TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 355, Number 6, Pages 2385–2396
S 0002-9947(03)03262-8
Article electronically published on January 27, 2003

FINITENESS THEOREMS FOR POSITIVE DEFINITE n-REGULAR QUADRATIC FORMS

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Abstract. An integral quadratic form f of m variables is said to be n-regular if f globally represents all quadratic forms of n variables that are represented by the genus of f. For any $n \geq 2$, it is shown that up to equivalence, there are only finitely many primitive positive definite integral quadratic forms of $n+3$ variables that are n-regular. We also investigate similar finiteness results for almost n-regular and spinor n-regular quadratic forms. It is shown that for any $n \geq 2$, there are only finitely many equivalence classes of primitive positive definite spinor or almost n-regular quadratic forms of $n+2$ variables. These generalize the finiteness result for 2-regular quaternary quadratic forms proved by Earnest (1994).

1. Introduction

A positive definite integral quadratic form f is called regular if f represents all integers that are represented by the genus of f. Regular quadratic forms were first studied systematically by Dickson in [4] where the term "regular" was coined. In the last chapter of his doctoral thesis [17], Watson showed by arithmetic arguments that there are only finitely many equivalence classes of primitive positive definite regular ternary quadratic forms. He did so by providing explicit bounds on the prime power divisors of the discriminant of those regular ternary quadratic forms. Watson never published this part of the thesis. Instead, in a later article [18] he brought in various analytic tools to prove that the size of the set of exceptional integers of a primitive positive definite ternary quadratic form grows with the discriminant, and hence only finitely many classes of ternary quadratic forms are regular. Although [18] is considerably shorter, neither its method nor result leads to an effective upper bound for the discriminants of those regular ternary quadratic forms.

The problem of enumerating the equivalence classes of the primitive positive definite regular ternary quadratic forms was recently resurrected by Kaplansky and his collaborators [12]. Their algorithm relies on the complete list of those regular ternary quadratic forms with square-free discriminant [20] and a method of descent set forth by Watson in [17]. This method of descent involves a collection of transformations which change a regular ternary form to another one with smaller

Received by the editors July 13, 2002 and, in revised form, November 19, 2002.
2000 *Mathematics Subject Classification.* Primary 11E12, 11E20.
Key words and phrases. Regular integral quadratic forms.
The research of the first author is partially supported by the National Security Agency and the National Science Foundation.
The second author was supported by a postdoctoral fellowship program from the Korea Science and Engineering Foundation (KOSEF).

discriminant and simpler local structure, and it is this method which enables Watson to obtain the explicit discriminant bounds for regular ternary quadratic forms.

The study of higher-dimensional analogs of regular quadratic forms is first initiated by Earnest in [6]. A positive definite quadratic form f of m variables is called n-regular if f represents all quadratic forms of n variables that are represented by the genus of f. By [14, Cor 6.4.1], we see that any n-regular primitive positive definite quadratic forms of $n+1$ variables must have class number 1 and such forms exist only when $n \leq 10$ [19]. In [6], Earnest showed that there exist only finitely many inequivalent 2-regular primitive positive definite quaternary quadratic forms. His method is an extension of Watson's analytic argument, which seems to be inadequate for other higher-dimensional situations. In this paper, we turn the stage back to an arithmetic setting and bring back Watson's transformations into the arsenal. In the next section, we will explain how these transformations affect an n-regular quadratic form. They are not as effective as they were in the case of regular ternary quadratic forms, but together with tools for bounding the successive minima we develop in Section 3, they enable us to prove the following finiteness result in Sections 4 and 5.

Theorem 1.1. *For any integer $n \geq 2$, there exist only finitely many inequivalent primitive positive definite n-regular quadratic forms of $n+3$ variables.*

Note that Theorem 1.1 does not hold for $n = 1$. Indeed, in [7] Earnest produced an infinite family of inequivalent primitive positive definite regular quaternary quadratic forms, and a complete characterization of those that are equivalent to diagonal forms is made available by B. M. Kim [13].

A quadratic form f is said to be almost n-regular if f represents all but finitely many of those quadratic forms of n variables that are represented by the genus of f. Since an almost n-regular quadratic form is also $(n-1)$-regular (Lemma 6.1), therefore we can conclude, as a consequence of Theorem 1.1, that for any $n \geq 3$, there exist only finitely many inequivalent primitive positive definite almost n-regular quadratic forms of $n+2$ variables. In Section 6, we show by a separate argument that the same conclusion holds for $n = 2$.

Theorem 1.2. *For any $n \geq 2$, there exist only finitely many inequivalent primitive positive definite almost n-regular quadratic forms of $n+2$ variables.*

Since an n-regular quadratic form is by definition almost n-regular, we therefore prove that for any $n \geq 2$, there are only finitely many inequivalent primitive positive definite n-regular quadratic forms of $n+2$ variables, and hence generalize Earnest's finiteness result on the 2-regular quaternary quadratic forms [6]. We also show that Theorem 1.2 is not true for $n = 1$ by exhibiting an infinite family of inequivalent primitive positive definite almost 1-regular ternary quadratic forms. It turns out that these almost 1-regular ternary quadratic forms share some nice properties, which will be investigated in a future paper.

Another type of regularity condition we are interested in is the spinor regularity. A quadratic form f of m variables is said to be spinor n-regular if f represents all quadratic forms of n variables that are represented by the proper spinor genus of f. It is known that if f is spinor n-regular and $m - n \geq 3$, then f is in fact n-regular [10]. From [14, Cor 6.4.1], [19] and [9], we know that there are only finitely many n for which primitive positive definite spinor n-regular quadratic forms of $n+1$ variables exist. Using an analytic argument, which is an extension of Watson's in

[18], Benham and his collaborators [1] showed that there exist only finitely many inequivalent primitive positive definite spinor 1-regular ternary quadratic forms. Explicit bounds for the discriminants of those quadratic forms are derived by the first author and Earnest in [2]. In this paper, we will generalize the finiteness result in [1] via an arithmetic argument to quadratic forms of $n+2$ variables for any $n \geq 2$.

Theorem 1.3. *For any $n \geq 2$, there are only finitely many inequivalent primitive positive definite spinor n-regular quadratic forms of $n+2$ variables.*

The subsequent discussion will be conducted in the better adapted geometric language of quadratic spaces and lattices, and any unexplained notation and terminology can be found in [16] or [14]. The term lattice will always refer to an integral $\mathbb{Z}$-lattice on an m-dimensional positive definite quadratic space over $\mathbb{Q}$. The scale and the norm ideal of a lattice M are denoted by $\mathfrak{s}(M)$ and $\mathfrak{n}(M)$ respectively. For any positive rational number a, M^a is the lattice whose quadratic map is scaled by a. The successive minima of M are denoted by $\mu_1(M) \leq \cdots \leq \mu_m(M)$. For each $k \leq m$, a $k \times k$ section of M is defined to be the primitive sublattice of M spanned by the first k vectors in a Minkowski basis of M. Since we want to include nonclassic integral quadratic forms in our discussion, therefore it will be assumed that every lattice is even, and in particular L is always an *even primitive* lattice, i.e., $\mathfrak{n}(L) = 2\mathbb{Z}$.

In what follows, a real-valued function of several variables is said to be bounded if it is bounded above by a constant that is independent of those variables. A family of isometry classes of lattices can be shown to be finite if the discriminants of those lattices are proved to be bounded. This can be done by bounding their successive minima because of the inequality $dM \leq \mu_1(M) \cdots \mu_m(M)$ from reduction theory (see, for example, [6]).

Let p be a prime. The group of units in $\mathbb{Z}_p$ is denoted by $\mathbb{Z}_p^\times$. Unless confusion arises, we will simply use Δ to denote a nonsquare element in $\mathbb{Z}_p^\times$. When we discuss lattices over $\mathbb{Z}_p$, $\mathbb{H}$ denotes the hyperbolic plane and $\mathbb{A}$ stands for the binary anisotropic $\mathbb{Z}_p$-lattice $\langle 1, -\Delta \rangle$ when $p > 2$, and $A(2,2)$ if $p = 2$.

2. Watson's Transformations

For any positive integer m, define

$$\Lambda_m(L) = \{x \in L : Q(x+z) \equiv Q(z) \mod m \text{ for all } z \in L\}.$$

For example, $\Lambda_2(L)$ is simply L itself. But if $\mathfrak{s}(L) = 2\mathbb{Z}$, then

$$\Lambda_4(L) = \{x \in L : Q(x) \in 4\mathbb{Z}\}.$$

Let $\lambda_m(L)$ be the even primitive lattice obtained from $\Lambda_m(L)$ by scaling V by a suitable rational number. Note that the scaling factor depends on the lattice structure of L_p for all $p \mid m$. For any prime p, we write $L_p = M_p \perp N_p$ where M_p is the leading Jordan component and $\mathfrak{s}(N_p) \subseteq p\mathfrak{s}(M_p)$.

Lemma 2.1. *Suppose M_p is unimodular and $\mathfrak{n}(N_p) \subseteq 2p\mathbb{Z}_p$. Then*

$$\Lambda_{2p}(L)_p = pM_p \perp N_p.$$

Furthermore, if L is n-regular and M_p is anisotropic, then $\lambda_{2p}(L)$ is also n-regular.

Proof. Let x be a vector in $\Lambda_{2p}(L)_p$. Write $x = y + z$ where $y \in M_p$ and $z \in N_p$. If $y \notin pM_p$, then there exists $y' \in M_p$ such that $B(y, y') = 1$. So, $Q(x+y') - Q(y') = Q(x) + 2 \in 2p\mathbb{Z}_p$. But $Q(x) \in 2p\mathbb{Z}_p$ and thus $Q(x + y') \not\equiv Q(y') \bmod 2p$, which is a contradiction. The first assertion follows immediately.

For the second assertion, it suffices to show that $\Lambda_{2p}(L)$ is n-regular. Let ℓ be a lattice of rank n represented by the genus of $\Lambda_{2p}(L)$. Then $\mathfrak{n}(\ell) \subseteq p\mathbb{Z}$, and $\ell \longrightarrow L$ because L is n-regular. Without loss of generality, we may assume that $\ell \subseteq L$. Let $x \in \ell_p$. Then $Q(x) \in 2p\mathbb{Z}_p$ and, since M_p is anisotropic, $x \in pM_p \perp N_p = \Lambda_{2p}(L)_p$. Therefore $\ell_q \subseteq \Lambda_{2p}(L)_q$ for all primes q, i.e., $\ell \subseteq \Lambda_{2p}(L)$. □

The following lemmas describe the action of the λ_4-transformation when $\mathfrak{s}(L) = 2\mathbb{Z}$. Their proofs are straightforward and are left to the readers.

Lemma 2.2. *If L is n-regular and $\mathfrak{s}(L) = 2\mathbb{Z}$, then $\lambda_4(L)$ is also n-regular.*

Lemma 2.3. *Suppose $\mathfrak{s}(L) = 2\mathbb{Z}$ and $\mathfrak{s}(N_2) \subseteq 8\mathbb{Z}_2$.*

(1) *If $\mathrm{rank}(M_2) \geq 3$, then $\lambda_4(L)_2$ is split by a unimodular $\mathbb{Z}_2$-lattice.*
(2) *If $\mathrm{rank}(M_2) = 2$, then*
$$\lambda_4(L)_2 \cong \begin{cases} M_2 \perp N_2^{\frac{1}{2}} & \text{if } \frac{dM}{4} \equiv 1 \bmod 4, \\ \mathbb{P} \perp N_2^{\frac{1}{4}} & \text{if } \frac{dM}{4} \equiv 3 \bmod 4, \end{cases}$$
where $\mathbb{P}$ is an even binary unimodular $\mathbb{Z}_2$-lattice.
(3) *If $\mathrm{rank}(M_2) = 1$, then $\lambda_4(L)_2 \cong M_2 \perp N_2^{\frac{1}{4}}$.*

Lemma 2.4. *If $\mathrm{rank}(M_2) = 1$ and $N_2 = J_2 \perp K_2$ where J_2 is a 4-modular $\mathbb{Z}_2$-lattice and $\mathfrak{s}(K_2) \subseteq 8\mathbb{Z}_2$, then*
$$\lambda_4(L)_2 \cong \begin{cases} M_2^2 \perp N_2^{\frac{1}{2}} & \text{if } J_2 \text{ is proper}, \\ M_2 \perp N_2^{\frac{1}{4}} & \text{if } J_2 \text{ is improper}. \end{cases}$$

Theorem 2.5. *Suppose L is an n-regular lattice of rank ≥ 4. For any prime p, there exists an n-regular even primitive lattice L' such that L'_p represents every element in $2\mathbb{Z}_p$ and L'_q is isometric to L_q up to a scaling factor for all $q \neq p$.*

Proof. It suffices to show that L' can be obtained from L by means of a finite number of those λ_{2p}-transformations. The second assertion follows immediately since $\Lambda_{2p}(L)_q = L_q$ for all $q \neq p$. The case for $p > 2$ is quite straightforward and we leave its proof to the reader.

For $p = 2$, we first treat the case when L_2 has a unimodular component but $\mathbb{H} \not\longrightarrow L_2$. Therefore, the unimodular component of L_2 is $\mathbb{A}$ and Lemma 2.1 applies. After applying the λ_4-transformation a finite number of times, we obtain an n-regular lattice M whose 2-adic structure is isometric to $\mathbb{A} \perp N_2$ with $\mathfrak{s}(N_2) = 2\mathbb{Z}_2$ or $4\mathbb{Z}_2$. If $\mathfrak{s}(N_2) = 2\mathbb{Z}_2$, then either $\mathbb{H} \longrightarrow M_2$ or $\lambda_4(M)_2$ represents $\mathbb{H}$ or $\mathbb{A} \perp \mathbb{A}^2$, and hence we may take L' to be M or $\lambda_4(M)$.

When $\mathfrak{s}(N_2) = 4\mathbb{Z}$, let us write $M_2 = \mathbb{A} \perp J_2 \perp K_2$ where J_2 is a 4-modular $\mathbb{Z}_2$-lattice and $\mathfrak{s}(K_2) \subseteq 8\mathbb{Z}_2$. If J_2 is improper, then $\lambda_4(M)$ has a unimodular component of rank ≥ 4 and we are done. So we suppose that J_2 is proper. If $\mathbb{A} \perp J_2$ is isotropic, then applying the λ_4-transformation twice we obtain an even primitive lattice L' such that $\mathbb{H} \longrightarrow L'_2$. However, if $\mathbb{A} \perp J_2$ is anisotropic and $K_2 = 0$, then $\mathbb{A} \perp J_2$ must be isometric to $\mathbb{A} \perp \langle 4, 12\rangle$. It becomes $\mathbb{A} \perp \mathbb{A}^2$

after applying λ_4 twice more. When $K_2 \neq 0$, we apply λ_4 twice and observe that $\lambda_4^2(M)_2 \cong \mathbb{A} \perp J_2 \perp K_2^{\frac{1}{4}}$. Therefore, we may continue to apply λ_4^2 until we obtain a lattice whose 2-adic structure is one of those already discussed.

Now suppose the scale of the leading Jordan component of L_2 is $2\mathbb{Z}_2$. From Lemma 2.3, we may assume that the rank of M_2 is 1. Then by Lemma 2.3 (3) and Lemma 2.4, after applying a finite number of λ_4-transformations we arrive at an even primitive lattice whose 2-adic Jordan splitting has a leading component of rank ≥ 2. We are done since this situation is already covered by the previous discussion. $\square$

Corollary 2.6. *Let L be an n-regular lattice of rank $m \geq 4$. There exists an n-regular lattice $\lambda(L)$ of rank m which represents all positive even integers and $d\lambda(L) \mid dL$.*

Remark 2.7. In Theorem 2.5, if L_q is also isotropic, then we can make sure that $\mathbb{H} \longrightarrow L'_q$.

3. Some Lemmas

In this section, we collect some technical lemmas that help bound the successive minima of an n-regular lattice.

Lemma 3.1. *Let W be a binary space over $\mathbb{Q}$ of discriminant δ and let $k \longrightarrow W$. Let q be a prime such that $p \nmid 2k\delta$. If $-\delta$ is not a square at q, then W does not represent any integer a with $\mathrm{ord}_q(a)$ odd.*

Proof. If Q is the quadratic map on W, then kQ is the norm form from $\mathbb{Q}(\sqrt{-\delta})$ to $\mathbb{Q}$. The conditions imposed on q imply that q is inert in $\mathbb{Q}(\sqrt{-\delta})$. Therefore, if a is an integer represented by W, $\mathrm{ord}_q(a)$ must be even. $\square$

Lemma 3.2. *Let ℓ be an even ternary lattice. Then ℓ cannot represent all positive even integers.*

Proof. It is easy to see that ℓ is anisotropic for at least one prime p. Since any ternary space over $\mathbb{Q}_p$ does not represent any elements in one particular square class of $\mathbb{Q}_p^\times$, there must be a positive even integer α that is not represented by ℓ. $\square$

If L represents all positive even integers, the above two lemmas would produce bounds on the first four successive minima of L. We cannot go beyond the fourth minimum since there are quaternary lattices that represent all even positive integers. However, if L is also n-regular with $n \geq 2$, then the following lemmas would give us a procedure to bound the other minima of L.

Lemma 3.3. *Let M be a $k \times k$ section of L for some $k < m$. If $\mathfrak{n}(M^\perp) \subseteq a\mathbb{Z}$, then $\mu_{k+1}(L) \geq \frac{a}{(dM)^2}$.*

Proof. Let δ be the discriminant of M. Then $[L : M \perp M^\perp] \leq \delta$. Let $y \in L$ so that $Q(y) = \mu_{k+1}(L)$. Then $\delta y = x + z$ where $x \in M$ and $z \in M^\perp \setminus \{0\}$. Therefore, $\delta^2 Q(y) \geq Q(z) \geq a$. $\square$

Lemma 3.4. *Let ℓ be a lattice of rank $k \geq 3$ and let L be a $(k-1)$-regular lattice of rank $> k$. If $\ell \longrightarrow L$, then $\mu_{k+1}(L) \leq C$ where $C = C(\ell)$ is a constant depending only on ℓ.*

Proof. Let M be a $k \times k$ section of L. If $\ell \not\longrightarrow M$, then it is clear that $\mu_{k+1}(L) \le \mu_k(\ell)$. Hence we may assume that $\ell \longrightarrow M$. Since $dM \le d\ell$, the number of possible isometry classes of M is finite. Now fix a sublattice $K := \langle a \rangle \perp \langle b \rangle \perp K_1$ of M of rank $k-1$ and choose a set of primes $S := \{q_1, q_2, r_1, r_2\}$ satisfying:

(1) for all $q \in S$, $q \in (\mathbb{Z}_p^\times)^2$ for all $p \mid (dK)(dM)$ and $q \equiv 1 (\mathrm{mod}\, 8)$,
(2) $\left(\dfrac{q_i}{r_i}\right) = \left(\dfrac{q_1}{q_2}\right) = -1$,
(3) $\left(\dfrac{q_j}{r_i}\right) = \left(\dfrac{r_1}{r_2}\right) = 1$, whenever $i \neq j$.

Note that the set S depends only on ℓ and M. Suppose $\mathrm{ord}_{q_i}(\mathfrak{n}(M^\perp)) = \beta_i$ for $i = 1, 2$ and $\langle q_i^{\beta_i} \epsilon_i \rangle \longrightarrow M_{q_i}^\perp$. Set $s_i = 1$ if ϵ_i is a square and $s_i = r_i$ otherwise. Let α_i be the smallest odd positive integer greater than or equal to β_i and put

$$\tilde{K} := \langle a q_1^{\alpha_1} s_1 \rangle \perp \langle b q_2^{\alpha_2} s_2 \rangle \perp K_1.$$

It is easy to check that $\tilde{K}_{q_i} \not\longrightarrow M_{q_i}$ and $\tilde{K}_p \longrightarrow L_p$ for all p. Therefore,

$$\mu_{k+1}(L) \le \mu_k(\tilde{K}) \le \max(C_1 q_1^{\alpha_1}, C_2 q_2^{\alpha_2}, C_3) := A$$

where C_1, C_2, C_3, and hence A, depend only on ℓ and M.

Now choose another set of primes $S' = \{q_1', q_2', r_1', r_2'\}$ satisfying conditions (1), (2) and (3), and define α_1' and α_2' accordingly. By Lemma 3.3,

$$q_1^{\beta_1} q_2^{\beta_2} (q_1')^{\beta_1'} (q_2')^{\beta_2'} \le (dM)^2 A.$$

Hence $(q_1')^{\alpha_1'}$ and $(q_2')^{\alpha_2'}$ are bounded by some constant depending only on ℓ and M. Now, we can use q_1', q_2' and repeat the argument used in the last paragraph to get the desired constant C. □

Lemma 3.5. *Let ℓ be a lattice of rank $k \ge 5$ and L be a $(k-2)$-regular lattice of rank $> k$. If $\ell \longrightarrow L$, then $\mu_{k+1}(L) \le C$ where $C = C(\ell)$ is a constant depending only on ℓ.*

Proof. Let M be a $k \times k$ section of L. As in Lemma 3.4, we may assume that $\ell \longrightarrow M$. Fix a sublattice $\langle a \rangle \perp \langle b \rangle \perp \langle c \rangle \perp K_1$ of M of rank $k-2$ and choose two primes q_1, q_2 to satisfy the following:

(1) $q_i \in (\mathbb{Z}_p^\times)^2$ for all $p \mid (dK)(dL)$ and $q_i \equiv 1 (\mathrm{mod}\, 8)$,
(2) $\left(\dfrac{q_2}{q_1}\right) = -1$.

Define α_i as in the proof of Lemma 3.4. If we put

$$\tilde{K} := \langle q_1^{\alpha_1} q_2^{\alpha_2} a \rangle \perp \langle b q_1^{\alpha_1} \rangle \perp \langle c q_2^{\alpha_2} \rangle \perp K_1,$$

then $\tilde{K}_{q_i} \not\longrightarrow M_{q_i}$ and $\tilde{K}_p \longrightarrow L_p$ for all p. We then proceed as in the proof of Lemma 3.4. □

4. n-Regular Lattices

Throughout this section, L is an n-regular lattice of rank $n+3$ where $n \ge 2$ although at the end only the proof of Theorem 1.1 for $n \ge 3$ will be given.

Lemma 4.1. *If L represents all positive even integers, then dL is bounded.*

Proof. Using Lemmas 3.1 and 3.2, we can show that the first four successive minima of L are bounded. Since L is also k-regular for any $k \leq n$ [6], we can apply Lemma 3.4 to bound $\mu_5(L)$, and then apply Lemma 3.5 to bound the other minima. □

If $q \mid d\lambda(L)$ (see Corollary 2.6), then q is bounded because $\lambda(L)$ represents all positive even integers. However, some odd prime divisor p of dL might not divide $d\lambda(L)$. This happens only when one of the following holds:

- $L_p \cong \langle a \rangle \perp N_p$ where $a \in \mathbb{Z}_p^\times$ and $\mathfrak{s}(N_p) \subseteq p^2\mathbb{Z}_p$;
- $L_p \cong \langle 1, -\Delta \rangle \perp N_p$ where $\mathfrak{s}(N_p) \subseteq p^2\mathbb{Z}_p$.

Proposition 4.2. *If p divides the discriminant of some n-regular lattice, then p is bounded.*

Proof. We may assume that $p > 2$ and L_q represents all elements in $2\mathbb{Z}_q$ for all $q \neq p$, and L_p is one of the two possibilities described above.

We first treat the case when $L_p \cong \langle a \rangle \perp N_p$. Let

$$P = \{2t : 1 \leq t \leq p-1, 2t \longrightarrow L_p\}.$$

Note that $|P| = (p-1)/2$ and $\min P \leq p+1$. Let M be the sublattice of L spanned by all $v \in L$ such that $Q(v) \in P$. If $\operatorname{rank}(M) = k \geq 3$, then $p^{2(k-1)} \leq dM \leq (p+1)(2p-2)^{k-1}$, which is not possible for any prime p. When $\operatorname{rank}(M) = 2$, then $dM \leq 2(p^2-1)$. However, since $p^2 \mid dM$ and $dM \equiv 0$ or $3 \bmod 4$, we must have $3p^2 \mid dM$. As a result, $3p^2 \leq 2(p^2-1)$, which is not possible either. Lastly, if $\operatorname{rank}(M) = 1$, then all integers in P fall into a single square class. So, $|P| \leq \sqrt{p}$ and hence $p \leq 5$.

Now, suppose $L_p \cong \langle 1, -\Delta \rangle \perp N_p$. It is clear that L represents every integer in the set $Y = \{2, 4, \ldots, 2(p-1)\}$. Let G be the sublattice of L spanned by the vectors $u \in L$ with $Q(u) \in Y$. Since $p \geq 3$, therefore G represents 2 and 4, and hence $\operatorname{rank}(G) \geq 2$. If $\operatorname{rank}(G) = 4$, then $p^4 \leq dG \leq 8(2p-2)^2$ and thus $p = 3$. If $\operatorname{rank}(G) = 3$, then $p^2 \leq 8(2p-2)$ and hence $p \leq 13$. If G is binary, then G must be isometric to one of the following: $[2,0,2], [2,1,2], [2,0,4]$, or $[2,1,4]$. These four binaries do not represent 4, 6, 6, and 16 respectively. Therefore, $p \leq 7$. □

Remark 4.3. Observe that in the proof of Lemma 4.1, the first four successive minima are already bounded once L represents all positive even integers. Therefore, Proposition 4.2 holds if L is 1-regular and $\operatorname{rank}(L) = 4$. This will be useful later in Sections 6 and 7.

Proof of Theorem 1.1 ($n \geq 3$). By Proposition 4.2, it suffices to fix a prime divisor p of dL and bound $\operatorname{ord}_p(dL)$. For any $q \neq p$, the λ_{2q}-transformations do not change $\operatorname{ord}_p(dL)$. Therefore we may assume that L_q represents all elements in $2\mathbb{Z}_q$ for any $q \neq p$. Since L_p represents at least one square class of integers in $2\mathbb{Z}_p^\times$, $\mu_1(L), \mu_2(L)$ and $\mu_3(L)$ are bounded. We then apply Lemma 3.4 to bound $\mu_4(L)$ and $\mu_5(L)$ (note that $n \geq 3$). The rest of the minima can be bounded by applying Lemma 3.5. □

5. 2-Regular Lattices of Rank 5

This section is devoted solely to the proof of the special case $n = 2$ of Theorem 1.1. We are informed that this special case was also treated by Y. C. Chung in his Ph.D. thesis [3]. However, it is believed that our proof is independent of Chung's and contains some arguments that will be useful for the proof of Theorem 1.2 in

Section 6. Moreover, we feel that by including this section, this paper will be more complete and self-contained.

Within this section, L is a 2-regular lattice of rank 5. It is clear that the first three minima of L are bounded. Using Lemma 3.4, we can also bound $\mu_4(L)$. Let M be a 4×4 section of L. Note that the number of possibilities for M is finite.

Lemma 5.1. *For each prime p, there exists a binary sublattice Γ_p of M_p such that $\Gamma_p^\epsilon \cong \Gamma_p$ for all $\epsilon \in \mathbb{Z}_p^\times$.*

Proof. If $p > 2$, then it is clear that M_p contains some binary modular sublattices and we may take Γ_p to be any one of them.

When $p = 2$ and one of the Jordan components of M_2 is improper, then we simply take Γ_2 to be a binary improper modular sublattice of M_2. So, we may assume that all Jordan components of M_2 are proper. Let $\{v_1, v_2, v_3, v_4\}$ be an orthogonal basis for M_2 and $a_i = \mathrm{ord}_2(Q(v_i))$ for $i = 1, 2, 3, 4$. If three of the a_i have the same parity, then M_2 contains a modular sublattice of rank ≥ 3, and hence Γ_2 can be chosen to be a binary improper modular sublattice of M_2. Therefore, we can further assume that two of the a_i are even and the other two are odd. Then M_2 contains a sublattice of the form $J \perp K$ where J and K are binary proper modular sublattices, and $\mathfrak{s}(K) = 2\mathfrak{s}(J)$. If $J = 2^a J'$ with $dJ' \equiv 3 \bmod 4$, then we take Γ_2 to be J. Otherwise let $J = \mathbb{Z}_2[u_1, u_2]$ and $K = \mathbb{Z}_2[u_3, u_4]$ where the u_i are mutually orthogonal. Then we can take Γ_2 to be $\mathbb{Z}_2[u_1, u_2 + u_3]$. □

Proof of Theorem 1.1 ($n = 2$). We first assume that dM is not a square. Let q be an odd prime so that dM is a nonsquare unit in $\mathbb{Z}_q$. Then $M_q \cong \langle 1, -1\rangle \perp \langle 1, -\Delta\rangle$. Let Γ_q be a sublattice of M_q that is isometric to $\langle 1, -\Delta\rangle$. For any $p \mid dM$, let Γ_p be a binary sublattice satisfying the conclusion of Lemma 5.1. Let a be the positive generator of $\mathfrak{n}(M^\perp)$ and α be the smallest odd integer greater than or equal to $\mathrm{ord}_q(a)$

By [11, Lemma 1.6], there exists a binary sublattice N of M such that

(1) $N_p \cong \Gamma_p$ for all $p \mid qdM$;
(2) $\mathrm{ord}_p(dN) \leq 1$ for all $p \nmid qdM$.

Let $N' = N^{q^\alpha}$. For any $p \mid dM$, $N'_p \cong N_p$ by our choice of Γ_p. If $p \nmid dM$ and $p \neq q$, then M_p is unimodular and N'_p represents a unit in $\mathbb{Z}_p$, and thus $N'_p \longrightarrow M_p$. At q, $N'_q \cong \langle q^\alpha, -q^\alpha\Delta\rangle$ is represented by L_p by [15, Theorem 1]. So, $N' \longrightarrow \mathrm{gen}(L)$ and thus $N' \longrightarrow L$. However, $N' \not\longrightarrow M$ since $N'_q \not\longrightarrow M_q$.

By Lemma 3.3, we have

$$\frac{a}{(dM)^2} \leq \mu_5(L) \leq q^\alpha \mu_2(N).$$

Since N depends only on M, we can conclude that $a \leq Aq^\alpha$ for some constant A depending only on M. Now, choose another prime q' so that dM is a nonsquare in $\mathbb{Z}_{q'}^\times$, and repeat the above argument. We then obtain an inequality of the form

$$\mu_5(L) \leq C(q')^{\alpha'},$$

where C depends only on M. But at the same time we also have

$$q^{\mathrm{ord}_q(a)} \cdot (q')^{\mathrm{ord}_{q'}(a)} \leq Aq^\alpha.$$

Since $\alpha - \mathrm{ord}_q(a) = 0$ or 1, we see that $(q')^{\alpha'}$ is bounded and hence $\mu_5(L)$ is also bounded.

Henceforth, we assume that dM is a square. Fix a prime p. It suffices to derive a bound for $p^{\mathrm{ord}_p(a)}$ since such a bound would imply that dL is also bounded. Suppose that there exists a prime $q \neq p$ at which M_q is anisotropic. Then $q \mid dM$ and hence q is bounded. Fix a full sublattice $M' := \langle u\rangle \perp \langle v\rangle \perp K$ of M. Write $u = q^{\mathrm{ord}_q(u)}u'$ and $v = q^{\mathrm{ord}_q(v)}v'$ with $q \nmid u'v'$.

For the sake of convenience, let us introduce the following notation. For a $\mathbb{Z}_q$-lattice G, let $\mathfrak{l}(G)$ be the norm ideal of the last Jordan component of a Jordan splitting of G. In what follows, let β be the smallest positive integer satisfying: (i) $\beta \geq \mathrm{ord}_q(\mathfrak{l}(M_q \perp M_q^{\perp}))$ and (ii) $\beta \equiv \mathrm{ord}_q(uv) \bmod 2$. Choose a prime s to satisfy the following:

(1) $\left(\dfrac{s}{q}\right) = \left(\dfrac{-u'v'}{q}\right)$,

(2) $\left(\dfrac{s}{\ell}\right) = \left(\dfrac{q^{\beta}}{\ell}\right)$, for all primes $\ell \mid dM'$ and $\ell \neq q$.

The choice of s can be made independently of β and L. If we set $N = \langle u, q^{\beta+\delta}vs\rangle$, where $\delta = 0$ if $q > 2$ and $\delta = 4$ otherwise, then $N \longrightarrow \mathrm{gen}(L)$ [15, Theorems 1 and 3] but $N_q \not\longrightarrow M_q$. By Lemma 3.3, we obtain

$$q^{\mathrm{ord}_q(a)} \cdot p^{\mathrm{ord}_p(a)} \leq B \max\{u, q^{\beta+\delta}vs\},$$

where B is a quantity depending only on M. However, the definition of β shows that either $\beta - \mathrm{ord}_q(\mathfrak{l}(M_q))$ or $\beta - \mathrm{ord}_q(a)$ is 0 or 1. Therefore, $p^{\mathrm{ord}_p(a)}$ is bounded.

So, from now on, we may assume that M_q is isotropic at all $q \neq p$. Then M_p must be anisotropic by a Hasse symbol calculation, and thus $p \mid dM$. We can apply the argument used in the last paragraph to show that $q^{\mathrm{ord}_q(a)}$ is bounded for all $q \neq p$, and hence the primes dividing dL are bounded. Since the λ_{2q}-transformations do not change $\mathrm{ord}_p(dL)$, we may apply them to L, if necessary, and further assume that $\mathbb{H} \longrightarrow L_q$ for all $q \neq p$ (Remark 2.7).

We may also assume that $\mathrm{ord}_p(\mathfrak{l}(L_p)) \geq \mathrm{ord}_p(\mathfrak{l}(M_p))+4$; otherwise $\mathrm{ord}_p(\mathfrak{l}(L_p))$ is bounded and we are done. Then M_p is represented by the orthogonal complement of the last Jordan component of a Jordan decomposition of L_p [15, Theorems 1 and 3]. Therefore, the last Jordan component of L_p has rank 1, and its orthogonal complement is anisotropic. Since $pL \subseteq \Lambda_{2p}(L) \subseteq L$, the first four minima of $\lambda_{2p}(L)$ are bounded. So, we may apply the λ_{2p}-transformation a bounded number of times and further assume that $L_p \cong \mathbb{A} \perp \mathbb{A}^p \perp \langle p^k\epsilon\rangle$ for some $\epsilon \in \mathbb{Z}_p^{\times}$. Note that $k \leq \mathrm{ord}_p(dM) - 2 + \mathrm{ord}_p(a)$ and dL is divisible by $2p$. Let η be the smallest even integer greater than or equal to $\max\{k, \mathrm{ord}_p(a)\}$. It is clear that $\eta - \mathrm{ord}_p(a)$ is bounded.

Let $r > 2dL$ be a prime such that

$$r \in -(\mathbb{Z}_q^{\times})^2 \text{ for all } q \mid dL.$$

Choose another prime t such that $t > 2p^{\eta-\mathrm{ord}_p(a)}r\,dM$, $t \equiv 1 \bmod 8$, and

$$t \in (\mathbb{Z}_q^{\times})^2 \text{ for all } q \mid 2pr.$$

Then $-r$ is a square modulo t and there exist positive integers b $(< t)$ and W such that $tW = b^2 + p^{\eta}r$. Let N be the binary lattice

$$\left\langle \begin{matrix} 2t & 2b \\ 2b & 2W \end{matrix} \right\rangle.$$

Then $N \longrightarrow \mathrm{gen}(L)$ but $\mathbb{Q}_pN \not\longrightarrow \mathbb{Q}_pM$; hence

$$p^{\mathrm{ord}_p(a)} \le dM \cdot \max\{2t, 2W\}.$$

If $2t \ge 2W$, then we are done; otherwise from $tW = b^2 + p^\eta r$ we see that

$$p^{\mathrm{ord}_p(a)} \le \frac{2t^2 dM}{t - 2p^{\eta - \mathrm{ord}_p(a)} r\, dM}.$$

Therefore, $\mathrm{ord}_p(a)$ is bounded by a constant independent of L. □

6. Almost n-Regular Lattices

Lemma 6.1. *Suppose L is an almost n-regular lattice with $n \ge 2$. Then L is an $(n-1)$-regular lattice.*

Proof. Let N be a lattice of rank $n-1$ which is represented by $\mathrm{gen}(L)$. Then there exists $K \in \mathrm{gen}(L)$ such that $N \subseteq K$. Choose $v \in K$ to be orthogonal to N. Then $N \perp \mathbb{Z}[av]$ is represented by $\mathrm{gen}(L)$ for all $a \in \mathbb{Z}$. Consequently, there exists $a_0 \in \mathbb{Z}$ such that $N \subseteq N \perp \mathbb{Z}[a_0v]$ is represented by L. □

It is not hard to see that Lemmas 2.1 and 2.2 hold when "n-regular" is replaced by "almost n-regular". Indeed, if $E_n(L)$ is the set of rank n lattices that are represented by $\mathrm{gen}(L)$ but not by L itself, then $E_n(\Lambda_{2p}(L)) \subseteq E_n(L)$ when L satisfies the conditions in Lemmas 2.1 or 2.2.

Proof of Theorem 1.2. Lemma 6.1 and Theorem 1.1 together imply that Theorem 1.2 holds for any $n \ge 3$. Henceforth we assume that L is an almost 2-regular quaternary lattice. By Remark 4.3, the prime divisors of dL are bounded. Hence it suffices to bound $\mathrm{ord}_p(dL)$ for a fixed prime p. By applying suitable Watson's transformations, we may assume that for any $q \ne p$, L_q represents $\mathbb{H}$ or $\mathbb{A} \perp \mathbb{P}^q$ for some binary (improper if $q = 2$) unimodular $\mathbb{Z}_q$-lattice $\mathbb{P}$. In particular, L_q represents all elements in $2\mathbb{Z}_q$ for any $q \ne p$.

It is clear that the first three successive minima of L are bounded. Let M be a 3×3 section of L. Suppose M_q is anisotropic at some $q \ne p$. Then $q \mid dM$ and hence q is bounded. Then we can find a bounded $\alpha \in \mathbb{Z}$ such that $\alpha \longrightarrow \mathrm{gen}(L)$ but $\alpha \not\longrightarrow M_q$. Since L is 1-regular (Lemma 6.1), $\mu_4(L)$ is bounded and we are done. So, we may assume that M_q is isotropic for all $q \ne p$ and hence M_p is anisotropic. In particular, $\mathbb{H} \longrightarrow L_q$ for all $q \ne p$ by Remark 2.7. As in the last section, we may further assume that $L_p \cong \mathbb{A} \perp \langle 2p\epsilon \rangle \perp \langle p^k\delta \rangle$, where $\epsilon, \delta \in \mathbb{Z}_p^\times$. If a is the positive generator of $\mathfrak{n}(M^\perp)$, then $k + 2 - \mathrm{ord}_p(a) \le \mathrm{ord}_p(dM)$.

The final step of the proof depends on the parity of k and whether $p > 2$ or $p = 2$. We give the details for the case $p = 2$ and $k \equiv 1 \bmod 2$, and leave the other cases to the reader. First of all, choose a prime $r > 2dL$ that satisfies:

$$r \equiv \epsilon(10\epsilon - \delta) \bmod 8 \quad \text{and} \quad r \in -(\mathbb{Z}_q^\times)^2 \quad \text{for all odd } q \text{ dividing } dL.$$

Then choose another prime $t > 2^{\mathrm{ord}_2(dM)} r dM$ such that

$$t \equiv \delta \bmod 8 \quad \text{and} \quad r \in -(\mathbb{Z}_t^\times)^2.$$

Although δ and ϵ depend on L_2, there are only finitely many square classes in $\mathbb{Z}_2^\times$ containing δ or ϵ, and the number of possible pairs of r and t is bounded. As in

Section 5, there exist integers W and $b < t$ such that $tW = b^2 + 2^{k+1}r$. Let N be the lattice

$$\begin{pmatrix} 2t & 2b \\ 2b & 2W \end{pmatrix}.$$

Then $N \longrightarrow \mathrm{gen}(L)$ but $\mathbb{Q}_2 N \not\longrightarrow \mathbb{Q}_2 M$. If $N \longrightarrow L$, then we can argue as in Section 5 that k and $\mathrm{ord}_2(a)$ are bounded. Now, observe that $N_2 \cong \langle 2\delta\rangle \perp \langle 2^{k+2}(10\epsilon - \delta)\rangle$. Let $\{x, y\}$ be an orthogonal basis of N_2 that gives that splitting. For each integer $\gamma > 0$, construct a lattice $N(\gamma)$ by specifying its local completions as

$$N(\gamma)_q = \begin{cases} N_q & \text{if } q \neq 2\ , \\ \mathbb{Z}_2[x, 2^\gamma y] & \text{if } q = 2\ . \end{cases}$$

Since L is almost 2-regular, there exists γ such that $N(\gamma) \longrightarrow L$. If $\sigma : N(\gamma) \longrightarrow L$ is a representation, then $\sigma(x)$ must be primitive in L_2 but $\sigma(2^\gamma y)$ is never primitive. In other words, $N(\gamma - 1)$ is also represented by L. By repeating this argument we see that $N \longrightarrow L$.

The other cases require different constructions of N but proceed in a similar manner. For $p = 2$ and $k \equiv 0 \bmod 2$, N_2 should be isometric to $\langle 4\epsilon\rangle \perp \langle 2^k\delta\rangle$. When $p > 2$, we want N_p to be $\langle p\epsilon\rangle \perp \langle p^k\delta\rangle$ if $k \equiv 1 \bmod 2$, and $\langle -\delta\rangle \perp \langle p^k\delta\rangle$ otherwise. □

Example 6.2. Let $K(n)$ be the lattice $\langle 1\rangle \perp \langle 1\rangle \perp \langle 3^{2n+1}\rangle$. An integer $3^a t$ with $3 \nmid t$ is represented by $\mathrm{gen}(K(n))$ if and only if (i) a is even or (ii) a is odd, $a \geq 2n+1$, and t is a square mod 3.

Since $K := K(0)$ is regular [12], K represents all those integers satisfying (i) or (ii). Furthermore, in (ii) any representation of $3^a t$ must be inside $K(n)$. The same happens in (i) when $a > 2n$. However, if $a \leq 2n$, then t must be primitively represented by $\mathrm{gen}(K(n))$ which has only one spinor genus. By [5], $t \longrightarrow K(n)$ for all sufficiently large t. Consequently, $K(n)$ is almost 1-regular and Theorem 1.2 does not hold for $n = 1$.

7. Spinor n-Regular Lattices

Lemma 7.1. *Let L be a spinor n-regular lattice of rank $n+2$ for some $n \geq 2$. Then L is $(n-1)$-regular.*

Proof. Let N be a lattice of rank $n-1$ which is represented by $\mathrm{gen}(L)$. Then N is represented by $\mathrm{spn}^+(L)$ because $\mathrm{rank}(L) - \mathrm{rank}(N) = 3$, see [10]. The rest of the proof is the same as that of Lemma 6.1. □

It is clear that any lattice in $\mathrm{spn}^+(\Lambda_{2p}(L))$ is of the form $\Lambda_{2p}(K)$ for some $K \in \mathrm{spn}^+(L)$. Therefore, Lemmas 2.1 and 2.2 hold for spinor n-regular lattices.

Proof of Theorem 1.3. Let L be a spinor n-regular lattice of rank $n+2$ for some $n \geq 2$. By Lemma 7.1 and Theorem 1.2, it suffices to treat the special case $n = 2$.

Fix a prime p. Let M be a 3×3 section of L. As in the last section, dM is bounded and we may assume that M_q is isotopic for all $q \neq p$. Furthermore, $\mathbb{H} \longrightarrow L_q$ for any $q \neq p$, and $L_p \cong \mathbb{A} \perp \langle 2p\epsilon\rangle \perp \langle p^k\delta\rangle$. All these together imply that $\mathrm{gen}(L)$ coincides with $\mathrm{spn}^+(L)$ and hence L is regular. Therefore the proof of Theorem 1.2 presented in the last section prevails here. Note that since L is regular, the binary lattice N constructed there is represented by L. □

Acknowledgement

The authors would like to thank Professor John Hsia for his suggestions and encouragement.

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