TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 355, Number 6, Pages 2305–2325
S 0002-9947(03)03253-7
Article electronically published on January 31, 2003

FOUR-WEIGHT SPIN MODELS AND JONES PAIRS

ADA CHAN, CHRIS GODSIL, AND AKIHIRO MUNEMASA

Abstract. We introduce and discuss Jones pairs. These provide a generalization and a new approach to the four-weight spin models of Bannai and Bannai. We show that each four-weight spin model determines a "dual" pair of association schemes.

1. Jones Pairs

The space of $k \times k$ matrices acts on itself in three distinct ways: if $C \in M_k(\mathbb{F})$, we can define endomorphisms X_C, Δ_C and Y_C by

$$X_C(M) = CM, \qquad \Delta_C(M) = C \circ M, \qquad Y_C(M) = MC^T.$$

If A and B are $k \times k$ matrices, we say (A, B) is a *one-sided Jones pair* if X_A and Δ_B are invertible and

$$X_A \Delta_B X_A = \Delta_B X_A \Delta_B. \tag{1.1}$$

We call this the *braid relation* for reasons that will become clear as we proceed. We note that X_A is invertible if and only if A is, and Δ_B is invertible if and only if the Schur inverse $B^{(-)}$ is defined. (Recall that if B and C are matrices of the same order, then their Schur product $B \circ C$ is defined by the condition

$$(B \circ C)_{i,j} = B_{i,j} C_{i,j}$$

and $B \circ B^{(-)} = J$.) We will see that each one-sided Jones pair determines a representation of the braid group B_3, and this is one of the reasons we are interested in Jones pairs. The pair (I, J) forms a trivial but useful example. A one-sided Jones pair (A, B) is *invertible* if $A^{(-)}$ and B^{-1} both exist.

We observe that X_A and Y_A commute and that

$$Y_A \Delta_B Y_A = \Delta_B Y_A \Delta_B \tag{1.2}$$

if and only if (A, B^T) is a one-sided Jones pair. A pair of matrices (A, B) is a *Jones pair* if both (A, B) and (A, B^T) are one-sided Jones pairs. From a Jones pair we obtain a family of representations of the braid groups B_r, for all r.

In this paper we describe the basic theory of Jones pairs. We find that if $A^{(-)}$ exists, then $(A, A^{(-)})$ is a Jones pair if and only if A is a spin model in the sense of

Received by the editors November 9, 2001.
2000 *Mathematics Subject Classification.* Primary 05E30; Secondary 20F36.
Support from a National Sciences and Engineering Council of Canada operating grant is gratefully acknowledged by the second author.

Jones [10], and that invertible Jones pairs correspond to the 4-weight spin models due to Bannai and Bannai [1]. The theory we develop includes the basic theory of these spin models. Finally, we prove that each invertible Jones pair (A, B) of $n \times n$ matrices determines a dual pair of association schemes, one built from A and the other from B.

2. Representations of the Braid Group

The braid group B_n on n strands has $n-1$ generators $\sigma_1, \dots, \sigma_{n-1}$ that satisfy the relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

and, if $|i-j| > 1$,

$$\sigma_i \sigma_j = \sigma_j \sigma_i.$$

We will show how we can use Jones pairs to construct a class of representations of the braid group and, in certain cases, obtain invariants of oriented links. These constructions are due to Jones [10], who did not feel the need to write out proofs. We did, and so we record them here. Nonetheless, these proofs play no role in the sections that follow (and there will be no exam).

Let $V^{\otimes m}$ denote the tensor product of m copies of the n-dimensional vector space V. Let (A, B) be a pair of $n \times n$ matrices. Let e_i denote the ith standard basis vector of $\mathbb{F}^n$ and let $E_{i,j}$ be the matrix $e_i e_j^T$. Let g_{2i-1} be the element of $\mathrm{End}(V^{\otimes m})$ obtained by applying A to the ith tensor factor of $V^{\otimes m}$. We let g_{2i} denote the element of $\mathrm{End}(V^{\otimes m})$ such that

$$g_{2i}(e_{r_1} \otimes \cdots \otimes e_{r_m}) = B_{r_i, r_{i+1}} (e_{r_1} \otimes \cdots \otimes e_{r_m}).$$

We remark that (A, B) is a one-sided Jones pair if and only if the map sending σ_1 to g_1 and σ_2 to g_2 determines a representation of B_3. Our next result is from Jones [10, Section 3.3].

2.1. **Lemma.** *Let (A, B) be a pair of $n \times n$ matrices and let the endomorphisms g_i be defined as above. If $r \geq 4$ and $m \geq \lfloor (r+1)/2 \rfloor$, the following are equivalent.*

(i) *(A, B) is a Jones pair.*

(ii) *The mapping that assigns σ_i in B_r to g_i defines a representation of B_r in* $\mathrm{End}(V^{\otimes m})$. □

Proof. It suffices to show that, for each positive integer $r \geq 4$, we have

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad (i = 1, \dots, r-2) \tag{2.1}$$

if and only if (1.1) and (1.2) hold. By the definition of the action of g_i, we see that the equations (2.1) are equivalent to

$$g_1 g_2 g_1 = g_2 g_1 g_1, \quad g_2 g_3 g_2 = g_3 g_2 g_3 \tag{2.2}$$

on $V \otimes V$. If we identify $V \otimes V$ with $M_k(\mathbb{F})$ via the map $\phi : e_i \otimes e_j \mapsto E_{ij}$, then

$$\begin{aligned}
\phi g_1(e_i \otimes e_j) &= \phi(Ae_i \otimes e_j) \\
&= \phi(\sum_{h=1}^{n} A_{hi} e_h \otimes e_j) \\
&= \sum_{h=1}^{n} A_{hi} E_{hj} \\
&= AE_{ij} \\
&= X_A(E_{ij}), \\
\phi g_2(e_i \otimes e_j) &= \phi(B_{ij} e_i \otimes e_j) \\
&= B_{ij} E_{ij} \\
&= \Delta_B(E_{ij}), \\
\phi g_3(e_i \otimes e_j) &= \phi(e_i \otimes Ae_j) \\
&= \phi(\sum_{h=1}^{n} A_{hj} e_i \otimes e_h) \\
&= \sum_{h=1}^{n} A_{hj} E_{ih} \\
&= \sum_{h=1}^{n} A^T_{jh} E_{ih} \\
&= E_{ij} A^T \\
&= Y_A(E_{ij}).
\end{aligned}$$

Therefore, we have shown that

$$g_1 = \phi^{-1} X_A \phi, \quad g_2 = \phi^{-1} \Delta_B \phi, \quad g_3 = \phi^{-1} Y_A \phi,$$

and hence the equations (2.2) are equivalent to (1.1) and (1.2). □

Although representations of the braid group are interesting in their own right, the work we have discussed is motivated by the relation with link invariants. The following lemma gives a sufficient condition for a Jones pair to yield a link invariant. (We comment on the constraints on A and B following the proof of Lemma 7.2.)

2.2. **Lemma.** *Let (A, B) be a Jones pair of $n \times n$ matrices such that we have*

$$A_{i,i} = (A^{-1})_{i,i} = 1/\sqrt{n}$$

and

$$\sum_{j=1}^{n} B_{ij} = \sum_{j=1}^{n} B^{-1}_{ij} = \sqrt{n},$$

for $i = 1, \dots, n$. If $g_1, \dots, g_{r-1}$ are defined as above and $h \in \langle g_1, \dots, g_{r-2} \rangle$, then

$$\operatorname{tr}(h g_{r-1}^{\pm 1}) = \frac{1}{n} \operatorname{tr}(h) \operatorname{tr}(A).$$

Proof. Let $r' = \lfloor (r+1)/2 \rfloor$ and let $\mathcal{V}$ denote the tensor product of r' copies of V. Suppose first that r is even. If $\alpha \in \{1,\dots,n\}^{r'-1}$, denote by e_α the vector $e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_{r'-1}}$. Then the set

$$\{e_\alpha \otimes e_i \mid \alpha \in \{1,\dots,n\}^{r'-1},\ 1 \le i \le n\}$$

forms a basis of $\mathcal{V}$. Since h leaves the subspace

$$\langle e_\alpha \otimes e_i \mid \alpha \in \{1,\dots,n\}^{r'-1} \rangle$$

invariant, we can write

$$h(e_\alpha \otimes e_i) = \sum_\beta h^i_{\beta,\alpha} e_\beta \otimes e_i,$$

where $h^i_{\beta,\alpha} \in \mathbb{F}$. Now we have

$$\operatorname{tr}(h) = \sum_{i=1}^n \sum_\alpha h^i_{\alpha,\alpha}.$$

Since

$$\begin{aligned} hg^{\pm 1}_{r-1}(e_\alpha \otimes e_i) &= h(e_\alpha \otimes \sum_{j=1}^n (A^{\pm 1})_{ji} e_j) \\ &= \sum_{j=1}^n (A^{\pm 1})_{ji} \sum_\beta h^j_{\beta,\alpha} e_\beta \otimes e_j, \end{aligned}$$

we have

$$\begin{aligned} \operatorname{tr}(hg^{\pm 1}_{r-1}) &= \sum_{i=1}^n \sum_\alpha (A^{\pm 1})_{ii} h^i_{\alpha,\alpha} \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \sum_\alpha h^i_{\alpha,\alpha} \\ &= \frac{1}{n} \operatorname{tr}(h) \operatorname{tr}(A). \end{aligned}$$

This proves the lemma when r is even.

Next suppose r is odd. For $\alpha \in \{1,\dots,n\}^{r'-2}$, we denote by e_α the vector $e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_{r'-2}}$. Then the set

$$\{e_\alpha \otimes e_i \otimes e_j \mid \alpha \in \{1,\dots,n\}^{r'-2},\ 1 \le i,j \le n\}$$

forms a basis of $\mathcal{V}$. Since h acts as the identity on the last tensor factor of $\mathcal{V}$, we can write

$$h(e_\alpha \otimes e_i \otimes e_j) = \sum_{k=1}^n \sum_\beta h_{(\beta,k),(\alpha,i)} e_\beta \otimes e_k \otimes e_j,$$

where $h_{(\beta,k),(\alpha,i)} \in \mathbb{F}$ are independent of j. Now we have

$$\operatorname{tr}(h) = n \sum_{i=1}^n \sum_\alpha h_{(\alpha,i),(\alpha,i)}.$$

Since

$$\begin{aligned} hg_{r-1}^{\pm 1}(e_\alpha \otimes e_i \otimes e_j) &= B_{ij}^{\pm 1} h(e_\alpha \otimes e_i \otimes e_j) \\ &= B_{ij}^{\pm 1} \sum_{k=1}^{n} \sum_{\beta} h_{(\beta,k),(\alpha,i)} e_\beta \otimes e_k \otimes e_j, \end{aligned}$$

we have

$$\operatorname{tr}(hg_{r-1}^{\pm 1}) = \sum_{i,j=1}^{n} \sum_{\alpha} B_{ij}^{\pm 1} h_{(\alpha,i),(\alpha,i)} \tag{2.3}$$

$$= \sum_{i=1}^{n} \sum_{\alpha} h_{(\alpha,i),(\alpha,i)} \sum_{j=1}^{n} B_{ij}^{\pm 1}. \tag{2.4}$$

Given the constraints on B and $B^{(-)}$, we see that the lemma holds when r is odd. □

3. Further Properties

We develop some basic properties of one-sided Jones pairs. We begin with an alternative form of the definition.

Equation (1.1) is equivalent to the condition that, for all matrices M in $M_k(\mathbb{F})$,

$$A(B \circ (AM)) = B \circ (A(B \circ M)). \tag{3.1}$$

If M and N are elements of $M_k(\mathbb{F})$, then $\operatorname{tr}(M^T N)$ is a nondegenerate bilinear form on $M_k(\mathbb{F})$. If $Y \in \operatorname{End}(M_k(\mathbb{F}))$, denote its adjoint relative to this form by Y^T, and call it the transpose of Y. It is easy to verify that

$$X_A^T = X_{A^T}, \qquad \Delta_B^T = \Delta_B.$$

3.1. **Lemma.** *If (A,B) is a one-sided Jones pair, then so are*

(a) (A^T, B),
(b) $(A^{-1}, B^{(-)})$,
(c) $(D^{-1}AD, B)$, *where D is invertible and diagonal,*
(d) (A, BP), *where P is a permutation matrix,*
(e) $(P^{-1}AP, P^{-1}BP)$, *where P is a permutation matrix,*
(f) $(\lambda A, \lambda B)$, *for any nonzero λ in $\mathbb{F}$.*

Proof. The first claim follows by taking the transpose of equation (1.1), and the second by taking the inverse and noting that

$$X_A^{-1} = X_{A^{-1}}, \qquad \Delta_B^{-1} = \Delta_{B^{(-)}}.$$

If D is diagonal and invertible, then

$$X_{D^{-1}} X_C X_D = X_{D^{-1}CD}, \qquad X_{D^{-1}} \Delta_C X_D = \Delta_C.$$

Then (c) follows by conjugating (1.1) above by X_D.

Next, if P is a permutation matrix, then

$$(BP) \circ M = (B \circ MP^{-1})P$$

and

$$(B \circ C)P = BP \circ CP;$$

hence,

$$\begin{aligned} A(BP \circ (AM)) &= A(B \circ (AMP^{-1}))P \\ &= (B \circ A(B \circ (MP^{-1})))P \\ &= BP \circ A(B \circ (MP^{-1}))P \\ &= BP \circ A(BP \circ (M)). \end{aligned}$$

We leave (e) as an exercise, while (f) is truly trivial. □

4. Eigenvectors

Note that we can rewrite the braid relation in the equivalent form

$$\Delta_B^{-1} X_A \Delta_B = X_A \Delta_B X_A^{-1},$$

from which we see that the endomorphisms X_A and Δ_B are similar. Since

$$\Delta_B E_{i,j} = B_{i,j} E_{i,j},$$

$E_{i,j}$ is an eigenvector for the operator Δ_B, with eigenvalue $B_{i,j}$. Since X_A and Δ_B are similar, we conclude that X_A is diagonalizable, and its eigenvalues are the entries of B.

4.1. **Lemma.** *The matrices A and B form a one-sided Jones pair if and only if for all i and j we have*

$$A(Ae_i \circ Be_j) = B_{i,j}(Ae_i \circ Be_j).$$

Proof. Put $M = E_{i,j}$ in the second form, (3.1), of the braid relation. This yields

$$A(B \circ (AE_{i,j})) = B \circ (A(B \circ E_{i,j})). \tag{4.1}$$

Observe that

$$B \circ (AE_{i,j}) = B \circ (Ae_i e_j^T) = (Ae_i \circ Be_j)e_j^T$$

and $B \circ E_{i,j} = B_{i,j} E_{i,j}$. Therefore,

$$B \circ (A(B \circ E_{i,j})) = B_{i,j} B \circ (AE_{i,j}) = B_{i,j}(Ae_i \circ Be_j)e_j^T$$

and hence the lemma follows from (4.1). □

4.2. **Lemma.** *If (A, B) is a one-sided Jones pair, then B has constant column sum.*

Proof. We have

$$A(Ae_i \circ Be_j) = B_{i,j}(Ae_i \circ Be_j).$$

Since $B^{(-)}$ exists and A is invertible, the vectors

$$Ae_r \circ Be_j, \quad r = 1, \ldots, n,$$

are linearly independent, and therefore they form a basis of $\mathbb{F}^n$ consisting of eigenvectors of A. Consequently, the entries $B_{r,j}$ in the jth column of B provide a complete list of the eigenvalues of A. Thus each column of B sums to $\mathrm{tr}(A)$. □

It follows from this lemma that if (A, B^T) is a Jones pair, then the row sums of B are also constant. The argument used will also show that, if (A, B) is a one-sided Jones pair and $A^{(-)}$ and B^{-1} exist, then the row sums of B are constant.

4.3. **Lemma.** *Let (A, B) be a pair of $n \times n$ matrices and let D_j be the diagonal matrix whose rth diagonal entry is $B_{r,j}$. Then (A, B) is a one-sided Jones pair if and only if*

$$AD_jA = D_jAD_j$$

for $j = 1, \dots, n$.

Proof. By Lemma 4.1 we have that (A, B) is a one-sided Jones pair if and only if

$$A\, Ae_i \circ Be_j = B_{i,j}\, Ae_i \circ Be_j$$

for all i and j. Since $Be_j \circ x = D_jx$, this is equivalent to

$$AD_j\, Ae_i = B_{i,j}D_jAe_i,$$

from which the lemma follows. □

A representation of B_3 over $\mathbb{F}$ is given by assigning invertible $n \times n$ matrices A_1 and A_2 over $\mathbb{F}$ to the generators σ_1 and σ_2 of B_3, such that the braid condition holds:

$$A_1A_2A_1 = A_2A_1A_2.$$

It follows from Lemma 4.3 that if (A, B) is a one-sided Jones pair and D_j is the diagonal matrix whose rth diagonal entry is $B_{r,j}$, then the pair of matrices (A, D_j) provide a representation of B_3.

Conversely, suppose A and D are invertible matrices that provide a representation of B_3, and that D is diagonal. Set $B = DJ$. Then $(DJ) \circ M = DM$ for any matrix M; therefore,

$$A(B \circ (AM)) = A(DJ \circ (AM)) = ADAM = DADM = B \circ (A(B \circ M)).$$

Thus, (A, D) determines a representation of B_3 if and only if (A, DJ) is a one-sided Jones pair. To get an invertible one-sided Jones pair containing A, we require n linearly independent diagonal matrices D_i such that (A, D_i) gives a representation of B_3 for $i = 1, \dots, n$.

5. The Exchange Lemma

It is trivially true that, if $Ae_i \circ Be_j$ is an eigenvector for A, then so is $Be_j \circ Ae_i$. Somewhat surprisingly, this yields a very useful "Exchange Lemma".

5.1. **Lemma.** *If A, B, C and Q, R, S are elements of $M_k(\mathbb{F})$, then*

$$X_A\Delta_BX_C = \Delta_QX_R\Delta_S$$

if and only if

$$X_A\Delta_CX_B = \Delta_RX_Q\Delta_{S^T}.$$

Proof. The first relation holds if and only if

$$A(B \circ (CE_{i,j})) = Q \circ (R(S \circ E_{i,j})). \tag{5.1}$$

We have

$$B \circ (CE_{i,j}) = (Ce_i \circ Be_j)e_j^T$$

and, therefore,

$$A(B \circ (CE_{i,j})) = A(Ce_i \circ Be_j)e_j^T.$$

Furthermore,

$$Q \circ (R(S \circ E_{i,j})) = S_{i,j}(Re_i \circ Qe_j)e_j^T.$$

Consequently, our first relation is equivalent to the system of equations

$$A(Ce_i \circ Be_j) = S_{i,j}(Re_i \circ Qe_j).$$

To complete the argument, note that

$$A(Ce_i \circ Be_j)e_i^T = A(C \circ (BE_{j,i}))$$

and

$$S_{i,j}(Re_i \circ Qe_j)e_i^T = S_{i,j}(R \circ QE_{j,i}) = R \circ Q(S^T \circ E_{j,i}).$$

We conclude that (5.1) holds if and only if

$$A(C \circ (BE_{j,i})) = R \circ Q(S^T \circ E_{j,i}),$$

which is equivalent to the second of the two relations in the lemma. □

We note one corollary of this lemma.

5.2. **Corollary.** *Let A and B be matrices such that A^{-1} and $B^{(-)}$ exist. Then (A, B) is a one-sided Jones pair if and only if*

$$X_A \Delta_A X_B = \Delta_A X_B \Delta_{B^T}.$$

Proof. Apply the exchange lemma to the braid relation $X_A \Delta_B X_A = \Delta_B X_A \Delta_B$. □

This immediately yields the following:

5.3. **Lemma.** *Let A and B be invertible and Schur invertible matrices. Then (A, B) is a one-sided Jones pair if and only if*

$$\Delta_{A^{(-)}} X_A \Delta_A = X_B \Delta_{B^T} X_{B^{-1}}.$$

6. Duality

Let A and B be two $n \times n$ matrices. We define $\mathcal{N}_{A,B}$ to be the space of $n \times n$ matrices such that $Ae_i \circ Be_j$ is an eigenvector for all i and j; this is an algebra under matrix multiplication. We call it the *Nomura algebra* of the pair (A, B). If $R \in \mathcal{N}_{A,B}$ and S is the $n \times n$ matrix such that

$$R\, Ae_i \circ Be_j = S_{i,j}\, Ae_i \circ Be_j,$$

then we denote S by $\Theta_{A,B}(R)$. We denote the image of $\mathcal{N}_{A,B}$ under $\Theta_{A,B}$ by $\mathcal{N}'_{A,B}$, and observe that this is a commutative algebra under Schur multiplication. In all cases, $I \in \mathcal{N}_{A,B}$ and $J = \Theta_{A,B}(I) \in \mathcal{N}'_{A,B}$. If (A, B) is a one-sided Jones pair, then $\mathcal{N}_{A,B}$ contains all polynomials in A and $\mathcal{N}'_{A,B}$ contains all "Schur polynomials" in B.

6.1. **Lemma.** *Let (A, B) be a one-sided Jones pair and let Λ denote the operator $X_A \Delta_B X_A = \Delta_B X_A \Delta_B$. Then Λ^2 commutes with X_A and Δ_B. In addition, $\Lambda^{-1} X_A \Lambda = \Delta_B$ and $\Lambda^{-1} \Delta_B \Lambda = X_A$.*

Proof. For the first claim, note that

$$\Lambda^2 = (X_A \Delta_B)^3 = (X_A \Delta_B X_A)^2,$$

whence Λ^2 commutes with $X_A \Delta_B$ and $X_A \Delta_B X_A$. So Λ^2 commutes with X_A and Δ_B. Next

$$X_A \Lambda = X_A \Delta_B X_A \Delta_B = \Lambda \Delta_B$$

and

$$\Delta_B \Lambda = \Delta_B X_A \Delta_B X_A = \Lambda X_A.$$ □

We cannot prove that conjugation by Λ swaps $\mathcal{N}_{A,B}$ and $\mathcal{N}'_{A,B}$, but the following is a very useful consolation.

6.2. **Theorem.** *Let (A, B) be a pair of $n \times n$ matrices. Then $R \in \mathcal{N}_{A,B}$ and $\Theta_{A,B}(R) = S$ if and only if $X_R \Delta_B X_A = \Delta_B X_A \Delta_S$. If A is invertible and B is Schur invertible, then $\Theta_{A,B}$ is an isomorphism.*

Proof. We have

$$X_R \Delta_B X_A (E_{i,j}) = R\,(Ae_i \circ Be_j)\, e_j^T$$

and

$$\Delta_B X_A \Delta_S (E_{i,j}) = S_{i,j}\,(Ae_i \circ Be_j)\, e_j^T,$$

from which the first claim follows.

Since Δ_B and Δ_S commute, we see that

$$X_R \Delta_B X_A = \Delta_B X_A \Delta_S$$

if and only if

$$X_R \Delta_B X_A \Delta_B = \Delta_B X_A \Delta_B \Delta_S,$$

i.e., if and only if $X_R \Lambda = \Lambda \Delta_S$. If A^{-1} and $B^{(-)}$ exist, then Λ is invertible, whence it follows that $\Theta_{A,B}$ is an isomorphism. □

Since $\mathcal{N}'_{A,B}$ is commutative, we see that $\mathcal{N}_{A,B}$ is a commutative algebra. We list three equivalent forms of the first part of this theorem for later use.

6.3. **Corollary.** *If $R \in \mathcal{N}_{A,B}$ and $\Theta_{A,B}(R) = S$, then*

(a) $X_{B^T} \Delta_A X_{R^T} = \Delta_{S^T} X_{B^T} \Delta_A$,
(b) $\Delta_R X_{B^{(-)}} \Delta_{B^T} = X_A \Delta_{A^{-1}} X_S$,
(c) $\Delta_{B^T} X_{B^{(-)T}} \Delta_R = X_{S^T} \Delta_{A^{-1}} X_{A^T}$.

Proof. We have

$$X_R \Delta_B X_A = \Delta_B X_A \Delta_S, \tag{6.1}$$

and by using the exchange identity,

$$X_R \Delta_A X_B = \Delta_A X_B \Delta_{S^T}.$$

Taking the transpose of each side, we get

$$X_{B^T} \Delta_A X_{R^T} = \Delta_{S^T} X_{B^T} \Delta_A,$$

which yields (a).

Next rewrite (6.1) as

$$\Delta_{B^{(-)}} X_R \Delta_B = X_A \Delta_S X_{A^{-1}}$$

and apply the exchange lemma to get (b). Taking the transpose of each side yields (c). □

7. Type II Matrices

An $n \times n$ matrix A is a *type II matrix* if $A^{(-)}$ exists and

$$AA^{(-)T} = nI.$$

Hadamard matrices provide one important class of examples. When discussing type II matrices we will assume implicitly that n is coprime to the characteristic of our underlying field $\mathbb{F}$. The algebra $\mathcal{N}_{A,A^{(-)}}$ is known as the *Nomura algebra* of A. We will usually denote it by $\mathcal{N}_A$ (rather than $\mathcal{N}_{A,A^{(-)}}$), and similarly we abbreviate $\Theta_{A,A^{(-)}}$ to Θ_A.

We recall that a one-sided Jones pair (A, B) is *invertible* if $A^{(-)}$ and B^{-1} both exist. Somewhat surprisingly, invertibility implies that A and B are type II matrices.

7.1. **Theorem.** *Suppose (A, B) is a one-sided Jones pair. If B^{-1} exists, then the diagonal of A is constant and A and B are type II matrices.*

Proof. We use Corollary 6.3, which implies that, since $A \in \mathcal{N}_{A,B}$ and $\Theta_{A,B}(A) = B$,

$$\Delta_{B^T} X_{B^{(-)T}} \Delta_A = X_{B^T} \Delta_{A^{-1}} X_{A^T}.$$

Apply both sides to I. On the left we get

$$\begin{aligned}\Delta_{B^T} X_{B^{(-)T}} \Delta_A(I) &= B^T \circ (B^{(-)T}(A \circ I)) \\ &= (B^T \circ B^{(-)T})(A \circ I) \\ &= J(A \circ I)\end{aligned}$$

and, on the right,

$$X_{B^T} \Delta_{A^{-1}} X_{A^T}(I) = B^T(A^{-1} \circ A^T).$$

Therefore,

$$J(A \circ I) = B^T(A^{-1} \circ A^T).$$

Since (A, B) is a one-sided Jones pair, $B^T\mathbf{1} = \beta\mathbf{1}$, where $\beta = \operatorname{tr}(A)$. If B is invertible, then $\beta \neq 0$ and we have

$$A^{-1} \circ A^T = (B^T)^{-1} J(A \circ I) = \beta^{-1} J(A \circ I).$$

The sum of the entries in the ith column of $A^{-1} \circ A^T$ is

$$\sum_r (A^{-1})_{r,i}(A^T)_{r,i} = \sum_r (A^{-1})_{r,i} A_{i,r} = 1,$$

from which it follows that all columns of $J(A \circ I)$ are equal. Therefore, the diagonal of A is constant, and so $A^{-1} \circ A^T$ is a multiple of J. Since the columns of $A^{-1} \circ A^T$ sum to 1, we conclude that $nA^{-1} \circ A^T = J$, and therefore A is a type II matrix.

Now, we prove that B is type II. We know that A and B are both invertible and Schur invertible. So, from Lemma 5.3,

$$\Delta_{A^{(-)}} X_A \Delta_A = X_B \Delta_{B^T} X_{B^{-1}}.$$

Applying each side to I, we find that

$$A^{(-)} \circ (A(A \circ I)) = B(B^T \circ B^{-1}).$$

Since the diagonal of A is constant, this implies that $B(B^T \circ B^{-1})$ is a multiple of J; hence,

$$B^T \circ B^{-1} = cB^{-1}J,$$

for some c. Here we can write the right side as DJ, where D is diagonal. However, arguing as before, all rows of $B^T \circ B^{-1}$ sum to 1, and it follows that B is type II. □

7.2. **Lemma.** *If (A,B) is a Jones pair and $A^{(-)}$ exists, then B^{-1} exists (and A and B are type II matrices).*

Proof. We use Corollary 6.3(b) with $A = R$ and $B = S$ to get

$$\Delta_A X_{B^{(-)}} \Delta_{B^T} = X_A \Delta_{A^{-1}} X_B.$$

Applying each side of this to J yields

$$A \circ (B^{(-)} B^T) = A(A^{-1} \circ (BJ)).$$

Since (A,B) is a Jones pair, the row sums of B are all equal. So the right side here is a multiple of I. Since the diagonal of $B^{(-)}B^T$ is constant and $A^{(-)}$ exists, $B^{(-)}B^T$ must be a scalar matrix. Therefore, B is invertible (and type II). □

It follows that if (A,B) is an invertible Jones pair, then all rows and all columns of B have the same sum. From the proof of Lemma 4.2, this sum is $\mathrm{tr}(A)$ and, from Theorem 7.1, if B is invertible, then the diagonal of A is constant. From Lemma 3.1(b), we know that $(A^{-1}, B^{(-)})$ is a one-sided Jones pair if (A,B) is. Therefore, by Lemma 3.1(b), we conclude that if (A,B) is an invertible Jones pair then there is a nonzero scalar λ such that the pair $(\lambda A, \lambda B)$ satisfies the conditions of Lemma 2.2.

The next result is due to Jaeger, Matsumoto and Nomura [7]. We include a short new proof of it here, using the exchange lemma.

7.3. **Lemma.** *Let A be a type II matrix. Then we have $R \in \mathcal{N}_A$ if and only if $\Theta_A(R) \in \mathcal{N}_{A^T}$. Furthermore, if $R \in \mathcal{N}_A$, then $\Theta_{A^T}(\Theta_A(R)) = nR^T$.*

Proof. Applying Corollary 6.3(b) (with $B = A^{(-)}$) we have that

$$X_A \Delta_{A^{-1}} X_S = \Delta_R X_A \Delta_{A^{(-)T}}$$

and, therefore,

$$X_S \Delta_{A^T} X_{A^{-1}} = \Delta_{A^{-1(-)}} X_{A^{-1}} \Delta_R. \tag{7.1}$$

Since A is type II, we have $nA^{-1} = A^{(-)T}$, and so (7.1) yields that $S \in \mathcal{N}_{A^{-1}}$ and $\Theta_{A^{-1}}(S) = nR$. Finally, $\mathcal{N}_{A^{-1},A^{-1(-)}} = \mathcal{N}_{A^{(-)T},A^T}$ and, therefore, $\Theta_{A^{-1}}(S) = \Theta_{A^T}(S)^T$. □

Since A^T is type II if A is, this lemma implies that Θ_{A^T} maps $\mathcal{N}_{A^T}$ into $\mathcal{N}_A$. If we apply the exchange lemma to the transpose of (7.1), we see that $R^T \in \mathcal{N}_A$ and $\Theta_A(R^T) = \Theta_A(R)^T$. Therefore, $\mathcal{N}_A$ and $\mathcal{N}_{A^T}$ are closed under transposes. Since

$$\Theta_A(Q) \circ \Theta_A(R) = \Theta_A(QR),$$

we also find that $\mathcal{N}_{A^T}$ is closed under the Schur product, as well as under multiplication. Since I and J both lie in $\mathcal{N}_{A^T}$, it follows that $\mathcal{N}_{A^T}$ is the Bose-Mesner algebra of an association scheme. Since $\mathcal{N}_A$ is the image of $\mathcal{N}_{A^T}$ under Θ_{A^T}, it too is Schur-closed and is the Bose-Mesner algebra of a second association scheme. (These schemes necessarily form a dual pair, but we do not stop to discuss this.)

7.4. **Lemma.** *Let (A,B) be a pair of type II matrices of the same order. Suppose the diagonal of A is constant, and all row sums of B are equal. If $A \in \mathcal{N}_{A,B}$, then there is a scalar c such that (A, cB) is a one-sided Jones pair.*

Proof. If A^{-1} exists and $A \in \mathcal{N}_{A,B}$, then $A^{-1} \in \mathcal{N}_{A,B}$. Hence there is a matrix S such that

$$X_{A^{-1}} \Delta_B X_A = \Delta_B X_A \Delta_S$$

and from the exchange lemma it follows that

$$X_{A^{-1}} \Delta_A X_B = \Delta_A X_B \Delta_{S^T}.$$

If we apply both sides of this equality to J, we find that

$$A^{-1}(A \circ (BJ)) = A \circ (BS^T).$$

Suppose $BJ = \beta J$. Then the left side above is equal to βI, and, therefore, $\beta A^{(-)} \circ I = BS^T$. If $A \circ I = \alpha I$, this shows that

$$\Theta_{A,B}(A^{-1}) = \frac{\beta}{\alpha} B^{-T} = \frac{\beta}{n\alpha} B^{(-)}$$

and, consequently,

$$\Theta_{A,B}(A) = \frac{n\alpha}{\beta} B.$$

This implies that

$$X_A \Delta_B X_A = \frac{n\alpha}{\beta} \Delta_B X_A \Delta_B,$$

from which the result follows (with $c = n\alpha\beta^{-1}$). □

If, in the context of this lemma, we set B equal to $A^{(-)}$, then $\mathcal{N}_{A,B}$ is the Bose-Mesner algebra of an association scheme. So the assumption $A \in \mathcal{N}_{A,B}$ implies that the diagonal of A is constant and the column sums of B are equal. Therefore, Lemma 7.4 extends Proposition 9 of Jaeger, Matsumoto and Nomura [7]. (This proposition asserts that cA is a spin model for some nonzero c if and only if $A \in \mathcal{N}_A$.)

8. Gauge Equivalence

We show here that each member of an invertible Jones pair almost determines the other.

If D is an invertible diagonal matrix, we call $D^{-1}JD$ a *dual permutation matrix*. We note that if $A^{(-)} \circ C = D^{-1}JD$, then

$$C = A \circ (D^{-1}JD) = D^{-1}AD.$$

Thus, $A^{(-)} \circ C$ is a dual permutation matrix if and only if A and C are diagonally equivalent. The Schur inverse of a dual permutation matrix is a dual permutation matrix. (The concept of dual permutation matrix comes from Jaeger and Nomura [8].)

8.1. **Lemma.** *If A, C and M are Schur invertible matrices and $X_A \Delta_M = \Delta_M X_C$, then $C^{(-)} \circ A$ is a dual permutation matrix. If B, C and M are invertible matrices and $\Delta_B X_M = X_M \Delta_C$, then CB^{-1} is a permutation matrix.*

Proof. Suppose $X_A \Delta_M = \Delta_M X_C$. Applying each side of this equality to $E_{i,j}$ yields

$$M_{i,j}\, Ae_i = Ce_i \circ Me_j$$

and, since $C^{(-)}$ exists, we get

$$Me_j = M_{i,j}(C^{(-)} \circ A)e_i. \tag{8.1}$$

Therefore, each column of $(C^{(-)} \circ A)$ is a multiple of Me_1, and so $C^{(-)} \circ A = D_1 J D_2$, for some invertible diagonal matrices D_1 and D_2. Next, equation (8.1) implies that

$$M_{i,j} = e_i^T M e_j = M_{i,j}\, e_i^T (C^{(-)} \circ A) e_i$$

and, since $M_{i,j} \neq 0$, this implies that $e_i^T (C^{(-)} \circ A) e_i = 1$. Hence, $D_1 D_2 = I$ and $C^{(-)} \circ A$ is a dual permutation matrix.

Now, suppose $\Delta_B X_M = X_M \Delta_C$. Then we get

$$M e_i \circ B e_j = C_{i,j}\, M e_i;$$

so, if we multiply both sides of this by $(B^{-1})_{j,k}$ and sum over j, we get

$$M_{k,i} e_k = M e_i \circ e_k = \left(\sum_k C_{i,j} (B^{-1})_{j,k} \right) M e_i = (CB^{-1})_{i,k}\, M e_i. \tag{8.2}$$

This implies that each column of M is a multiple of some vector e_r. Since M is invertible, no column is zero; so we also see that, for the value of i there is at most one index k such that $(CB^{-1})_{i,k} \neq 0$. If $M_{k,i} \neq 0$, then (8.2) gives

$$M_{k,i} = (CB^{-1})_{i,k}\, e_k^T M e_i = (CB^{-1})_{i,k}\, M_{k,i}$$

and thus $(CB^{-1})_{i,k} = 1$. Since CB^{-1} is invertible, we conclude that it is a permutation matrix. $\square$

8.2. **Corollary.** *Let (A, B) be an invertible one-sided Jones pair. If (C, B) is also an invertible one-sided Jones pair, then there is an invertible diagonal matrix D such that $C = D^{-1} A D$.*

Proof. By Corollary 5.2, if (A, B) and (C, B) are invertible one-sided Jones pairs, then

$$X_A \Delta_A X_B = \Delta_A X_B \Delta_{B^T}, \qquad X_C \Delta_C X_B = \Delta_C X_B \Delta_{B^T}$$

and, consequently,

$$X_A \Delta_A \Delta_{C^{(-)}} X_{C^{-1}} = \Delta_A \Delta_{C^{(-)}}.$$

Therefore,

$$X_A \Delta_{C^{(-)} \circ A} = \Delta_{C^{(-)} \circ A} X_C$$

and, by Lemma 8.1, this yields that $C^{(-)} \circ A$ is a dual permutation matrix. Hence, A and C are diagonally equivalent. $\square$

8.3. **Corollary.** *Let (A, B) be an invertible one-sided Jones pair. If (A, C) is also an invertible one-sided Jones pair, then there is a permutation matrix P such that $C = BP$.*

Proof. Now we have

$$X_A \Delta_A X_B = \Delta_A X_B \Delta_{B^T}, \qquad X_A \Delta_A X_C = \Delta_A X_C \Delta_{C^T},$$

and, consequently,

$$X_{C^{-1}} X_B = \Delta_{C^{(-)T}} X_{C^{-1}} X_B \Delta_{B^T}.$$

Hence,

$$\Delta_{C^T} X_{C^{-1}B} = X_{C^{-1}B} \Delta_{B^T}$$

and Lemma 8.1 implies that $B^T C^{-T}$ is a permutation matrix. $\square$

These results have interesting consequences. We saw that (A^T, B) is a one-sided Jones pair if (A, B) is. So we may deduce that if (A, B) is an invertible one-sided Jones pair, then there is a diagonal matrix C such that $C^{-1}AC = A^T$. Further, if we are working over $\mathbb{C}$ and $C^{-1}AC = A^T$, then there is a diagonal matrix C_1 such that $C_1^2 = C$; then we have

$$C_1^{-1}AC_1 = C_1A^TC_1^{-1} = (C_1^{-1}AC_1)^T$$

and so $C_1^{-1}AC_1$ is symmetric.

If (A, B) is an invertible Jones pair, then so is (A, B^T), whence it follows that $B^T = BP$ for some permutation matrix P. Since

$$B = (B^T)^T = (BP)^T = P^TB^T = P^TBP,$$

we see that P must commute with B. If P has odd order, then there is an integer r such that $P^{2r} = P$. Suppose $Q = P^r$. Then Q commutes with B and $Q^TB^T = BQ$. Therefore, BQ is symmetric.

(The facts that if (A, B) is an invertible Jones pair, then $A^T = C^{-1}AC$ for some diagonal matrix C and $B^T = BP$ for some permutation matrix P are due to Jaeger [6], but our proofs are new, and simpler.)

9. Four-Weight Spin Models

A *four-weight spin model* is a 5-tuple $(W_1, W_2, W_3, W_4; D)$ where W_1, W_2, W_3, W_4 are $n \times n$ complex matrices and D is a square root of n such that

$$W_1 \circ W_3^T = J, \quad W_2 \circ W_4^T = J, \tag{9.1}$$

$$W_1W_3 = nI, \quad W_2W_4 = nI, \tag{9.2}$$

and

$$\sum_{h=1}^{n} (W_1)_{k,h}(W_1)_{h,i}(W_4)_{h,j} = D(W_4)_{i,j}(W_1)_{k,i}(W_4)_{k,j}, \tag{9.3}$$

$$\sum_{h=1}^{n} (W_1)_{h,k}(W_1)_{i,h}(W_4)_{j,h} = D(W_4)_{j,i}(W_1)_{i,k}(W_4)_{j,k}. \tag{9.4}$$

Note that (9.3) and (9.4) are equivalent to (3a) and (3b) in [1, p. 1] respectively. The original spin models due to Jones [10] are referred to as *two-weight* spin models.

9.1. **Theorem.** *Suppose that $A, B \in M_n(\mathbb{C})$ and $D^2 = n$. Then the following are equivalent.*

(i) (A, B) *is an invertible Jones pair.*
(ii) $(DA, nB^{-1}, DA^{-1}, B; D)$ *is a four-weight spin model.*

Proof. Write $A = D^{-1}W_1$ and $B = W_4$. Then (9.3) is equivalent to (4.1) and holds if and only if (A, B) is a one-sided Jones pair. Observe next that (9.4) is obtained from (9.3) by replacing W_1 and W_4 by their transposes. This implies that (9.4) holds if and only if (A^T, B^T) is a one-sided Jones pair which, by Lemma 3.1, is equivalent to (A, B^T) being a one-sided Jones pair. This shows that (ii) implies (i). The converse is easy, given Theorem 7.1. □

Thus, invertible Jones pairs are equivalent to four-weight spin models. Our treatment shows that most of the theory developed in [2] and [1] holds under the weaker assumption that (A, B) is a one-sided Jones pair.

Four-weight spin models were introduced as a generalization of the generalized spin models of Kawagoe, Munemasa and Watatani [11]. In our terms, Kawagoe et al defined a generalized spin model to be an $n \times n$ type II matrix A such that $(A^T, \sqrt{n}A^{(-)})$ is a one-sided Jones pair. We use the exchange lemma to show that such a pair must be two-sided. (This shows that four-weight spin models are indeed a generalization of generalized spin models, as noted in Bannai and Bannai [1].)

9.2. **Lemma.** *If A is type II and $(A^T, \sqrt{n}A^{(-)})$ is a one-sided Jones pair, then it is a Jones pair.*

Proof. By Corollary 5.2,

$$X_{A^T}\Delta_{A^T}X_{A^{(-)}} = \Delta_{A^T}X_{A^{(-)}}\Delta_{\sqrt{n}A^{(-)T}}.$$

Inverting each side of this, we find that

$$X_{A^{(-)-1}}\Delta_{A^{(-)T}}X_{A^{-T}} = \frac{1}{\sqrt{n}}\Delta_{A^T}X_{A^{(-)-1}}\Delta_{A^{(-)T}}.$$

Since $A^{-T} = \frac{1}{n}A^{(-)}$, we get

$$X_{A^T}\Delta_{A^{(-)T}}X_{A^{-T}} = \frac{1}{\sqrt{n}}\Delta_{A^T}X_{A^T}\Delta_{A^{(-)T}},$$

which gives

$$\Delta_{\sqrt{n}A^{(-)T}}X_{A^T}\Delta_{\sqrt{n}A^{(-)T}} = X_{A^T}\Delta_{\sqrt{n}A^{(-)T}}X_{A^T}.$$

This implies that $(A^T, \sqrt{n}A^{(-)T})$ is a one-sided Jones pair. □

10. Algebras and Bijections

Given a one-sided Jones pair we have a number of algebras including $\mathcal{N}_{A,B}$, $\mathcal{N}'_{A,B}$ and the Bose-Mesner algebra $\mathcal{N}_A$. In this section we study some of the relations between these.

10.1. **Theorem.** *Let A, B and C be type II matrices with the same order. If $F \in \mathcal{N}_{A,B}$ and $G \in \mathcal{N}_{B^{(-)},C}$, then $F \circ G \in \mathcal{N}_{A,C}$ and*

$$\Theta_{A,C}(F \circ G) = n^{-1}\Theta_{A,B}(F)\,\Theta_{B^{(-)},C}(G).$$

Proof. If $X_F\Delta_B X_A = \Delta_B X_A \Delta_{F'}$, then, by Corollary 6.3(b),

$$\Delta_F X_{B^{(-)}}\Delta_{B^T} = X_A\Delta_{A^{-1}}X_{F'}. \tag{10.1}$$

Similarly, applying Corollary 6.3(b) to $X_G\Delta_C X_{B^{(-)}} = \Delta_C X_{B^{(-)}}\Delta_{G'}$, we get

$$\Delta_G X_{C^{(-)}}\Delta_{C^T} = X_{B^{(-)}}\Delta_{B^{(-)-1}}X_{G'} = n^{-1}X_{B^{(-)}}\Delta_{B^T}X_{G'}. \tag{10.2}$$

From (10.1), we have

$$\Delta_F = X_A\Delta_{A^{-1}}X_{F'}(X_{B^{(-)}}\Delta_{B^T})^{-1},$$

which, combined with (10.2), yields

$$\Delta_F\Delta_G X_{C^{(-)}}\Delta_{C^T} = n^{-1}X_A\Delta_{A^{-1}}X_{F'}X_{G'},$$

and, therefore,

$$\Delta_{F\circ G}X_{C^{(-)}}\Delta_{C^T} = n^{-1}X_A\Delta_{A^{-1}}X_{F'G'}.$$

By applying the exchange lemma to this we find that

$$\Delta_{C^{(-)}}X_{F\circ G}\Delta_C = n^{-1}X_A\Delta_{F'G'}X_{A^{-1}},$$

whence
$$X_{F\circ G}\Delta_C X_A = n^{-1}\Delta_C X_A \Delta_{F'G'}.$$
Therefore, $F\circ G \in \mathcal{N}_{A,C}$ and
$$\Theta_{A,C}(F\circ G) = n^{-1}\Theta_{A,B}(F)\,\Theta_{B^{(-)},C}(G). \qquad \square$$

10.2. **Lemma.** *Suppose A and B are type II matrices with the same order. If $G\in\mathcal{N}_{A,B}$, then $G^T\in\mathcal{N}_{A^{(-)},B^{(-)}}$ and $\Theta_{A^{(-)},B^{(-)}}(G^T)=\Theta_{A,B}(G)$.*

Proof. Suppose $R\in\mathcal{N}_{A,B}$ and $X_R\Delta_B X_A = \Delta_B X_A \Delta_S$. Then
$$X_{A^{-1}}\Delta_{B^{(-)}}X_R = \Delta_S X_{A^{-1}}\Delta_{B^{(-)}}$$
and, taking the transpose of this, we find that
$$X_{R^T}\Delta_{B^{(-)}}X_{A^{-T}} = \Delta_{B^{(-)}}X_{A^{-T}}\Delta_S.$$
Since A is type II, we have $nA^{-T}=A^{(-)}$, and we conclude that $R^T\in\mathcal{N}_{A^{(-)},B^{(-)}}$ and $\Theta_{A^{(-)},B^{(-)}}(R^T)=S$. $\square$

As an example, suppose A is type II and $G\in\mathcal{N}_A$. Then by the lemma,
$$G^T\in\mathcal{N}_{A^{(-)},A}=\mathcal{N}_A$$
and
$$\Theta_{A,A^{(-)}}(G)=\Theta_{A^{(-)},A}(G^T)=\Theta_{A,A^{(-)}}(G^T)^T,$$
which implies that $\Theta_A(G^T)=\Theta_A(G)^T$.

10.3. **Theorem.** *Suppose A and B are type II matrices of the same order. If $F\in\mathcal{N}_A$, $G\in\mathcal{N}_{A,B}$ and $H\in\mathcal{N}_B$, then $F\circ G$ and $G\circ H$ lie in $\mathcal{N}_{A,B}$ and*
$$\Theta_{A,B}(F\circ G)=n^{-1}\Theta_A(F)\,\Theta_{A,B}(G),$$
$$\Theta_{A,B}(G\circ H)=n^{-1}\Theta_{A,B}(G)\,\Theta_B(H)^T.$$

Proof. If we apply Theorem 10.1 to the triple $(A,A^{(-)},B)$, we get the first claim.

For the second claim, note that if $H\in\mathcal{N}_B$, then we have $H\in\mathcal{N}_{B^{(-)}}$ and $\Theta_{B^{(-)}}(H)=\Theta_B(H)^T$. Now, if we apply Theorem 10.1 to the triple (A,B,B), then we have
$$\Theta_{A,B}(G\circ H)=n^{-1}\Theta_{A,B}(G)\,\Theta_{B^{(-)}}(H),$$
from which the second assertion follows. $\square$

This corollary shows that $\mathcal{N}_A\circ\mathcal{N}_{A,B}\subseteq\mathcal{N}_{A,B}$ and $\mathcal{N}_{A^T}\mathcal{N}'_{A,B}\subseteq\mathcal{N}'_{A,B}$. We will see below that equality holds.

10.4. **Theorem.** *Suppose A and B are type II matrices of the same order. If F and G lie in $\mathcal{N}_{A,B}$, then $F\circ G^T\in\mathcal{N}_A\cap\mathcal{N}_B$ and*
$$\Theta_A(F\circ G^T)=n^{-1}\Theta_{A,B}(F)\,\Theta_{A,B}(G)^T,$$
$$\Theta_B(F\circ G^T)=n^{-1}\Theta_{A,B}(F)^T\,\Theta_{A,B}(G).$$

Proof. If $G\in\mathcal{N}_{A,B}$, then, by Lemma 10.2, $G^T\in\mathcal{N}_{A^{(-)},B^{(-)}}$ and
$$\Theta_{A^{(-)},B^{(-)}}(G^T)=\Theta_{A,B}(G).$$
If we now apply Theorem 10.1 to the triple $(A,B,A^{(-)})$, we find that $F\circ G^T\in\mathcal{N}_A$ and that $\Theta_A(F\circ G^T)$ is as stated. For the remaining claims, apply Theorem 10.1 with the triple $(B,A,B^{(-)})$. $\square$

Our next result is an easy consequence of Theorem 10.4. It implies that if (A, B) is an invertible one-sided Jones pair and A is symmetric, then either $A \circ A$ is a linear combination of I and J, or $\mathcal{N}_A$ is the Bose-Mesner algebra of an association scheme with at least two classes. (We will discuss the first case again in the final section.) This corollary also appears in Huang and Guo [5].

10.5. **Corollary.** *If (A, B) is an invertible one-sided Jones pair, then $A \circ A^T \in \mathcal{N}_A$ and $\Theta_A(A \circ A^T) = n^{-1}BB^T$.* □

Among other important consequences, our next result implies that if (A, B) is an invertible Jones pair, then $\mathcal{N}_A$ and $\mathcal{N}_{A,B}$ have the same dimension.

10.6. **Theorem.** *Suppose A and B are type II matrices of the same order. If $\mathcal{N}_{A,B}$ contains a Schur invertible matrix G, then $\mathcal{N}_A = \mathcal{N}_B$,*

(a) *$G \circ \mathcal{N}_A = \mathcal{N}_{A,B}$ and $G^T \circ \mathcal{N}_{A,B} = \mathcal{N}_A$.*

(b) *If $H = \Theta_{A,B}(G)$, then $\mathcal{N}_{A^T}H = \mathcal{N}'_{A,B}$ and $\mathcal{N}'_{A,B}H^T = \mathcal{N}_{A^T}$.*

Proof. By Theorem 10.3 we see that $\mathcal{N}_A \circ G \subseteq \mathcal{N}_{A,B}$, while Theorem 10.4 implies that $\mathcal{N}_{A,B} \circ G^T \subseteq \mathcal{N}_A$. Since $G^{(-)}$ exists, Schur multiplication by G is injective, and so (a) follows.

Theorem 10.3 also implies that $G \circ \mathcal{N}_B \subseteq \mathcal{N}_{A,B}$, while Theorem 10.4 implies that $G^T \circ \mathcal{N}_{A,B} \subseteq \mathcal{N}_B$. Hence, $G \circ \mathcal{N}_B = \mathcal{N}_{A,B}$ and $G^T \circ \mathcal{N}_{A,B} = \mathcal{N}_B$, and, therefore, $\mathcal{N}_A = \mathcal{N}_B$.

By Theorem 10.3,

$$\Theta_{A,B}(G \circ \mathcal{N}_A) = n^{-1}\Theta_A(\mathcal{N}_A)H.$$

By Lemma 7.3, $\Theta_A(\mathcal{N}_A) = \mathcal{N}_{A^T}$ and so the first part of the second claim follows. From Theorem 10.4 we see that

$$\Theta_A(\mathcal{N}_{A,B} \circ G^T) = n^{-1}\mathcal{N}'_{A,B}H^T,$$

whence the second part of the second claim follows. □

If (A, B) is an invertible one-sided Jones pair, then A is a Schur invertible element of $\mathcal{N}_{A,B}$. Thus, we have immediately:

10.7. **Corollary.** *If (A, B) is an invertible one-sided Jones pair, then we have $A^T \circ \mathcal{N}_{A,B} = \mathcal{N}_A$ and $\mathcal{N}_{A,B} = A \circ \mathcal{N}_A$.* □

Since $\mathcal{N}_A$ is a Bose-Mesner algebra, this result implies that all matrices in $\mathcal{N}_{A,B}$ have constant diagonal. Since $\mathcal{N}_A$ is closed under transposes, we also see that if A is symmetric, then $\mathcal{N}_{A,B}$ is closed under transposes and (by Lemma 10.2) that $\mathcal{N}_{A,B} = \mathcal{N}_{A^{(-)},B^{(-)}}$.

10.8. **Corollary.** *If (A, B) is an invertible one-sided Jones pair, then we have $\dim \mathcal{N}_A = \dim \mathcal{N}_{A,B}$ and $\mathcal{N}_{A^T} = \mathcal{N}_A = \mathcal{N}_B$.*

Proof. Given our hypothesis, A and A^T are Schur invertible, and it follows from the first part of Theorem 10.6 that $\dim \mathcal{N}_A = \dim \mathcal{N}_{A,B}$ and that $\mathcal{N}_A = \mathcal{N}_B$. Since (A^T, B) is an invertible one-sided Jones pair if (A, B) is, we also have $\mathcal{N}_{A^T} = \mathcal{N}_B$. □

Etsuko Bannai [2] proved that $\mathcal{N}_A = \mathcal{N}_B = \mathcal{N}_{A^T} = \mathcal{N}_{B^T}$ for four-weight spin models; this extended unpublished work by H. Guo and T. Huang, who had shown that $\mathcal{N}_A = \mathcal{N}_{A^T}$. Note that if (A, B) is an invertible Jones pair, then (A, B^T) is an invertible Jones pair; so our previous result also implies that $\mathcal{N}_A = \mathcal{N}_{B^T}$ in this case.

10.9. **Corollary.** *If (A, B) is an invertible Jones pair and $F \in \mathcal{N}_A = \mathcal{N}_B$, then $\Theta_B(F)^T = B^{-1}\Theta_A(F)B$.*

Proof. We have seen that $\mathcal{N}_A = \mathcal{N}_B$ and, therefore, $A \circ \mathcal{N}_B = \mathcal{N}_{A,B}$. Suppose $F \in \mathcal{N}_B$. Then by Theorem 10.3,

$$\Theta_{A,B}(A \circ F) = n^{-1}B\Theta_B(F)^T$$

and

$$\Theta_{A,B}(A \circ F) = n^{-1}\Theta_A(F)B.$$

Hence, the result follows. □

This result implies that $B^{-1}\mathcal{N}_A B = \mathcal{N}_A$. Similarly, if (A, B) is an invertible Jones pair, then we also find that $B^{-T}\mathcal{N}_A B^T = \mathcal{N}_A$. In this case $B^T = BP$ for some permutation matrix P, and $P^T\mathcal{N}_A P = \mathcal{N}_A$.

11. A Dual Pair of Schemes

Let A and B be type II matrices of the same order, and let W be the matrix defined by

$$W := \begin{pmatrix} A & B^{(-)} \\ -A & B^{(-)} \end{pmatrix}.$$

Then it is easy to verify that W is a type II matrix; we are going to describe its Nomura algebra.

11.1. **Theorem.** *If A and B are type II matrices of the same order and*

$$W = \begin{pmatrix} A & B^{(-)} \\ -A & B^{(-)} \end{pmatrix},$$

then $\mathcal{N}_W$ consists of the matrices

$$\begin{pmatrix} F+R & F-R \\ F-R & F+R \end{pmatrix},$$

where $F \in \mathcal{N}_A \cap \mathcal{N}_B$ and $R \in \mathcal{N}_{A,B} \cap \mathcal{N}_{A^{(-)},B^{(-)}}$.

Proof. Suppose M, N, P and Q are $n \times n$ matrices. Then the matrix

$$Z := \begin{pmatrix} M & N \\ P & Q \end{pmatrix}$$

lies in $\mathcal{N}_W$ if and only if all of the following vectors is a set of eigenvectors for Z:

$$\begin{pmatrix} Ae_i \circ A^{(-)}e_j \\ Ae_i \circ A^{(-)}e_j \end{pmatrix}, \quad \begin{pmatrix} B^{(-)}e_i \circ Be_j \\ B^{(-)}e_i \circ Be_j \end{pmatrix}, \quad \begin{pmatrix} Ae_i \circ Be_j \\ -Ae_i \circ Be_j \end{pmatrix}, \quad \begin{pmatrix} A^{(-)}e_i \circ B^{(-)}e_j \\ -A^{(-)}e_i \circ B^{(-)}e_j \end{pmatrix}.$$

The first of these four sets of vectors is a set of eigenvectors for Z if and only if both $M + N$ and $P + Q$ lie in $\mathcal{N}_A$ and

$$\Theta_A(M + N) = \Theta_A(P + Q).$$

This last condition implies that $M + N = P + Q$; we may assume that both sums are equal to F, where $F \in \mathcal{N}_A$. Similarly, the second set of vectors consists of eigenvectors for Z if and only if $M + N$ and $P + Q$ lie in $\mathcal{N}_B$, and, therefore, $F \in \mathcal{N}_A \cap \mathcal{N}_B$.

The third and fourth sets of vectors are eigenvectors for Z if and only if $M-N$ and $P-Q$ lie in $\mathcal{N}_{A,B} \cap \mathcal{N}_{A^{(-)},B^{(-)}}$ and

$$\Theta_{A,B}(M-N) = \Theta_{A,B}(P-Q),$$

whence $M-N = P-Q = R$. □

This result shows that $\mathcal{N}_W$ is the direct sum of two subspaces. The first consists of matrices of the form

$$\begin{pmatrix} F & F \\ F & F \end{pmatrix} \tag{11.1}$$

where $F \in \mathcal{N}_A \cap \mathcal{N}_B$. This set of matrices is closed under multiplication and Schur multiplication (but does not contain I). The second subspace consists of the matrices

$$\begin{pmatrix} R & -R \\ -R & R \end{pmatrix} \tag{11.2}$$

where $R \in \mathcal{N}_{A,B} \cap \mathcal{N}_{A^{(-)},B^{(-)}}$.

11.2. **Corollary.** *Suppose $F \in \mathcal{N}_A \cap \mathcal{N}_B$ and $R \in \mathcal{N}_{A,B} \cap \mathcal{N}_{A^{(-)},B^{(-)}}$. If*

$$Z := \frac{1}{2}\begin{pmatrix} F+R & F-R \\ F-R & F+R \end{pmatrix},$$

then

$$\Theta_W(Z) = \begin{pmatrix} \Theta_A(F) & \Theta_{A,B}(R) \\ \Theta_{B^{(-)},A^{(-)}}(R) & \Theta_{B^{(-)}}(F) \end{pmatrix}.$$

Proof. Assume A and B are $n \times n$ matrices and $1 \le i, j \le n$. Then, for example,

$$We_{i+n} \circ W^{(-)}e_j = \begin{pmatrix} B^{(-)}e_i \circ A^{(-)}e_j \\ -B^{(-)}e_i \circ A^{(-)}e_j \end{pmatrix}.$$

This determines the $(2,1)$-block of $\Theta_W(Z)$, and the other blocks can be found in a similar way. □

Observe that $\Theta_{B^{(-)}}(F) = (\Theta_B(F))^T$ and so, by Corollary 10.9,

$$\Theta_{B^{(-)}}(F) = B^{-1}\Theta_A(F)B.$$

By Lemma 10.2 we have that $\mathcal{N}_{A^{(-)},B^{(-)}} = \mathcal{N}_{A,B}^T$ and

$$\Theta_{A^{(-)},B^{(-)}}(R) = \Theta_{A,B}(R^T).$$

We can now state one of the main conclusions of our paper.

11.3. **Corollary.** *Let (A,B) be an invertible Jones pair of $n \times n$ matrices. Assume A is symmetric and $\dim \mathcal{N}_A = m$. Let W be the $2n \times 2n$ type II matrix defined above. Then $\mathcal{N}_W$ is the Bose-Mesner algebra of an imprimitive association scheme with $2m-1$ classes that contains the matrix*

$$\begin{pmatrix} A & -A \\ -A & A \end{pmatrix}.$$

The image of this under Θ_W is the following matrix in the dual scheme (with Bose-Mesner algebra $\mathcal{N}_{W^T}$):

$$\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}.$$

Proof. Since (A, B) is an invertible Jones pair, $\mathcal{N}_A = \mathcal{N}_B$. Hence, the space of matrices in (11.1) has dimension m. Since A is symmetric, $\mathcal{N}_{A,B}$ is closed under transposes and therefore equals $\mathcal{N}_{A^{(-)},B^{(-)}}$. Hence the space of matrices in (11.2) also has dimension m, and consequently $\dim \mathcal{N}_W = 2m$.

Because $A \in \mathcal{N}_{A,B}$, it follows that

$$\begin{pmatrix} A & -A \\ -A & A \end{pmatrix} \in \mathcal{N}_W$$

and then the previous corollary implies that

$$\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \in \mathcal{N}_{W^T}. \qquad \square$$

12. Dimension Two

We define the *dimension* of a Jones pair (A, B) to be the dimension of $\mathcal{N}_{A,B}$ and we define the *degree* to be the number of distinct entries in B or, equivalently, the number of distinct eigenvalues of A. Since $A \in \mathcal{N}_{A,B}$, the degree is bounded above by the dimension. Since $\mathcal{N}_{A,B}$ contains I and A, the dimension of a pair is at least two, unless $A = I$ (and then $B = J$).

Suppose (A, B) is an invertible Jones pair of $n \times n$ matrices with dimension two, and that A is symmetric. Since $\dim \mathcal{N}_A = \dim \mathcal{N}_{A,B}$, we see that $\mathcal{N}_A$ is the span of I and J; so $\mathcal{N}_A$ is the Bose-Mesner algebra of an association scheme with one class. It follows that there are complex numbers a and b such that

$$A \circ A = aI + bJ.$$

Therefore, there is symmetric matrix C such that $C \circ I = 0$ and $C_{i,j} = \pm 1$ if $i \neq j$ and $A = \lambda I + \gamma C$. Furthermore, since $\mathcal{N}_{A,B}$ has dimension two, the minimal polynomial of A is quadratic. Hence, the minimal polynomial of C is quadratic and, since $(C^2) \circ I = (n-1)I$, there is an integer δ such that

$$C^2 - \delta C - (n-1)I = 0.$$

This implies that C is the matrix of a regular two-graph. (For more information on regular two-graphs, see Seidel's two surveys in [12], and for more on the connection with type II matrices, see [4].)

If A has quadratic minimal polynomial, then B has exactly two distinct entries and so is a linear combination of J and a $(0, 1)$-matrix N. In [3], Bannai and Sawano show that N must be the incidence matrix of a symmetric design, and characterize the designs that can arise in this way. It is well known that symmetric designs on n points correspond to bipartite distance regular graphs on $2n$ vertices with diameter three, and it is less well known that a formal dual of such a scheme is the association scheme associated to a regular two-graph.

Finally, if (A, B) is a one-sided Jones pair and A has quadratic minimal polynomial, then the algebra generated by X_A and Δ_B is a quotient of the Hecke algebra. It follows that the link invariant we obtain is a specialization of the homfly polynomial. (For this see Jones [9, Section 4].)

References

1. Eiichi Bannai and Etsuko Bannai, *Generalized generalized spin models (four-weight spin models)*, Pacific J. Math. **170** (1995), no. 1, 1–16. MR **97f:**57005
2. Etsuko Bannai, *Bose-Mesner algebras associated with four-weight spin models*, Graphs Combin. **17** (2001), no. 4, 589–598. MR **2003a:**05158
3. Etsuko Bannai and Mitsuhiro Sawano, *Symmetric designs attached to four-weight spin models*, Des. Codes Cryptogr. **25** (2002), no. 1, 73–90. MR **2003b:**05041
4. Chris D. Godsil, *Symmetric designs, two-graphs and type II matrices*, preprint (2001).
5. Tayuan Huang and Haitao Guo, *Some remarks on four-weight spin models*, Ann. Comb. **4** (2000), no. 3-4, 401–412, Conference on Combinatorics and Physics (Los Alamos, NM, 1998). MR **2002d:**82011
6. François Jaeger, *On four-weight spin models and their gauge transformations*, J. Algebraic Combin. **11** (2000), no. 3, 241–268.
7. François Jaeger, Makoto Matsumoto, and Kazumasa Nomura, *Bose-Mesner algebras related to type II matrices and spin models*, J. Algebraic Combin. **8** (1998), no. 1, 39–72. MR **99f:**05125
8. François Jaeger and Kazumasa Nomura, *Symmetric versus non-symmetric spin models for link invariants*, J. Algebraic Combin. **10** (1999), no. 3, 241–278. MR **2001g:**57021
9. V. F. R. Jones, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. (2) **126** (1987), no. 2, 335–388. MR **89c:**46092
10. ———, *On knot invariants related to some statistical mechanical models*, Pacific J. Math. **137** (1989), no. 2, 311–334. MR **89m:**57005
11. Kenichi Kawagoe, Akihiro Munemasa, and Yasuo Watatani, *Generalized spin models*, J. Knot Theory Ramifications **3** (1994), no. 4, 465–475. MR **96g:**57009
12. J. J. Seidel, *Geometry and Combinatorics*, Academic Press Inc., Boston, MA, 1991, Selected works of J. J. Seidel, Edited and with a preface by D. G. Corneil and R. Mathon. MR **92m:**01098

Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

Current address: Department of Mathematics, California Institute of Technology, Pasadena, California 91106

E-mail address: `ssachan@alumni.uwaterloo.ca`

Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

E-mail address: `chris@dibbler.uwaterloo.ca`

Graduate School of Mathematics, Kyushu University, Fukuoka, 812-8581, Japan

E-mail address: `munemasa@math.kyushu-u.ac.jp`