TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 355, Number 6, Pages 2397–2411
S 0002-9947(02)03155-0
Article electronically published on September 25, 2002

ON RAMANUJAN'S CONTINUED FRACTION FOR $(q^2;q^3)_\infty/(q;q^3)_\infty$

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Abstract. The continued fraction in the title is perhaps the deepest of Ramanujan's q-continued fractions. We give a new proof of this continued fraction, more elementary and shorter than the only known proof by Andrews, Berndt, Jacobsen, and Lamphere. On page 45 in his lost notebook, Ramanujan states an asymptotic formula for a continued fraction generalizing that in the title. The second main goal of this paper is to prove this asymptotic formula.

1. Introduction

In his lost notebook [12, p. 45], Ramanujan offers several interesting and deep theorems about the continued fraction

$$\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty} = \frac{1}{1} - \frac{q}{1+q} - \frac{q^3}{1+q^2} - \frac{q^5}{1+q^3} - \cdots, \tag{1.1}$$

where $|q|<1$ and where, here, and in the sequel, we use the standard notation

$$(a)_0 := (a;q)_0 := 1, \qquad (a)_n := (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \qquad \text{if } n \geq 1,$$

and

$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n, \qquad |q|<1.$$

The continued fraction (1.1) is due to Ramanujan and is found in his second notebook [11, p. 290]. Of the many q-continued fractions found by Ramanujan, (1.1) is, by far, the most difficult to prove. Up until the present, the only known proof was found by Andrews, Berndt, L. Jacobsen, and R. L. Lamphere [6], [8, p. 46, Entry 19] in 1992 and uses a deep theorem of Andrews [1]. The many other q-continued fractions found by Ramanujan fall under a general hierarchy and have been established by several authors by using q-difference relations for basic hypergeometric series. In 1936, A. Selberg [13] was the first mathematician to systematically derive several q-continued fractions (mostly due to Ramanujan but hidden from the public

Received by the editors October 5, 2001.
2000 *Mathematics Subject Classification.* Primary 33Dxx; Secondary 11B65, 11A55, 30B70.
The first author was supported in part by grant DMS-9206993 from the National Science Foundation.
The second author was supported in part by grant MDA904-00-1-0015 from the National Security Agency.
The fourth author was supported in part by the postdoctoral fellowship program from the Korea Science and Engineering Foundation, and by a grant from the Number Theory Foundation.

in his notebooks) from general theorems. Ramanujan, himself, in his lost notebook [12], stated such general theorems, with the most elegant one first established by Andrews [3]. See [5] for further citations to the literature. One primary purpose of this paper is to provide a new, short, and more direct proof of (1.1).

One of the claims made by Ramanujan on page 45 of [12] is an asymptotic formula for (1.1) as $q \to 1^-$. This asymptotic formula and a similar asymptotic formula for another continued fraction found on the same page were established by Berndt and Sohn [9], who deduced the results from a general theorem that they established. The second major purpose of this paper is to prove another asymptotic formula related to that for (1.1) found on the same page. In fact, the continued fraction is slightly more general than (1.1). Although both (1.1) and its generalization do not converge for $q > 1$, Ramanujan claims that his asymptotic formula is valid as $q \to 1$ from both directions. However, the continued fraction satisfies a simple difference equation, which is given by Ramanujan immediately preceding the asymptotic formula. Thus, Ramanujan's asymptotic formula should be more properly interpreted as an asymptotic formula for solutions of this difference equation, which does not have a unique solution. Therefore, a sequence of arbitrary constants arises in Ramanujan's asymptotic formula.

On page 45 of [12], Ramanujan also claims that, for $\omega = e^{2\pi i/3}$ and $|q| < 1$,

(1.2)
$$\lim_{n\to\infty}\left(\frac{1}{1} - \frac{1}{1+q} - \frac{1}{1+q^2} - \cdots - \frac{1}{1+q^n+a}\right) = -\omega^2\left(\frac{\Omega-\omega^{n+1}}{\Omega-\omega^{n-1}}\right)\cdot\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty},$$

where

$$\Omega := \frac{1-a\omega^2}{1-a\omega}\,\frac{(\omega^2 q;q)_\infty}{(\omega q;q)_\infty}. \tag{1.3}$$

Note that, when $a = 0$ and q is replaced by q^{-1}, the continued fraction on the left side of (1.2) equals

$$\frac{1}{1} - \frac{1}{1+q^{-1}} - \frac{1}{1+q^{-2}} - \frac{1}{1+q^{-3}} - \cdots, \tag{1.4}$$

which, by an equivalence transformation, is equal to the continued fraction in (1.1). Because of the appearance of the limiting variable n on the right side of (1.2), Ramanujan's claim is meaningless as it stands. However, if we let $n \to \infty$ in each of the three residue classes modulo 3, then, in each of these three cases, the limit exists. With this interpretation, the authors established a proof of Ramanujan's claim in [7]. In particular, if $q > 1$, the continued fraction in (1.1) has three limit points, and so it would not be possible in any way to prescribe values to the constants mentioned at the close of the previous paragraph.

2. Preliminary Results

For our new proof of (1.1), we need the following result from Ramanujan's lost notebook [12, p. 43], which was first proved by Andrews [4].

Lemma 2.1. *Let* $\omega = e^{2\pi i/3}$. *Then*

$$\sum_{n=0}^{\infty}\frac{(-\omega)^n q^{n(n+1)/2}(\omega q)_n}{(q^3;q^3)_n} = (\omega q)_\infty (q^2;q^3)_\infty.$$

We also need an analogue of this result, (2.3) below, which we will establish with the same tools as Andrews used to prove Lemma 2.1.

Lemma 2.2. *For any complex numbers a, b, with $b \neq 0$,*

$$\frac{1}{(aq)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^n q^{n(n+1)/2}}{(q)_n(aq/b)_n} - \frac{1}{(aq)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^n q^{n(n+1)/2+n}}{(q)_n(aq/b)_n}$$
$$= \frac{b}{a}\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n} b^{-n} q^{n(3n-1)/2}}{(q)_n(aq/b)_n(aq)_n} - \frac{b}{a}\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n} b^{-n} q^{n(3n+1)/2}}{(q)_n(aq/b)_n(aq)_n}. \tag{2.1}$$

Proof. We need a limiting case of Watson's q-analogue of Whipple's theorem given in Andrews's paper [4, eq. (2.3)],

$$\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n} b^{-n} q^{n(3n+1)/2}}{(q)_n(aq/b)_n(aq)_n} = \frac{1}{(aq)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^n q^{n(n+1)/2}}{(q)_n(aq/b)_n}. \tag{2.2}$$

By replacing b by bq and multiplying both sides by $(1-a/b)$, we obtain

$$\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n} b^{-n}(1-aq^n/b) q^{n(3n-1)/2}}{(q)_n(aq/b)_n(aq)_n} = \frac{1}{(aq)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^n(1-aq^n/b) q^{n(n+1)/2}}{(q)_n(aq/b)_n}$$

or

$$\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n} b^{-n} q^{n(3n-1)/2}}{(q)_n(aq/b)_n(aq)_n} - \frac{a}{b}\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n} b^{-n} q^{n(3n+1)/2}}{(q)_n(aq/b)_n(aq)_n}$$
$$= \frac{1}{(aq)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^n q^{n(n+1)/2}}{(q)_n(aq/b)_n} - \frac{a}{b}\frac{1}{(aq)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^n q^{n(n+1)/2+n}}{(q)_n(aq/b)_n}.$$

Using (2.2) above, we deduce (2.1). □

To maintain the flow of the proof of (1.1), we place here an application of Lemma 2.2 which will be needed.

If we set $a = \omega$ and $b = \omega^2$ in (2.1), we find that, after some simplification,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-\omega)^n q^{n(n+1)/2}}{(q)_n(\omega^2 q)_n} - \sum_{n=0}^{\infty}\frac{(-\omega)^n q^{n(n+1)/2+n}}{(q)_n(\omega^2 q)_n} &= \omega(\omega q)_\infty\left\{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(3n-1)/2}}{(q^3;q^3)_n} - \sum_{n=0}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{(q^3;q^3)_n}\right\}\\ &= \omega(\omega q)_\infty\left\{(q;q^3)_\infty - (q^2;q^3)_\infty\right\},\end{aligned} \tag{2.3}$$

by letting $N \to \infty$ in the q-binomial theorem, as found in [2, pp. 35–36], with q replaced by q^3 and z replaced by q and q^2, respectively, in [2].

By employing an argument similar to that used by Andrews [4] to prove Lemma 2.1, we can utilize Lemma 2.2 to prove the following lemma, which we use in [7].

Lemma 2.3. *Let $\omega = e^{2\pi i/3}$. Then*

$$-\frac{\omega}{(\omega q)_\infty}\left\{\sum_{n=0}^{\infty}\frac{(-\omega)^n q^{n(n+1)/2}}{(q)_n(\omega^2 q)_n} + \omega\sum_{n=0}^{\infty}\frac{(-\omega)^n q^{n(n+1)/2+n}}{(q)_n(\omega^2 q)_n}\right\} = (q;q^3)_\infty. \tag{2.4}$$

Proof. Letting $a=\omega$ and $b=\omega^2$ in Lemma 2.2, we obtain

$$\sum_{n=0}^{\infty}\frac{(-\omega)^n q^{n(n+1)/2}}{(q)_n(\omega^2 q)_n}-\sum_{n=0}^{\infty}\frac{(-\omega)^n q^{n(n+1)/2+n}}{(q)_n(\omega^2 q)_n} = \omega(\omega q)_\infty\left\{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(3n-1)/2}}{(q^3;q^3)_n}-\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{(q^3;q^3)_n}\right\}. \tag{2.5}$$

From (2.2), we see that

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{(q)_n(\omega q)_n(\omega^2 q)_n}=\frac{1}{(\omega q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n\omega^n q^{n(n+1)/2}}{(q)_n(\omega^2 q)_n}. \tag{2.6}$$

By combining (2.5) and (2.6), we find that

$$-\frac{\omega}{(\omega q)_\infty}\left\{\sum_{n=0}^{\infty}\frac{(-\omega)^n q^{n(n+1)/2}}{(q)_n(\omega^2 q)_n}+\omega\sum_{n=0}^{\infty}\frac{(-\omega)^n q^{n(n+1)/2+n}}{(q)_n(\omega^2 q)_n}\right\} =\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(3n-1)/2}}{(q^3;q^3)_n}. \tag{2.7}$$

By letting $N\to\infty$ in the q-binomial theorem, as found in [2, pp. 35–36], with q replaced by q^3 and $z=q$ in [2], we find that the right side of (2.7) equals $(q;q^3)_\infty$, and so (2.4) is established. □

3. Proof of Ramanujan's Continued Fraction (1.1)

Lemma 3.1. *Let*

$$F(a,b)=\sum_{n=0}^{\infty}\frac{(-1)^n a^n q^{n(n+1)/2}}{(q)_n(aq/b)_n}. \tag{3.1}$$

Then $F(a,b)$ satisfies the recurrence relations,

$$\text{(i)}\quad F(a,b)=(1-a/b)F(a,bq)+(a/b)F(aq,bq), \tag{3.2}$$

$$\text{(ii)}\quad F(a,b)=F(aq,bq)-\frac{aq}{1-aq/b}F(aq,b). \tag{3.3}$$

Proof. (i) We first show that $(1-a/b)F(a,bq)=F(a,b)-(a/b)F(aq,bq)$. To that end,

$$\begin{aligned}(1-a/b)F(a,bq)&=(1-a/b)\sum_{n=0}^{\infty}\frac{(-1)^n a^n q^{n(n+1)/2}}{(q)_n(a/b)_n}\\ &=\sum_{n=0}^{\infty}\frac{(-1)^n a^n q^{n(n+1)/2}(1-aq^n/b)}{(q)_n(aq/b)_n}\\ &=\sum_{n=0}^{\infty}\frac{(-1)^n a^n q^{n(n+1)/2}}{(q)_n(aq/b)_n}-\frac{a}{b}\sum_{n=0}^{\infty}\frac{(-1)^n(aq)^n q^{n(n+1)/2}}{(q)_n(aq/b)_n}\\ &=F(a,b)-(a/b)F(aq,bq).\end{aligned}$$

(ii) Next, we consider $F(a,b)-F(aq,bq)$. Accordingly,

$$\begin{aligned}F(a,b)-F(aq,bq) &= \sum_{n=0}^{\infty}\frac{(-1)^n a^n q^{n(n+1)/2}}{(q)_n(aq/b)_n} - \sum_{n=0}^{\infty}\frac{(-1)^n (aq)^n q^{n(n+1)/2}}{(q)_n(aq/b)_n}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n a^n q^{n(n+1)/2}(1-q^n)}{(q)_n(aq/b)_n}\\ &= \sum_{n=1}^{\infty}\frac{(-1)^n a^n q^{n(n+1)/2}}{(q)_{n-1}(aq/b)_n}\\ &= \frac{-aq}{1-aq/b}\sum_{n=0}^{\infty}\frac{(-1)^n (aq)^n q^{n(n+1)/2}}{(q)_n(aq^2/b)_n}\\ &= \frac{-aq}{1-aq/b}F(aq,b).\end{aligned}$$

□

We are now ready to prove (1.1).

Theorem 3.2. *For* $|q|<1$,

$$\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty} = \frac{1}{1}\, - \,\frac{q}{1+q}\, - \,\frac{q^3}{1+q^2}\, - \,\frac{q^5}{1+q^3}\, - \cdots.$$

Proof. By replacing a by aq in (3.2) and (3.3), we obtain

$$F(aq,b) = (1-aq/b)F(aq,bq) + (aq/b)F(aq^2,bq), \tag{3.4}$$

$$F(aq,b) = F(aq^2,bq) - \frac{aq^2}{1-aq^2/b}F(aq^2,b). \tag{3.5}$$

Solving (3.4) for $F(aq,bq)$ and substituting the resulting equality in (3.3), we find that

$$F(a,b) = \frac{1-aq}{1-aq/b}F(aq,b) - \frac{aq/b}{1-aq/b}F(aq^2,bq). \tag{3.6}$$

Solving (3.5) for $F(aq^2,bq)$ and substituting the result in (3.6), we deduce that

$$F(a,b) = \frac{1-aq-aq/b}{1-aq/b}F(aq,b) - \frac{aq/b}{(1-aq/b)}\frac{aq^2}{(1-aq^2/b)}F(aq^2,b).$$

Hence, from the last equality and iteration, we find that

$$\begin{aligned}\frac{(1-aq/b)F(a,b)}{F(aq,b)} &= (1-aq-aq/b) - \frac{aq}{b}\frac{aq^2}{(1-aq^2/b)}\frac{F(aq^2,b)}{F(aq,b)}\\ &= (1-aq-aq/b) - \frac{a^2q^3}{b}\frac{1}{(1-aq^2/b)\dfrac{F(aq,b)}{F(aq^2,b)}}\\ &= (1-aq-aq/b)\\ &\quad - \frac{a^2q^3}{b}\frac{1}{1-aq^2-aq^2/b} - \frac{a^2q^5}{b}\frac{1}{(1-aq^3/b)\dfrac{F(aq^2,b)}{F(aq^3,b)}}.\end{aligned} \tag{3.7}$$

We now specialize the iterative process above by setting $a=\omega$ and $b=\omega^2$. Then the last equalities, upon simplification, reduce to

(3.8)
$$\begin{aligned}\frac{(1-\omega^2 q)F(\omega,\omega^2)}{F(\omega q,\omega^2)} &= 1-\omega q-\omega^2 q-\cfrac{q^3}{(1-\omega^2q^2)\cfrac{F(\omega q,\omega^2)}{F(\omega q^2,\omega^2)}}\\ &= 1+q-\frac{q^3}{1-\omega q^2-\omega^2 q^2}-\cfrac{q^5}{(1-\omega^2q^3)\cfrac{F(\omega q^2,\omega^2)}{F(\omega q^3,\omega^2)}}\\ &= 1+q-\frac{q^3}{1+q^2}-\frac{q^5}{1-\omega q^3-\omega^2q^3}-\cfrac{q^7}{(1-\omega^2q^4)\cfrac{F(\omega q^3,\omega^2)}{F(\omega q^4,\omega^2)}}\\ &= 1+q-\frac{q^3}{1+q^2}-\frac{q^5}{1+q^3}-\frac{q^7}{1+q^4}-\cdots.\end{aligned}$$

On the other hand, by (3.3) and (2.3), we obtain

$$\begin{aligned}-\frac{\omega q}{1-\omega^2 q}F(\omega q,\omega^2) &= F(\omega,\omega^2)-F(\omega q,\omega^2 q)\\ &= \omega(\omega q)_\infty\{(q;q^3)_\infty-(q^2;q^3)_\infty\},\end{aligned}$$

and from Lemma 2.1, we see that

$$F(\omega,\omega^2)=(\omega q)_\infty(q^2;q^3)_\infty.$$

Hence,

$$\begin{aligned}\frac{(1-\omega^2 q)F(\omega,\omega^2)}{F(\omega q,\omega^2)} &= \frac{q(\omega q)_\infty(q^2;q^3)}{(\omega q)_\infty\{(q^2;q^3)_\infty-(q;q^3)_\infty\}}\\ &= \frac{q}{1-(q;q^3)_\infty/(q^2;q^3)_\infty}.\end{aligned}\tag{3.9}$$

By (3.8) and (3.9), we conclude that, after some manipulation,

$$\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty}=\frac{1}{1}-\frac{q}{1+q}-\frac{q^3}{1+q^2}-\frac{q^5}{1+q^3}-\cdots.\tag{3.10}$$

This completes the proof, except that it remains to show that the iterative process in (3.8) does indeed converge and that it converges to $(q^2;q^3)_\infty/(q;q^3)_\infty$.

First, the continued fraction in (1.1) is equivalent to a continued fraction of the form

$$\frac{c_1}{1}+\frac{c_2}{1}+\frac{c_3}{1}+\cdots,$$

where, for $n\geq 3$,

$$c_n=-q^{2n-3}(1+q^{n-2})^{-1}(1+q^{n-1})^{-1}.$$

By Worpitzky's theorem [10, p. 35], the continued fractions (1.1) and (3.8) therefore converge. Second,

$$(1-\omega^2q^n)\frac{F(\omega q^{n-1},\omega^2)}{F(\omega q^n,\omega^2)}=1+O(q^n),$$

as $n\to\infty$, which is sufficient to show that the continued fraction (3.8) converges to what is claimed in (3.10). □

Theorem 3.2 can, in fact, be generalized, as we demonstrate in the next corollary. However, in contrast to Theorem 3.2, it does not seem possible to find a product representation for the continued fraction.

Corollary 3.3. *Let $|q|<1$ and suppose that a is any complex number such that $1+aq^n\neq 0$ for any positive integer n. Then, if $F(a,b)$ is defined by* (3.1),

$$\frac{a^2q\,F(a\omega q,\omega^2)}{(1-a\omega^2q)F(a\omega,\omega^2)}=\frac{a^2q}{1+aq}-\frac{a^2q^3}{1+aq^2}-\frac{a^2q^5}{1+aq^3}-\cdots. \tag{3.11}$$

Proof. In (3.7), replace a by $a\omega$ and set $b=\omega^2$. Then take the reciprocal of both sides and multiply both sides of the resulting equality by a^2q. Equality (3.11) then immediately follows. □

Representations for the Rogers–Ramanujan continued fraction and the generalized Rogers–Ramanujan continued fraction also follow from (3.7).

Corollary 3.4. *Let $|q|<1$ and suppose that a is any complex number such that $1+aq^n\neq 0$ for any positive integer n. Then*

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(-a)^nq^{n(n+3)/2}}{(q)_n(-aq)_{n+1}}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-a)^nq^{n(n+1)/2}}{(q)_n(-aq)_n}}=\frac{1}{1}+\frac{a^2q^3}{1}+\frac{a^2q^5}{1}+\frac{a^2q^7}{1}+\cdots. \tag{3.12}$$

Proof. In (3.7), set $b=-1$. Taking the reciprocal of both sides, we complete the proof. □

To obtain from (3.12) the generalized Rogers–Ramanujan continued fraction in its usual form, replace a^2 by a^2/q and then replace q^2 by q. The Rogers–Ramanujan continued fraction is the special case $a=1$ of the latter result.

4. An Asymptotic Expansion

In [9], Berndt and Sohn established an asymptotic expansion, as $q\to 1^-$, for the continued fraction (1.1); this asymptotic series is found on page 45 in Ramanujan's lost notebook [12]. Elsewhere on page 45, Ramanujan gives an asymptotic expansion for a continued fraction which generalizes that of (1.1), but as we remarked in the Introduction, Ramanujan evidently derived his result from a recurrence relation, (4.2) below, satisfied by the continued fraction. Since Ramanujan claims that his asymptotic formula is valid for both positive and negative values of x, where $q=e^{-x}$, his assertion must be interpreted as an asymptotic expansion for solutions of (4.2). Because (4.2) does not have a unique solution, his asymptotic series includes a sequence $\phi_0,\phi_1,\phi_2,\ldots$ of arbitrary constants. In this section, we establish this unusual asymptotic series claimed by Ramanujan.

Theorem 4.1. *Let*

$$u_\lambda:=\frac{1}{1+e^{(\lambda+1)x}}-\frac{1}{1+e^{(\lambda+2)x}}-\cdots. \tag{4.1}$$

Then

$$u_\lambda+\frac{1}{u_{\lambda-1}}=1+e^{\lambda x} \tag{4.2}$$

and, as $x \to 0$,

(4.3)

$$u_\lambda = 1 - \frac{\phi_0}{1-\lambda\phi_0} + x\left(\frac{\lambda+1}{2} + \frac{\phi_1 + (\lambda^2-1)(\frac{1}{2} - \frac{2}{3}\lambda\phi_0 + \frac{1}{4}\lambda^2\phi_0^2)}{(1-\lambda\phi_0)^2}\right)$$
$$+ x^2\left(\frac{\lambda(\lambda+1)(\lambda+2)}{12} - \frac{\phi_2 + \lambda(\lambda^2-1)(\lambda^2-4)\left(\frac{1}{45} - \frac{\lambda\phi_0}{36} + \frac{(\lambda^2+\frac{1}{3})\phi_0^2}{112}\right)}{(1-\lambda\phi_0)^2}\right.$$
$$\left. - \frac{\lambda}{(1-\lambda\phi_0)^3}\left(\phi_1 + \frac{\lambda^2-1}{6}(1-\frac{1}{2}\lambda\phi_0)\right)^2\right) + \cdots,$$

where $\phi_0, \phi_1, \phi_2, \ldots$ are independent of λ.

Proof. From (4.1),

$$u_{\lambda-1} = \frac{1}{1+e^{\lambda x}} - \frac{1}{1+e^{(\lambda+1)x}} - \frac{1}{1+e^{(\lambda+2)x}} + \cdots.$$

Hence,

$$\frac{1}{u_{\lambda-1}} = 1 + e^{\lambda x} - \frac{1}{1+e^{(\lambda+1)x}} - \frac{1}{1+e^{(\lambda+2)x}} + \cdots = 1 + e^{\lambda x} - u_\lambda,$$

which proves (4.2).

To prove (4.3), we shall use the recurrence relation (4.2) and the method of successive approximations. We restrict our attention to solutions of (4.2) which have asymptotic expansions of the form

$$u_\lambda = c_0(\lambda) + c_1(\lambda)x + c_2(\lambda)x^2 + \cdots,$$

an assumption evidently also made by Ramanujan. We first calculate $c_0(\lambda)$. Now, from (4.2), the constant terms yield

$$c_0(\lambda) + \frac{1}{c_0(\lambda-1)} = 2.$$

Set

$$c_0(\lambda) = 1 + f(\lambda).$$

Then

$$1 + f(\lambda) + \frac{1}{1+f(\lambda-1)} = 2,$$

or

$$f(\lambda)(1+f(\lambda-1)) = f(\lambda-1). \tag{4.4}$$

Next put

$$g(\lambda) = \frac{1}{f(\lambda)}.$$

Then, from (4.4), we easily deduce that

$$g(\lambda) - g(\lambda-1) = 1. \tag{4.5}$$

This is an inhomogeneous linear recurrence relation which has the characteristic root 1. Thus, the general homogeneous solution is

$$g(\lambda) = c \cdot 1^\lambda = c.$$

Since 1 is the characteristic root, a particular inhomogeneous solution has the form $k\lambda$. Hence, from (4.5),

$$k\lambda - k(\lambda-1) = 1.$$

Hence, $k=1$, and the general solution for the linear recurrence relation (4.5) is $g(\lambda) = c+\lambda$. Thus,

$$f(\lambda) = \frac{1}{c+\lambda} \quad \text{and} \quad c_0(\lambda) = 1 + \frac{1}{c+\lambda}.$$

Ramanujan sets $c = -1/\phi_0$. Thus,

$$c_0(\lambda) = 1 + \frac{1}{\lambda - 1/\phi_0} = 1 - \frac{\phi_0}{1-\phi_0\lambda}. \tag{4.6}$$

For our second approximation, from (4.2), we have

$$c_0(\lambda) + c_1(\lambda)x + \frac{1}{c_0(\lambda-1)+c_1(\lambda-1)x} = 2+\lambda x,$$

or

$$(c_0(\lambda)+c_1(\lambda)x)(c_0(\lambda-1)+c_1(\lambda-1)x)+1 = (2+\lambda x)(c_0(\lambda-1)+c_1(\lambda-1)x).$$

Equate coefficients of x to obtain

$$c_0(\lambda-1)c_1(\lambda) + c_0(\lambda)c_1(\lambda-1) = 2c_1(\lambda-1) + \lambda c_0(\lambda-1). \tag{4.7}$$

From (4.6) and (4.7), we have

$$\frac{(1-\phi_0\lambda)c_1(\lambda)}{1+\phi_0-\phi_0\lambda} + \frac{(1-\phi_0\lambda-\phi_0)c_1(\lambda-1)}{1-\phi_0\lambda} = \frac{\lambda(1-\phi_0\lambda)}{1-\phi_0\lambda+\phi_0} + 2c_1(\lambda-1). \tag{4.8}$$

Now

$$c_1(\lambda-1)\left(\frac{1-\phi_0\lambda-\phi_0}{1-\phi_0\lambda} - 2\right) = c_1(\lambda-1)\left(\frac{-1+\phi_0\lambda-\phi_0}{1-\phi_0\lambda}\right). \tag{4.9}$$

Thus, from (4.8) and (4.9),

$$c_1(\lambda)\left(\frac{1-\phi_0\lambda}{1+\phi_0-\phi_0\lambda}\right) + c_1(\lambda-1)\left(\frac{-1+\phi_0\lambda-\phi_0}{1-\phi_0\lambda}\right) = \frac{\lambda(1-\phi_0\lambda)}{1-\phi_0\lambda+\phi_0}. \tag{4.10}$$

Set

$$f_1(\lambda) = (1-\phi_0\lambda)c_1(\lambda). \tag{4.11}$$

Then, from (4.10) and (4.11),

$$\frac{f_1(\lambda)}{1+\phi_0-\phi_0\lambda} - \frac{f_1(\lambda-1)}{1-\phi_0\lambda} = \frac{\lambda(1-\phi_0\lambda)}{1-\phi_0\lambda+\phi_0}.$$

Multiply both sides by $(1-\phi_0\lambda)(1+\phi_0-\phi_0\lambda)$ to deduce that

$$(1-\phi_0\lambda)f_1(\lambda) - (1+\phi_0-\phi_0\lambda)f_1(\lambda-1) = \lambda(1-\phi_0\lambda)^2. \tag{4.12}$$

Set

$$g_1(\lambda) = (1-\phi_0\lambda)f_1(\lambda). \tag{4.13}$$

Hence, from (4.12) and (4.13),

$$g_1(\lambda) - g_1(\lambda-1) = \lambda(1-\phi_0\lambda)^2, \tag{4.14}$$

which has the characteristic root 1, and so the general homogeneous solution is

$$\phi_1 \cdot 1^\lambda = \phi_1,$$

for an arbitrary constant ϕ_1. Since 1 is a homogeneous solution, a particular solution for the recurrence relation (4.14) has the form

$$g_1(\lambda) = f_1\lambda + f_2\lambda^2 + f_3\lambda^3 + f_4\lambda^4. \tag{4.15}$$

Substitute (4.15) into (4.14) to find that

$$\begin{aligned} f_1 - f_2 + f_3 - f_4 + \lambda(2f_2 - 3f_3 + 4f_4) + \lambda^2(3f_3 - 6f_4) + \lambda^3(4f_4) \\ = \lambda(1 - 2\phi_0\lambda + \phi_0^2\lambda^2). \end{aligned}$$

Equate coefficients of like powers of λ to deduce that

$$f_1 = \frac{1}{2} - \frac{1}{3}\phi_0, \quad f_2 = \frac{1}{2} - \phi_0 + \frac{1}{4}\phi_0^2, \quad f_3 = -\frac{2}{3}\phi_0 + \frac{1}{2}\phi_0^2, \quad \text{and} \quad f_4 = \frac{1}{4}\phi_0^2.$$

Substitute these values into (4.15) to find that

$$\begin{aligned} g_1(\lambda) &= \left(\frac{1}{2} - \frac{1}{3}\phi_0\right)\lambda + \left(\frac{1}{2} - \phi_0 + \frac{1}{4}\phi_0^2\right)\lambda^2 + \left(-\frac{2}{3}\phi_0 + \frac{1}{2}\phi_0^2\right)\lambda^3 + \frac{1}{4}\phi_0^2 \\ &= \frac{1}{2}(\lambda+1)(1 - 2\phi_0\lambda + \phi_0^2\lambda^2) + \frac{\lambda^2 - 1}{2} - \frac{2}{3}\phi_0\lambda^3 + \frac{2}{3}\lambda\phi_0 + \frac{1}{4}\phi_0^2\lambda^4 - \frac{1}{4}\phi_0^2\lambda^2 \\ &= \frac{1}{2}(\lambda+1)(1-\phi_0\lambda)^2 + (\lambda^2 - 1)\left(\frac{1}{2} - \frac{2}{3}\lambda\phi_0 + \frac{1}{4}\phi_0^2\lambda^2\right), \end{aligned}$$

by elementary algebra. Hence, the general solution for the recurrence relation (4.14) is

$$\phi_1 + \frac{1}{2}(\lambda+1)(1-\phi_0\lambda)^2 + (\lambda^2 - 1)\left(\frac{1}{2} - \frac{2}{3}\lambda\phi_0 + \frac{1}{4}\phi_0^2\lambda^2\right). \tag{4.16}$$

From (4.11), (4.13), and (4.16),

$$c_1(\lambda) = \frac{g_1(\lambda)}{(1-\phi_0\lambda)^2} = \frac{1}{2}(\lambda+1) + \frac{\phi_1 + (\lambda^2-1)\left(\frac{1}{2} - \frac{2}{3}\lambda\phi_0 + \frac{1}{4}\phi_0^2\lambda^2\right)}{(1-\phi_0\lambda)^2},$$

as claimed by Ramanujan.

To calculate the coefficient of x^2, write $u_\lambda = c_0(\lambda) + c_1(\lambda)x + c_2(\lambda)x^2$. Then, from (4.2),

$$c_0(\lambda) + c_1(\lambda)x + c_2(\lambda)x^2 + \frac{1}{c_0(\lambda-1) + c_1(\lambda-1)x + c_2(\lambda-1)x^2} = 2 + \lambda x + \frac{\lambda^2x^2}{2}$$

or

$$\begin{aligned} (c_0(\lambda) + c_1(\lambda)x + c_2(\lambda)x^2)\cdot(c_0(\lambda-1) + c_1(\lambda-1)x + c_2(\lambda-1)x^2) + 1 \\ = \left(2 + \lambda x + \frac{\lambda^2x^2}{2}\right)(c_0(\lambda-1) + c_1(\lambda-1)x + c_2(\lambda-1)x^2). \end{aligned} \tag{4.17}$$

Equate coefficients of x^2 to deduce that

$$\begin{aligned} c_0(\lambda)c_2(\lambda-1) + c_1(\lambda)c_1(\lambda-1) + c_2(\lambda)c_0(\lambda-1) \\ = 2c_2(\lambda-1) + \lambda c_1(\lambda-1) + \frac{\lambda^2}{2}c_0(\lambda-1), \end{aligned}$$

or

$$c_2(\lambda)c_0(\lambda-1) + c_2(\lambda-1)(c_0(\lambda)-2) = c_0(\lambda-1)\frac{\lambda^2}{2} + c_1(\lambda-1)\lambda - c_1(\lambda)c_1(\lambda-1). \tag{4.18}$$

Recall that

$$c_0(\lambda)=\frac{1-\phi_0\lambda-\phi_0}{1-\phi_0\lambda}\quad\text{and}\quad c_0(\lambda)-2=\frac{-1+\phi_0\lambda-\phi_0}{1-\phi_0\lambda}. \tag{4.19}$$

Set

$$c_2(\lambda)(1-\phi_0\lambda)=f_2(\lambda). \tag{4.20}$$

Now from (4.18), (4.19), and (4.20),

$$\begin{aligned}c_2(\lambda)\frac{1-\phi_0\lambda}{1+\phi_0-\phi_0\lambda}+c_2(\lambda-1)\frac{-1+\phi_0\lambda-\phi_0}{1-\phi_0\lambda}&=\frac{f_2(\lambda)}{1+\phi_0-\phi_0\lambda}-\frac{f_2(\lambda-1)}{1-\phi_0\lambda}\\&=\frac{\lambda^2}{2}\frac{1-\phi_0\lambda}{1+\phi_0-\phi_0\lambda}+\lambda c_1(\lambda-1)-c_1(\lambda)c_1(\lambda-1).\end{aligned} \tag{4.21}$$

Set

$$g_2(\lambda)=f_2(\lambda)(1-\phi_0\lambda)=c_2(\lambda)(1-\phi_0\lambda)^2. \tag{4.22}$$

Then, after multiplying both sides of (4.21) by $(1-\phi_0\lambda)(1+\phi_0-\phi_0\lambda)$ and using (4.22), we find that

$$\begin{aligned}&g_2(\lambda)-g_2(\lambda-1)\\&=(1-\phi_0\lambda)(1+\phi_0-\phi_0\lambda)\left(\frac{\lambda^2}{2}\frac{1-\phi_0\lambda}{1+\phi_0-\phi_0\lambda}+\lambda c_1(\lambda-1)-c_1(\lambda)c_1(\lambda-1)\right).\end{aligned} \tag{4.23}$$

Recall that

$$c_1(\lambda)=\frac{g_1(\lambda)}{(1-\phi_0\lambda)^2} \tag{4.24}$$

with

$$g_1(\lambda)=\frac{1}{2}(\lambda+1)(1-\phi_0\lambda)^2+(\lambda^2-1)\left(\frac{1}{2}-\frac{2}{3}\lambda\phi_0+\frac{1}{4}\phi_0^2\lambda^2\right)+\phi_1. \tag{4.25}$$

The right-hand side of (4.23) is not a polynomial in λ. However, by making a judicious change of variable, we will be able to determine the general solution of the recurrence relation (4.23). Now multiply both sides of (4.23) by $(1-\phi_0\lambda)(1+\phi_0-\phi_0\lambda)$. Then

$$\begin{aligned}&g_2(\lambda)(1-\phi_0\lambda)(1+\phi_0-\phi_0\lambda)-g_2(\lambda-1)(1-\phi_0\lambda)(1+\phi_0-\phi_0\lambda)\\&=(1-\phi_0\lambda)^2(1+\phi_0-\phi_0\lambda)^2\left(\frac{\lambda^2}{2}\frac{1-\phi_0\lambda}{1+\phi_0-\phi_0\lambda}+\lambda c_1(\lambda-1)-c_1(\lambda)c_1(\lambda-1)\right)\\&=\frac{\lambda^2}{2}(1-\phi_0\lambda)^3(1+\phi_0-\phi_0\lambda)+\lambda c_1(\lambda-1)(1-\phi_0\lambda)^2(1+\phi_0-\phi_0\lambda)^2\\&\qquad-c_1(\lambda)c_1(\lambda-1)(1-\phi_0\lambda)^2(1+\phi_0-\phi_0\lambda)^2\\&=\frac{\lambda^2}{2}(1-\phi_0\lambda)^3(1+\phi_0-\phi_0\lambda)+\lambda(1-\phi_0\lambda)^2g_1(\lambda-1)-g_1(\lambda)g_1(\lambda-1),\end{aligned} \tag{4.26}$$

where we have used (4.24) in the last step. Set

$$h_2(\lambda)=g_2(\lambda)(1-\phi_0\lambda). \tag{4.27}$$

Then we can rewrite (4.26) as

$$\begin{aligned}(4.28)\qquad & h_2(\lambda)(1-\phi_0\lambda+\phi_0)-h_2(\lambda-1)(1-\phi_0\lambda)\\ &=\frac{\lambda^2}{2}(1-\phi_0\lambda)^3(1+\phi_0-\phi_0\lambda)+\lambda(1-\phi_0\lambda)^2g_1(\lambda-1)-g_1(\lambda)g_1(\lambda-1).\end{aligned}$$

We see that the general solution of

$$h_2(\lambda)(1-\phi_0\lambda+\phi_0)-h_2(\lambda-1)(1-\phi_0\lambda)=0$$

is

$$c(1-\phi_0\lambda),$$

where c is a constant, since the characteristic root for the corresponding homogeneous recurrence relation, $g_2(\lambda)-g_2(\lambda-1)=0$, equals 1.

Note from (4.25) that $g_1(\lambda)$ is a polynomial of degree 4 in λ, and so the right-hand side of (4.28) is a polynomial of degree 8 in λ. For a particular solution to the recurrence relation (4.28), let

$$h_2(\lambda)=\sum_{i=0}^{8}g_i\lambda^i. \tag{4.29}$$

Then the general solution is

$$\begin{aligned}(4.30)\qquad & c(1-\phi_0\lambda)+\sum_{i=0}^{8}g_i\lambda^i\\ &=(c+g_0)(1-\phi_0\lambda)+g_0\phi_0\lambda+\sum_{i=1}^{8}g_i\lambda^i\\ &=(c+g_0)(1-\phi_0\lambda)+(g_0\phi_0+g_1)\lambda+\sum_{i=2}^{8}g_i\lambda^i\\ &=\phi_2(1-\phi_0\lambda)+g_1^*\lambda+\sum_{i=2}^{8}g_i\lambda^i,\end{aligned}$$

where $\phi_2=c+g_0$ and $g_1^*=g_0\phi_0+g_1$. From this observation, we do not need to consider the constant term, and so we only need to find a particular solution of the form

$$h_2(\lambda)=\sum_{i=1}^{8}e_i\lambda^i. \tag{4.31}$$

By (4.31), (4.28), and (4.25), we have the system of equations

$$\phi_1^2+e_1-e_2+e_3-e_4+e_5-e_6+e_7-e_8=0,$$

$$\begin{aligned}\lambda(&-\phi_1-\frac{2}{3}\phi_0\phi_1+2e_2+\phi_0e_2-3e_3-\phi_0e_3+4e_4+\phi_0e_4-5e_5-\phi_0e_5\\ &+6e_6+\phi_0e_6-7e_7-\phi_0e_7+8e_8+\phi_0e_8)=0,\end{aligned}$$

$$\begin{aligned}\lambda^2(&-\frac{1}{4}-\frac{\phi_0}{6}+\frac{\phi_0^2}{9}+\phi_1+2\phi_0\phi_1+\frac{1}{2}\phi_0^2\phi_1-\phi_0e_2+3e_3+3\phi_0e_3-6e_4-4\phi_0e_4\\ &+10e_5+5\phi_0e_5-15e_6-6\phi_0e_6+21e_7+7\phi_0e_7-28e_8-8\phi_0e_8)=0,\end{aligned}$$

$$\lambda^3(-\frac{1}{2}+\frac{2}{3}\phi_0+\frac{7\phi_0^2}{12}-\frac{\phi_0^3}{6}-\frac{4}{3}\phi_0\phi_1-\phi_0^2\phi_1-2\phi_0e_3+4e_4+6\phi_0e_4$$
$$-10e_5-10\phi_0e_5+20e_6+15\phi_0e_6-35e_7-21\phi_0e_7+56e_8+28\phi_0e_8)=0,$$

$$\lambda^4(\frac{1}{4}+\frac{5\phi_0}{3}-\frac{29\phi_0^2}{36}-\frac{2\phi_0^3}{3}+\frac{\phi_0^4}{16}+\frac{1}{2}\phi_0^2\phi_1-3\phi_0e_4+5e_5+10\phi_0e_5$$
$$-15e_6-20\phi_0e_6+35e_7+35\phi_0e_7-70e_8-56\phi_0e_8)=0,$$

$$\lambda^5(-\frac{2\phi_0}{3}-\frac{25\phi_0^2}{12}+\frac{\phi_0^3}{2}+\frac{\phi_0^4}{4}-4\phi_0e_5+6e_6$$
$$+15\phi_0e_6-21e_7-35\phi_0e_7+56e_8+70\phi_0e_8)=0,$$

$$\lambda^6(\frac{25\phi_0^2}{36}+\frac{7\phi_0^3}{6}-\frac{\phi_0^4}{8}-5\phi_0e_6+7e_7+21\phi_0e_7-28e_8-56\phi_0e_8)=0,$$

$$\lambda^7(-\frac{\phi_0^3}{3}-\frac{\phi_0^4}{4}-6\phi_0e_7+8e_8+28\phi_0e_8)=0,$$

and

$$\lambda^8(\frac{\phi_0^4}{16}-7\phi_0e_8)=0.$$

If we solve this system of equations by using *Mathematica*, we find that

$$e_1=\frac{1}{420}(21-5\phi_0^2+140\phi_1-420\phi_1^2),\quad e_2=\frac{1}{1260}(315-343\phi_0+15\phi_0^3-210\phi_0\phi_1),$$
$$e_3=\frac{1}{36}(9-27\phi_0+13\phi_0^2-12\phi_1),\quad e_4=-\frac{1}{144}\phi_0(80-108\phi_0+21\phi_0^2-24\phi_1),$$
$$e_5=\frac{1}{180}(-980\phi_0^2-45\phi_0^3),\quad e_6=\frac{1}{360}(28\phi_0-45\phi_0^3),$$
$$e_7=-\frac{11\phi_0^2}{252},\quad \text{and}\quad e_8=\frac{\phi_0^3}{112}.$$

From the equalities above, (4.31), (4.30), (4.27), and (4.22), the coefficient of x^2, after rearrangement, is equal to

$$\left(\frac{\lambda(\lambda+1)(\lambda+2)}{12}-\frac{\phi_2+\lambda(\lambda^2-1)(\lambda^2-4)\left(\frac{1}{45}-\frac{\lambda\phi_0}{36}+\frac{(\lambda^2+\frac{1}{3})\phi_0^2}{112}\right)}{(1-\lambda\phi_0)^2}\right.$$
$$\left.-\frac{\lambda}{(1-\lambda\phi_0)^3}\left(\phi_1+\frac{\lambda^2-1}{6}(1-\frac{1}{2}\lambda\phi_0)\right)^2\right),$$

as claimed by Ramanujan. □

Ramanujan claims that if $x<0$, the coefficients $\phi_0,\phi_1,\phi_2,\phi_3,\ldots$ are arbitrary, but that if $x>0$, then $\phi_1=\phi_2=\phi_3=\cdots=0$ and

$$\phi_0=\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}(3x)^{1/3}e^{-G(x)},$$

where $G(x)$ has the asymptotic expansion, as $x\to 0^+$,

$$G(x)\sim a_2x^2+a_4x^4+a_6x^6+\cdots,$$

with the coefficients a_ν given by

$$a_\nu = \frac{4\Gamma(\nu)\zeta(\nu)L(\nu+1,\chi)}{(2\pi/\sqrt{3})^{2\nu+1}}.$$

Here $\zeta(s)$ denotes Riemann's zeta function, and $L(s,\chi)$ denotes the Dirichlet L-function associated with the character $\chi(n) = \left(\frac{n}{3}\right)$, where $\left(\frac{n}{3}\right)$ denotes the Legendre symbol. By rearrangement, Ramanujan is asserting that

$$\frac{1}{1-u_\lambda} \sim \frac{1}{\phi_0}, \tag{4.32}$$

as $x \to 0^+$. Note that, when $\lambda = 0$, the continued fraction in (1.1) is equal to $1/(1-u_0)$, with $q = e^{-x}$. In this case, the asymptotic formula (4.32) is identical to the aforementioned asymptotic formula also found on page 45 of [12] and proved by Berndt and Sohn in [9]. However, if $\lambda > 0$, the method of proof used in [9] does not generalize, and so in this particular situation we cannot verify Ramanujan's claim. Note that if we set $q = e^{-x}$ and $a = e^{-\lambda x}$ in Corollary 3.3, the continued fraction there is equivalent to the continued fraction $e^{-\lambda x}u_\lambda$, where u_λ is defined by (4.3). As remarked at the beginning of this section, the constants $\phi_0, \phi_1, \phi_2, \dots$ are indeed arbitrary when $x < 0$, because (4.2) does not have a unique solution.

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