p_0^* \leq^* p_0$, $q_1 \leq p_1$, $S \subseteq T^{p^*} \restriction [V_\kappa]^n$ such that

(1) $\forall k < n\, \exists \xi < \mathrm{l}(\bar{E})\ \mathrm{Suc}_S(\langle \bar{\nu}_1, \dots, \bar{\nu}_k \rangle) \in E_{\mathrm{mc}(p^*)}(\xi)$,

(2) $\forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in S \; q_1 {}^\frown (p^*)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle} \in D$.

Set

$$E = \{r \in P_{\bar{E}} \mid \exists q_1 \; q_1 {}^\frown r \in D, q_1 \leq p_1\}.$$

This $E$ is dense open in $P_{\bar{E}}/P_{\bar{\epsilon}}$: Let $r \in P_{\bar{E}}/P_{\bar{\epsilon}}$. Then $p_1 {}^\frown r \in P_{\bar{E}}$. By density of $D$, there is $q_1 {}^\frown s \in D$ such that $q_1 \leq p_1$, $s \leq r$. By the definition of $E$, $s \in E$. Hence $E$ is dense. Openness of $E$ is immediate from the openness of $D$.

By Claim 6.6 there are $p_0^* \leq^* p_0$, $S'$, $n < \omega$ such that

(1) $\forall k < n \, \exists \xi < \mathrm{l}(\bar{E}) \; \mathrm{Suc}_{S'}(\langle \bar{\nu}_1, \dots, \bar{\nu}_k \rangle) \in E_{\mathrm{mc}(p^*)}(\xi)$,
(2) $\forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in S' \; (p_0^*)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle} \in E$.

This means that $\forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in S'$ there is $q_1(\bar{\nu}_1, \dots, \bar{\nu}_n) \leq p_1$ such that

$$\forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in S' \; q_1(\bar{\nu}_1, \dots, \bar{\nu}_n) {}^\frown (p_0^*)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle} \in D.$$

Since $|P_{\bar{\epsilon}}| < \kappa$, $q_1(\bar{\nu}_1, \dots, \bar{\nu}_n)$ is in fact almost always constant. Hence, by shrinking $S'$ to $S$ and letting $q_1$ be this constant value, we get

$$\forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in S \; q_1 {}^\frown (p_0^*)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle} \in D.$$

With this, we have finished the first part of the proof. We use this claim for all conditions in $P_{\bar{\epsilon}}$.

Let $P_{\bar{\epsilon}} = \{p_1^\zeta \mid \zeta < \lambda\}$ where $\lambda < \kappa$.

We construct by induction a $\leq^*$-decreasing sequence $\langle p_0^\zeta \mid \zeta < \lambda \rangle$. Set

$$p_0^0 = p_0.$$

Assume we have constructed $\langle p_0^\zeta \mid \zeta < \zeta_0 \rangle$.

If $\zeta_0$ is limit, choose $p_0^{\zeta_0} \leq^* p_0^\zeta$ for all $\zeta < \zeta_0$.

If $\zeta_0 = \zeta + 1$, use the first part of the proof on $p_1^\zeta {}^\frown p_0^\zeta$ to construct $p_0^{\zeta_0}$.

When the induction terminates we have $\langle p_0^\zeta \mid \zeta < \lambda \rangle$. Choose

$$\forall \zeta < \lambda \; p_0^* \leq^* p_0^\zeta.$$

Let

$$D_{\bar{\epsilon}} = \{q_1 \in P_{\bar{\epsilon}} \mid \exists n \; \exists S \; \forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in S \; q_1 {}^\frown (p_0^*)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle} \in D\}.$$

$D_{\bar{\epsilon}}$ is dense open: Let $q_1 \in P_{\bar{\epsilon}}$. Then there is $\zeta$ such that $q_1 = p_1^\zeta$. By the induction we have that there are $n$, $S$, $r_1 \leq q_1$ such that

$$\forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in S \; r_1 {}^\frown (p_0^{\zeta_0})_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle} \in D.$$

By the openness of $D$ we get

$$\forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in S \; r_1 {}^\frown (p_0^*)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle} \in D.$$

Hence

$$r_1 \in D_{\bar{\epsilon}}.$$

Since $D_{\bar{\epsilon}}$ is dense open, we can use Claim 6.6. Hence there are $p_1^* \leq p_1$, $S^1$, $n_1$ such that

$$\forall \langle \bar{\nu}_1, \dots, \bar{\nu}_{n_1} \rangle \in S^1 \; (p_1^*)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_{n_1} \rangle} \in D_{\bar{\epsilon}}.$$

This means that

$$\forall\langle\bar\nu_{1,1},\dots,\bar\nu_{1,n_1}\rangle\in S^1\ \exists S\ \exists n\ \forall\langle\bar\nu_{0,1},\dots,\bar\nu_{0,n}\rangle\in S \\ (p_1^*)_{\langle\bar\nu_{1,1},\dots,\bar\nu_{1,n_1}\rangle}{}^\frown(p_0^*)_{\langle\bar\nu_{0,1},\dots,\bar\nu_{0,n}\rangle}\in D,$$

which is what we need to prove. □

Finally, we add the last touch.

*Proof of Theorem 6.1.* The proof is done by induction on $k$. The case $k=1$ is Claim 6.7. We assume, then, that the theorem is proved for $k$ and prove it for $k+1$.

Then $p=p_{k+1}{}^\frown p_k{}^\frown\cdots{}^\frown p_0$. Let $\bar\epsilon$ be such that $p_{k+1}\in P_{\bar\epsilon}$. We just repeat the proof of Claim 6.7 with $P_{\bar\epsilon}$ and use the induction hypotheses to conclude the proof. □

## 7. Prikry's Condition

**Theorem 7.1.** *Let $p\in P_{\bar E}$, and let $\sigma$ be a formula in the forcing language. Then there is a $p^*\le^* p$ such that $p^*\parallel\sigma$.*

*Proof.* The set $\{q\in P_{\bar E}\mid q\parallel\sigma\}$ is dense open. Assuming $p=p_k{}^\frown\cdots{}^\frown p_0$ and using 6.1, we get that there is a $q\le^* p$ such that

$$\exists S^k\ \exists n_k\forall\langle\bar\nu_{k,1},\dots,\bar\nu_{k,n_k}\rangle\in S^k\ \dots\exists S^0\ \exists n_0\forall\langle\bar\nu_{0,1},\dots,\bar\nu_{0,n_0}\rangle\in S^0 \\ (p_k^*)_{\langle\bar\nu_{k,1},\dots,\bar\nu_{k,n_k}\rangle}{}^\frown\cdots{}^\frown(p_0^*)_{\langle\bar\nu_{0,1},\dots,\bar\nu_{0,n_0}\rangle}\parallel\sigma.$$

Recall that we really should write

$$\begin{aligned}&S^{k-1}(\bar\nu_{k,1},\dots,\bar\nu_{k,n_k}),\\&S^{k-2}(\bar\nu_{k,1},\dots,\bar\nu_{k,n_k},\bar\nu_{k-1,1},\dots,\bar\nu_{k-1,n_k}),\\&\vdots\end{aligned}$$

In order to avoid (too much) clutter, we use the following convention in the proof. When we write

$$\vec\nu\in\prod_{1\le l\le k}S^l$$

we mean that

$$\begin{aligned}&\langle\bar\nu_{k,1},\dots,\bar\nu_{k,n_k}\rangle\in S^k,\\&\vdots\\&\langle\bar\nu_{1,1},\dots,\bar\nu_{1,n_1}\rangle\in S^1,\end{aligned}$$

and $r(\vec\nu)$ is

$$(q_k)_{\langle\bar\nu_{k,1},\dots,\bar\nu_{k,n_k}\rangle}{}^\frown\cdots{}^\frown(q_1)_{\langle\bar\nu_{1,1},\dots,\bar\nu_{1,n_1}\rangle}.$$

We start by naming $q_0$ as $q_0^{n_0}$ and $T^{q_0}$ as $T^{0,n_0}$. For $\langle\bar\nu_1,\dots,\bar\nu_{n_0-1}\rangle\in S^0$ set

$$A^0=\{\langle\bar\nu_{n_0}\rangle\in\mathrm{Suc}_{T^{0,n_0}}(\langle\bar\nu_1,\dots,\bar\nu_{n-1}\rangle)\mid r(\vec\nu)^\frown(q_0^{n_0})_{\langle\bar\nu_1,\dots,\bar\nu_{n_0}\rangle}\Vdash\sigma\},$$
$$A^1=\{\langle\bar\nu_{n_0}\rangle\in\mathrm{Suc}_{T^{0,n_0}}(\langle\bar\nu_1,\dots,\bar\nu_{n_0-1}\rangle)\mid r(\vec\nu)^\frown(q_0^{n_0})_{\langle\bar\nu_1,\dots,\bar\nu_{n_0}\rangle}\Vdash\neg\sigma\}.$$

Note that

$$\begin{aligned}&\mathrm{Suc}_{S^0}(\langle\bar{\nu}_1,\dots,\bar{\nu}_{n_0-1}\rangle)\subseteq A^0\cup A^1,\\&A^0\cap A^1=\emptyset.\end{aligned}$$

Hence, there is a $\xi<\mathrm{l}(\bar{E})$ such that one and only one of the following is true:

(1) $A^0\in E_{\mathrm{mc}(q_0^{n_0})}(\xi)$,
(2) $A^1\in E_{\mathrm{mc}(q_0^{n_0})}(\xi)$.

In either case, using Lemma 4.8, we can shrink $T^{0,n_0}_{\langle\bar{\nu}_1,\dots,\bar{\nu}_{n_0-1}\rangle}$ and get a condition $q_0'$ such that $r(\vec{\nu})^\frown q'_{\langle\bar{\nu}_1,\dots,\bar{\nu}_{n_0-1}\rangle}\parallel\sigma$.

So now we shrink $T^{0,n_0}_{\langle\bar{\nu}_1,\dots,\bar{\nu}_{n_0-1}\rangle}$ for all $\langle\bar{\nu}_1,\dots,\bar{\nu}_{n_0-1}\rangle\in S^0$, and we call this tree $T^{0,n_0-1}$. The name of the condition $q_0^{n_0}$ with $T^{0,n_0-1}$ substituted for $T^{0,n_0}$ is $q_0^{n_0-1}$.

$q_0^{n_0-1}$ satisfies

$$\forall\langle\bar{\nu}_1,\dots,\bar{\nu}_{n_0-1}\rangle\in S^0\ r(\vec{\nu})^\frown(q_0^{n_0-1})_{\langle\bar{\nu}_1,\dots,\bar{\nu}_{n_0-1}\rangle}\parallel\sigma.$$

We are now in the same position as we were when setting $q_0^{n_0}$. So by repeating the above arguments we get

$$p_0\geq^* q_0=q_0^{n_0}\geq^* q_0^{n_0-1}\geq^*\cdots\geq^* q_0^1\geq^* q_0^0$$

such that for each $l=n_0,n_0-1,\dots,1,0$,

$$\forall\langle\bar{\nu}_1,\dots,\bar{\nu}_l\rangle\in S^0\ r(\vec{\nu})^\frown(q_0^l)_{\langle\bar{\nu}_1,\dots,\bar{\nu}_l\rangle}\parallel\sigma.$$

Specifically, we get

$$r(\vec{\nu})^\frown(q_0^0)\parallel\sigma.$$

Of course $q_0^0$ depends on $\vec{\nu}$. Note that we got from $q_0^{n_0}$ to $q_0^0$ only by shrinking the trees. So, we repeat this process for all $\vec{\nu}$, calling the resulting condition $q_0^0(\vec{\nu})$. So we have

$$\forall\vec{\nu}\in\prod_{1\leq l\leq k}S^l\ r(\vec{\nu})^\frown q_0^0(\vec{\nu})\parallel\sigma.$$

By setting

$$T^{p_0^*}=\bigcap_{\vec{\nu}}T^{q_0^0(\vec{\nu})}$$

and letting $p_0^*$ be $q_0$ with $T^{p_0^*}$ substituted for $T^{q_0}$, we get

$$\forall\vec{\nu}\in\prod_{1\leq l\leq k}S^l\ r(\vec{\nu})^\frown p_0^*\parallel\sigma.$$

We are in the same position as in the beginning of the proof. So in the same way we can generate $p_1^*$ from $p_1$ and so on, until we have

$$p_k^*{}^\frown\cdots{}^\frown p_0^*\parallel\sigma.$$

□

## 8. Properness

The notions of $\langle N, P\rangle$-generic and properness, as defined by Shelah [9], are as follows:

**Definition 8.1.** Let $N \prec H_\chi$ be such that

(1) $|N| = \omega$,
(2) $P \in N$.

Then $p \in P$ is called $\langle N, P\rangle$-*generic* if

$$p \Vdash \ulcorner \forall D \in \widehat{N}\ D \text{ is dense open in } \widehat{P} \implies D \cap \widetilde{G} \cap \widehat{N} \neq \emptyset \urcorner.$$

**Definition 8.2.** A forcing notion $P$ is called *proper* if for all $N \prec H_\chi$ such that

(1) $|N| = \omega$,
(2) $P \in N$,

and for all $q \in P \cap N$ there is a $p \leq q$ that is $\langle N, P\rangle$-generic.

We adapt these definitions for our needs (namely, larger submodels), keeping the original names:

**Definition 8.3.** Let $N \prec H_\chi$ be such that

(1) $|N| = \kappa$,
(2) $N \supseteq V_\kappa$,
(3) $N \supseteq N^{<\kappa}$,
(4) $P \in N$.

Then $p \in P$ is called $\langle N, P\rangle$-*generic* if

$$p \Vdash \ulcorner \forall D \in \widehat{N}\ D \text{ is dense open in } \widehat{P} \implies D \cap \widetilde{G} \cap \widehat{N} \neq \emptyset \urcorner.$$

**Definition 8.4.** A forcing notion $P$ is called *proper* if for all $N \prec H_\chi$ such that

(1) $|N| = \kappa$,
(2) $N \supseteq V_\kappa$,
(3) $N \supseteq N^{<\kappa}$,
(4) $P \in N$,

and for all $q \in P \cap N$ there is a $p \leq q$ that is $\langle N, P\rangle$-generic.

It was brought to our attention by the referee that Woodin initiated the use of proper forcing arguments to show cardinal preservation in Radin-style forcing.

**Claim 8.5.** *Let $p \in P_{\bar{E}}$ and $N \prec H_\chi$ be such that*

(1) $|N| = \kappa$,
(2) $N \supseteq V_\kappa$,
(3) $N \supseteq N^{<\kappa}$,
(4) $P_{\bar{E}} \in N$,
(5) $p \in P_{\bar{E}} \cap N$.

*Then there is $p^* \leq^* p$ such that $p^*$ is $\langle N, P_{\bar{E}}\rangle$-generic.*

*Proof.* Let $p = p_{k(p)} \frown \cdots \frown p_1 \frown p_0$.

Let $\langle D_\xi \mid \xi < \kappa\rangle$ be an enumeration of all dense open subsets of $P_{\bar{E}}$ that are in $N$. Note that for $\xi_0 < \kappa$ we have that $\langle D_\xi \mid \xi < \xi_0\rangle \in N$.

We now start an induction on $\xi$ in which we build

$$\langle \bar{\alpha}^\xi, u^\xi \mid \xi < \kappa\rangle.$$

The construction is done ensuring that $\langle \bar{\alpha}^\xi, u^\xi \mid \xi < \xi_0 \rangle \in N$ for all $\xi_0 < \kappa$. We start by setting

$$\begin{aligned} u^0 &= p_0 \setminus \{T^{p_0}\}, \\ \bar{\alpha}^0 &= \mathrm{mc}(p_0), \\ T^0 &= T^{p_0} \restriction \pi^{-1}_{\bar{\alpha}^0,0}\{\bar{\nu} \mid \kappa^0(\bar{\nu}) \text{ is inaccessible}\}, \end{aligned}$$

and taking an increasing enumeration in $N$,

$$\{\kappa^0(\bar{\nu}) \mid \langle \bar{\nu} \rangle \in T^0\} = \langle \tau_\xi \mid \xi < \kappa \rangle.$$

Assume then that we have

$$\langle \bar{\alpha}^\xi, u^\xi \mid \xi < \xi_0 \rangle.$$

The construction splits now according to whether $\xi_0$ is limit or successor. In both cases the work is done inside $N$.

If $\xi_0$ is limit, choose $\bar{\alpha}^{\xi_0} >_{\bar{\mathrm{E}}} \bar{\alpha}^\xi$ for all $\xi < \xi_0$ and set

$$u^{\xi_0} = \bigcup_{\xi<\xi_0} u^\xi \cup \left\{\langle \bar{\alpha}^{\xi_0}, t \rangle\right\} \text{ where } \kappa^0(t) = \tau_{\xi_0}.$$

If $\xi_0 = \xi + 1$, for each $\bar{\nu}_1, \ldots, \bar{\nu}_n$ such that $\kappa^0(\bar{\nu}_1) < \cdots < \kappa^0(\bar{\nu}_n) = \tau_\xi$ we set

$$\begin{aligned} S(\bar{\nu}_1, \ldots, \bar{\nu}_n) = \Big( &\prod_{\bar{\alpha} \in \operatorname{supp} u^\xi} \{\bar{\mu}_1 \mid \kappa^0(\bar{\mu}_1) = \kappa^0(\bar{\nu}_1)\}\Big) \\ &\times \Big( \prod_{\bar{\alpha} \in \operatorname{supp} u^\xi} \{\bar{\mu}_2 \mid \kappa^0(\bar{\mu}_2) = \kappa^0(\bar{\nu}_2)\}\Big) \\ &\vdots \\ &\times \Big( \prod_{\bar{\alpha} \in \operatorname{supp} u^\xi} \{\bar{\mu}_n \mid \kappa^0(\bar{\mu}_n) = \kappa^0(\bar{\nu}_n)\}\Big) \\ &\times \left\{\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle\right\}. \end{aligned}$$

Let

$$S = \bigcup_{\kappa^0(\bar{\nu}_1) < \cdots < \kappa^0(\bar{\nu}_n) = \tau_\xi} S(\bar{\nu}_1, \ldots, \bar{\nu}_n)$$

and fix an enumeration of $S$,

$$S = \langle s^{\xi_0,\rho} \mid \rho < \tau_{\xi_0} \rangle.$$

We do induction on $\rho$, which builds

$$\langle \bar{\alpha}^{\xi_0,\rho}, u_0^{\xi_0,\rho}, T_0^{\xi_0,\rho} \mid \rho < \tau_{\xi_0} \rangle,$$

from which we build $\langle \bar{\alpha}^{\xi_0}, u^{\xi_0} \rangle$. Set

$$\begin{aligned} \bar{\alpha}^{\xi_0,0} &= \bar{\alpha}^\xi, \\ u_0^{\xi_0,0} &= u_0^\xi. \end{aligned}$$

Assume we have constructed $\langle \bar{\alpha}^{\xi_0,\rho}, u_0^{\xi_0,\rho}, T_0^{\xi_0,\rho} \mid \rho < \rho_0 \rangle$.

If $\rho_0$ is limit, set

$$\forall \rho < \rho_0 \ \bar{\alpha}^{\xi_0,\rho_0} >_{\bar{\mathrm{E}}} \bar{\alpha}^{\xi_0,\rho},$$
$$u^{\xi_0,\rho_0} = \bigcup_{\rho<\rho_0} u^{\xi_0,\rho} \cup \left\{\langle \bar{\alpha}^{\xi_0,\rho_0}, t\rangle\right\} \text{ where } \kappa^0(t) = \tau_{\xi_0}.$$

We set $T^{\xi_0,\rho_0}$ equal to anything we like, since we do not use it later.

If $\rho_0 = \rho + 1$, let $s^{\xi_0,\rho}(n(s)+1) = \langle \bar{\nu}_1, \dots, \bar{\nu}_n\rangle$ and set

$$\begin{aligned}
u'' &= (u_0^{\xi_0,\rho})_{\langle s^{\xi_0,\rho}\rangle},\\
T_0'' &= \pi^{-1}_{\bar{\alpha}^{\xi},\bar{\alpha}^0}(T^0_{\langle \bar{\nu}_1,\dots,\bar{\nu}_n\rangle}),\\
T_1'' &= \pi^{-1}_{\mathrm{mc}(u_1''),\bar{\nu}_n}(T^0_{\langle \bar{\nu}_1,\dots,\bar{\nu}_{n-1}\rangle}(\bar{\nu}_n)),\\
&\vdots\\
T_{n-1}'' &= \pi^{-1}_{\mathrm{mc}(u_{n-1}''),\bar{\nu}_2}(T^0_{\langle \bar{\nu}_1\rangle}(\bar{\nu}_2)),\\
T_n'' &= \pi^{-1}_{\mathrm{mc}(u_n''),\bar{\nu}_1}(T^0(\bar{\nu}_1)).
\end{aligned}$$

Take the enumeration

$$\{D_\sigma \mid \sigma < \tau_\xi\} \times \{q \mid q \le p_{k(p)} \frown \dots \frown p_1 \frown u''_{n(s)} \cup \{T_n''\} \frown \dots \frown u_1'' \cup \{T_1''\}\}$$
$$= \langle\langle E^{\xi_0,\rho_0,\zeta}, q^{\xi_0,\rho_0,\zeta}\rangle \mid \zeta < \tau_{\xi_0}\rangle.$$

We start an induction on $\zeta$. Set

$$\bar{\alpha}^{\xi_0,\rho_0,0} = \bar{\alpha}^{\xi_0,\rho},$$
$$u_0^{\xi_0,\rho_0,0} = u^{\xi_0,\rho}.$$

Assume we have constructed $\langle \bar{\alpha}^{\xi_0,\rho_0,\zeta}, u_0^{\xi_0,\rho_0,\zeta}, T_0^{\xi_0,\rho_0,\zeta} \mid \zeta < \zeta_0\rangle$.

If $\zeta_0$ is limit, take

$$\forall \zeta < \zeta_0 \ \bar{\alpha}^{\xi_0,\rho_0,\zeta_0} >_{\bar{\mathrm{E}}} \bar{\alpha}^{\xi_0,\rho_0,\zeta},$$
$$u_0^{\xi_0,\rho_0,\zeta_0} = \bigcup_{\zeta<\zeta_0} u_0^{\xi_0,\rho_0,\zeta} \cup \left\{\langle \bar{\alpha}^{\xi_0,\rho_0,\zeta_0}, t\rangle\right\} \text{ where } \kappa^0(t) = \tau_{\xi_0}.$$

We set $T^{\xi_0,\rho_0,\zeta}$ equal to whatever we want, since no use of it is made later.

If $\zeta_0 = \zeta + 1$, we set

$$u'' = (u_0^{\xi_0,\rho_0,\zeta})_{s^{\xi_0,\rho}},$$
$$T_0'' = \pi^{-1}_{\bar{\alpha}^{\xi_0,\rho_0,\zeta},\bar{\alpha}^0} T^0_{\langle \bar{\nu}_1,\dots,\bar{\nu}_n\rangle}.$$

If there is

$$u_0' \le^* u_0'' \cup \{T_0''\}$$

such that

$$q^{\xi_0,\rho_0,\zeta} \cup u_0' \in E^{\xi_0,\rho_0,\zeta},$$

then set

$$\begin{aligned}
\bar{\alpha}^{\xi_0,\rho_0,\zeta_0} &= \mathrm{mc}(u_0'),\\
u_0^{\xi_0,\rho_0,\zeta_0} &= u_0^{\xi_0,\rho_0,\zeta} \cup (u_0' \setminus (u_0'' \cup \left\{T^{u_0'}\right\})),\\
T^{\xi_0,\rho_0,\zeta_0} &= T^{u_0'};
\end{aligned}$$

otherwise set

$$\begin{aligned}\bar{\alpha}^{\xi_0,\rho_0,\zeta_0} &= \bar{\alpha}^{\xi_0,\rho_0,\zeta},\\ u_0^{\xi_0,\rho_0,\zeta_0} &= u_0^{\xi_0,\rho_0,\zeta},\\ T_0^{\xi_0,\rho_0,\zeta_0} &= T_0''.\end{aligned}$$

When the induction on $\zeta$ terminates, we have $\langle \bar{\alpha}^{\xi_0,\rho_0,\zeta}, u_0^{\xi_0,\rho_0,\zeta}, T^{\xi_0,\rho_0,\zeta} \mid \zeta < \tau_{\xi_0}\rangle$.

We continue with the induction on $\rho$. We set

$$\begin{aligned}&\forall \zeta < \tau_{\xi_0}\ \bar{\alpha}^{\xi_0,\rho_0} >_{\bar{\mathrm{E}}} \bar{\alpha}^{\xi_0,\rho_0,\zeta},\\ &u_0^{\xi_0,\rho_0} = \bigcup_{\zeta<\tau_{\xi_0}} u_0^{\xi_0,\rho_0,\zeta} \cup \left\{\langle \bar{\alpha}^{\xi_0,\rho_0}, t\rangle\right\} \text{ where } \kappa^0(t) = \tau_{\xi_0}.\end{aligned}$$

When the induction on $\rho$ terminates, we have $\langle \bar{\alpha}^{\xi_0,\rho}, u_0^{\xi_0,\rho}, T^{\xi_0,\rho} \mid \rho < \tau_{\xi_0}\rangle$. We continue with the induction on $\xi$. We set

$$\begin{aligned}&\forall \rho < \tau_{\xi_0}\ \bar{\alpha}^{\xi_0} >_{\bar{\mathrm{E}}} \bar{\alpha}^{\xi_0,\rho},\\ &u_0^{\xi_0} = \bigcup_{\rho<\tau_{\xi_0}} u_0^{\xi_0,\rho} \cup \left\{\langle \bar{\alpha}^{\xi_0}, t\rangle\right\} \text{ where } \kappa^0(t) = \tau_{\xi_0}.\end{aligned}$$

When the induction on $\xi$ terminates we have $\langle \bar{\alpha}^{\xi}, u_0^{\xi} \mid \xi < \kappa\rangle$. We note that this sequence is not in $N$. Let

$$\begin{aligned}&\forall \xi < \kappa\ \bar{\alpha}^* >_{\bar{\mathrm{E}}} \bar{\alpha}^{\xi},\\ &p_0^* = \bigcup_{\xi<\kappa} u_0^{\xi} \cup \{\langle \bar{\alpha}^*, t\rangle\} \text{ where } \kappa^0(t) = \max p_0^0.\end{aligned}$$

We construct a series of trees, $R^n$, and $T^{p_0^*}$ is $\bigcap_{n<\omega} R^n$. Then

$$\mathrm{Lev}_0(R^0) = \pi^{-1}_{\bar{\alpha}^*,\bar{\alpha}^0}\,\mathrm{Lev}_0(T^0).$$

Let us consider $\langle \bar{\nu}_1\rangle \in \mathrm{Lev}_0(R^0)$. There is a $\xi$ such that $\kappa^0(\bar{\nu}_1) = \tau_\xi$. We set

$$\begin{aligned}s(0) &= \{\langle \bar{\alpha}, \pi_{\bar{\alpha}^*,\bar{\alpha}}(\bar{\nu}_1)\rangle \mid \bar{\alpha} \in \operatorname{supp} p_0^*\},\\ s(1) &= \left\{\langle \pi_{\bar{\alpha}^*,\bar{\alpha}^0}(\bar{\nu}_1)\rangle\right\}.\end{aligned}$$

Let $\xi_0 = \xi + 1$. By our construction there is a $\rho$ such that

$$(u_0^{\xi_0,\rho_0})_{\langle s\rangle} = (u_0^{\xi_0,\rho_0})_{\langle s^{\xi_0,\rho}\rangle},$$

where $\rho_0 = \rho + 1$. We set

$$R^1_{\langle \bar{\nu}_1\rangle} = \pi^{-1}_{\bar{\alpha}^*,\bar{\alpha}^{\xi_0,\rho_0}}(T^{\xi_0,\rho_0}) \cap \pi^{-1}_{\bar{\alpha}^*,\bar{\alpha}^0}(T^0_{\langle \pi_{\bar{\alpha}^*,\bar{\alpha}^0}(\bar{\nu}_1)\rangle}).$$

Assume that we have constructed $R^n$. We set the first $n$ levels of $R^{n+1}$ to be the same as the first $n$ levels of $R^n$, and we complete the tree as follows. Let us consider $\langle \bar{\nu}_1, \ldots, \bar{\nu}_n\rangle \in R^n$. There is a $\xi$ such that $\kappa^0(\bar{\nu}_n) = \tau_\xi$. We take $s$ as follows:

$$\begin{aligned}&\forall 1 \le k \le n\ s(k) = \{\langle \bar{\alpha}, \pi_{\bar{\alpha}^*,\bar{\alpha}}(\bar{\nu}_k)\rangle \mid \bar{\alpha} \in \operatorname{supp} p_0^*\},\\ &s(n+1) = \left\{\langle \pi_{\bar{\alpha}^*,\bar{\alpha}^0}(\bar{\nu}_1), \ldots, \pi_{\bar{\alpha}^*,\bar{\alpha}^0}(\bar{\nu}_n)\rangle\right\}.\end{aligned}$$

Let $\xi_0 = \xi + 1$. By our construction there is a $\rho$ such that

$$(u_0^{\xi_0,\rho_0})_{\langle s\rangle} = (u_0^{\xi_0,\rho_0})_{\langle s^{\xi_0,\rho}\rangle},$$

where $\rho_0 = \rho + 1$. We set

$$R^{n+1}_{\langle \bar{\nu}_1,\dots,\bar{\nu}_n\rangle} = \pi^{-1}_{\bar{\alpha}^*,\bar{\alpha}^{\xi_0,\rho_0}}(T^{\xi_0,\rho_0}) \cap \pi^{-1}_{\bar{\alpha}^*,\bar{\alpha}^0}(T^0_{\langle \pi_{\bar{\alpha}^*,\bar{\alpha}^0}(\bar{\nu}_1),\dots,\pi_{\bar{\alpha}^*,\bar{\alpha}^0}(\bar{\nu}_n)\rangle}).$$

After $\omega$ stages we set

$$T^{p_0^*} = \bigcap_{n<\omega} R^n.$$

We finish the construction by setting

$$p^* = p_{k(p)} {}^\frown \cdots {}^\frown p_1 {}^\frown p_0^*.$$

We show that $p^*$ is as required.

Let $G$ be $P_{\bar{E}}$-generic such that $p^* \in G$. Let $D \in N$ be dense open in $P_{\bar{E}}$. We want to show that $D \cap G \cap N \neq \emptyset$.

Choose $q {}^\frown r_0 \in D \cap G$ such that

$$\begin{aligned} &r_0 \leq^* p_0'', \\ &q \leq p_{k(p)} {}^\frown \cdots {}^\frown p_1 {}^\frown p_n'' {}^\frown \cdots {}^\frown p_1'', \end{aligned}$$

where

$$\begin{aligned} &\langle \bar{\nu}_1,\dots,\bar{\nu}_n\rangle \in \operatorname{dom} T^{p_0^*}, \\ &\kappa^0(\bar{\nu}_n) = \tau_\xi, \\ &D \in \langle D_\zeta \mid \zeta < \tau_\xi\rangle, \\ &p'' = (p_0^*)_{\langle \bar{\nu}_1,\dots,\bar{\nu}_n\rangle}. \end{aligned}$$

We take $s$ to be

$$\begin{aligned} &\forall 1 \leq k \leq n \ s(k) = \{\langle \bar{\alpha}, \pi_{\bar{\alpha}^*,\bar{\alpha}}(\bar{\nu}_k)\rangle \mid \bar{\alpha} \in \operatorname{supp} p_0^*\}, \\ &s(n+1) = \left\{\langle \pi_{\bar{\alpha}^*,\bar{\alpha}^0}(\bar{\nu}_1),\dots,\pi_{\bar{\alpha}^*,\bar{\alpha}^0}(\bar{\nu}_n)\rangle\right\}. \end{aligned}$$

We get that

$$(p_0^*)_{\langle \bar{\nu}_1,\dots,\bar{\nu}_n\rangle} = (p_0^*)_{\langle s\rangle}.$$

We let $\xi_0 = \xi + 1$. Recall the enumeration of $S$ in the construction. There is a $\rho$ such that

$$(p_0^* \setminus \{\langle \mathrm{mc}(p_0^*), (p_0^*)^{\mathrm{mc}}\rangle\})_{\langle s\rangle} \cup \{\langle \mathrm{mc}(p_0^*), (p_0^*)^{\mathrm{mc}}\rangle\} = (p_0^*)_{\langle s^{\xi_0,\rho}\rangle}.$$

We let $\rho_0 = \rho + 1$. Considering the construction of $T^{p_0^*}$, we see that

$$T^{p_0^*}_{\langle \bar{\nu}_1,\dots,\bar{\nu}_n\rangle} \leq T^{\xi_0,\rho_0};$$

hence

$$(p_0^*)_{\langle s\rangle} \leq^* (u_0^{\xi_0,\rho_0})_{\langle s^{\xi_0,\rho_0}\rangle}.$$

We note that

$$\forall 1 \leq k \leq n \ p_k'' = ((p_0^*)_{\langle s\rangle})_k.$$

Recalling that $q$ was chosen so that

$$q \leq p_{k(p)} {}^\frown \cdots {}^\frown p_1 {}^\frown p_n'' {}^\frown \cdots {}^\frown p_1'',$$

we conclude that there is a $\zeta$ such that $q = q^{\xi_0,\rho_0,\zeta}$ and $D = E^{\xi_0,\rho_0,\zeta}$. That is,

$$E^{\xi_0,\rho_0,\zeta} \ni q^{\xi_0,\rho_0,\zeta} {}^\frown r_0 \leq^* q^{\xi_0,\rho_0,\zeta} {}^\frown ((u^{\xi_0,\rho_0,\zeta})_{\langle s^{\xi_0,\rho}\rangle})_0.$$

Note that this is an answer to the question we asked in the construction. Hence, because $N$ is elementary, there was such a condition in $N$. Hence

$$q^{\xi_0,\rho_0,\zeta} {}^\frown ((u^{\xi_0,\rho_0,\zeta})_{\langle s^{\xi_0,\rho}\rangle})_0 \in D \cap N.$$

The last point to note is that

$$q^{\xi_0,\rho_0,\zeta} {}^\frown ((u^{\xi_0,\rho_0,\zeta})_{\langle s^{\xi_0,\rho}\rangle})_0 \geq^* .q^{\xi_0,\rho_0,\zeta} {}^\frown p_0'' \geq^* q {}^\frown r_0 \in G.$$

Hence

$$q^{\xi_0,\rho_0,\zeta} {}^\frown ((u^{\xi_0,\rho_0,\zeta})_{\langle s^{\xi_0,\rho}\rangle})_0 \in G.$$

□

**Corollary 8.6.** *$P_{\bar{E}}$ is proper.*

## 9. Cardinals in $V^{P_{\bar{E}}}$

**Lemma 9.1.** *$\kappa^+$ remains a cardinal in $V^{P_{\bar{E}}}$.*

*Proof.* The proof really has no connection to the specific structure of $P_{\bar{E}}$. It is an exercise in properness.

Let

$$p \Vdash \ulcorner \widetilde{f}{:}\widehat{\kappa} \to \widehat{\kappa^+} \urcorner.$$

Choose $\chi$ large enough so that $H_\chi$ contains everything we are interested in. Take $N \prec H_\chi$ such that

(1) $p, P_{\bar{E}}, \widetilde{f} \in N$,
(2) $|N| = \kappa$,
(3) $N \supseteq V_\kappa$,
(4) $N \supseteq N^{<\kappa}$.

By Corollary 8.6 there is a $q \leq p$ that is $\langle N, P_{\bar{E}}\rangle$-generic. Let us set

$$\lambda = N \cap \kappa^+;$$

hence $\lambda$ is an ordinal $< \kappa^+$.

Let $G$ be $P_{\bar{E}}$-generic with $q \in G$. The $\langle N, P_{\bar{E}}\rangle$-genericity ensures us that for all $\xi < \kappa$, $\widetilde{f}(\xi)^{N[G]} \in N$ and $\widetilde{f}(\xi)^{N[G]} = \widetilde{f}(\xi)^{H_\chi[G]}$. Hence $\operatorname{ran} \widetilde{f}^{V[G]} \subseteq \lambda$. That is,

$$q \Vdash \ulcorner \widetilde{f} \text{ is bounded in } \kappa^+ \urcorner.$$

□

**Lemma 9.2.** *No cardinals $> \kappa$ are collapsed by $P_{\bar{E}}$.*

*Proof.* $\kappa^+$ is not collapsed by Lemma 9.1. No cardinals $\geq \kappa^{++}$ are collapsed, since $P_{\bar{E}}$ satisfies $\kappa^{++}$-c.c. □

**Lemma 9.3.** *Let $\xi < \kappa$, and let $\zeta$ be the ordinal such that $\kappa^0(\bar{E}_G(\zeta)) \leq \xi < \kappa^0(\bar{E}_G(\zeta+1))$. Then $\mathcal{P}(\xi) \cap V[G] = \mathcal{P}(\xi) \cap V[G{\restriction}\zeta]$.*

*Proof.* Take $p = p_n {}^\frown \cdots {}^\frown p_{k+1} {}^\frown p_k {}^\frown \cdots {}^\frown p_0 \in G$ such that $\bar{E}(p_{k+1}) = \bar{E}_G(\zeta)$, $\bar{E}(p_k) = \bar{E}_G(\zeta+1)$. We know that $V[G] = V[G/p]$. So we work in $P_{\bar{E}}/p$. Set $p^l = p_n {}^\frown \cdots {}^\frown p_{k+1}$, $p^h = \{\langle \bar{E}_G(\zeta), \emptyset\rangle\} {}^\frown p_k {}^\frown \cdots {}^\frown p_0$. Then $P_{\bar{E}}/p = P_{\bar{E}}{\restriction}p^l \times P_{\bar{E}}{\restriction}p^h$. Note that $\langle P_{\bar{E}}/p^h, \leq^*\rangle$ is $\kappa^0(\bar{E}_G(\zeta+1))$-closed. In particular, it is $\xi^+$-closed.

Let $A \in V[G]$, $A \subseteq \xi$. Choose $\widetilde{A}$, a canonical $P_{\bar{E}}/p$-name for $A$. Let $q \in P_{\bar{E}}/p^h$. By induction we construct $\langle q_\tau \mid \tau < \xi\rangle$ satisfying

(1) $\tau_0 < \tau_1 \implies q_{\tau_1} \leq^* q_{\tau_0}$, and
(2) $q_\tau \parallel \ulcorner \widehat{\tau} \in \widetilde{A} \urcorner$.

Choose $q_\xi \leq^* q_\tau$ for all $\tau < \xi$.

By a density argument we can construct a $P{\restriction}p^l$-name $\widetilde{B}$ such that $A = \widetilde{A}[G] = \widetilde{B}[G{\restriction}p^l]$. □

**Corollary 9.4.** *No cardinals $\leq \kappa$ are collapsed by $P_{\bar{E}}$.*

*Proof.* Let $G \subseteq P_{\bar{E}}$ be generic. Assume $\lambda < \kappa$ is a collapsed cardinal. Let $\mu = |\lambda|^{V[G]}$. We have $\mu < \lambda$, and there is an $A \in \mathcal{P}(\mu)^{V[G]}$ that codifies the order type $\lambda$. Let $\zeta$ be the unique ordinal such that $\kappa^0(\bar{E}_G(\zeta)) \leq \mu < \kappa^0(\bar{E}_G(\zeta+1))$. By Lemma 9.3, $A \in V[G{\restriction}\zeta]$. Hence $\lambda$ is collapsed already in $V[G{\restriction}\zeta]$. However, by Lemma 9.2, $P_{\bar{E}_G(\zeta)}$ collapses no cardinals above $\kappa^0(\bar{E}_G(\zeta))$, a contradiction.

So, no cardinal $< \kappa$ is collapsed. Since $\kappa$ is a limit of cardinals that are not collapsed, it is not collapsed. □

We have just shown

**Theorem 9.5.** *No cardinals are collapsed in $V^{P_{\bar{E}}}$.*

## 10. Properties of $\kappa$ in $V^{P_{\bar{E}}}$

**Theorem 10.1.** *If $\mathrm{l}(\bar{E}) = \kappa^+$, then $V^{P_{\bar{E}}} \vDash \ulcorner \kappa$ is regular$\urcorner$.*

*Proof.* Let $\lambda < \kappa$, and let $\widetilde{f}$ be such that

$$\Vdash_{P_{\bar{E}}} \ulcorner \widetilde{f} : \widehat{\lambda} \to \widehat{\kappa} \urcorner.$$

Let

$$D_0 = \{p \mid \exists i \; p \Vdash \ulcorner \widetilde{f}(0) = \widehat{i} \urcorner\}.$$

Since $D_0$ is a dense open set, we can invoke Theorem 6.1 to get $p'^0$, $n_0$, $S^0 \subset T^{p'^0}$ such that

$$\forall k < n_0 \, \forall \langle \nu_1, \ldots, \nu_k \rangle \in S'^0 \, \exists \xi < \mathrm{l}(\bar{E}) \;\; \mathrm{Suc}_S(\langle \nu_1, \ldots, \nu_k \rangle) \in E_{\mathrm{mc}(p'^0)}(\xi),$$
$$\forall \langle \nu_1, \ldots, \nu_{n_0} \rangle \in S'^0 \;\; (p'^0)_{\langle \nu_1, \ldots, \nu_{n_0} \rangle} \in D_0.$$

Let us set

$$A'_0 = \{(p'^0)_{\langle \nu_1, \ldots, \nu_{n_0} \rangle} \mid \langle \nu_1, \ldots, \nu_{n_0} \rangle \in S'^0\}.$$

$A_0$ is an anti-chain. By shrinking $T^{p'^0}$ as was done in the proof of Theorem 7.1, we can make $A_0$ into a maximal anti-chain below $p'^0$. Since $\lambda < \kappa$ and $\langle P^*_{\bar{E}}, \leq^* \rangle$ is $\kappa$-closed, we can construct a $\leq^*$-decreasing sequence

$$p'^0 \geq^* p'^1 \geq^* \cdots \geq^* p'^\tau \geq^* \cdots, \qquad \tau < \lambda,$$

and $n_\tau$, $S'^\tau \subseteq T^{p'^\tau}$ such that

$$\forall k < n_\tau \, \forall \langle \nu_1, \ldots, \nu_k \rangle \in S'^\tau \, \exists \xi < \mathrm{l}(\bar{E}) \;\; \mathrm{Suc}_{S'^\tau}(\langle \nu_1, \ldots, \nu_k \rangle) \in E_{\mathrm{mc}(p'^\tau)}(\xi),$$
$$\forall \langle \nu_1, \ldots, \nu_{n_\tau} \rangle \in S'^\tau \;\; \exists i \; (p'^\tau)_{\langle \nu_1, \ldots, \nu_{n_\tau} \rangle} \Vdash \ulcorner \widetilde{f}(\widehat{\tau}) = \widehat{i} \urcorner,$$

and

$$A'_\tau = \{(p'^\tau)_{\langle \nu_1, \ldots, \nu_{n_\tau} \rangle} \mid \langle \nu_1, \ldots, \nu_{n_\tau} \rangle \in S'^\tau\}$$

is a maximal anti-chain below $p'^\tau$.

Let $p' \leq^* p'^\tau$ for all $\tau < \lambda$. We set $S^\tau = \pi^{-1}_{\mathrm{mc}(p'),\mathrm{mc}(p'^\tau)}(S'^\tau)$, and take $p^\tau$ to be $p'$ with $\pi^{-1}_{\mathrm{mc}(p'),\mathrm{mc}(p'^\tau)}(T^{p'^\tau})$ substituted for $T^{p'}$ and maybe shrunken a bit so that

$$A_\tau = \{(p^\tau)_{\langle \nu_1,\ldots,\nu_{n_\tau}\rangle} \mid \langle \nu_1,\ldots,\nu_{n_\tau}\rangle \in S^\tau\}$$

is a maximal anti-chain below $p^\tau$.

Let $p \leq^* p^\tau$ for all $\tau < \lambda$, and let $\widetilde{g}$ be the following $P_{\bar{E}}$-name:

$$\widetilde{g} = \bigcup_{\tau<\lambda} \{\langle \widehat{\langle \tau, i\rangle}, (p^\tau)_{\langle \nu_1,\ldots,\nu_{n_\tau}\rangle}\rangle \mid A_\tau \ni (p^\tau)_{\langle \nu_1,\ldots,\nu_{n_\tau}\rangle} \Vdash \ulcorner \widetilde{f}(\tau) = \widehat{i} \urcorner \}.$$

Then

$$p \Vdash \ulcorner \widetilde{f} = \widetilde{g} \urcorner.$$

Let $P^*$ be the following forcing notion:

$$P^* = \{q \leq_R p \mid q \in P_{\bar{E}}\}.$$

By Claim 5.2, $\langle P^*, \leq_R\rangle$ is sub-forcing of $\langle P_{\bar{E}}/p, \leq\rangle$. Hence, if $G$ is $P_{\bar{E}}$-generic, then $G^* = G \cap P^*$ is $P^*$-generic. $\widetilde{g}$ is in fact a $P^*$-name, and, as can be seen from its definition, $\widetilde{g}[G] = \widetilde{g}[G^*] \in V[G^*]$. So in order to complete the proof it is enough to show that $\Vdash_{P^*} \ulcorner \widetilde{g}$ is bounded$\urcorner$.

By Claim 5.3 there is $r \in R_{\mathrm{mc}(p)}$ such that $P^* \simeq R_{\mathrm{mc}(p)}/r$. Now we use the following fact about Radin forcing: When the measure sequence is of length $\kappa^+$, $\kappa$ is regular in the generic extension. Necessarily, $\Vdash_{P^*} \ulcorner \widetilde{g}$ is bounded$\urcorner$. $\square$

**Definition 10.2.** We say that $\tau < \mathrm{l}(\bar{E})$ is a *repeat point* of $\bar{E}$ if $P_{\bar{E}} = P_{\bar{E}\restriction\tau}$.

We note, again, our convention that when $\tau_1 < \tau_2$ we have $P_{\bar{E}\restriction\tau_2} \subseteq P_{\bar{E}\restriction\tau_1}$. The equality in the above definition is meant to be with this convention. We point out that if $\tau$ is a repeat point, then $P_{\bar{E}\restriction\tau} \in M$.

**Theorem 10.3.** *If $\bar{E}$ has a repeat point, then $V^{P_{\bar{E}}} \models \ulcorner\kappa$ is measurable$\urcorner$.*

*Proof.* Let $\tau$ be a repeat point of $\bar{E}$. The crucial observation regarding $\tau$ being a repeat point is as follows. Let $p = p_{n..0} \in P_{\bar{E}}$. Set a function $f(\bar{\nu}) = p_{\langle\bar{\nu}\rangle n+1..1}$ for all $\langle\bar{\nu}\rangle \in T^{p_0}$. Then $j(f)(\mathrm{mc}(p)\restriction\tau) \in P_{\bar{E}\restriction\tau}$. By our convention this means $j(f)(\mathrm{mc}(p)\restriction\tau) \in P_{\bar{E}}$. Moreover, $j(f)(\mathrm{mc}(p)\restriction\tau) = p$!

We start by showing that for each $p \in P_{\bar{E}}$, $\Vdash_{P_{\bar{E}}} \ulcorner \widetilde{A} \subseteq \widehat{\kappa}\urcorner$ there is $p^* \leq^* p$ such that $j(p^*)_{\langle \mathrm{mc}(p^*)\restriction\tau\rangle} \Vdash_{j(P_{\bar{E}})} \ulcorner \widehat{\kappa} \in j(\widetilde{A})\urcorner$. Note that $j(p)_{\langle \mathrm{mc}(p)\restriction\tau\rangle} = p^\frown(j(p)_{\langle \mathrm{mc}(p)\restriction\tau\rangle})_0$.

Choose $\chi$ large enough so that $H_\chi$ contains everything we are interested in. Take $N \prec H_\chi$ such that $p, \widetilde{A}, P_{\bar{E}} \in N$, $|N| = \kappa$, $N \supseteq V_\kappa$, $N \supseteq N^{<\kappa}$. By Corollary 8.6 there is a $p' \leq^* p$ that is $\langle N, P_{\bar{E}}\rangle$-generic.

Let us fix $\langle\bar{\nu}\rangle \in T^{p'}$. By Theorem 7.1 there is a $q \leq^* p'_{\langle\bar{\nu}\rangle}$ such that $q \Vdash_{P_{\bar{E}}} \ulcorner \kappa^0(\bar{\nu}) \in \widetilde{A}\urcorner$. Then, by the $N$-genericity of $p'$, there is $q' \in N$, $q' \leq^* p_{\langle \pi_{\mathrm{mc}(p'),\mathrm{mc}(p)}(\nu)\rangle}$, $q' \parallel q$, $q' \Vdash_{P_{\bar{E}}} \ulcorner \kappa^0(\bar{\nu}) \in \widetilde{A}\urcorner$. Hence there is a $q^* \leq^* q', p'_{\langle\bar{\nu}\rangle}$ such that

$$\operatorname{supp} q^*_0 = \operatorname{supp} p'_0, \quad q^* \Vdash_{P_{\bar{E}}} \ulcorner \kappa^0(\bar{\nu}) \in \widetilde{A}\urcorner.$$

So, for each $\langle\bar{\nu}\rangle \in T^{p'}$ let $q^*(\bar{\nu}) \leq^* p'_{\langle\bar{\nu}\rangle}$ be such that

$$\operatorname{supp} q^*_0(\bar{\nu}) = \operatorname{supp} p'_0, \quad q^*(\bar{\nu}) \Vdash_{P_{\bar{E}}} \ulcorner \kappa^0(\bar{\nu}) \in \widetilde{A}\urcorner.$$

Of course this means $j(q^*)(\mathrm{mc}(p')\restriction\tau) \Vdash_{j(P_{\bar{E}})} \ulcorner \kappa \in j(\widetilde{A})\urcorner$. Let us set $r_{n..0} = j(q^*)(\mathrm{mc}(p')\restriction\tau) \in j(P_{\bar{E}})$. By our convention $r_{n..1} \in P_{\bar{E}}$. Let us set $p^* = r_{n..1}$. We

have $j(p^*)_{\langle \mathrm{mc}(p^*)\restriction\tau\rangle} \leq^* j(q^*)(\mathrm{mc}(p'\restriction\tau))$; hence $j(p^*)_{\langle \mathrm{mc}(p^*)\restriction\tau\rangle} \Vdash_{j(P_{\bar{E}})} \ulcorner \kappa \in j(\widetilde{A})\urcorner$, as needed.

Let $\widetilde{U}$ be a $P_{\bar{E}}$-name defined by

$$p \Vdash_{P_{\bar{E}}} \ulcorner \widetilde{A} \in \widetilde{U}\urcorner \text{ if } \{q \in P_{\bar{E}} \mid j(q)_{\langle \mathrm{mc}(q)\restriction\tau\rangle} \Vdash_{j(P_{\bar{E}})} \ulcorner \kappa \in j(\widetilde{A})\urcorner\} \text{ is dense below } p.$$

We note that, since $\{q \in P_{\bar{E}} \mid j(q)_{\langle \mathrm{mc}(q)\restriction\tau\rangle} \Vdash_{j(P_{\bar{E}})} \ulcorner \kappa \in j(\widetilde{A})\urcorner\}$ is dense, we have

$$p \Vdash_{P_{\bar{E}}} \ulcorner \widetilde{A} \notin \widetilde{U}\urcorner \text{ if } \{q \in P_{\bar{E}} \mid j(q)_{\langle \mathrm{mc}(q)\restriction\tau\rangle} \Vdash_{j(P_{\bar{E}})} \ulcorner \kappa \notin j(\widetilde{A})\urcorner\} \text{ is dense below } p.$$

We show that $\widetilde{U}$ is a name for a measure on $\kappa$:

(1) $\Vdash_{P_{\bar{E}}} \ulcorner \emptyset \notin \widetilde{U}\urcorner$: Immediate due to $\Vdash_{j(P_{\bar{E}})} \ulcorner \kappa \notin j(\emptyset)\urcorner$.

(2) $p \Vdash_{P_{\bar{E}}} \ulcorner \widetilde{A} \notin \widetilde{U}\urcorner \implies p \Vdash_{P_{\bar{E}}} \ulcorner \kappa \setminus \widetilde{A} \in \widetilde{U}\urcorner$: So, there is a maximal anti-chain $X$ below $p$ such that $q \in X \implies j(q)_{\langle \mathrm{mc}(q)\restriction\tau\rangle} \Vdash_{j(P_{\bar{E}})} \ulcorner \kappa \notin j(\widetilde{A})\urcorner$. Then $q \in X$ implies $j(q)_{\langle \mathrm{mc}(q)\restriction\tau\rangle} \Vdash_{j(P_{\bar{E}})} \ulcorner \kappa \in j(\kappa \setminus \widetilde{A})\urcorner$. This means $p \Vdash_{P_{\bar{E}}} \ulcorner \kappa \setminus \widetilde{A} \in \widetilde{U}\urcorner$.

(3) $p \Vdash \ulcorner \widetilde{A} \in \widetilde{U},\ \widetilde{A} \subseteq \widetilde{B}\urcorner \implies p \Vdash \ulcorner \widetilde{B} \in \widetilde{U}\urcorner$: Obvious.

(4) $\lambda < \kappa$, $p \Vdash_{P_{\bar{E}}} \ulcorner \forall \xi < \lambda\, \widetilde{A}_\xi \in \widetilde{U}\urcorner \implies p \Vdash_{P_{\bar{E}}} \ulcorner \bigcap_{\xi<\lambda} \widetilde{A}_\xi \in \widetilde{U}\urcorner$: So, for all $\xi < \lambda$ there is a maximal anti-chain $X_\xi$ below $p$ such that $q \in X_\xi \implies j(q)_{\langle \mathrm{mc}(q)\restriction\tau\rangle} \Vdash_{j(P_{\bar{E}})} \ulcorner \kappa \in j(\widetilde{A}_\xi)\urcorner$. It is enough to show that $\{r \in P_{\bar{E}} \mid j(r)_{\langle \mathrm{mc}(r)\restriction\tau\rangle} \Vdash_{j(P_{\bar{E}})} \ulcorner \forall \xi < \lambda\ \kappa \in j(\widetilde{A}_\xi)\urcorner\}$ is dense below $p$.

Let $q = q_{n..0} \leq p$. Since $\{r \in P_{\bar{E}} \mid j(r)_{\langle \mathrm{mc}(r)\restriction\tau\rangle} \Vdash_{j(P_{\bar{E}})} \ulcorner \kappa \in j(\widetilde{A}_\xi)\urcorner\}$ is dense below $p$ for all $\xi < \lambda$, we can construct $\langle r^\xi \mid \xi \leq \lambda\rangle$, $\langle Y_\xi \mid \xi < \lambda\rangle$ such that:

- $\xi = 0$: $r^0 = q_0$,
- $\xi$ is a limit ordinal: $\forall \xi' < \xi\ r^\xi \leq^* r^{\xi'}$,
- $\xi = \xi' + 1$: $r^\xi \leq^* r^{\xi'}$, $Y_\xi$ is a maximal anti-chain below $q_{n..1}$, and, for all $s \in Y_\xi$, $s^\frown j(r^\xi)_{\langle \mathrm{mc}(q)\restriction\tau\rangle} \Vdash_{j(P_{\bar{E}})} \ulcorner \kappa \in j(\widetilde{A}_{\xi'})\urcorner$.

Then we have $q_{n..1}{}^\frown j(r^\lambda)_{\langle \mathrm{mc}(r^\lambda)\restriction\tau\rangle} \Vdash_{j(P_{\bar{E}})} \ulcorner \forall \xi < \lambda\ \kappa \in j(\widetilde{A}_\xi)\urcorner$.

□

We note that actually the measure $\widetilde{U}$ constructed in the above proof is normal. Moreover, we can construct an extender, lifting $E(0)$, by deciding formulas of the form $\alpha \in j(\widetilde{A})$ for $\alpha \in \operatorname{dom} E(0)$.

## 11. What Have We Proved?

We can sum everything up as follows:

We can control independently two properties of $\kappa$ in a generic extension. The first is the size of $2^\kappa$, which is controlled by $|\bar{E}|$. The second is how "big" we want $\kappa$ to be, which is controlled by $\mathrm{l}(\bar{E})$.

## 12. Generic by Iteration

Recall that if $R$ is Radin forcing generated from $j{:}V \to M$, then there are $\tau$ and $G \in V$ such that $G$ is $j_{0,\tau}(R)$-generic over $M_\tau$.

Our original aim was to find some form of this claim for our forcing. We have a partial result in this direction. Namely, when $\mathrm{l}(\bar{E}) = 1$ we have a generic filter in $V$ over an elementary submodel in $M_\omega$.

In this section we assume that $\mathrm{l}(\bar{E}) = 1$.

Let us take an iteration of $j = j_{0,1}$,

$$\langle\langle M_n \mid n < \omega\rangle, \langle j_{n,m} \mid n \leq m < \omega\rangle\rangle.$$

Choose $\chi$ large enough so that everything interesting is in $H_\chi$ (i.e., $P_{\bar{E}} \in H_\chi$), and set

$$\begin{aligned}
\kappa_n &= j_{0,n}(\kappa),\\
\bar{E}^n &= j_{0,n}(\bar{E}),\\
P^n &= j_{0,n}(P_{\bar{E}})(= P_{\bar{E}^n}),\\
\chi_n &= j_{0,n}(\chi).
\end{aligned}$$

**Definition 12.1** (when $\mathrm{l}(\bar{E}) = 1$)**.** We call $\langle N, p\rangle$ a *k-pair* if

(1) $M_k \vDash \ulcorner N \prec H^{M_k}_{\chi_k} \urcorner$,
(2) $M_k \vDash \ulcorner |N| = \kappa_k \urcorner$,
(3) $M_k \vDash \ulcorner N \supseteq V^{M_k}_{\kappa_k} \urcorner$,
(4) $M_k \vDash \ulcorner N \supseteq N^{<\kappa_k} \urcorner$,
(5) $p \in P^{k+1} \cap j_{k,k+1}(N)$, and
(6) if $D \in N$ is dense open in $P^k$, then there are $n$, $S \leq T^p$ such that

$$\forall \langle \nu_1, \ldots, \nu_n\rangle \in S\ (p)_{\langle \nu_1,\ldots,\nu_n\rangle} \in j_{k,k+1}(D).$$

**Claim 12.2.** *Let $N \in M_k$, $k < \omega$ and $q \in j_{k,k+1}(N) \cap P^{k+1}$ be such that $M_k$ satisfies*

(1) $N \prec H^{M_k}_{\chi_k}$,
(2) $|N| = \kappa_k$,
(3) $N \supseteq V^{M_k}_{\kappa_k}$,
(4) $N \supseteq N^{<\kappa_k}$,
(5) $P^k \in N$.

*Then there is a $p \in j_{k,k+1}(N) \cap P^{k+1}$ such that*

(1) *$p \leq^* q$, and*
(2) *$\langle N, p\rangle$ is a k-pair.*

*Proof.* Let $\langle D^k_\xi \mid \xi < \kappa_k\rangle$ be an enumeration, in $M_k$, of all the dense open subsets of $P^k$ that are in $N$. Set $N_{k+1} = j_{k,k+1}(N)$. Since we have

$$\begin{aligned}
&\langle j_{k,k+1}(D^k_\xi) \mid \xi < \kappa_k\rangle \subseteq N_{k+1},\\
&\langle j_{k,k+1}(D^k_\xi) \mid \xi < \kappa_k\rangle \in M_{k+1},\\
&M_{k+1} \vDash \ulcorner N_{k+1} \supseteq N^{<\kappa_{k+1}}_{k+1} \urcorner,
\end{aligned}$$

we get that

$$\langle j_{k,k+1}(D^k_\xi) \mid \xi < \kappa_k\rangle \in N_{k+1}.$$

Starting with $q$, we construct in $N_{k+1}$ a $\leq^*$-decreasing sequence $\langle p_\xi \mid \xi < \kappa_k\rangle$ using Theorem 6.1. Note that we have no problem at the limit stages, since $N_{k+1} \vDash \ulcorner\langle P^{k+1}, \leq^*\rangle$ is $\kappa_{k+1}$-closed$\urcorner$. Now choose $p \in P^{k+1} \cap N_{k+1}$ such that $\forall \xi < \kappa_k\ p \leq^* p_\xi$. We get that $\langle N, p\rangle$ is a $k$-pair. □

**Definition 12.3.** We call $\langle \vec{N}, \vec{p}\rangle$ a *$P^\omega$-generic approximation sequence* if

$$\langle \vec{N}, \vec{p}\rangle = \langle\langle N_k, p^{k+1}\rangle \mid k_0 \leq k < \omega\rangle$$

and for all $k_0 \leq k < \omega$,

(1) $M_k \vDash \ulcorner N_k \prec H^{M_k}_{\chi_k} \urcorner$,
(2) $M_k \vDash \ulcorner |N_k| = \kappa_k \urcorner$,
(3) $M_k \vDash \ulcorner N_k \supseteq V^{M_k}_{\kappa_k} \urcorner$,
(4) $M_k \vDash \ulcorner N_k \supseteq N^{<\kappa_k} \urcorner$,
(5) $P^k \in N_k$,
(6) $\langle N_k, p^{k+1} \rangle$ is a $k$-pair,
(7) $j_{k_1,k_2}(N_{k_1}) = N_{k_2}$,
(8) $p^{k+2} \leq^* (j_{k+1,k+2}(p^{k+1}))_{\langle \mathrm{mc}(p^{k+1}) \rangle}$.

**Definition 12.4.** Let $\langle \vec{N}, \vec{r} \rangle$ be a $P^\omega$-generic approximating sequence. Then

$$G(\langle \vec{N}, \vec{r} \rangle) = \{p \in P^\omega \mid \exists k \; j_{k,\omega}(r^k) \leq^* p\}.$$

**Claim 12.5.** *Let $k_0 < \omega$, $q \in P^{k_0} \cap N_{k_0}$, and assume for all $k_0 \leq k < \omega$,*

(1) $P^k \in N_k$,
(2) $j_{k,k+1}(N_k) = N_{k+1}$,
(3) $M_k \vDash \ulcorner N_k \prec H^{M_k}_{\chi_k} \urcorner$,
(4) $M_k \vDash \ulcorner |N_k| = \kappa_k \urcorner$,
(5) $M_k \vDash \ulcorner N_k \supseteq V^{M_k}_{\kappa_k} \urcorner$,
(6) $M_k \vDash \ulcorner N_k \supseteq N_k^{<\kappa_k} \urcorner$.

*Then there is a $P^\omega$-generic approximating sequence*

$$\langle \vec{N}, \vec{p} \rangle = \langle \langle N_k, p^{k+1} \rangle \mid k_0 \leq k < \omega \rangle$$

*such that $p^{k_0+1} \leq^* j_{k_0,k_0+1}(q)$.*

*Proof.* We construct the $p^{k+1}$ by induction. We set $p^{k_0} = q$.

Assume that we have constructed $\langle p^{k'} \mid k_0 \leq k' \leq k \rangle$. We set $q^{k+1} = j_{k,k+1}(p^k)_{\langle \mathrm{mc}(p^k) \rangle}$. Invoke Claim 12.2 to get $p^{k+1} \leq^* q^{k+1}$ such that $\langle N_k, p^{k+1} \rangle$ is a $k$-pair.

When the induction terminates we have $\langle \langle N_k, p^{k+1} \rangle \mid k_0 \leq k < \omega \rangle$, as required. □

**Claim 12.6.** *Let $k_0 < \omega$, and assume that*

(1) $M_{k_0} \vDash \ulcorner N_{k_0} \prec H^{M_{k_0}}_{\chi_{k_0}} \urcorner$,
(2) $M_{k_0} \vDash \ulcorner |N_{k_0}| = \kappa_{k_0} \urcorner$,
(3) $M_{k_0} \vDash \ulcorner N_{k_0} \supseteq V^{M_{k_0}}_{\kappa_{k_0}} \urcorner$,
(4) $M_{k_0} \vDash \ulcorner N_{k_0} \supseteq N_{k_0}^{<\kappa_{k_0}} \urcorner$,
(5) $P^{k_0} \in N_{k_0}$,
(6) $q \in P^{k_0} \cap N_{k_0}$.

*Then there is a $P^\omega$-generic approximating sequence*

$$\langle \vec{N}, \vec{p} \rangle = \langle \langle N_k, p^{k+1} \rangle \mid k_0 \leq k < \omega \rangle$$

*such that $p^{k_0+1} \leq^* j_{k_0,k_0+1}(q)$.*

*Proof.* We set $N_k = j_{k_0,k}(N_{k_0})$ for all $k_0 < k < \omega$, and then we invoke Claim 12.5. □

**Theorem 12.7.** *Assume*

(1) $M_\omega \vDash \ulcorner N \prec H^{M_\omega}_{\chi_\omega} \urcorner$,
(2) $M_\omega \vDash \ulcorner |N| = \kappa_\omega \urcorner$,

(3) $M_\omega \models \ulcorner N \supseteq V_{\kappa_\omega}^{M_\omega} \urcorner$,
(4) $M_\omega \models \ulcorner N \supseteq N^{<\kappa_\omega} \urcorner$,
(5) $P^\omega \in N$,
(6) $q \in P^\omega \cap N$.

*Then there is, in $V$, a filter $G \subseteq P^\omega$ such that*

(1) *$q \in G$, and*
(2) *$\forall D \in N$ $D$ is dense open in $P^\omega \implies G \cap D \cap N \neq \emptyset$.*

*Proof.* We find $\langle \vec{N}, \vec{p} \rangle$, a $P^\omega$-generic approximating sequence. $G(\langle \vec{N}, \vec{p} \rangle)$ is the required filter.

Find $k_0$ and $N_{k_0}$, $q^{k_0}$ such that

$$\begin{aligned} &P^{k_0} \in N_{k_0}, \\ &j_{k_0,\omega}(N_{k_0}) = N, \\ &j_{k_0,\omega}(q^{k_0}) = q. \end{aligned}$$

Invoke Claim 12.6 to get from $N^{k_0}$ and $q^{k_0}$ a generic approximating sequence $\langle \vec{N}, \vec{p} \rangle$.

Let $D \in N$ be dense open in $P^\omega$.

Find $k \geq k_0$ and $D^k$ such that $j_{k,\omega}(D^k) = D$. As usual, set $D^{k+l} = j_{k,k+l}(D^k)$. By construction there is an $n$ such that

$$\forall \langle \nu_1, \ldots, \nu_n \rangle \in T^{p^{k+1}} \; (p^{k+1})_{\langle \nu_1, \ldots, \nu_n \rangle} \in D^{k+1} \cap N_{k+1},$$

which means that

$$j_{k+1,k+1+n}(p^{k+1})_{\langle \mathrm{mc}(p^{k+1}), \ldots, j_{k+1,k+n}(\mathrm{mc}(p^{k+1})) \rangle} \in D^{k+1+n} \cap N_{k+1+n}.$$

Hence

$$j_{k+1,\omega}(p^{k+1})_{\langle \mathrm{mc}(p^{k+1}), \ldots, j_{k+1,k+n}(\mathrm{mc}(p^{k+1})) \rangle} \in D \cap N. \tag{12.7.1}$$

Since $\langle \vec{N}, \vec{p} \rangle$ is a $P^\omega$-generic approximation sequence, it satisfies

$$p^{k+1+n} \leq^* j_{k+1,k+1+n}(p^{k+1})_{\langle \mathrm{mc}(p^{k+1}), \ldots, j_{k+1,k+1+n}(\mathrm{mc}(p^{k+1})) \rangle}.$$

Hence

$$j_{k+1+n,\omega}(p^{k+1+n}) \leq^* j_{k+1,\omega}(p^{k+1})_{\langle \mathrm{mc}(p^{k+1}), \ldots, j_{k+1,k+n}(\mathrm{mc}(p^{k+1})) \rangle},$$

giving us that

$$j_{k+1,\omega}(p^{k+1})_{\langle \mathrm{mc}(p^{k+1}), \ldots, j_{k+1,k+n}(\mathrm{mc}(p^{k+1})) \rangle} \in G(\langle \vec{N}, \vec{p} \rangle); \tag{12.7.2}$$

so $G(\langle \vec{N}, \vec{p} \rangle) \cap D \cap N \neq \emptyset$ by (12.7.1) and (12.7.2). □

## 13. Concluding Remarks

(1) A definition of repeat point that depends only on the extender sequence and is equivalent to the one we gave (which mentions $P_{\bar{E}}$) will probably be useful.
(2) It is not completely clear what $\mathrm{l}(\bar{E})$ should be in order to make sure that $\bar{E}$ has a repeat point.
(3) A finer analysis in the case of measurability and stronger properties is needed—for example, extending the elementary embedding to the generic extension, and not just constructing a normal ultrafilter.
(4) We do not know how to get a generic by iteration when $\mathrm{l}(\bar{E}) > 1$.

(5) Making this forcing more "precise" by adding "gentle" collapses so that we get a prescribed behaviour on *all* cardinals below $\kappa$ in the generic extension is in preparation.

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