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ON GINZBURG'S BIVARIANT CHERN CLASSES

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ABSTRACT. The convolution product is an important tool in geometric representation theory. Ginzburg constructed the "bivariant" Chern class operation from a certain convolution algebra of Lagrangian cycles to the convolution algebra of Borel-Moore homology. In this paper we prove a "constructible function version" of one of Ginzburg's results; motivated by its proof, we introduce another bivariant algebraic homology theory $s\mathbb{AH}$ on smooth morphisms of nonsingular varieties and show that the Ginzburg bivariant Chern class is the unique Grothendieck transformation from the Fulton-MacPherson bivariant theory of constructible functions to this new bivariant algebraic homology theory, modulo a reasonable conjecture. Furthermore, taking a hint from this conjecture, we introduce another bivariant theory $\mathbb{GF}$ of constructible functions, and we show that the Ginzburg bivariant Chern class is the unique Grothendieck transformation from $\mathbb{GF}$ to $s\mathbb{AH}$ satisfying the "normalization condition" and that it becomes the Chern-Schwartz-MacPherson class when restricted to the morphisms to a point.

§1. INTRODUCTION

The present work is motivated by reading V. Ginzburg's survey article [G2], Chriss and Ginzburg's book [CG] and H. Nakajima's survey article (in Japanese) [N3] (cf. its original paper [N4]).

In [N1] Nakajima gives a representation of the Kac-Moody algebra, in which the key ingredients are the constructible functions and their pullbacks and pushforwards. Also, in [N2] Nakajima gives a representation of the Heisenberg algebra on the homology groups of the Hilbert scheme and, for that purpose, the key ingredients are the Borel-Moore homology groups and the convolution product on the homology groups. One motivation of the present paper is to see whether or not one could use the theory of the Chern-Schwartz-MacPherson class transformation $c_* : F \to H_*$ from the constructible function functor to the homology theory to obtain some kind of connections, if any, between the above representations of the Kac-Moody algebra and the Heisenberg algebra.

In [G1] Ginzburg introduced the notion of the "bivariant" Chern class from the abelian group of certain Lagrangian cycles satisfying some special conditions to the Borel-Moore homology group and showed that it is "convolutive", i.e., it commutes with the convolution product. In this paper we show a "constructible

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function version" of Ginzburg's result and also, in a special case, the bivariant Chern class from the Fulton-MacPherson bivariant group of constructible functions to the homology is uniquely determined and is the same as Ginzburg's bivariant Chern class.

Also, motivated by this result, we introduce another kind of bivariant homology theory on morphisms of nonsingular varieties, and we show the unique existence of the bivariant Chern class, modulo a reasonable conjecture. Although we have not been able to resolve the conjecture, suggested by this question, we introduce another bivariant theory of constructible functions and we show that the Ginzburg bivariant Chern class is the unique Grothendieck transformation between the above two new bivariant theories satisfying the normalization condition that this transformation is merely the original Chern-Schwartz-MacPherson transformation when restricted to the morphisms to a point.

§2. Convolution in Borel-Moore homology

The notion of *convolution* (product) is important and ubiquitous in geometric representation theory. Here we recall the convolution on the Borel-Moore homology theory.

In this paper the homology theory $H_*(X)$ is the Borel-Moore homology group of a locally compact Hausdorff space X, i.e., the ordinary (singular) cohomology group of the pair $(\bar{X}, \infty)$ where $\bar{X} = X \cup \infty$ is the one-point compactification of X.

For any closed subsets X and X' in a smooth manifold m, we have the cup product

$$\cup : H^p(M, M \setminus X) \otimes H^q(M, M \setminus X') \to H^{p+q}(M, M \setminus (X \cap X')),$$

which implies, by the Alexander duality isomorphism

$$H^\bullet(M, M \setminus A) \cong H_{\dim M - \bullet}(A),$$

the following intersection product:

$$\cdot : H_i(X) \otimes H_j(X') \to H_{i+j-\dim M}(X \cap X').$$

Let M_1, M_2, M_3 be smooth oriented manifolds. Let $p_{ij} : M_1 \times M_2 \times M_3 \to M_i \times M_j$ be the canonical projections. Let $Z \subset M_1 \times M_2$ and $Z' \subset M_2 \times M_3$ be closed subsets and assume that the restricted map

$$p_{13} : p_{12}^{-1}(Z) \cap p_{23}^{-1}(Z') \to M_1 \times M_3$$

is proper. Then its image is denoted by $Z \circ Z'$, i.e., the composite of the two correspondences Z and Z' (see Fulton's book [F]). With this set-up, the convolution

$$\bigstar : H_i(Z) \otimes H_j(Z') \to H_{i+j-\dim M}(Z \circ Z')$$

is defined by

$$\alpha \bigstar \beta := p_{13*}(p_{12}^*\alpha \cdot p_{23}^*\beta).$$

In particular, when $M_1 = M_2 = M_3 = M$, for any closed subvariety $Z \subset M \times M$, we have $Z \circ Z = Z$ and therefore $H_*(Z)$ is a convolution algebra.

As one can see in the above construction of convolution, as long as the operations of product, pullback and pushforward are available on certain algebraic objects defined on (topological) spaces, one can always define a convolution product.

§3. Ginzburg's bivariant Chern class

In [G1] Ginzburg defined the "bivariant" Chern class c^{biv} from the abelian group of Lagrangian cycles satisfying some special conditions to the Borel-Moore homology group and also showed that the bivariant Chern class is convolutive; i.e., the bivariant Chern class commutes with the convolution product. The situation of interest to us in this paper is the following result of Ginzburg:

Theorem (3.1) ([G2, Theorem 6.7]). *The map $c^{\mathrm{biv}} : L(X \times Y) \to H_*(X \times Y)$ commutes with convolution, i.e., the following diagram commutes:*

$$\begin{array}{ccc} L(X_1 \times X_2) \otimes L(X_2 \times X_3) & \xrightarrow{\star} & L(X_1 \times X_3) \\ {\scriptstyle c^{\mathrm{biv}} \otimes c^{\mathrm{biv}}}\downarrow & & \downarrow{\scriptstyle c^{\mathrm{biv}}} \\ H_*(X_1 \times X_2) \otimes H_*(X_2 \times X_3) & \xrightarrow[\star]{} & H_*(X_1 \times X_3). \end{array}$$

Here all X_1, X_2, X_3 are nonsingular varieties.

The construction or definition of c^{biv} given in [G1] is not direct, but in his survey article [G2] he gives an explicit description of the Chern class operation c^{biv}. It assigns to a Lagrangian cycle, i.e., $\Lambda_Y, Y \subset X_1 \times X_2$, the *relative Chern-Mather class of the fibers of the projection* $p_Y : Y \to X_2$. The projection p_Y is the restriction of the projection $p_2 : X_1 \times X_2 \to X_2$ to the subvariety Y. Let $\nu : \widehat{Y} \to Y$ be the Nash blow-up and $\widehat{TY}$ the tautological Nash tangent bundle over $\widehat{Y}$. Then the above relative Chern-Mather class is defined by

$$c^{\mathrm{biv}}(\Lambda_Y) := i_{Y*}\nu_*\Big(c(\widehat{TY} - \nu^* p_Y^* TX_2) \cap [\widehat{Y}]\Big)$$

where $i_Y : Y \to X_1 \times X_2$ is the inclusion. Then it follows from the projection formula and from $p_Y = p_2 \circ i_Y$ that

$$\begin{aligned} c^{\mathrm{biv}}(\Lambda_Y) &= i_{Y*}\Big(\frac{1}{p_Y^* c(TX_2)} \cap c^M(Y)\Big) \\ &= p_2^* s(TX_2) \cap i_{Y*} c^M(Y). \end{aligned}$$

Here $s(TX_2)$ denotes the Segre class of the tangent bundle of the manifold X_2.

This Ginzburg bivariant Chern class can be defined for a constructible function and for any morphism $\pi : X \to S$ from a possibly singular variety X to a smooth variety S. So we define

$$c_*^{\mathrm{biv}} = \pi^* s(TS) \cap c_* : F(X) \to H_*(X, \mathbb{Q})$$

where $c_* : F(X) \to H_*(X; \mathbb{Q})$ is the usual Chern-Schwartz-MacPherson class from the abelian group $F(X)$ of constructible functions to the homology group with rational coefficients [M]. In this section we shall give a "constructible function" version of the above Theorem (3.1).

Since, from now on we make full use of basic properties of the Fulton-MacPherson bivariant theory, we recall some necessary things from [FM]. A bivariant theory $\mathbb{B}$ on a category $\mathcal{C}$ with values in an abelian category is an assignment to each morphism $X \xrightarrow{f} Y$ in the category $\mathcal{C}$ a graded abelian group $\mathbb{B}(X \xrightarrow{f} Y)$ that is equipped with the following three basic operations:

(Product operations): For morphisms $f : X \to Y$ and $g : Y \to Z$, the product operation

$$\bullet : \mathbb{B}(X \xrightarrow{f} Y) \otimes \mathbb{B}(Y \xrightarrow{g} Z) \to \mathbb{B}(X \xrightarrow{gf} Z)$$

is defined.

(Pushforward operations): For morphisms $f : X \to Y$ and $g : Y \to Z$ with f proper, the pushforward operation

$$f_* : \mathbb{B}(X \xrightarrow{gf} Z) \to \mathbb{B}(Y \xrightarrow{g} Z)$$

is defined.

(Pullback operations): For a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y, \end{array}$$

the pullback operation

$$g^* : \mathbb{B}(X \xrightarrow{f} Y) \to \mathbb{B}(X' \xrightarrow{f'} Y')$$

is defined.

These three operations are required to satisfy the following seven axioms (see [FM, Part I, §2.2] for details):

(B-1) product is associative,
(B-2) pushforward is functorial,
(B-3) pullback is functorial,
(B-4) product and pushforward commute,
(B-5) product and pullback commute,
(B-6) pushforward and pullback commute, and
(B-7) projection formula.

Let $\mathbb{B}, \mathbb{B}'$ be two bivariant theories on a category C. Then a *Grothendieck transformation* from $\mathbb{B}$ to $\mathbb{B}'$

$$\gamma : \mathbb{B} \to \mathbb{B}'$$

is a collection of homomorphisms

$$\mathbb{B}(X \to Y) \to \mathbb{B}'(X \to Y)$$

for a morphism $X \to Y$ in the category C that preserves the above three basic operations; i.e.,

(i) $\gamma(\alpha \bullet_{\mathbb{B}} \beta) = \gamma(\alpha) \bullet_{\mathbb{B}'} \gamma(\beta)$,
(ii) $\gamma(f_*\alpha) = f_*\gamma(\alpha)$, and
(iii) $\gamma(g^*\alpha) = g^*\gamma(\alpha)$.

We recall some bivariant theories of constructible functions and the bivariant homology theory.

The constructible function functor F itself can be a bivariant theory as follows: For *any* morphism $f : X \to Y$ the group $s\mathbb{F}(X \to Y)$ is defined by

$$s\mathbb{F}(X \xrightarrow{f} Y) := F(X).$$

We define the following operations of product, pushforward and pullback:

(Product operation):

$$\bullet : s\mathbb{F}(X \xrightarrow{f} Y) \otimes s\mathbb{F}(Y \xrightarrow{g} Z) \to s\mathbb{F}(X \xrightarrow{gf} Z)$$

is defined by

$$\alpha \bullet \beta := \alpha \cdot f^*\beta.$$

(Pushforward operation):

$$f_* : s\mathbb{F}(X \xrightarrow{gf} Z) \to s\mathbb{F}(Y \xrightarrow{g} Z)$$

is the pushforward

$$f_* : F(X) \to F(Y).$$

(Pullback operation): For a fiber square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y, \end{tikzcd}$$

the pullback operation

$$g^* : s\mathbb{F}(X \xrightarrow{f} Y) \to s\mathbb{F}(X' \xrightarrow{f'} Y')$$

is the functional pullback

$$g'^* : F(X) \to F(X').$$

Then $s\mathbb{F}$ becomes a bivariant theory with these three operations, i.e., they satisfy the seven axioms of the bivariant theory. This bivariant theory $s\mathbb{F}$ shall be called *the simple bivariant theory of constructible functions*.

The axioms (B-2) and (B-3) clearly hold and one can prove that the above three operations satisfy the other five axioms, using the following three properties, which will be used often in later sections:

3.2. For a fiber square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y, \end{tikzcd}$$

the following diagram commutes:

$$\begin{tikzcd} F(Y') \arrow[r, "f'^*"] \arrow[d, "g_*"'] & F(X') \arrow[d, "g'_*"] \\ F(Y) \arrow[r, "f^*"] & F(X). \end{tikzcd}$$

3.3. For a morphism $f : X \to Y$ and constructible functions $\alpha, \beta \in F(Y)$ we have

$$f^*(\alpha \cdot \beta) = f^*\alpha \cdot f^*\beta.$$

3.4. (Projection formula). For a morphism $f : X \to Y$ and constructible functions $\alpha \in F(Y)$ and $\beta \in F(X)$ we have

$$f_*(f^*\alpha \cdot \beta) = \alpha \cdot f_*\beta.$$

Let $\mathbb{H}$ be the Fulton-MacPherson bivariant homology theory constructed from the singular cohomology theory H^*. For a morphism $f : X \to Y$, choose a morphism $\phi : X \to \mathbb{R}^n$ such that $\Phi := (f, \phi) : X \to Y \times \mathbb{R}^n$ is a closed embedding. Then the ith bivariant homology group $\mathbb{H}^i(X \xrightarrow{f} Y)$ is defined by

$$\mathbb{H}^i(X \xrightarrow{f} Y) := H^{i+n}(Y \times \mathbb{R}^n, Y \times \mathbb{R}^n \setminus X_\phi),$$

where X_ϕ is defined to be the image of the morphism $\Phi = (f, \phi)$. The definition is independent of the choice of ϕ. Note that instead of taking the Euclidean space $\mathbb{R}^n$ we can take a manifold M so that $i : X \to M$ is a closed embedding and then consider the graph embedding $f \times i : X \to Y \times M$ (see [FM] and [B]). See [FM, §3.1] for more details of $\mathbb{H}$.

The above simple bivariant theory $s\mathbb{F}$ is not a good one in the sense that one looks for a bivariant version of the Chern-Schwartz-MacPherson class. In fact, there does not exist a Grothendieck transformation $\gamma^{\mathrm{s}} : s\mathbb{F} \to \mathbb{H}$ such that $\gamma^{\mathrm{s}}(1\!\!1_\pi) = c(TX) \cap [X]$ for X smooth, where $\pi : X \to pt$ and $1\!\!1_\pi = 1\!\!1_X$ ([Y2, Theorem (3.2)]). At the moment, the best bivariant group of constructible functions is the Fulton-MacPherson bivariant theory of constructible functions, i.e., $\mathbb{F}(X \xrightarrow{f} Y)$ consisting of all the constructible functions on X that satisfy the local Euler condition with respect to f (see [B], [FM], [Sa], [Z]). Here a constructible function $\alpha \in F(X)$ is said to satisfy the *local Euler condition with respect to* f if for any point $x \in X$ and for any local embedding $(X, x) \to (\mathbf{C}^N, 0)$ the following equality holds:

$$\alpha(x) = \chi\big(B_\epsilon \cap f^{-1}(z); \alpha\big),$$

where B_ϵ is a sufficiently small open ball of the origin 0 with radius ϵ and z is any point close to $f(x)$ (cf. [B], [Sa]). The three operations on $\mathbb{F}$ are the same as above in $s\mathbb{F}$ and it is known that these three operations are well defined for $\mathbb{F}$ (e.g., see [BY], [Sa], [Z]). Note that $\mathbb{F}(X \xrightarrow{\mathrm{id}_X} X)$ consists of all locally constant functions and $\mathbb{F}(X \to pt) = F(X)$.

Theorem (3.5) (Brasselet's theorem [B]). *On the category of compact complex analytic varieties and cellular morphisms, there exists a Grothendieck transformation*

$$\gamma^{\mathrm{Br}} : \mathbb{F} \to \mathbb{H}$$

satisfying the normalization condition that $\gamma^{\mathrm{Br}}(1\!\!1_\pi) = c(TX) \cap [X]$ *for* X *smooth, where* $\pi : X \to pt$ *and* $1\!\!1_\pi = 1\!\!1_X$.

Let $\mathbb{B}$ be any bivariant theory defined on the category of algebraic varieties. Then the $\mathbb{B}$-convolution product

$$\bigstar_{\mathbb{B}} : \mathbb{B}(X_1 \times_S X_2 \xrightarrow{\pi_1^{12}} X_1) \otimes \mathbb{B}(X_2 \times_S X_3 \xrightarrow{\pi_2^{23}} X_2) \to \mathbb{B}(X_1 \times_S X_3 \xrightarrow{\pi_1^{13}} X_1)$$

is defined by

$$\alpha \bigstar_{\mathbb{B}} \beta := \big(\pi_{13}^{123}\big)_* \Big((\pi_2^{12})^* \beta \bullet_{\mathbb{B}} \alpha\Big).$$

Here we use the following diagram:

$$\begin{array}{ccccccc}
X_1 \times_S X_3 & \xleftarrow{\pi_{13}^{123}} & X_1 \times_S X_2 \times_S X_3 & \xrightarrow{\pi_{23}^{123}} & X_2 \times_S X_3 & \xrightarrow{\pi_3^{23}} & X_3 \\
{\scriptstyle \pi_1^{13}}\downarrow & & \downarrow{\scriptstyle \pi_{12}^{123}} & & \downarrow{\scriptstyle \pi_2^{23}} & & {\scriptstyle \pi_0^{3}}\downarrow \\
X_1 & \xleftarrow{\pi_1^{12}} & X_1 \times_S X_2 & \xrightarrow[\pi_2^{12}]{} & X_2 & \xrightarrow[\pi_0^{2}]{} & S \\
 & & {\scriptstyle \pi_1^{12}}\downarrow & & \downarrow{\scriptstyle \pi_0^{2}} & & \\
 & & X_1 & \xrightarrow[\pi_0^{1}]{} & S & &
\end{array}$$

Then it is clear that any Grothendieck transformation $\gamma : \mathbb{B} \to \mathbb{B}'$ between any two bivariant theories is convolutive, i.e., preserves the convolution product:

$$\gamma(\alpha \bigstar_{\mathbb{B}} \beta) = \gamma(\alpha) \bigstar_{\mathbb{B}'} \gamma(\beta),$$

because the Grothendieck transformation commutes with the bivariant theoretic operations of product, pullback and pushforward.

From now on we set the base variety S as a point. Let $p : X \times Y \to X$ be the projection to the first factor. Then we can show the following theorem, which is a "constructible function version" of Theorem (3.1):

Theorem (3.6). *For nonsingular varieties X_1, X_2, X_3, the homomorphism $c_*^{\mathrm{biv}} : \mathbb{F}(X \times Y \to X) \to H_*(X \times Y)$ is convolutive; i.e., the following diagram commutes:*

$$\begin{array}{ccc}
\mathbb{F}(X_1 \times X_2 \to X_1) \otimes \mathbb{F}(X_2 \times X_3 \to X_2) & \xrightarrow{\bigstar} & \mathbb{F}(X_1 \times X_3 \to X_1) \\
{\scriptstyle c_*^{\mathrm{biv}} \otimes c_*^{\mathrm{biv}}}\downarrow & & \downarrow{\scriptstyle c_*^{\mathrm{biv}}} \\
H_*(X_1 \times X_2) \otimes H_*(X_2 \times X_3) & \xrightarrow[\bigstar]{} & H_*(X_1 \times X_3).
\end{array}$$

Proof. Since projection maps are always cellular, we can use Brasselet's bivariant Chern class $\gamma^{\mathrm{Br}} : \mathbb{F} \to \mathbb{H}$ and we can get the following commutative diagram as in the above observation:

$$\begin{array}{ccc}
\mathbb{F}(X_1 \times X_2 \to X_1) \otimes \mathbb{F}(X_2 \times X_3 \to X_2) & \xrightarrow{\bigstar_{\mathbb{F}}} & \mathbb{F}(X_1 \times X_3 \to X_1) \\
{\scriptstyle \gamma^{\mathrm{Br}} \otimes \gamma^{\mathrm{Br}}}\downarrow & & \downarrow{\scriptstyle \gamma^{\mathrm{Br}}} \\
\mathbb{H}(X_1 \times X_2 \to X_1) \otimes \mathbb{H}(X_2 \times X_3 \to X_2) & \xrightarrow{\bigstar_{\mathbb{H}}} & \mathbb{H}(X_1 \times X_3 \to X_1).
\end{array}$$

Since $\mathbb{H}(X_i \times X_j \xrightarrow{\pi_i^{ij}} X_i) = H^*(X_i \times X_j)$ for any i, j, it follows from the definitions of the bivariant theoretic operations on the bivariant homology theory $\mathbb{H}$ that the $\mathbb{H}$-convolution product $\bigstar_{\mathbb{H}}$ is described by

$$A \bigstar_{\mathbb{H}} B = \mathcal{D}^{-1}_{X_1 \times X_3} \left(\pi_{13}^{123}\right)_* \mathcal{D}_{X_1 \times X_2 \times X_3} \Big(\left(\pi_{12}^{123}\right)^* A \cup \left(\pi_{23}^{123}\right)^* B \Big).$$

For a manifold Z, $\mathcal{D}_Z : H^*(Z) \cong H_*(Z)$ denotes the Poincaré duality isomorphism. On the other hand, by the definition, the convolution of the (Borel-Moore) homology

$$\bigstar_{BM} : H_*(X_1 \times X_2) \otimes H_*(X_2 \times X_3) \to H_*(X_1 \times X_3)$$

is given by

$$a\bigstar_{BM}b = \left(\pi_{13}^{123}\right)_*\mathcal{D}_{X_1\times X_2\times X_3}\left(\left(\pi_{12}^{123}\right)^*\mathcal{D}_{X_1\times X_2}^{-1}(a) \cup \left(\pi_{23}^{123}\right)^*\mathcal{D}_{X_2\times X_3}^{-1}(b)\right).$$

Therefore, we can see that the following diagram commutes:

$$\begin{CD}
\mathbb{H}(X_1\times X_2 \to X_1)\otimes \mathbb{H}(X_2\times X_3\to X_2) @>{\bigstar_{\mathbb{H}}}>> \mathbb{H}(X_1\times X_3\to X_1)\\
@V{\mathcal{D}_{X_1\times X_2}\otimes\mathcal{D}_{X_2\times X_3}}VV @VV{\mathcal{D}_{X_1\times X_3}}V\\
H_*(X_1\times X_2)\otimes H_*(X_2\times X_3) @>{\bigstar_{BM}}>> H_*(X_1\times X_3).
\end{CD}$$

It follows from these two commutative diagrams that the homomorphism

$$\mathcal{D}_{X\times Y}\circ\gamma^{\mathrm{Br}}: \mathbb{F}(X\times Y\to X)\to H_*(X\times Y)$$

is convolutive. Now we want to show that for any bivariant constructible function $\alpha\in\mathbb{F}(X\times Y\xrightarrow{p} X)$ we have

$$\mathcal{D}_{X\times Y}\circ\gamma^{\mathrm{Br}}(\alpha) = p^*s(TX)\cap c_*(\alpha),$$

which completes the proof of the theorem. Since $1\!\!1_X\in F(X)=\mathbb{F}(X\to pt)$, we have

$$\alpha = \alpha\bullet 1\!\!1_X \in F(X\times Y) = \mathbb{F}(X\times Y\to pt).$$

To avoid some possible confusion, the homomophism $\gamma^{\mathrm{Br}}:\mathbb{F}(X\to Y)\to\mathbb{H}(X\to Y)$ shall be denoted by $\gamma^{\mathrm{Br}}_{X\to Y}$. Hence, we have

$$\gamma^{\mathrm{Br}}_{X\times Y\to pt}(\alpha) = \gamma^{\mathrm{Br}}_{X\times Y\to X}(\alpha)\bullet\gamma^{\mathrm{Br}}_{X\to pt}(1\!\!1_X). \tag{3.6.1}$$

Here we note that the homomorphism $\gamma^{\mathrm{Br}}_{Z\to pt}:\mathbb{F}(Z\to pt)\to\mathbb{H}(Z\to pt)$ is expressed by

$$\gamma^{\mathrm{Br}}_{Z\to pt} = \mathcal{A}_Z^{-1}\circ c_*,$$

where $\mathcal{A}_Z: H^*(M, M\setminus Z)\to H_*(Z)$ is the Alexander duality isomorphism with Z being embedded into a (any, in fact) manifold M. In particular, if Z is nonsingular, $\mathcal{A}_Z=\mathcal{D}_Z$; so we have

$$\gamma^{\mathrm{Br}}_{Z\to pt} = \mathcal{D}_Z^{-1}\circ c_*.$$

Therefore (3.6.1) becomes

$$\mathcal{D}^{-1}_{X\times Y}(c_*(\alpha)) = \gamma^{\mathrm{Br}}_{X\times Y\to X}(\alpha)\bullet\mathcal{D}_X^{-1}(c_*(1\!\!1_X).$$

Since $\mathcal{D}_X^{-1}(c_*(1\!\!1_X)) = c(TX)$, we have

$$\mathcal{D}^{-1}_{X\times Y}(c_*(\alpha)) = \gamma^{\mathrm{Br}}_{X\times Y\to X}(\alpha)\cup p^*c(TX),$$

which implies that

$$\gamma^{\mathrm{Br}}_{X\times Y\to X}(\alpha) = p^*s(TX)\cup\mathcal{D}^{-1}_{X\times Y}(c_*(\alpha)).$$

Therefore, we get that

$$\mathcal{D}_{X\times Y}\circ\gamma^{\mathrm{Br}}(\alpha) = p^*s(TX)\cap c_*(\alpha).$$

This completes the proof of the theorem. □

For any varieties X, Y, we set

$$\widetilde{\mathbb{F}}(X\times Y) := 1\!\!1_X\times F(Y) = \{1\!\!1_X\times\alpha \mid \alpha\in F(Y)\}.$$

Then, Theorem (3.6) is a generalization of the following naïve version, which can be proved without appealing to the Brasselet theorem:

Theorem (3.7) ([Y4, Theorem (3.2)]). *Let X be a nonsingular variety and Y an arbitrary variety. Then the homomorphism $c_*^{\mathrm{biv}} : \widetilde{\mathbb{F}}(X \times Y) \to H_*(X \times Y)$ defined by $c_*^{\mathrm{biv}}(\mathbb{1}_X \times \alpha) = [X] \times c_*(\alpha)$ is convolutive; i.e., the following diagram commutes:*

$$\begin{array}{ccc} \widetilde{\mathbb{F}}(X_1 \times X_2) \otimes \widetilde{\mathbb{F}}(X_2 \times X_3) & \xrightarrow{\star} & \widetilde{\mathbb{F}}(X_1 \times X_3) \\ {\scriptstyle c_*^{\mathrm{biv}} \otimes c_*^{\mathrm{biv}}} \downarrow & & \downarrow {\scriptstyle c_*^{\mathrm{biv}}} \\ H_*(X_1 \times X_2) \otimes H_*(X_2 \times X_3) & \xrightarrow[\star]{} & H_*(X_1 \times X_3). \end{array}$$

§4. A bivariant algebraic homology theory and a conjecture

Motivated by Theorem (3.6) and its proof, we introduce another kind of bivariant homology theory for morphisms of nonsingular varieties.

Definition (4.1). For a morphism $f : X \to Y$ of nonsingular varieties, we define an abelian group $p\mathbb{H}(X \xrightarrow{f} Y)$ simply by

$$p\mathbb{H}(X \xrightarrow{f} Y) := H_*(X).$$

The three operations are defined on $p\mathbb{H}$ as follows:

(Product operation):

$$\bullet : p\mathbb{H}(X \xrightarrow{f} Y) \otimes p\mathbb{H}(Y \xrightarrow{g} Z) \to p\mathbb{H}(X \xrightarrow{gf} Z)$$

is defined by, for $\alpha \in p\mathbb{H}(X \xrightarrow{f} Y)$, $\beta \in p\mathbb{H}(Y \xrightarrow{g} Z)$,

$$\alpha \bullet \beta := f^*(\mathcal{D}_Y^{-1}(\beta)) \cap \alpha.$$

(Pushforward operation):

$$f_* : p\mathbb{H}(X \xrightarrow{gf} Z) \to p\mathbb{H}(Y \xrightarrow{g} Z)$$

is the usual homology pushforward

$$f_* : H_*(X) \to H_*(Y).$$

(Pullback operation): For a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'} \downarrow & & \downarrow {\scriptstyle f} \\ Y' & \xrightarrow{g} & Y, \end{array}$$

the pullback operation

$$g^* : p\mathbb{H}(X \xrightarrow{f} Y) \to p\mathbb{H}(X' \xrightarrow{f'} Y')$$

is the Gysin homomorphism

$$g'^{!} : H_*(X) \to H_*(X'),$$

i.e., $g'^{!} = \mathcal{D}_{X'} g'^* \mathcal{D}_X^{-1}$.

Proposition (4.2). *The above $p\mathbb{H}$ satisfies the five axioms (B-1), (B-2), (B-3), (B-4) and (B-5) of the bivariant theory.*

The proof is straightforward and is left to the reader. It is not clear whether the axioms (B-6) and (B-7) are satisfied or not. In this sense $p\mathbb{H}$ shall be called a *"pre-bivariant" homology theory.*

As to [(B-6) pushforward and pullback commute], for a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{h''} & X \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{h'} & Y \\ {\scriptstyle g'}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ Z' & \xrightarrow{h} & Z \end{array}$$

and for $\alpha \in p\mathbb{H}(X \xrightarrow{gf} Z)$, it is required that

$$f'_* h^*(\alpha) = h^* f_*(\alpha), \quad \text{namely } f'_* {h''}^!(\alpha) = {h'}^! f_*(\alpha).$$

As to [(B-7) projection formula], given

$$\begin{array}{ccccc} X' & \xrightarrow{g'} & X & & \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} & & \\ Y' & \xrightarrow{g} & Y & \xrightarrow{h} & Z \end{array}$$

and for $\alpha \in p\mathbb{H}(X \xrightarrow{f} Z)$ and $\beta \in p\mathbb{H}(Y' \xrightarrow{hg} Z)$, it is required that

$$g'_*(g^*\alpha \bullet \beta) = \alpha \bullet g_*(\beta),$$

which asks if the following equality holds:

$$\mathcal{D}_X^{-1}(\alpha) \cap g'_* {f'}^!(\beta) = \mathcal{D}_X^{-1}(\alpha) \cap f^! g_*(\beta).$$

Indeed,

$$\begin{aligned} g'_*(g^*\alpha \bullet \beta) &= g'_*\Big({f'}^* \mathcal{D}_{Y'}^{-1}(\beta) \cap g^*\alpha \Big) \\ &= g'_*\Big({f'}^* \mathcal{D}_{Y'}^{-1}(\beta) \cap {g'}^!\alpha \Big) \\ &= g'_*\left(\Big({f'}^* \mathcal{D}_{Y'}^{-1}(\beta) \cup {g'}^* \mathcal{D}_X^{-1}(\alpha) \Big) \cap [X'] \right) \\ &= g'_*\left({g'}^* \big(\mathcal{D}_{Y'}^{-1}(\alpha)\big) \cap \Big({f'}^* \big(\mathcal{D}_X^{-1}(\beta)\big) \cap [X'] \Big) \right) \\ &= \mathcal{D}_X^{-1}(\alpha) \cap g'_*\Big({f'}^* \big(\mathcal{D}_{Y'}^{-1}(\beta)\big) \cap [X'] \Big) \\ &= \mathcal{D}_X^{-1}(\alpha) \cap g'_* {f'}^!(\beta) \end{aligned}$$

and

$$\begin{aligned}\alpha \bullet g_*\beta &= f^*\big(\mathcal{D}_Y^{-1}(g_*\beta)\big) \cap \alpha \\ &= \Big(f^*\big(\mathcal{D}_Y^{-1}(g_*\beta)\big) \cup \mathcal{D}_X^{-1}(\alpha)\Big) \cap [X] \\ &= \mathcal{D}_X^{-1}(\alpha) \cap \Big(f^*\big(\mathcal{D}_Y^{-1}(g_*\beta)\big) \cap [X]\Big) \\ &= \mathcal{D}_X^{-1}(\alpha) \cap f^! g_*(\beta).\end{aligned}$$

However, if we consider the algebraic homology group, i.e., the homology classes represented by algebraic cycles, instead of the broader homology group, we can get a bivariant theory on the category of smooth morphisms of nonsingular varieties. Before going further, first we recall the following theorem (Verdier-Riemann-Roch theorem for Chern class) and corollary for later uses.

Theorem (4.3) (cf. [FM], [Y1]). *For a smooth morphism $f : X \to Y$ of possibly singular varieties X and Y, the following diagram commutes:*

$$\begin{array}{ccc} F(Y) & \xrightarrow{c_*} & H_*(Y) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle c(T_f)\cap f^!} \\ F(X) & \xrightarrow[c_*]{} & H_*(X). \end{array}$$

This theorem follows from the following commutative diagrams: the left-hand-side commutative diagram is [Y1, Theorem (2.2)] and the commutativity of the right-hand-side diagram follows from [F, Prop. 19.1.2] and [F, Example 19.2.1]:

$$\begin{array}{ccccc} F(Y) & \xrightarrow{c_*} & A(Y) & \xrightarrow{c\ell} & H_*(Y) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle c(T_f)\cap f^*} & & \downarrow{\scriptstyle c(T_f)\cap f^!} \\ F(X) & \xrightarrow[c_*]{} & A(X) & \xrightarrow[c\ell]{} & H_*(X). \end{array}$$

Remark (4.3.1). For a more general Verdier-Riemann-Roch theorem for Chern class, see [Sch2].

Corollary (4.4). *For a smooth morphism $f : X \to Y$ of nonsingular varieties X and Y and for any constructible function $\beta \in F(Y)$, the following equality holds:*

$$c_*(f^*\beta) = f^*\Big(s(TY) \cup \mathcal{D}_Y^{-1}(c_*(\beta))\Big) \cap c_*(X).$$

Proof. First we have to show that for a smooth morphism $f : X \to Y$ of nonsingular varieties X and Y the Gysin homomorphism $f^! : H_*(Y) \to H_*(X)$ is merely the usual Gysin homomorphism $\mathcal{D}_X f^* \mathcal{D}_Y^{-1}$. It may be clear, but we give a proof for the sake of the reader. We follow [F, Example 19.2.1]. Let $i : X \to Y \times X$ be the graph, i.e., $i(x) = (f(x), x)$ and let $p : Y \times X \to Y$ be the projection; thus, we have $f = p \circ i$. Let $u_{X,Y\times X} \in H^*(Y \times X, Y \times X \setminus X)$ be the *orientation class* (see [F, §19.2]). Then by the definition ([F, Example 19.2.1]) we have, for $\alpha \in H_*(Y)$,

$$f^!(\alpha) = u_{X,Y\times X} \cap (\alpha \times [X]).$$

What we want to show is that

$$u_{X,Y\times X} \cap (\alpha \times [X]) = \mathcal{D}_X f^* \mathcal{D}_Y^{-1}(\alpha).$$

This can be seen as follows:

$$\begin{aligned}
&u_{X,Y\times X}\cap(\alpha\times[X])\\
&\quad=u_{X,Y\times X}\cap\Big(\big(\mathcal{D}_Y^{-1}(\alpha)\times 1\big)\cap\big([Y]\times[X]\big)\Big)\\
&\quad=u_{X,Y\times X}\cap\Big(p^*\mathcal{D}_Y^{-1}(\alpha)\cap[Y\times X]\Big)\\
&\quad=\Big(u_{X,Y\times X}\cup(p^*\mathcal{D}_Y^{-1}(\alpha)\Big)\cap[Y\times X]\\
&\quad=i^*p^*\mathcal{D}_Y^{-1}(\alpha)\cap\big(u_{X,Y\times X}\cap[Y\times X]\big)\quad\text{(by [F, §19.1, (8), p. 371])}\\
&\quad=f^*\mathcal{D}_Y^{-1}(\alpha)\cap[X]\quad\text{(by [F, §19.1, (9), p. 373])}\\
&\quad=\mathcal{D}_X f^*\mathcal{D}_Y^{-1}(\alpha).
\end{aligned}$$

Now the statement of the corollary is straightforward. Indeed,

$$\begin{aligned}
c_*(f^*\beta)&=c(Tf)\cap f^!c_*(\beta)\\
&=\frac{c(TX)}{f^*c(TY)}\cap\Big(f^*\mathcal{D}_Y^{-1}(c_*(\beta))\cap[X]\Big)\\
&=(f^*s(TY)\cup f^*\mathcal{D}_Y^{-1}(c_*(\beta))\cup c(TX))\cap[X]\\
&=f^*\Big(s(TY)\cup\mathcal{D}_Y^{-1}(c_*(\beta))\Big)\cap c_*(X).
\end{aligned}$$

□

Definition (4.5). For a morphism $f:X\to Y$ of nonsingular varieties, we define an abelian group $s\mathbb{AH}(X\xrightarrow{f}Y)$ simply by

$$s\mathbb{AH}(X\xrightarrow{f}Y):=AH_*(X),$$

and the three operations of product, pushforward and pullback are defined on $s\mathbb{AH}$ as above in Definition (4.1).

Now, using Corollary (4.4) we can show the following theorem.

Theorem (4.6). *The above $s\mathbb{AH}$ is a bivariant theory on the category of smooth morphisms of nonsingular varieties, i.e., satisfies the seven axioms of the bivariant theory.*

$s\mathbb{AH}$ shall be called *the simple bivariant algebraic homology theory.*

Proof of Theorem (4.6). All we have to do is to show "pullback followed by pushforward = pushforward followed by pullback" for the fiber square.

The algebraic homology group $AH_*(X)$ of the variety X is the image of the cycle map from the Chow homology group to the singular homology (see [F]):

$$c\ell:A(X)\to H_*(X).$$

Then we observe that the algebraic homology group is merely the image of the Chern-Schwartz-MacPherson class homomorphism $c_*:F(X)\to H_*(X)$; i.e.,

$$AH_*(X)=\mathrm{Image}(c\ell:A(X)\to H_*(X))=\mathrm{Image}(c_*:F(X)\to H_*(X)).$$

This is due to the fact that the original Chern-Schwartz-MacPherson class homomorphism c_* factors through the Chow homology group (see [F, Example 19.1.7]):

$$c_*:F(X)\xrightarrow{c_*}A(X)\xrightarrow{c\ell}H_*(X).$$

To see this, just observe that $c_* : F(X) \to A(X)$ is surjective, because for any subvariety $W \subset X$,

$$c_*(\mathbb{1}_W) = [W] + \text{lower-dimensional classes.}$$

Now for the fiber square (of smooth morphisms of nonsingular varieties)

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y, \end{array}$$

and for the algebraic homology class $c_*(\alpha) \in AH_*(X) \subset H_*(X)$ with $\alpha \in F(X)$, we want to show that

$$f'_*{g'}^!(c_*(\alpha)) = g^! f_*(c_*(\alpha)).$$

This can be shown as follows. It follows from Theorem (4.3) that we get

$$c(T_{g'}) \cap {g'}^!(c_*(\alpha)) = c_*({g'}^*\alpha),$$

which implies that

$${g'}^!(c_*(\alpha)) = s(T_{g'}) \cap c_*({g'}^*\alpha).$$

Here we note that $T_{g'} = {f'}^* T_g$ as virtual bundles; therefore, we get that

$$\begin{aligned}
f'_*{g'}^!(c_*(\alpha)) &= f'_*\Big(s(T_{g'}) \cap c_*({g'}^*\alpha)\Big) \\
&= f'_*\Big({f'}^* s(T_g) \cap c_*({g'}^*\alpha)\Big) \\
&= s(T_g) \cap f'_* c_*({g'}^*\alpha) \quad \text{(by the projection formula)} \\
&= s(T_g) \cap c_*(f'_*{g'}^*\alpha) \quad \text{(by MacPherson's theorem)} \\
&= s(T_g) \cap c_*(g^* f_*\alpha) \quad (\text{since } f'_*{g'}^*\alpha = g^* f_*\alpha) \\
&= s(T_g) \cap \Big(g^*\big(s(TY) \cup \mathcal{D}_Y^{-1}(c_*(f_*\alpha))\big) \cap c_*(Y')\Big) \quad \text{(by Corollary (4.4))} \\
&= \Big(s(T_g) \cup g^* s(TY) \cup g^*\mathcal{D}_Y^{-1}(c_*(f_*\alpha))\Big) \cap c_*(Y') \\
&= \Big(s(TY') \cup g^*\mathcal{D}_Y^{-1}(c_*(f_*\alpha))\Big) \cap c_*(Y') \\
&= g^*\mathcal{D}_Y^{-1}(c_*(f_*\alpha)) \cap \Big(s(TY') \cap c_*(Y')\Big) \\
&= g^*\mathcal{D}_Y^{-1}(c_*(f_*\alpha)) \cap [Y'] \quad (\text{since } c_*(Y') = c(TY') \cap [Y']) \\
&= g^!(c_*(f_*\alpha)) \\
&= g^! f_*(c_*(\alpha)) \quad \text{(by MacPherson's theorem).}
\end{aligned}$$

□

Now we go on to the question of existence of a Grothendieck transformation from the bivariant constructible function $\mathbb{F}$ to the simple bivariant algebraic homology theory $s\mathbb{AH}$.

Proposition (4.7). *If there exists a Grothendieck transformation*

$$\gamma^{\mathrm{s}} : \mathbb{F} \to s\mathbb{AH}$$

satisfying the "normalization condition" that, when restricted to the morphisms of nonsingular varieties to a point γ^{s}*, it is merely the Chern-Schwartz-MacPherson* *class $c_* : F \to H_*$, then γ^{s} is unique and for a smooth morphism $f : X \to Y$ of nonsingular varieties X, Y and for a bivariant constructible function $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$ we have*

$$\gamma^{\mathrm{s}}(\alpha) = f^* s(TY) \cap c_*(\alpha).$$

Proof. As in the previous section, for $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$ we denote the homology class $\gamma^{\mathrm{s}}(\alpha)$ by $\gamma^{\mathrm{s}}_{X\to Y}(\alpha)$. Since $1\!\!1_Y \in F(Y) = \mathbb{F}(Y \to pt)$, we have

$$\alpha = \alpha \bullet 1\!\!1_Y \in F(X) = \mathbb{F}(X \to pt).$$

Hence, we get that

$$\gamma^{\mathrm{s}}_{X\to pt}(\alpha) = \gamma^{\mathrm{s}}_{X\to Y}(\alpha) \bullet \gamma^{\mathrm{s}}_{Y\to pt}(1\!\!1_Y).$$

Since the assignment $\gamma^{\mathrm{s}}_{X\to pt} : F(X) = \mathbb{F}(X \to pt) \to AH_*(X) = s\mathbb{AH}(X \to pt)$ is the Chern-Schwartz-MacPherson transformation c_*, we get that

$$c_*(\alpha) = \gamma^{\mathrm{s}}_{X\to Y}(\alpha) \bullet c_*(Y),$$

which implies, since $c_*(Y) = c(TY) \cap [Y]$, that by the definition of the product we have

$$c_*(\alpha) = f^* c(TY) \cap \gamma^{\mathrm{s}}_{X\to Y}(\alpha),$$

from which we get

$$\gamma^{\mathrm{s}}_{X\to Y}(\alpha) = f^* s(TY) \cap c_*(\alpha).$$

□

Remark (4.8). In the above proposition, the "normalization condition" cannot be weakened into the "expected" normalization condition that for a morphism $\pi : X \to pt$ of a nonsingular variety X to a point pt,

$$\gamma^{\mathrm{s}}(1\!\!1_X) = c(TX) \cap [X],$$

from which one might come to the conclusion that $\gamma^{\mathrm{s}}_{X\to pt} : \mathbb{F}(X \to pt) \to s\mathbb{AH}(X \to pt)$ has to be $c_* : F(X) \to H_*(X)$ by resolution of singularities. However, in our category only smooth morphisms of nonsingular varieties are allowed, but not any resolution of singularities.

As to the converse, we can show at least the following:

Proposition (4.9). *The "Ginzburg" transformation $\gamma^{\mathrm{Gin}} : \mathbb{F} \to s\mathbb{AH}$ defined by, for a bivariant constructible function $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$,*

$$\gamma^{\mathrm{Gin}}(\alpha) = f^* s(TY) \cap c_*(\alpha),$$

satisfies the commutativity with the pushforward operation and the pullback operation.

Proof. Since it is easy to see that it satisfies the commutativity with pushforward, we have only to show the commutativity with pullback. For a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y, \end{array}$$

and for a bivariant constructible function $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$, by the definition and from Corollary (4.4) we have

$$\begin{aligned}
\gamma^{\mathrm{Gin}}(g^*\alpha) &= \gamma^{\mathrm{Gin}}({g'}^!\alpha) \\
&= {f'}^* s(TY') \cap c_*({g'}^*\alpha) \\
&= {f'}^* s(TY') \cap {g'}^*\Big(s(TX) \cup \mathcal{D}_X^{-1}(c_*(\alpha))\Big) \cap c_*(X').
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
g^*\gamma^{\mathrm{Gin}}(\alpha) &= g^*\Big(f^*s(TY) \cap c_*(\alpha)\Big) \\
&= \mathcal{D}_{X'}{g'}^*\mathcal{D}_X^{-1}\Big(f^*s(TY) \cap c_*(\alpha)\Big) \\
&= {g'}^*\Big(f^*s(TY) \cup \mathcal{D}_X^{-1}(c_*(\alpha))\Big) \cap [X'] \\
&= \Big({g'}^*f^*s(TY) \cup {g'}^*\mathcal{D}_X^{-1}(c_*(\alpha))\Big) \cap [X'] \\
&= {g'}^*f^*s(TY) \cup \frac{{f'}^*s(TY')}{{g'}^*f^*s(TY)} \cup \frac{{f'}^*c(TY')}{{g'}^*f^*c(TY)} \cap \Big({g'}^*\mathcal{D}_X^{-1}(c_*(\alpha)) \cap [X']\Big) \\
&= {f'}^*s(TY') \cap \Big(\frac{{f'}^*c(TY')}{{g'}^*f^*c(TY)} \cap {g'}^*\mathcal{D}_X^{-1}(c_*(\alpha)) \cap [X']\Big).
\end{aligned}$$

Here we observe the following equality as virtual bundles in the K-group:

$$ {g'}^*(TX - f^*TY) = TX' - {f'}^*TY', $$

which implies the following equality of Chern classes

$$ \frac{{f'}^*c(TY')}{{g'}^*f^*c(TY)} = \frac{c(TX')}{{g'}^*c(TX')}. $$

Therefore, we get

$$\begin{aligned}
g^*\gamma^{\mathrm{Gin}}(\alpha) &= {f'}^*s(TY') \cap \Big(\frac{c(TX')}{{g'}^*c(TX')} \cap {g'}^*\mathcal{D}_X^{-1}(c_*(\alpha)) \cap [X']\Big) \\
&= {f'}^*s(TY') \cap \Big({g'}^*\big(s(TX') \cup \mathcal{D}_X^{-1}(c_*(\alpha))\big) \cap c_*(X')\Big) \\
&= \gamma^{\mathrm{Gin}}(g^*\alpha).
\end{aligned}$$

$\square$

In the proof of Proposition (4.9), in fact, the constructible function α does not need to be a bivariant constructible function in the sense of Fulton and MacPherson, but it can be any one. However, as to the commutativity with the product, one needs to solve a problem which seems to be heavily connected to a topological nature of "local Euler condition" of a bivariant constructible function. First, for bivariant constructible functions $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$ and $\beta \in \mathbb{F}(Y \xrightarrow{g} Z)$ we compute both $\gamma^{\mathrm{Gin}}(\alpha \bullet \beta)$ and $\gamma^{\mathrm{Gin}}(\alpha) \bullet \gamma^{\mathrm{Gin}}(\beta)$:

$$\begin{aligned}
\gamma^{\mathrm{Gin}}(\alpha \bullet \beta) &= \gamma^{\mathrm{Gin}}(\alpha \cdot f^*\beta) = (gf)^* s(TZ) \cap c_*(\alpha \cdot f^*\beta);\\
\gamma^{\mathrm{Gin}}(\alpha) \bullet \gamma^{\mathrm{Gin}}(\beta) &= f^*\Big(\mathcal{D}_Y^{-1}\big(g^* s(TZ) \cap c_*(\beta)\big)\Big) \cap \big(f^* s(TY) \cap c_*(\alpha)\big)\\
&= f^*\Big(g^* s(TZ) \cup \mathcal{D}_Y^{-1}(c_*(\beta))\Big) \cap \big(f^* s(TY) \cap c_*(\alpha)\big)\\
&= (gf)^* s(TZ) \cap \left(f^*\Big(s(TY) \cup \mathcal{D}_Y^{-1}(c_*(\beta))\Big) \cap c_*(\alpha)\right).
\end{aligned}$$

Therefore, it follows that $\gamma^{\mathrm{Gin}}(\alpha \bullet \beta) = \gamma^{\mathrm{Gin}}(\alpha) \bullet \gamma^{\mathrm{Gin}}(\beta)$ holds if and only if for a bivariant constructible function $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$ the following equality holds:

$$c_*(\alpha \cdot f^*\beta) = f^*\Big(s(TY) \cup \mathcal{D}_Y^{-1}(c_*(\beta))\Big) \cap c_*(\alpha). \tag{4.10}$$

Since a smooth morphism $f : X \to Y$ is an Euler morphism, i.e., $\mathbb{1}_X$ belongs to the bivariant group $\mathbb{F}(X \to Y)$, we can rewrite the formula in Corollary (4.4) as follows:

$$c_*(\mathbb{1}_X \cdot f^*\beta) = f^*\Big(s(TY) \cup \mathcal{D}_Y^{-1}(c_*(\beta))\Big) \cap c_*(\mathbb{1}_X).$$

So we make the following

Conjecture (4.11). *The above formula* (4.10) *is correct.*

Putting aside the problem of whether the above conjecture is correct or not, by introducing a new bivariant theory of constructible functions, which takes the above equality (4.10) as the definition, we will show that the Ginzburg bivariant Chern class is the unique Grothendieck transformation from this new bivariant theory of constructible functions to the bivariant theory $s\mathbb{AH}$.

Definition (4.12). For a smooth morphism $f : X \to Y$ of nonsingular varieties X, Y, we define

$$\mathbb{GF}(X \xrightarrow{f} Y)$$

to be the set of all constructible functions $\alpha \in F(X)$ satisfying the condition that for any fiber square of smooth morphisms

$$\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
Y' & \xrightarrow{g} & Y,
\end{array}$$

and for any constructible function $\beta' \in F(Y')$, the following equality holds:

$$c_*({g'}^*\alpha \cdot {f'}^*\beta') = {f'}^*\Big(s(TY') \cup \mathcal{D}_{Y'}^{-1}(c_*(\beta'))\Big) \cap c_*({g'}^*\alpha). \tag{4.12.1}$$

Theorem (4.13). *The above $\mathbb{GF}$ becomes a bivariant theory of constructible functions with the same operations as in the bivariant theory $s\mathbb{F}$ of constructible functions. Furthermore, we have that $\mathbb{GF}(X \to pt) = F(X)$.*

Proof. It suffices to show that the three operations are well defined.

(i) The well-definedness of the product $\bullet$: Let $\alpha \in \mathbb{GF}(X \xrightarrow{f} Y)$ and $\beta \in \mathbb{GF}(Y \xrightarrow{g} Z)$; we want to show that $\alpha \bullet \beta \in \mathbb{GF}(X \xrightarrow{gf} Z)$. Namely, we want to show that for the following fiber squares

$$
\begin{array}{ccc}
X' & \xrightarrow{h''} & X \\
{\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
Y' & \xrightarrow{h'} & Y \\
{\scriptstyle g'}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
Z' & \xrightarrow{h} & Z
\end{array}
\tag{4.13.1}
$$

and for any constructible function $\gamma \in F(Z')$ the following equality holds:

$$c_*({h''}^*(\alpha \bullet \beta) \cdot (g'f')^*\gamma) = (g'f')^*\Big(s(TZ') \cup \mathcal{D}_{Z'}^{-1}(c_*(\gamma))\Big) \cap c_*({h''}^*(\alpha \bullet \beta)).$$

Indeed,

$$
\begin{aligned}
&c_*({h''}^*(\alpha \bullet \beta) \cdot (g'f')^*\gamma) \\
&= c_*({h''}^*(\alpha \cdot f^*\beta) \cdot {f'}^*{g'}^*\gamma) \\
&= c_*({h''}^*\alpha \cdot {h''}^* f^*\beta \cdot {f'}^*{g'}^*\gamma) \\
&= c_*({h''}^*\alpha \cdot {f'}^*{h'}^*\beta \cdot {f'}^*{g'}^*\gamma) \quad (\text{since } {h''}^*f^*\beta = {f'}^*{h'}^*\beta\ ) \\
&= c_*({h''}^*\alpha \cdot {f'}^*({h'}^*\beta \cdot {g'}^*\gamma)) \\
&= {f'}^*\Big(s(TY') \cup \mathcal{D}_{Y'}^{-1}(c_*({h'}^*\beta \cdot {g'}^*\gamma))\Big) \cap c_*({h''}^*\alpha) \quad (\text{since } \alpha \in \mathbb{GF}(X \xrightarrow{f} Y)).
\end{aligned}
$$

Since $\beta \in \mathbb{GF}(Y \xrightarrow{g} Z)$, the above equation continues as follows:

$$
\begin{aligned}
&= {f'}^*\left(s(TY') \cup \mathcal{D}_{Y'}^{-1}\Big({g'}^*\big(s(TZ') \cup \mathcal{D}_{Z'}^{-1}(c_*(\gamma))\big) \cap c_*({h'}^*\beta)\Big)\right) \cap c_*({h''}^*\alpha) \\
&= {f'}^*\left(s(TY') \cup {g'}^*\big(s(TZ') \cup \mathcal{D}_{Z'}^{-1}(c_*(\gamma))\big) \cup \mathcal{D}_{Y'}^{-1}(c_*({h'}^*\beta))\right) \cap c_*({h''}^*\alpha) \\
&= {f'}^*{g'}^*\big(s(TZ') \cup \mathcal{D}_{Z'}^{-1}(c_*(\gamma))\big) \cap \left(\Big({f'}^*s(TY') \cup \mathcal{D}_{Y'}^{-1}(c_*({h'}^*\beta))\Big) \cap c_*({h''}^*\alpha)\right) \\
&= (g'f')^*\big(s(TZ') \cup \mathcal{D}_{Z'}^{-1}(c_*(\gamma))\big) \cap c_*({h''}^*\alpha \cdot {f'}^*{h'}^*\beta) \quad (\text{since } \alpha \in \mathbb{GF}(X \xrightarrow{f} Y)) \\
&= (g'f')^*\big(s(TZ') \cup \mathcal{D}_{Z'}^{-1}(c_*(\gamma))\big) \cap c_*({h''}^*\alpha \cdot {h''}^*f^*\beta) \quad (\text{since } {h''}^*f^*\beta = {f'}^*{h'}^*\beta\ ) \\
&= (g'f')^*\big(s(TZ') \cup \mathcal{D}_{Z'}^{-1}(c_*(\gamma))\big) \cap c_*({h''}^*(\alpha \bullet \beta)).
\end{aligned}
$$

(ii) The well-definedness of the pushforward: Let $\alpha \in \mathbb{GF}(X \xrightarrow{gf} Z)$. We want to show that $f_*\alpha \ \in \mathbb{GF}(Y \xrightarrow{g} Z)$. Namely, we want to show that for the above fiber diagram (4.13.1) and for any constructible function $\gamma \in F(Z')$ the following equality holds:

$$c_*({h'}^*f_*\alpha \cdot {g'}^*\gamma) = {g'}^*\Big(s(TZ') \cup \mathcal{D}_{Z'}^{-1}(c_*(\gamma))\Big) \cap c_*({h'}^*f_*\alpha).$$

Indeed,

$$
\begin{aligned}
& c_*(h'^* f_*\alpha \cdot g'^*\gamma) \\
& = c_*(f'_* h''^*\alpha \cdot g'^*\gamma) \quad (\text{since } h'^* f_*\alpha = f'_* h''^*\alpha) \\
& = c_*(f'_*(h''^*\alpha \cdot f'^* g'^*\gamma)) \quad (\text{by the projection formula (3.3) in §3}) \\
& = f'_* c_*(h''^*\alpha \cdot (g'f')^*\gamma) \quad (\text{by the MacPherson theorem}) \\
& = f'_*\Big((g'f')^*\big(s(TZ') \cup \mathcal{D}_{Z'}^{-1}(c_*(\gamma))\big) \cap c_*(h''^*\alpha)\Big) \quad (\text{since } \alpha \in \mathbb{GF}(X \xrightarrow{gf} Z)) \\
& = g'^*\Big(s(TZ') \cup \mathcal{D}_{Z'}^{-1}(c_*(\gamma))\Big) \cap f'_* c_*(h''^*\alpha) \quad (\text{by the projection formula}) \\
& = g'^*\Big(s(TZ') \cup \mathcal{D}_{Z'}^{-1}(c_*(\gamma))\Big) \cap c_*(f'_* h''^*\alpha) \quad (\text{by the MacPherson theorem}) \\
& = g'^*\Big(s(TZ') \cup \mathcal{D}_{Z'}^{-1}(c_*(\gamma))\Big) \cap c_*(h'^* f_*\alpha) \quad (\text{since } f'_* h''^*\alpha = h'^* f_*\alpha).
\end{aligned}
$$

(iii) The well-definedness of the pullback: This is obvious.

Finally, we want to show that $\mathbb{GF}(X \to pt) = F(X)$. For this, consider the following diagram:

$$
\begin{array}{ccc}
X \times Y & \xrightarrow{p} & X \\
{\scriptstyle q}\downarrow & & \downarrow \\
Y & \longrightarrow & pt
\end{array}
$$

where X and Y are both nonsingular varieties. We want to show that for any constructible functions $\alpha \in F(X)$ and $\beta \in F(Y)$ the following equality holds:

$$c_*(p^*\alpha \cdot q^*\beta) = q^*\Big(s(TY) \cup \mathcal{D}_Y^{-1}(c_*(\beta))\Big) \cap c_*(p^*\alpha).$$

Indeed, since $p^*\alpha \cdot q^*\beta = \alpha \times \beta$, it follows from the cross product formula of the Chern-Schwartz-MacPherson class, $c_*(\gamma \times \delta) = c_*(\gamma) \times c_*(\delta)$ ([K] and also see [KY]) that

$$c_*(p^*\alpha \cdot q^*\beta) = c_*(\alpha) \times c_*(\beta).$$

On the other hand, letting 1_W denote the Poincaré dual of the fundamental class $[W]$, we have

$$
\begin{aligned}
& q^*\Big(s(TY) \cup \mathcal{D}_Y^{-1}(c_*(\beta))\Big) \cap c_*(p^*\alpha) \\
& = \Big(1_X \times \big(s(TY) \cup \mathcal{D}_Y^{-1}(c_*(\beta))\big)\Big) \cap c_*(\alpha \times 1\!\!1_Y) \\
& = \big(1_X \cap c_*(\alpha)\big) \times \Big(\big(s(TY) \cup \mathcal{D}_Y^{-1}(c_*(\beta))\big) \cap c_*(Y)\Big) \\
& = c_*(\alpha) \times \big(\mathcal{D}_Y^{-1}(c_*(\beta)) \cap [Y]\big) \\
& = c_*(\alpha) \times c_*(\beta).
\end{aligned}
$$

□

Remark (4.14). (i) It is quite natural to think that one might be able to define another bivariant theory $\mathbb{GF}'$ of constructible functions by simply requiring that $\alpha \in \mathbb{GF}'(X \xrightarrow{f} Y)$ be defined so that α satisfies the "conjectural equality" (4.10), i.e., that for any constructible function $\beta \in F(Y)$ the following holds:

$$c_*(\alpha \cdot f^*\beta) = f^*\Big(s(TY) \cup \mathcal{D}_Y^{-1}(c_*(\beta))\Big) \cap c_*(\alpha).$$

With this definition one can see that the product operation and the pushforward operation are well defined and also that $\mathbb{GF}'(X \to pt) = F(X)$, but the well-definedness of the pullback is not clear at all. In fact, this drawback leads us to the above definition of $\mathbb{GF}$.

(ii) The above proof of Theorem (4.13) is a more direct or detailed proof, but one can give a quicker proof "bivariant-theoretically": Indeed, first we notice that $\alpha \in \mathbb{GF}(X \to Y)$ means that for the above fiber square and for any constructible function $\beta \in F(Y')$ the following holds:

$$\gamma^{\mathrm{Gin}}(g^\star \alpha \bullet \beta') = \gamma^{\mathrm{Gin}}(g^\star \alpha) \bullet \gamma^{\mathrm{Gin}}(\beta'),$$

where the morphism considered for $\gamma^{\mathrm{Gin}}(\beta')$ is of course the morphism $Y' \to pt$ and thus $\gamma^{\mathrm{Gin}}(\beta')$ is simply $c_*(\beta')$, or equivalently, for any morphism $g' : Y' \to Z'$, the equality $\gamma^{\mathrm{Gin}}(g^\star \alpha \bullet \beta') = \gamma^{\mathrm{Gin}}(g^\star \alpha) \bullet \gamma^{\mathrm{Gin}}(\beta')$ holds and in this case $\gamma^{\mathrm{Gin}}(\beta')$ is simply ${g'}^* s(TZ') \cap c_*(\beta')$. For the sake of simplicity we delete the superscript Gin. Then the well-definedness of the product can be shown as follows: We need to show that for any constructible function $\delta \in F(Z')$,

$$\gamma(h^\star(\alpha \bullet \beta) \bullet \delta) = \gamma(h^\star(\alpha \bullet \beta)) \bullet \gamma(\delta).$$

Indeed,

$$\begin{aligned}
\gamma(h^\star(\alpha \bullet \beta) \bullet \delta) &= \gamma\Big(({h'}^\star \alpha \bullet h^\star \beta) \bullet \delta\Big) \\
&= \gamma\Big({h'}^\star \alpha \bullet (h^\star \beta \bullet \delta)\Big) \\
&= \gamma({h'}^\star \alpha) \bullet \gamma(h^\star \beta \bullet \delta) \quad (\text{since } \alpha \in \mathbb{GF}(X \to Y)) \\
&= \gamma({h'}^\star \alpha) \bullet \gamma(h^\star \beta) \bullet \gamma(\delta) \quad (\text{since } \beta \in \mathbb{GF}(Y \to Z)) \\
&= \gamma({h'}^\star \alpha \bullet h^\star \beta) \bullet \gamma(\delta) \quad (\text{since } \alpha \in \mathbb{GF}(X \to Y) \text{ again}) \\
&= \gamma(h^\star(\alpha \bullet \beta)) \bullet \gamma(\delta).
\end{aligned}$$

Similarly, the other well-definednesses can be proved.

Now it is easy to see that we have the following theorem:

Theorem (4.15). *The Ginzburg transformation*

$$\gamma^{\mathrm{Gin}} : \mathbb{GF} \to s\mathbb{AH}$$

is the unique Grothendieck transformation such that it becomes the Chern-Schwartz-MacPherson class $c_* : F \to H_*$ when restricted to the morphisms to a point.

Remark (4.16). We note that Conjecture (4.11) means that $\mathbb{F}(X \to Y) \subset \mathbb{GF}(X \to Y)$, and also that if $f : X \to Y$ is a cellular morphism with Y being nonsingular, then we have

$$\mathbb{F}(X \xrightarrow{f} Y) \subset \mathbb{GF}(X \xrightarrow{f} Y),$$

which follows from the Brasselet theorem (i.e., Theorem (3.5)) and our previous uniqueness theorem [Y5].

Remark (4.17). In [Sch1] J. Schürmann has studied characteristic classes in a more micro-local analytic approach and has obtained interesting results; for instance, he gave a solution to a conjecture posed by Ginzburg in [G1]. For a further study on Ginzburg's bivariant Chern class, see [Y6], [Y7]; see also [Sch3] for a further study from a more general viewpoint.

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