TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 355, Number 6, Pages 2341–2384
S 0002-9947(03)03266-5
Article electronically published on February 4, 2003

INDUCTION THEOREMS OF SURGERY OBSTRUCTION GROUPS

MASAHARU MORIMOTO

Dedicated to Professor Anthony Bak for his sixtieth birthday

ABSTRACT. Let G be a finite group. It is well known that a Mackey functor $\{H \mapsto M(H)\}$ is a module over the Burnside ring functor $\{H \mapsto \Omega(H)\}$, where H ranges over the set of all subgroups of G. For a fixed homomorphism $w : G \to \{-1, 1\}$, the Wall group functor $\{H \mapsto L_n^h(\mathbb{Z}[H], w|_H)\}$ is not a Mackey functor if w is nontrivial. In this paper, we show that the Wall group functor is a module over the Burnside ring functor as well as over the Grothendieck-Witt ring functor $\{H \mapsto \mathrm{GW}_0(\mathbb{Z}, H)\}$. In fact, we prove a more general result, that the functor assigning the equivariant surgery obstruction group on manifolds with middle-dimensional singular sets to each subgroup of G is a module over the Burnside ring functor as well as over the special Grothendieck-Witt ring functor. As an application, we obtain a computable property of the functor described with an element in the Burnside ring.

1. Introduction

Dress' induction theory ([10], [11], [12]) of Mackey functors has been useful for algebraic computation of Wall's surgery obstruction groups ([27]) with trivial orientation homomorphisms and related groups (cf. [6], [13], [14]) as well as for applications in transformation groups (e.g. [16], [18], [25], [26]). In this paper, we develop induction theory for surgery obstruction groups appearing in [4], [5] and [19], which allows nontrivial orientation homomorphisms, and by using this generalization and [22, Theorem 1.1] we can construct various group actions on smooth manifolds (e.g. [4], [15], [16], [17], [20], [21], [24]).

Throughout this paper, let G be a finite group, $\mathcal{S}(G)$ the set of all subgroups of G, and R a principal ideal domain (possibly a commutative field). Hence R is a commutative ring and any finitely generated projective R-module is free over R. An R-module is always assumed to be finitely generated over R, unless otherwise stated.

Let $\mathrm{GW}_0(R, G)$ denote the Grothendieck-Witt ring in A. Dress [11]. It is well known that the functor $H \mapsto \mathrm{GW}_0(R, H)$, $H \in \mathcal{S}(G)$, with canonical correspondence of morphisms is a Green functor, which is a special case of Theorem 11.3 since $\mathrm{GW}_0(R, G) = \mathrm{GW}_0(R, G, \emptyset)$. Let $\mathcal{C}(G)$ denote the set of all cyclic subgroups

Received by the editors January 1, 2002.
2000 *Mathematics Subject Classification.* Primary 19G12, 19G24, 19J25; Secondary 57R67.
Key words and phrases. Induction, restriction, Burnside ring, Grothendieck group, Witt group, equivariant surgery.
Partially supported by a Grant-in-Aid for Scientific Research (Kakenhi).

of G. By [11, Theorem 1], the functor $H \mapsto \mathrm{GW}_0(R, H)$ is $\mathcal{C}(G)$-hypercomputable in the sense of A. Bak [2]. Let $w : G \to \{-1, 1\}$ be a homomorphism and $n = 2k$ an even integer. If w is nontrivial, the Wall group functor $H \mapsto L_n^h(R[H], w|_H)$ ([27]), $H \in \mathcal{S}(G)$, is not a Mackey functor. Since $L_n^h(R[G], w) = \mathrm{WQ}_0(\boldsymbol{A}, \emptyset)$ with $\boldsymbol{A} = (R, G, \emptyset, \emptyset, (-1)^k, w)$, Propositions 12.7 and 2.6 imply that the Wall group functor is a w-Mackey functor in the sense of Definition 2.2 and a module over the Burnside ring functor. Furthermore, the Wall group functor is a module over the functor $H \mapsto \mathrm{GW}_0(R, H)$, which is a special case of Theorem 12.10. Thus, we obtain the theorem:

Theorem 1.1. *Let $w : G \to \{-1, 1\}$ be a homomorphism and n an even integer. Then the Wall group functor $H \mapsto L_n^h(R[H], w|_H)$, $H \in \mathcal{S}(G)$, is $\mathcal{C}(G)$-hypercomputable.*

The main purpose of this paper is to study the induction–restriction theory of the equivariant surgery obstruction group $\mathrm{SWQ}_0(R, G, Q, S, \Theta_G)$ obtained by Bak and Morimoto [5], which consists of equivalence classes of special λ-quadratic $R[G]$-modules. This surgery obstruction group is determined by a datum

$$\mathcal{D} = (R, G, Q, S, \lambda, w, \Theta_G, \rho^{(2)}).$$

The ingredient λ stands for a symmetry, namely either 1 or -1. Let $G(2)$ denote the subset of G consisting of all elements of order 2. An element $g \in G(2)$ is called *λ-symmetric* or *λ-quadratic* if $g = \lambda w(g) g^{-1}$ or $g = -\lambda w(g) g^{-1}$, respectively. The ingredients Q and S are conjugation-invariant subsets of $G(2)$ consisting of λ-quadratic elements and λ-symmetric ones, respectively. Let $\mathfrak{P}(S)$ denote the set of all subsets of S. In a general case, Θ_G stands for a finite G-set and $\rho^{(2)}$ is a G-map $\Theta_G \to \mathfrak{P}(S)$. In the case where S and Θ_G are both empty and $\lambda = (-1)^k$, the group $\mathrm{SWQ}_0(\mathbb{Z}, G, Q, S, \Theta_G)$ coincides with the Bak group $W_{2k}(\mathbb{Z}[G], \Gamma Q, w)$ (see [19]); if moreover Q is also empty, then the group is nothing but the Wall group $L_{2k}^h(\mathbb{Z}[G], w)$ (see [27]).

In the current section, since the case $\Theta_G = S$ has interesting applications (e.g. [4], [15], [16]), we let Θ_G and $\rho^{(2)}$ be the same as the set S and the map $s \mapsto \{s\}$, $s \in S$, respectively.

We detail the pairing

$$\mathrm{SGW}_0(\mathbb{Z}, G, S, S) \times \mathrm{SWQ}_0(\mathbb{Z}, G, Q, S, S) \longrightarrow \mathrm{SWQ}_0(\mathbb{Z}, G, Q, S, S)$$

in Sections 9 and 10, and show that $\mathrm{SWQ}_0(\mathbb{Z}, G, Q, S, S)$ is a module over the special Grothendieck-Witt ring $\mathrm{SGW}_0(\mathbb{Z}, G, S, S)$, which corrects the invalid description [15, page 513, lines 9–10] of the pairing.

The groups $\mathrm{GW}_0(R, G)$ and $L_n^h(R[G], w)$ with $n = 2k$ have the hyperelementary computability. Dress proved this fact by studying the index of the subgroup $I(\mathfrak{H}_\Sigma(G), \mathrm{GW}_0)$ of $\mathrm{GW}_0(R, G)$ ([11, Theorem 1]), which we call the *Dress index*. The theorem looks technical but is fundamental. It is natural to regard the Burnside ring as a generalization of the ring of integers in the theory of transformation groups. Thus, one expects that some computability of the groups $\mathrm{SGW}_0(\mathbb{Z}, G, S, S)$ and $\mathrm{SWQ}_0(\mathbb{Z}, G, Q, S, S)$ can be described with an element in the Burnside ring instead of the Dress index. The following theorems are obtained in this respect.

Let $1_{\Omega(G)}$ denote the unit of the Burnside ring $\Omega(G)$.

Theorem 1.2. *Let S be a conjugation-invariant subset of G consisting of elements of order 2, let $\mathcal{F}$ be a conjugation-invariant set of subgroups of G such that*

$$S \times S \subset \bigcup_{H \in \mathcal{F}} H \times H,$$

and let β be an element of the Burnside ring $\Omega(G)$ such that

$$\mathrm{Res}^G_H \beta = 1_{\Omega(H)} \quad \textit{for any } H \in \mathcal{F}.$$

If $\mathcal{F}$ contains all 2-hyperelementary (resp. cyclic) subgroups of G, then, for an arbitrary element $x \in \mathrm{SGW}_0(R, G, S, S)$,

$$(1_{\Omega(G)} - \beta)^2 x = 0$$

(resp. $(1_{\Omega(G)} - \beta)^{2k+3} x = 0$, where $|G| = 2^k m$ with m odd).

We say that R is *square identical* if

$$r^2 \equiv r \mod 2R \quad \text{for all } r \in R. \tag{1.1}$$

Theorem 1.3. *Let S, β and $\mathcal{F}$ be as in the theorem above. Suppose that R is square identical, and each element of S is λ-symmetric. Let Q be a conjugation-invariant subset of G consisting of λ-quadratic elements of order 2. If $\mathcal{F}$ contains all 2-hyperelementary (resp. cyclic) subgroups of G, then for an arbitrary element x of $\mathrm{SWQ}_0(R, G, Q, S, S)$,*

$$(1_{\Omega(G)} - \beta)^2 x = 0$$

(resp. $(1_{\Omega(G)} - \beta)^{2k+3} x = 0$, where $|G| = 2^k m$ with m odd).

Note that the datum $\mathcal{D} = (R, G, Q, S, \lambda, w, S, \rho^{(2)})$, where $\rho^{(2)} : S \to \mathfrak{P}(S)$ is the "identity map" $s \mapsto \{s\}$, yields the datum $\mathcal{D} = (R, H, Q \cap H, S \cap H, \lambda, w|_H, S \cap H, \rho^{(2)}|_{S \cap H})$ and determines the group $\mathrm{SWQ}_0(R, H, Q \cap H, S \cap H, S \cap H)$ for each subgroup H of G.

Theorem 1.4. *Let G be a nonsolvable group and let R, Q and S be as in the previous theorem. Then*

$$\mathrm{SWQ}_0(R, G, Q, S, S) = \sum_H \mathrm{Ind}^G_H \mathrm{SWQ}_0(R, H, Q \cap H, S \cap H, S \cap H),$$

and the restriction homomorphism

$$\mathrm{Res} : \mathrm{SWQ}_0(R, G, Q, S, S) \longrightarrow \bigoplus_H \mathrm{SWQ}_0(R, H, Q \cap H, S \cap H, S \cap H)$$

is injective, where H ranges over the set of all solvable subgroups of G.

Each of Theorems 1.2–1.4 is slightly generalized in Section 13.

The organization of the paper is as follows. In Section 2, we define a w-Mackey functor, a Green functor, and a module over a Green functor. In Section 3, we observe basic properties of Θ-positioned $R[G]$-modules, namely induction-restriction properties and the Mackey double coset formula. Section 4 is devoted to observing induction-restriction properties of Θ-positioned Hermitian $R[G]$-modules as well as defining their Grothendieck-Witt rings. In Section 5, we introduce the ∇-invariant of Θ-positioned Hermitian $R[G]$-modules and define the special Grothendieck-Witt groups. Similarly to Wall's surgery theory, $R[G]$-valued λ-Hermitian forms are indispensable objects to equivariant surgery theory on manifolds with middle-dimensional singular sets. Section 6 is devoted to observing induction-restriction properties of $R[G]$-valued λ-Hermitian modules. Sections 7 and 8 are devoted to defining the Witt groups and the special Witt groups of Θ-positioned quadratic $R[G]$-modules, respectively. The tensor product of a Hermitian $R[G]$-module and a quadratic $R[G]$-module is introduced in Section 9, and it is discussed with ∇-invariants in Section 10. Section 11 is devoted to showing that the Grothendieck-Witt rings and special Grothendieck-Witt rings are Green functors (possibly without unit). In Section 12 we show that the bifunctor assigning the H-surgery obstruction group to a subgroup H of G is a module over the special Grothendieck-Witt ring functor. In Section 13, we present applications relevant to G-surgery.

Acknowledgements. I wish to express my heartfelt gratitude to Anthony Bak for his numerous valuable suggestions to my research related to K-theory. If it had not been for his influence, this work would not have been done. In addition, I thank the referee for his comments on elaborating upon the manuscript.

2. Bifunctors, w-Mackey functors and Green functors

Let $\mathcal{G}$ denote the category whose objects are subgroups of G and whose morphisms are inclusions $j_{H,K} : H \to K$, where $H \subset K \subset G$, conjugations $c_{(H,g)} : H \to gHg^{-1}$; $a \mapsto gag^{-1}$, where $H \subset G$ and $g \in G$, and compositions of those maps. Let $\mathcal{A}$ stand for the category whose objects are abelian groups and whose morphisms are group homomorphisms. We denote by $\mathbb{Z}[\mathcal{S}(G)]$ the free abelian group generated by all elements of $\mathcal{S}(G)$; hence each element of $\mathbb{Z}[\mathcal{S}(G)]$ has the form $\sum_H n_H H$ with $n_H \in \mathbb{Z}$. Let $\Omega(G)$ denote the Burnside ring of G (cf. [7], [8], [9], [23]). In fact, $\Omega(G)$ is the free abelian group generated by all G-isomorphism classes $[G/H]$ of finite G-sets G/H with $H \in \mathcal{S}(G)$. Clearly, one has the canonical homomorphism from $\mathbb{Z}[\mathcal{S}(G)]$ to $\Omega(G)$ such that $H \mapsto [G/H]$. In this paper, we mean by a *bifunctor*

$$L = (L^*, L_*) : \mathcal{G}(G) \to \mathcal{A}$$

a pair consisting of a contravariant functor $L^* : \mathcal{G}(G) \to \mathcal{A}$ and a covariant functor $L_* : \mathcal{G}(G) \to \mathcal{A}$ such that $L_*(H) = L^*(H)$, which is written as $L(H)$, for all $H \in \mathcal{S}(G)$. If the context is clear, f^* and f_* stand for $L^*(f)$ and $L_*(f)$ respectively, and Res_H^K and Ind_H^K stand for $L^*(j_{H,K})$ and $L_*(j_{H,K})$ respectively. Each bifunctor $L = (L^*, L_*) : \mathcal{G} \to \mathcal{A}$ possesses the canonical pairing

$$(2.1)\qquad \mathbb{Z}[\mathcal{S}(G)] \times L(G) \longrightarrow L(G);\ \left(\sum_H n_H\ H,\ x\right) \longmapsto \sum_H n_H \mathrm{Ind}_H^G(\mathrm{Res}_H^G x),$$

for $n_H \in \mathbb{Z}$ and $x \in L(G)$. It is interesting to look for a sufficient condition so that the pairing (2.1) factors through a pairing

$$(2.2)\qquad \Omega(G) \times L(G) \longrightarrow L(G).$$

If L is a Mackey functor, then, as was seen in [7, Proposition 6.2.3], the pairing (2.1) factors through a pairing (2.2). In the case where the orientation homomorphism $w : G \to \{-1, 1\}$ is not trivial, the Wall group functor $H \mapsto L_n^h(\mathbb{Z}[H], w|_H)$, $H \in \mathcal{S}(G)$, is not a Mackey functor; however, it will turn out that the associated pairing (2.1) factors through (2.2).

Let $L : \mathcal{G} \to \mathcal{A}$ be a bifunctor. Note that the kernel of the canonical map $\mathbb{Z}[\mathcal{S}(G)] \to \Omega(G)$ is

$$\langle H - gHg^{-1} \mid H \in \mathcal{S}(G),\ g \in G\rangle_{\mathbb{Z}}.$$

If

$$(2.3)\qquad L_*(j_{H,G})L^*(j_{H,G}) = L_*(j_{gHg^{-1},G})L^*(j_{gHg^{-1},G}) \quad (\forall H \in \mathcal{S}(G),\ \forall g \in G),$$

then the pairing (2.1) factors through (2.2).

Proposition 2.1. *Suppose $L_*(c_{(gHg^{-1},g^{-1})}) = L^*(c_{(H,g)})$ for all $H \in \mathcal{S}(G)$ and $g \in G$. Then the equality (2.3) holds if and only if*

$$(1)\qquad L^*(c_{(G,g)})L_*(j_{H,G})L^*(j_{H,G}) = L_*(j_{H,G})L^*(j_{H,G})L^*(c_{(G,g)})$$

for all $H \in \mathcal{S}(G)$ and $g \in G$.

Proof. By definition, the diagrams

$$\begin{array}{ccc} L(G) & \xrightarrow{L^*(j_{H,G})} & L(H) \\ {\scriptstyle L^*(c_{(G,g^{-1})})}\downarrow & & \downarrow{\scriptstyle L^*(c_{(gHg^{-1},g^{-1})})} \\ L(G) & \xrightarrow[L^*(j_{gHg^{-1},G})]{} & L(gHg^{-1}) \end{array}$$

and

$$\begin{array}{ccc} L(H) & \xrightarrow{L_*(j_{H,G})} & L(G) \\ {\scriptstyle L_*(c_{(gHg^{-1},g^{-1})})}\uparrow & & \uparrow{\scriptstyle L_*(c_{(G,g^{-1})})} \\ L(gHg^{-1}) & \xrightarrow[L_*(j_{gHg^{-1},G})]{} & L(G) \end{array}$$

commute. By using the hypothesis above, we obtain the commutative diagram

$$\begin{array}{ccc} L(G) & \xrightarrow{L_*(j_{H,G})L^*(j_{H,G})} & L(G) \\ {\scriptstyle L^*(c_{(G,g^{-1})})}\downarrow & & \uparrow{\scriptstyle L_*(c_{(G,g^{-1})})} \\ L(G) & \xrightarrow[L_*(j_{gHg^{-1},G})L^*(j_{gHg^{-1},G})]{} & L(G). \end{array}$$

Thus (2.3) holds if and only if

$$L_*(j_{gHg^{-1},G})L^*(j_{gHg^{-1},G}) = L_*(c_{(G,g^{-1})})L_*(j_{gHg^{-1},G})L^*(j_{gHg^{-1},G})L^*(c_{(G,g^{-1})}),$$

namely

$$L^*(c_{(G,g^{-1})})L_*(j_{gHg^{-1},G})L^*(j_{gHg^{-1},G}) = L_*(j_{gHg^{-1},G})L^*(j_{gHg^{-1},G})L^*(c_{(G,g^{-1})}).$$

This concludes the proposition. □

Let $w : G \to \{-1, 1\}$ be a homomorphism. We introduce a slight generalization of a Mackey functor (cf. [2], [7]).

Definition 2.2. A bifunctor $M = (M^*, M_*)$ from $\mathcal{G}$ to $\mathcal{A}$ is called a *w-Mackey functor* if the following conditions (1)–(3) are fulfilled:

(1) $M_*(c_{(H,g)}) = M^*(c_{(gHg^{-1},g^{-1})})$ for all $H \in \mathcal{S}(G)$ and $g \in G$,

(2) $M^*(c_{(H,h)}) = w(h)id_{M(H)}$ (hence $M_*(c_{(H,h)}) = w(h)id_{M(H)}$) for all $H \in \mathcal{S}(G)$ and $h \in H$,

(3) $M^*(j_{K,G}) \circ M_*(j_{H,G})$ coincides with

$$\bigoplus_{KgH \in K\backslash G/H} M_*(j_{K\cap gHg^{-1},K}) \circ (w(g)M_*(c_{(H\cap g^{-1}Kg,g)})) \circ M^*(j_{H\cap g^{-1}Kg,H})$$

for any H, $K \in \mathcal{S}(G)$.

A w-Mackey functor for trivial w is an ordinary Mackey functor. We will see that if w is nontrivial, then the Wall group functor $H \mapsto L_n^h(\mathbb{Z}[H], w|_H)$ is not an ordinary Mackey functor but a w-Mackey functor (cf. Propositions 6.6, 6.7, 6.8, 12.4, 12.5 and 12.6). The next proposition is clear by definition.

Proposition 2.3. *If $M = (M^*, M_*)$ is a w-Mackey functor, then $L = (L^*, L_*)$, given so that $L(H) = M(H)$, $L^*(j_{H,K}) = M^*(j_{H,K})$, $L_*(j_{H,K}) = M_*(j_{H,K})$, $L^*(c(H,g)) = w(g)M^*(c(H,g))$ and $L_*(c(H,g)) = w(g)M_*(c(H,g))$ for all $H \subset K$ and $g \in G$, is a Mackey functor.*

In the case above, we say that L is the *Mackey functor associated with M*.

We use the term "Frobenius pairing" in a sense slightly more general than [7], where relevant bifunctors were assumed to be Mackey functors.

Definition 2.4. Let L, M and N be bifunctors from $\mathcal{G}$ to $\mathcal{A}$. A *pairing* $L \times M \to N$ is a family of biadditive maps

$$L(H) \times M(H) \longrightarrow N(H);\ (x, y) \longmapsto x \cdot y,$$

where H runs over $\mathcal{S}(G)$. Such a pairing is called a *Frobenius pairing* if the following conditions (1)–(3) are satisfied for any morphism $f : H \to K$ in $\mathcal{G}$:

(1) $N^*(f)(x \cdot y) = (L^*(f)x) \cdot (M^*(f)y)$ for all $x \in L(K)$, $y \in M(K)$,

(2) $x \cdot M_*(f)(y) = N_*(f)(L^*(f)(x) \cdot y)$ for all $x \in L(K)$, $y \in M(H)$,

(3) $L_*(f)(x) \cdot y = N_*(f)(x \cdot M^*(f)(y))$ for all $x \in L(H)$, $y \in M(K)$.

Each of (2), (3) is referred to as the *Frobenius reciprocity law*.

Let us recall the definition of a Green functor.

Definition 2.5. A Mackey functor $M = (M_*, M^*) : \mathcal{G} \to \mathcal{A}$ is called a *Green functor* if each $M(H)$, $H \in \mathcal{S}(G)$, is a ring with unit and the associated pairing $M \times M \to M$ is a Frobenius pairing. If the existence of the unit in $M(H)$ is not guaranteed, then M is referred as a *Green functor, possibly without unit*.

The Burnside ring functor $H \mapsto \Omega(G)$ is a Green functor. Let $U : \mathcal{G} \to \mathcal{A}$ be a Green functor. We mean by a *U-module* L (or a module L over U) a bifunctor $L : \mathcal{G} \to \mathcal{A}$ equipped with a Frobenius pairing $U \times L \to L$.

Proposition 2.6. *A w-Mackey functor M is a module over the Burnside ring functor.*

Proof. Let L be the Mackey functor associated with M in Proposition 2.3. By [7, Proposition 6.2.3], L is a module over the Burnside ring functor. Hence, L satisfies the equality (1) in Proposition 2.1. By using the relations between M and L in Proposition 2.3, we can check that M satisfies the equality (1) in Proposition 2.1, and furthermore that M is a module over the Burnside ring functor. □

Proposition 2.7. *A module over a Green functor is a module over the Burnside ring functor.*

Proof. Let $L = (L^*, L_*) : \mathcal{G} \to \mathcal{A}$ be a module over a Green functor $U = (U^*, U_*) : \mathcal{G} \to \mathcal{A}$. Then the associated pairing

$$\Omega(H) \times L(H) \longrightarrow L(H)$$

can be defined so that $a \cdot x = (a \cdot 1_{U(H)}) \cdot x$ for $a \in \Omega(H)$ and $x \in L(H)$, where $1_{U(H)}$ is the identity element of $U(H)$. It is straightforward to check the Frobenius reciprocity laws of the pairing. □

3. Θ-Positioned $R[G]$-modules

Let Θ be a finite G-set. A pair (M, α) consisting of an $R[G]$-module M and a G-map $\alpha : \Theta \to M$ is called a *Θ-positioned* $R[G]$*-module*. Let H and K be finite groups and $\varphi : H \to K$ a homomorphism. For a finite H-set X, we define the K-set $K \times_{H,\varphi} X$ as the quotient set of $K \times X$ with respect to the equivalence relation $\sim$ generated by $(k\varphi(h), x) \sim (k, hx)$, $h \in H$. The set $K \times_{H,\varphi} X$ is also denoted by $K \times_\varphi X$ or $K \times_H X$ if the context is clear. For an $R[H]$-module M, the $R[K]$-module $R[K] \otimes_{R[H],\varphi} M$ is defined as follows. Let $F(R[K] \times M)$ denote the R-free module with basis $R[K] \times M$ which may not be finitely generated over R.

Let T denote the R-submodule generated by all elements of the form

$$\begin{aligned}
&r(a, x) - (ra, x), \quad r(a, x) - (a, rx),\\
&(a + b, x) - (a, x) - (b, x), \quad (a, x + y) - (a, x) - (a, y), \quad \text{or}\\
&(a\varphi(h), x) - (a, hx),
\end{aligned}$$

where r ranges over R, a and b over $R[K]$, x and y over M, and h over H. Then $R[K] \otimes_{R[H],\varphi} M$ is defined to be the quotient module $F(R[K] \times M)/T$, which will also be denoted by $R[K] \otimes_\varphi M$ or $R[K] \otimes_{R[H]} M$. The element of the module represented by $(a, x) \in F(R[K] \times M)$ is denoted by $a \otimes_{R[H],\varphi} x$, which will also be written as $a \otimes_\varphi x$, $a \otimes_{R[H]} x$ or $a \otimes x$ if the context is clear. The K-action on $R[K] \otimes_{R[H],\varphi} M$ is given by $(k, a \otimes_{R[H],\varphi} x) \mapsto (ka) \otimes_{R[H],\varphi} x$.

Let Θ_H be a finite H-set, Θ_K a finite K-set, and $\psi : \Theta_H \to \Theta_K$ a φ-equivariant map, namely

$$\psi(ht) = \varphi(h)\psi(t) \quad (h \in H,\ t \in \Theta_H).$$

Let $\boldsymbol{\varphi}$ stand for the pair (φ, ψ).

For a Θ_K-positioned $R[K]$-module $\boldsymbol{N} = (N, \beta)$, we define the Θ_H-positioned $R[H]$-module $\boldsymbol{\varphi}^\# \boldsymbol{N} = (\varphi^\# N, \psi^\# \beta)$ so that the underlying R-module of $\varphi^\# N$ is the same as N but the H-action on $\varphi^\# N$ is given by $(h, x) \mapsto \varphi(h)x$ for $h \in H$, $x \in \varphi^\# N$, and $\psi^\# \beta : \Theta_H \to \varphi^\# N$ is given by $\psi^\# \beta(t) = \beta(\psi(t))$ for $t \in \Theta_H$.

Proposition 3.1. *Let $\varphi : H \to K$ and $\psi : \Theta_H \to \Theta_K$ be as above and let $\boldsymbol{N}_i = (N_i, \beta_i)$, $i = 1,\ 2$, be Θ_K-positioned $R[K]$-modules. Then $\boldsymbol{\varphi}^\# \boldsymbol{N}_1 \otimes_R \boldsymbol{\varphi}^\# \boldsymbol{N}_2 = \boldsymbol{\varphi}^\# (\boldsymbol{N}_1 \otimes_R \boldsymbol{N}_2)$; namely, $(\varphi^\# N_1 \otimes_R \varphi^\# N_2, \psi^\# \beta_1 \otimes_R \psi^\# \beta_2)$ is canonically isomorphic to $(\varphi^\# (N_1 \otimes_R N_2), \psi^\# (\beta_1 \otimes_R \beta_2))$.*

Proof. By definition, the underlying R-modules of $\varphi^\# N_1 \otimes_R \varphi^\# N_2$ and $\varphi^\# (N_1 \otimes_R N_2)$ are $N_1 \otimes_R N_2$. One can check without difficulties that the K-actions of the two modules coincide. Moreover, we have

$$(\psi^\# \beta_1 \otimes_R \psi^\# \beta_2)(t) = \beta_1(\psi(t)) \otimes_R \beta_2(\psi(t)) = \psi^\# (\beta_1 \otimes_R \beta_2)(t)$$

for all $t \in \Theta_H$. □

To the contrary, for a Θ_H-positioned $R[H]$-module $\boldsymbol{M} = (M, \alpha)$, we define the Θ_K-positioned $R[K]$-module $\boldsymbol{\varphi}_\# \boldsymbol{M} = (\varphi_\# M, \psi_\# \alpha)$ by $\varphi_\# M = R[K] \otimes_{R[H],\varphi} M$ and

$$\psi_\# \alpha(t) = \sum_{[k,t']} \{k \otimes_\varphi \alpha(t') \mid [k, t'] \in K \times_{H,\varphi} \Theta_H \text{ such that } k\psi(t') = t\} \quad \text{for } t \in \Theta_K.$$

The K-equivariance of the map $\psi_\#\alpha$ holds because, for $a \in K$ and $t \in \Theta_K$,

$$\begin{aligned}
\psi_\#\alpha(at) &= \sum_{[k,t'] \in K \times_{H,\varphi} \Theta_H} \{k \otimes_\varphi \alpha(t') \mid k\psi(t') = at\} \\
&= \sum_{[k,t'] \in K \times_{H,\varphi} \Theta_H} \{k \otimes_\varphi \alpha(t') \mid a^{-1}k\psi(t') = t\} \\
&= \sum_{[ak',t'] \in K \times_{H,\varphi} \Theta_H} \{ak' \otimes_\varphi \alpha(t') \mid k'\psi(t') = t\} \\
&= a \sum_{[ak',t'] \in K \times_{H,\varphi} \Theta_H} \{k' \otimes_\varphi \alpha(t') \mid k'\psi(t') = t\} \\
&= a \sum_{[k',t'] \in K \times_{H,\varphi} \Theta_H} \{k' \otimes_\varphi \alpha(t') \mid k'\psi(t') = t\} \\
&= a\psi_\#\alpha(t).
\end{aligned}$$

Proposition 3.2. *Let H be a subgroup of G, $\boldsymbol{M} = (M, \alpha)$ a Θ_H-positioned $R[H]$-module, g an element of G, and $\psi : \Theta_H \to \Theta_{gHg^{-1}}$ a $c_{H,g}$-equivariant bijection. Then the diagram*

$$\begin{array}{ccc}
\Theta_{gHg^{-1}} & \xrightarrow{\ \psi_\#\alpha\ } & c_{(H,g)_\#}M \\
& \searrow^{\psi^{-1\#}\alpha} & \downarrow f_0 \\
& & c^\#_{(gHg^{-1},g^{-1})}M
\end{array}$$

commutes, where $f_0 : c_{(H,g)_\#}M \to c^\#_{(gHg^{-1},g^{-1})}M$ is the $R[gHg^{-1}]$-isomorphism such that

$$f_0(e \otimes_{H,c_{(H,g)}} x) = x \quad \textit{for } x \in M.$$

Proof. Let t be an element of Θ_H. Then by definition we have $\psi_\#\alpha(\psi(t)) = e \otimes_{H,c_{(H,g)}} \alpha(t)$ and $\psi^{-1\#}\alpha(\psi(t)) = \alpha(t)$, which concludes the proposition. □

Proposition 3.3. *Let (H, Θ_H), (K, Θ_K), and $\boldsymbol{\varphi} = (\varphi, \psi)$ be as above. Then for a Θ_H-positioned $R[H]$-module (M, α) and a Θ_K-positioned $R[K]$-module (N, β), the Frobenius reciprocity law holds; namely, the following diagram commutes:*

$$\begin{array}{ccc}
\Theta_K & \xrightarrow{\ (\psi_\#\alpha)\otimes_R\beta\ } & (R[K] \otimes_{R[H],\varphi} M) \otimes_R N \\
& \searrow^{\psi_\#(\alpha\otimes_R\psi^\#\beta)} & \downarrow f \\
& & R[K] \otimes_{R[H],\varphi} (M \otimes_R \varphi^\# N),
\end{array}$$

where f is the canonical isomorphism such that $f((k \otimes_\varphi x) \otimes y) = k \otimes_\varphi (x \otimes k^{-1}y)$ for $k \in K$, $x \in M$ and $y \in N$.

The commutability above is referred to as $(\psi_\#\alpha) \otimes_R \beta = \psi_\#(\alpha \otimes_R \psi^\#\beta)$.

Proof. The proof runs as follows:

$$\begin{aligned}
((\psi_{\#}\alpha)\otimes_R \beta)(t) &= \sum_{[k,t']\in K\times_H\Theta_H} \{k\otimes_{\varphi}\alpha(t') \mid k\psi(t')=t\}\otimes\beta(t)\\
&= \sum_{[k,t']\in K\times_H\Theta_H} \{(k\otimes_{\varphi}\alpha(t'))\otimes\beta(t) \mid k\psi(t')=t\}\\
&= \sum_{[k,t']\in K\times_H\Theta_H} \{(k\otimes_{\varphi}\alpha(t'))\otimes k\beta(\psi(t')) \mid k\psi(t')=t\}\\
&\overset{f}{=} \sum_{[k,t']\in K\times_H\Theta_H} \{k\otimes_{\varphi}(\alpha(t')\otimes\beta(\psi(t'))) \mid k\psi(t')=t\}\\
&= \sum_{[k,t']\in K\times_H\Theta_H} \{k\otimes_{\varphi}(\alpha(t')\otimes(\psi^{\#}\beta)(t')) \mid k\psi(t')=t\}\\
&= \sum_{[k,t']\in K\times_H\Theta_H} \{k\otimes_{\varphi}(\alpha\otimes\psi^{\#}\beta)(t') \mid k\psi(t')=t\}\\
&= \psi_{\#}(\alpha\otimes_R\psi^{\#}\beta)(t).
\end{aligned}$$

□

Let H be a subgroup of G and g an element of G. Let $c_{(H,g)} : H \to gHg^{-1}$ stand for the conjugation map by g, i.e., $c_{(H,g)}(h) = ghg^{-1}$ for $h \in H$. Let Z be a finite G-set, Θ_H an H-invariant subset of Z, and $\Theta_{gHg^{-1}}$ a gHg^{-1}-invariant subset of Z such that $g\Theta_H = \Theta_{gHg^{-1}}$. Then the left translation by g, namely the map $\ell_{(H,g)} : \Theta_H \to \Theta_{gHg^{-1}}$; $t \mapsto gt$, is a $c_{(H,g)}$-equivariant bijection. Let $\boldsymbol{c}_{(H,g)}$ denote the pair $(c_{(H,g)}, \ell_{(H,g)})$. If the context is clear, then we abuse ${c_{(H,g)}}_{\#}$ for ${\ell_{(H,g)}}_{\#}$, and $c^{\#}_{(H,g)}$ for $\ell^{\#}_{(H,g)}$.

In the special case where $g \in H$, the conjugation map $c_{(H,g)}$ is a map from H to itself. Note that the map

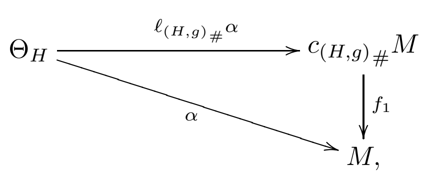
$$f_1 : {c_{(H,g)}}_{\#}M \longrightarrow M;\ e\otimes_{c_{(H,g)}} x \longmapsto gx$$

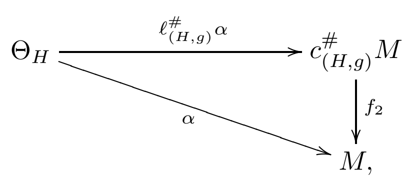
is an $R[H]$-isomorphism. In addition, the map

$$f_2 : c^{\#}_{(H,g)}M \longrightarrow M;\ x \longmapsto g^{-1}x$$

is an $R[H]$-isomorphism.

Proposition 3.4. *Let H be a subgroup of G and Θ_H a finite H-set. Then for any Θ_H-positioned $R[H]$-module (M,α) and $g \in H$, the following diagrams commute:*





where f_1 and f_2 are the $R[H]$-isomorphisms given above.

These commutabilities are referred to as $\ell_{(H,g)_\#}\alpha = \alpha$ (or $c_{(H,g)_\#}\alpha = \alpha$) and $\ell^\#_{(H,g)}\alpha = \alpha$ (or $c^\#_{(H,g)}\alpha = \alpha$), respectively.

Proof. The commutabilities follow from the equalities

$$\begin{aligned}
f_1(\ell_{(H,g)_\#}\alpha(t)) &= \sum_{[ghg^{-1},t']\in H\times_{H,c_{(H,g)}}\Theta_H} \{f_1(ghg^{-1}\otimes_{c_{(H,g)}}\alpha(t')) \mid ghg^{-1}(gt') = t\}\\
&= \sum_{[ghg^{-1},t']\in H\times_{H,c_{(H,g)}}\Theta_H} \{f_1(e\otimes_{c_{(H,g)}}\alpha(ht')) \mid ght' = t\}\\
&= \sum_{[e,t'']\in H\times_{H,c_{(H,g)}}\Theta_H} \{f_1(e\otimes_{c_{(H,g)}}\alpha(t'')) \mid gt'' = t\}\\
&= \sum_{[e,t'']\in H\times_{H,c_{(H,g)}}\Theta_H} \{g\alpha(t'') \mid gt'' = t\}\\
&= \sum_{[e,t'']\in H\times_{H,c_{(H,g)}}\Theta_H} \{\alpha(t) \mid gt'' = t\}\\
&= \alpha(t),
\end{aligned}$$

and

$$\begin{aligned}
f_2(\ell^\#_{(H,g)}\alpha(t)) &= f_2(\alpha(\ell_{(H,g)}(t)))\\
&= f_2(\alpha(gt))\\
&= g^{-1}\alpha(gt)\\
&= \alpha(t),
\end{aligned}$$

for $t \in \Theta_H$. □

Let Z be a finite G-set. Let $\mathcal{S}(G)$ and $\mathfrak{P}(Z)$ denote the set of all subgroups of G and the set of all subsets of Z, respectively. We regard $\mathcal{S}(G)$ as a G-set by conjugation, and $\mathfrak{P}(Z)$ has the canonical G-action. Let $\Theta : \mathcal{S}(G) \to \mathfrak{P}(G)$; $H \mapsto \Theta_H$, be a G-map. We say that Θ is *intersection preserving* if

(3.1) $$\Theta_H \cap \Theta_K = \Theta_{H\cap K} \quad \text{for all } H,\, K \in \mathcal{S}(G).$$

Let $H \subset K$ be subgroups of G. Then (3.1) implies $\Theta_H \subset \Theta_K$. Thus, the inclusion map $j_{H,K} : H \to K$ is automatically associated with the inclusion map $j_{\Theta_H,\Theta_K} : \Theta_H \to \Theta_K$, and hence yields the pair $\boldsymbol{j}_{H,K} = (j_{H,K}, j_{\Theta_H,\Theta_K})$.

Usually, we use Ind_H^K for $j_{H,K\#}$, $j_{\Theta_H,\Theta_K\#}$ and $\boldsymbol{j}_{H,K\#}$, and Res_H^K for $j^\#_{H,K}$, $j^\#_{\Theta_H,\Theta_K}$ and $\boldsymbol{j}^\#_{H,K}$, if the context is clear.

Next, let g be an element of G. Since Θ is a G-map, $\Theta_{gHg^{-1}} = g\Theta_H$ holds for any subgroup H of G.

Proposition 3.5. *Let $\Theta : \mathcal{S}(G) \to \mathfrak{P}(Z)$ be an intersection-preserving G-map. Then for arbitrary subgroups H and K of G, each Θ_H-positioned $R[H]$-module $\boldsymbol{M} = (M, \alpha)$ satisfies the Mackey double coset formula. Namely,*

$$\mathrm{Res}_K^G(\mathrm{Ind}_H^G\boldsymbol{M}) = \bigoplus_{KgH\in K\backslash G/H} \mathrm{Ind}^K_{K\cap gHg^{-1}}c_{(H\cap g^{-1}Kg,g)_\#}\mathrm{Res}^H_{H\cap g^{-1}Kg}\boldsymbol{M}.$$

More precisely, the following diagram commutes:

$$\begin{array}{ccc} \Theta_H & \xrightarrow{\ \gamma\ } & \bigoplus_{KgH \in K\backslash G/H} M(K,g,H) \\ & \searrow^{\mathrm{Res}_K^G \,\mathrm{Ind}_H^G \alpha} & \downarrow \omega \\ & & \mathrm{Res}_K^G(\mathrm{Ind}_H^G M), \end{array}$$

where

$$\begin{aligned} M(K,g,H) &= \mathrm{Ind}_{K\cap gHg^{-1}}^{K} c_{(H\cap g^{-1}Kg,g)_\#} \mathrm{Res}^H_{H\cap g^{-1}Kg} M \\ &= R[K] \otimes_{R[K\cap gHg^{-1}]} (R[K\cap gHg^{-1}] \otimes_{R[H\cap g^{-1}Kg], c_{(H\cap g^{-1}Kg,g)}} \mathrm{Res}^H_{H\cap g^{-1}Kg} M), \end{aligned}$$

$$\mathrm{Res}_K^G(\mathrm{Ind}_H^G M) = \mathrm{Res}_K^G(R[G] \otimes_{R[H]} M),$$

$$\gamma = \bigoplus_{KgH \in K\backslash G/H} \mathrm{Ind}^K_{K\cap gHg^{-1}} (\ell_{(H\cap g^{-1}Kg,g)_\#}(\mathrm{Res}^H_{H\cap g^{-1}Kg}\alpha)),$$

and ω *is the* $R[K]$*-isomorphism such that*

$$\omega(k \otimes (a \otimes_{c_{(H\cap g^{-1}Kg,g)}} x)) = kg \otimes (g^{-1}ag)x \quad \text{for } k\in K,\ a \in K\cap gHg^{-1},\ x \in M.$$

Proof. Let $\alpha : \Theta_H \to M$ be an H-map, and let $\{g_1, \dots, g_\ell\}$ be a complete set of representatives of $K\backslash G/H$. For $t \in \Theta_K$, we have

$$\begin{aligned} &(\mathrm{Res}^G_K \mathrm{Ind}^G_H \alpha)(t) \\ &\quad = \sum \{g \otimes \alpha(t') \mid [g,t'] \in G\times_H \Theta_H,\ gt' = t\} \\ &\quad = \sum_{j=1}^{\ell} \sum \{gg_j \otimes \alpha(t') \mid [gg_j,t'] \in Kg_jH \times_H \Theta_H,\ g\in K,\ gg_jt' = t\} \\ &\quad = \sum_{j=1}^{\ell} \sum \{gg_j \otimes \alpha(t') \mid [gg_j,t'] \in Kg_j \times_{H\cap g_j^{-1}Kg_j} \Theta_H,\ g\in K,\ gg_jt' = t\} \\ &\quad = \sum_{j=1}^{\ell} \sum \{gg_j \otimes \alpha(t') \mid [gg_j,t'] \in Kg_j \times_{H\cap g_j^{-1}Kg_j} \Theta_{H\cap g_j^{-1}Kg_j}, \\ &\qquad\qquad g\in K,\ gg_jt' = t\} \quad \text{in } \mathrm{Res}^G_K \mathrm{Ind}^G_H M \end{aligned}$$

and

$$\begin{aligned} &(\mathrm{Ind}^K_{K\cap g_jHg_j^{-1}} \ell_{(H\cap g_j^{-1}Kg_j,g_j)_\#} \mathrm{Res}^H_{H\cap g_j^{-1}Kg_j}\alpha)(t) \\ &\quad = \sum \{g \otimes \ell_{(H\cap g_j^{-1}Kg_j,g_j)_\#} \mathrm{Res}^H_{H\cap g_j^{-1}Kg_j}\alpha(t') \mid \\ &\qquad\qquad [g,t'] \in K \times_{K\cap g_jHg_j^{-1}} \Theta_{K\cap g_jHg_j^{-1}},\ gt' = t\} \\ &\quad = \sum \{g \otimes (e \otimes \alpha(g_j^{-1}t')) \mid [g,t'] \in K \times_{K\cap g_jHg_j^{-1}} \Theta_{K\cap g_jHg_j^{-1}},\ gt' = t\} \\ &\quad = \sum \{g \otimes (e \otimes \alpha(t'')) \mid [gg_j,t''] \in Kg_j \times_{H\cap g_j^{-1}Kg_j} \Theta_{H\cap g_j^{-1}Kg_j},\ gg_jt'' = t\} \\ &\quad \overset{\omega}{\cong} \sum \{gg_j \otimes \alpha(t'') \mid [gg_j,t''] \in Kg_j \times_{H\cap g_j^{-1}Kg_j} \Theta_{H\cap g_j^{-1}Kg_j},\ gg_jt'' = t\}. \end{aligned}$$

The proposition follows immediately from these equalities. □

4. Positioned Hermitian $R[G]$-modules

In this section we introduce the Grothendieck-Witt rings of Θ-positioned Hermitian $R[G]$-modules.

Definition 4.1. Let M be an $R[G]$-module. A map $B : M \times M \to R$ is called a *Hermitian form* on M if the following conditions (1)–(3) are satisfied:
(1) B is R-bilinear,
(2) B is G-invariant, namely $B(gx, gy) = B(x, y)$,
(3) B is symmetric, namely $B(x, y) = B(y, x)$,
for all x, $y \in M$ and $g \in G$. A couple (M, B) consisting of an $R[G]$-module M and a Hermitian form B on M is called a *Hermitian* $R[G]$*-module* (or simply *Hermitian module*).

A Hermitian $R[G]$-module (M, B) such that M is a free R-module is said to be *nonsingular* if the associated map

$$M \longrightarrow M^{\#} = \mathrm{Hom}_R(M, R);\ x \mapsto B(x, -)$$

is bijective.

Let H and K be finite groups and $\varphi : H \to K$ a monomorphism. A Hermitian $R[K]$-module (N, B) yields a Hermitian $R[H]$-module $(\varphi^{\#}N, \varphi^{\#}B)$ in a canonical way. By definition $\varphi^{\#}N$ is nothing but N as an R-module. The map $\varphi^{\#}B$: $\varphi^{\#}N \times \varphi^{\#}N \to R$ is also the same as $B : N \times N \to R$. Clearly, $\varphi^{\#}B$ is R-bilinear and symmetric. It is obvious that if B is nonsingular, then so is $\varphi^{\#}B$. Since

$$\varphi^{\#}B(hx, hy) = B(\varphi(h)x, \varphi(h)y) = B(x, y)$$

for $h \in H$, x, $y \in \varphi^{\#}N$, it follows that $\varphi^{\#}B$ is H-invariant.

Proposition 4.2. *Let $\varphi : H \to K$ be a monomorphism and let (N_i, B_i), $i = 1, 2$, be Hermitian $R[K]$-modules. Then*

$$(\varphi^{\#}N_1 \otimes_R \varphi^{\#}N_2, \varphi^{\#}B_1 \otimes_R \varphi^{\#}B_2) = (\varphi^{\#}(N_1 \otimes_R N_2), \varphi^{\#}(B_1 \otimes B_2)).$$

This proposition is obviously true.

Let (M, B) be a Hermitian $R[H]$-module. Then, by definition,

$$\varphi_{\#}M = R[K] \otimes_{R[H],\varphi} M.$$

We define the R-bilinear form

$$\varphi_{\#}B : \varphi_{\#}M \times \varphi_{\#}M \to R,$$

so that

$$\varphi_{\#}B(a \otimes_\varphi x, b \otimes_\varphi y) = \delta_{a\varphi(H), b\varphi(H)} B(x, \varphi^{-1}(a^{-1}b)y),$$

for a, $b \in K$ and x, $y \in M$, where $\delta_{a\varphi(H), b\varphi(H)} = 1$ if $a\varphi(H) = b\varphi(H)$, and $\delta_{a\varphi(H), b\varphi(H)} = 0$ otherwise. It is clear that $\varphi_{\#}B$ is K-invariant and symmetric. If B is nonsingular, then so is $\varphi_{\#}B$.

Proposition 4.3. *Let H be a subgroup of G, B a Hermitian form on an $R[H]$-module M, and g an element of G. Then the diagram*

$$\begin{array}{ccc} c_{(H,g)_\#}M \times c_{(H,g)_\#}M & & \\ \Big\downarrow f_0\times f_0 & \searrow^{c_{(H,g)_\#}B} & \\ c_{(gHg^{-1},g^{-1})^\#}M \times c_{(gHg^{-1},g^{-1})^\#}M & \xrightarrow[c_{(gHg^{-1},g^{-1})^\#}B]{} & R \end{array}$$

commutes, where f_0 is the canonical $R[gHg^{-1}]$-isomorphism (cf. Proposition 3.2).

The proof is straightforward.

Proposition 4.4. *Let $\varphi : H \to K$ be a monomorphism, and let B and B' be Hermitian forms on an $R[H]$-module M and an $R[K]$-module N, respectively. Then the following diagram commutes:*

$$\begin{array}{ccc} M_1 \times M_1 & & \\ \Big\downarrow f\times f & \searrow^{\varphi_\# B\otimes_R B'} & \\ M_2 \times M_2 & \xrightarrow[\varphi_\#(B\otimes_R \varphi^\# B')]{} & R, \end{array}$$

where $M_1 = (R[K] \otimes_{R[H],\varphi} M) \otimes_R N$, $M_2 = R[K] \otimes_{R[H],\varphi} (M \otimes_R \varphi^\# N)$, and f is the canonical isomorphism (cf. Proposition 3.3).

Proof. The commutability follows from

$$\begin{aligned} \varphi_\# B \otimes_R B'((a \otimes_\varphi x) \otimes u, (b \otimes_\varphi y) \otimes v) &= \varphi_\# B(a \otimes_\varphi x, (b \otimes_\varphi y))B'(u, v) \\ &= \delta_{a\varphi(H),b\varphi(H)} B(x, \varphi^{-1}(a^{-1}b)y)B'(u, v) \end{aligned}$$

and

$$\begin{aligned} &\varphi_\#(B \otimes_R \varphi^\# B')(a \otimes_\varphi (x \otimes a^{-1}u), b \otimes_\varphi (y \otimes b^{-1}v)) \\ &\qquad = \delta_{a\varphi(H),b\varphi(H)}(B \otimes_R \varphi^\# B')(x \otimes a^{-1}u, \varphi^{-1}(a^{-1}b)(y \otimes b^{-1}v)) \\ &\qquad = \delta_{a\varphi(H),b\varphi(H)} B(x, \varphi^{-1}(a^{-1}b)y)B'(a^{-1}u, \varphi(\varphi^{-1}(a^{-1}b))b^{-1}v) \\ &\qquad = \delta_{a\varphi(H),b\varphi(H)} B(x, \varphi^{-1}(a^{-1}b)y)B'(a^{-1}u, a^{-1}v) \\ &\qquad = \delta_{a\varphi(H),b\varphi(H)} B(x, \varphi^{-1}(a^{-1}b)y)B'(u, v), \end{aligned}$$

for $a, b \in K$, $x, y \in M$, and $u, v \in N$. □

Proposition 4.5. *Let H be a subgroup of G and (M, B) a Hermitian $R[H]$-module. Then for any $g \in H$, the following diagrams commute:*

$$\begin{array}{ccc} c_{(H,g)_\#}M \times c_{(H,g)_\#}M & & \\ \Big\downarrow f_1\times f_1 & \searrow^{c_{(H,g)_\#}B} & \\ M \times M & \xrightarrow[B]{} & R, \end{array}$$

$$\begin{array}{ccc} c^{\#}_{(H,g)}M \times c^{\#}_{(H,g)}M & & \\ \downarrow{\scriptstyle f_2 \times f_2} & \searrow^{c^{\#}_{(H,g)}B} & \\ M \times M & \xrightarrow[B]{} & R, \end{array}$$

where f_1 and f_2 are the canonical isomorphisms (cf. Proposition 3.4).

Proof. The commutability of the first diagram follows from

$$c_{(H,g)_{\#}}B(e \otimes x, e \otimes y) = B(x,y)$$

and

$$B(f_1(e \otimes x), f_1(e \otimes y)) = B(gx, gy) = B(x,y).$$

The commutability of the second diagram follows from

$$c_{(H,g)}{}^{\#}B(x,y) = B(x,y)$$

and

$$B(f_2(x), f_2(y)) = B(g^{-1}x, g^{-1}y) = B(x,y).$$

□

Proposition 4.6. *For any subgroups H and K of G, each Hermitian $R[H]$-module (M,B) satisfies the Mackey double coset formula. Namely,*

$$\mathrm{Res}^G_K \mathrm{Ind}^K_H B = \bigoplus_{KgH \in K\backslash G/H} \mathrm{Ind}^K_{K\cap gHg^{-1}} c_{(H\cap g^{-1}Kg,g)_{\#}} \mathrm{Res}^H_{H\cap g^{-1}Kg} B.$$

More precisely, the following diagram commutes:

$$\begin{array}{ccc} \left(\bigoplus_{KgH} M(K,g,H)\right) \times \left(\bigoplus_{KgH} M(K,g,H)\right) & & \\ \downarrow{\scriptstyle \omega\times\omega} & \searrow^{\bigoplus \mathrm{Ind}^K c_{(H\cap g^{-1}Kg,g)_{\#}} \mathrm{Res}^H B} & \\ \mathrm{Res}^G_K \mathrm{Ind}^K_H M \times \mathrm{Res}^G_K \mathrm{Ind}^K_H M & \xrightarrow[\mathrm{Res}^G_K \mathrm{Ind}^K_H B]{} & R, \end{array}$$

where KgH runs over $K\backslash G/H$,

$$M(K,g,H) = \mathrm{Ind}^K_{K\cap gHg^{-1}} c_{(H\cap g^{-1}Kg,g)_{\#}} \mathrm{Res}^H_{H\cap g^{-1}Kg} M,$$

and ω is the canonical isomorphism (cf. Proposition 3.5).

Proof. For u, $v \in R[K] \otimes_{R[K\cap gHg^{-1}]} c_{(H\cap g^{-1}Kg,g)_{\#}} \mathrm{Res}^H_{H\cap g^{-1}Kg} M$ with $u = a \otimes (e \otimes x)$ and $v = b \otimes (e \otimes x)$ respectively, where a, $b \in K$, x, $y \in \mathrm{Res}^H_{H\cap g^{-1}Kg} M$, we have

$$\begin{aligned} &\mathrm{Ind}^K_{K\cap gHg^{-1}} c_{(H\cap g^{-1}Kg,g)_{\#}} \mathrm{Res}^H_{H\cap g^{-1}Kg} B(u,v) \\ &\quad = \delta_{a(K\cap gHg^{-1}),b(K\cap gHg^{-1})} c_{(H\cap g^{-1}Kg,g)_{\#}} \mathrm{Res}^H_{H\cap g^{-1}Kg} B(e\otimes x, a^{-1}b(e\otimes y)) \\ &\quad = \delta_{a(K\cap gHg^{-1}),b(K\cap gHg^{-1})} B(x, g^{-1}a^{-1}bgy) \end{aligned}$$

and

$$\begin{aligned}\mathrm{Res}_K^G \mathrm{Ind}_H^G B(ag\otimes x, bg\otimes y) &= \delta_{agH,bgH} B(x,(ag)^{-1}bgy)\\ &= \delta_{agH,bgH} B(x, g^{-1}a^{-1}bgy)\\ &= \delta_{a(K\cap gHg^{-1}),b(K\cap gHg^{-1})} B(x, g^{-1}a^{-1}bgy).\end{aligned}$$

Thus we obtain the proposition. □

Definition 4.7. Let Θ be a finite G-set. A triple (M,B,α) consisting of a Hermitian $R[G]$-module (M,B) and a G-map $\alpha:\Theta\to M$ is called a *Θ-positioned Hermitian $R[G]$-module* (or simply *Θ-positioned Hermitian module*).

Let $\mathcal{H}(R,G,\Theta)$ stand for the family of all Θ-positioned Hermitian $R[G]$-modules (M,B,α) such that M is an R-free $R[G]$-module and $B:M\times M\to R$ is nonsingular. We say that α is *totally isotropic* (resp. *trivial*) if $B(\mathrm{Im}(\alpha),\mathrm{Im}(\alpha))=0$ (resp. $\mathrm{Im}(\alpha)=0$). We set

$$\mathcal{H}(R,G,\Theta)^{\text{t-iso}} = \{(M,B,\alpha)\in\mathcal{H}(R,G,\Theta) \mid \alpha \text{ is totally isotropic}\},$$
$$\mathcal{H}(R,G,\Theta)^{\text{triv}} = \{(M,B,\alpha)\in\mathcal{H}(R,G,\Theta) \mid \alpha \text{ is trivial}\}.$$

Let

$$\mathrm{KH}_0(R,G,\Theta),\ \ \mathrm{KH}_0(R,G,\Theta)^{\text{t-iso}}\ \text{ and }\ \mathrm{KH}_0(R,G)$$

denote the Grothendieck groups of $\mathcal{H}(R,G,\Theta)$, $\mathcal{H}(R,G,\Theta)^{\text{t-iso}}$ and $\mathcal{H}(R,G,\Theta)^{\text{triv}}$, respectively, under orthogonal sum.

Let $\boldsymbol{M}=(M,B,\alpha)$ be an object in $\mathcal{H}(R,G,\Theta)$. An R-direct summand, $R[G]$-submodule U of M is called a *Quillen submodule* of M if $U\subset U^{\perp}$ and $\mathrm{Im}(\alpha)\subset U$ both hold, where

$$U^{\perp}=\{x\in M \mid B(x,y)=0\ (\forall y\in U)\}.$$

In this case, $(\boldsymbol{M},U)$ is called a *Quillen pair.* If $\boldsymbol{M}\in\mathcal{H}(R,G,\Theta)$ admits a Quillen submodule, then $\boldsymbol{M}$ belongs to $\mathcal{H}(R,G,\Theta)^{\text{t-iso}}$ by definition. For a Quillen pair $(\boldsymbol{M},U)$, we have the well-defined map

$$B^{\perp}: U^{\perp}/U\times U^{\perp}/U\to R;\ B^{\perp}(x+U,y+U)=B(x,y)\quad (x,\ y\in U^{\perp}).$$

Proposition 4.8. *Let $(\boldsymbol{M},U)$, where $\boldsymbol{M}=(M,B,\alpha)$, be a Quillen pair. Then $U^{\perp}/U$ is an R-free $R[G]$-module and $B^{\perp}$ is a nonsingular Hermitian form on $U^{\perp}/U$.*

Proof. Since U is an R-direct summand of M, M factors to $M=U\oplus N$ as R-modules. It follows that U and N both are R-free, and so are $U^{\#}=\mathrm{Hom}_R(U,R)$ and M/U. Thus, the exact sequence

$$0\longrightarrow U^{\perp}/U\longrightarrow M/U\longrightarrow U^{\#}\longrightarrow 0$$

splits via R-homomorhisms, and hence $U^{\perp}/U$ is an R-direct summand of M/U. In particular, $U^{\perp}/U$ is R-free.

It is obvious that $B^{\perp}$ is R-bilinear, G-invariant and symmetric. So, it suffices to prove that $B^{\perp}$ is nonsingular. Since B is nonsingular, we can take an R-basis

$$\{u_1,\dots,u_m,y_1,\dots,y_n,v_1,\dots,v_m\}$$

of M so that $\{u_1,\dots,u_m\}$ is an R-basis of U, $y_j\in U^{\perp}$, and $B(v_i,u_j)=\delta_{i,j}$ and $B(v_i,y_j)=0$, where $\delta_{i,j}=1$ if $i=j$ and $\delta_{i,j}=0$ otherwise. Let V denote the R-submodule of M generated by $\{v_1,\dots,v_m\}$. There exist elements $z_1,\dots,z_n$ of M

such that $B(z_i, u_j) = 0$, $B(z_i, y_j) = \delta_{i,j}$ and $B(z_i, v_j) = 0$. Write z_i as $z_i = y'_i + v'_i$ with $y'_i \in U^\perp$ and $v'_i \in V$. Then

$$B(y'_i, y_j) = B(y'_i + v'_i, y_j) = B(z_i, y_j) = \delta_{i,j}.$$

This shows that $B^\perp : U^\perp/U \times U^\perp/U \to R$ is nonsingular. □

By the proposition, a Quillen pair $(\boldsymbol{M}, U)$ induces an object $(U^\perp/U, B^\perp, \mathrm{triv})$ of $\mathcal{H}(R, G, \Theta)$, where $\mathrm{triv} : \Theta \to U^\perp/U$ is the trivial map.

We define the Grothendieck-Witt groups

$$\mathrm{GW}_0(R, G, \Theta), \quad \mathrm{GW}_0(R, G, \Theta)^{\text{t-iso}}, \quad \mathrm{GW}_0(R, G)$$

by

$$\mathrm{GW}_0(R, G, \Theta) = \mathrm{KH}_0(R, G, \Theta)/\langle [\boldsymbol{M}] - [U^\perp/U, B^\perp, \mathrm{triv}] \rangle,$$
$$\mathrm{GW}_0(R, G, \Theta)^{\text{t-iso}} = \mathrm{KH}_0(R, G, \Theta)^{\text{t-iso}}/\langle [\boldsymbol{M}] - [U^\perp/U, B^\perp, \mathrm{triv}] \rangle,$$
$$\mathrm{GW}_0(R, G) = \mathrm{KH}_0(R, G)/\langle [\boldsymbol{M}] - [U^\perp/U, B^\perp, \mathrm{triv}] \rangle,$$

where $(\boldsymbol{M}, U)$ ranges over all Quillen pairs in $\mathcal{H}(R, G, \Theta)$, $\mathcal{H}(R, G, \Theta)^{\text{t-iso}}$ and $\mathcal{H}(R, G, \Theta)^{\text{triv}}$, respectively. By definition, there are canonical homomorphisms

$$\mathrm{GW}_0(R, G) \to \mathrm{GW}_0(R, G, \Theta)^{\text{t-iso}}$$

and

$$\mathrm{GW}_0(R, G, \Theta)^{\text{t-iso}} \to \mathrm{GW}_0(R, G, \Theta).$$

Proposition 4.9. *The homomorphisms*

$$\mathrm{GW}_0(R, G) \to \mathrm{GW}_0(R, G, \Theta)^{\text{t-iso}} \quad \textit{and} \quad \mathrm{GW}_0(R, G, \Theta)^{\text{t-iso}} \to \mathrm{GW}_0(R, G, \Theta)$$

are both injective. Moreover, the homomorphism $\mathrm{GW}_0(R, G) \to \mathrm{GW}_0(R, G, \Theta)^{\text{t-iso}}$ *is an isomorphism.*

Proof. Consider the homomorphism

$$\mathrm{GW}_0(R, G, \Theta) \to \mathrm{GW}_0(R, G)$$

assigning $[M, B, \mathrm{triv}]$ to $[M, B, \alpha]$. Since the composition

$$\mathrm{GW}_0(R, G) \to \mathrm{GW}_0(R, G, \Theta)^{\text{t-iso}} \to \mathrm{GW}_0(R, G, \Theta) \to \mathrm{GW}_0(R, G)$$

is the identity map, the homomorphisms

$$\mathrm{GW}_0(R, G) \to \mathrm{GW}_0(R, G, \Theta)^{\text{t-iso}} \ \text{ and } \ \mathrm{GW}_0(R, G) \to \mathrm{GW}_0(R, G, \Theta)$$

are injective.

Let $\boldsymbol{M} = (M, B, \alpha)$ be a Θ-positioned $R[G]$-Hermitian module such that α is totally isotropic. Then, let L denote the $R[G]$-submodule of M generated by $\alpha(\Theta)$, and set

$$U = \{x \in M \mid rx \in L \ \text{ for some } r \in R \text{ with } r \neq 0\}.$$

Then $B(U, U) = 0$, and U is an R-direct summand, $R[G]$-submodule of M. Thus, we have

$$[M, B, \alpha] = [U^\perp/U, B^\perp, \mathrm{triv}] \ \text{ in } GW_0(R, G, \Theta)^{\text{t-iso}}.$$

This implies that the canonical homomorphism $\mathrm{GW}_0(R, G) \to \mathrm{GW}_0(R, G, \Theta)^{\text{t-iso}}$ is surjective. □

For Θ-positioned Hermitian $R[G]$-modules $\boldsymbol{M}_1 = (M_1, B_1, \alpha_1)$ and $\boldsymbol{M}_2 = (M_2, B_2, \alpha_2)$, we define the tensor product $\boldsymbol{M}_1 \otimes_R \boldsymbol{M}_2$ over R as the Θ-positioned Hermitian $R[G]$-module $(M_1 \otimes_R M_2, B_1 \otimes_R B_2, \alpha_1 \otimes_R \alpha_2)$.

Proposition 4.10. *Let Θ be a finite G-set. Then $\mathrm{GW}_0(R, G, \Theta)$ and $\mathrm{GW}_0(R, G)$ ($= \mathrm{GW}_0(R, G, \Theta)^{\text{t-iso}}$) are commutative rings under the multiplication induced from the tensor product over R. Moreover, the rings $\mathrm{GW}_0(R, G, \Theta)$ and $\mathrm{GW}_0(R, G)$ possess units. Actually, the units of $\mathrm{GW}_0(R, G, \Theta)$ and $\mathrm{GW}_0(R, G)$ are the equivalence classes of*

$$(R, B : R \times R \to R, \alpha : \Theta \to R) \quad \text{and} \quad (R, B : R \times R \to R, \mathrm{triv} : \Theta \to R),$$

respectively, where G acts trivially on R, B is the map defined by $B(r_1, r_2) = r_1 r_2$ for r_1, $r_2 \in R$, and α is the map defined by $\alpha(t) = 1$ for $t \in \Theta$.

5. The special Grothendieck-Witt rings

Let S be a conjugation-invariant subset of

$$G(2) = \{g \in G \mid g^2 = e,\ g \neq e\}$$

and let $\mathfrak{P}(S)$ denote the set of all subsets of S. Then the G-action on S by conjugation yields a G-action on $\mathfrak{P}(S)$. Let Θ be a finite G-set and $\rho^{(2)} : \Theta \to \mathfrak{P}(S)$ a G-map.

For a G-map $\alpha : \Theta \to M$, where M is an $R[G]$-module, we define the map $\Delta_\alpha : S \to M$ by

$$\Delta_\alpha(s) = \sum_t \{\alpha(t) \mid t \in \Theta,\ \rho^{(2)}(t) \ni s\} \quad (s \in S). \tag{5.1}$$

Proposition 5.1. *The map Δ_α above is a G-map, namely $\Delta_\alpha(gsg^{-1}) = g\Delta_\alpha(s)$ for $g \in G$ and $s \in S$.*

Proof. The proof runs as follows:

$$\begin{aligned} g\Delta_\alpha(s) &= g \sum_t \{\alpha(t) \mid t \in \Theta,\ \rho^{(2)}(t) \ni s\} \\ &= \sum_t \{\alpha(gt) \mid t \in \Theta,\ \rho^{(2)}(t) \ni s\} \\ &= \sum_{t'} \{\alpha(t') \mid g^{-1}t' \in \Theta,\ \rho^{(2)}(g^{-1}t') \ni s\} \\ &= \sum_{t'} \{\alpha(t') \mid t' \in \Theta,\ \rho^{(2)}(t') \ni gsg^{-1}\} \\ &= \Delta_\alpha(gsg^{-1}). \end{aligned}$$

□

Let $\boldsymbol{M} = (M, B, \alpha)$ be an object in $\mathcal{H}(R, G, \Theta)$. We introduce a map

$$\nabla_{\boldsymbol{M}} : M \to \mathrm{Map}(S, R/2R),$$

which plays a key role in this paper. Define $\nabla_{\boldsymbol{M}}(x)(s) \in R/2R$ for $x \in M$ and $s \in S$ by

$$\nabla_{\boldsymbol{M}}(x)(s) = B(\Delta_\alpha(s) - x, sx). \tag{5.2}$$

Proposition 5.2. *The map $\nabla_{\boldsymbol{M}} : M \to \mathrm{Map}(S, R/2R)$ is a $\mathbb{Z}[G]$-homomorphism. Namely, the following hold:*

(1) $\nabla_{\boldsymbol{M}}(x + y)(s) = \nabla_{\boldsymbol{M}}(x)(s) + \nabla_{\boldsymbol{M}}(y)(s) \quad (x, y \in M,\ s \in S)$,

(2) $\nabla_{\boldsymbol{M}}(gx)(s) = \nabla_{\boldsymbol{M}}(x)(g^{-1}sg) \quad (x \in M,\ s \in S)$.

If R is square identical, $\nabla_{\boldsymbol{M}} : M \to \mathrm{Map}(S, R/2R)$ is an $R[G]$-homomorphism.

Proof. The formula (1) is obtained as follows:

$$\begin{aligned}\nabla_{\boldsymbol{M}}(x+y)(s) &= B(\Delta_\alpha(s)-(x+y), s(x+y)) \\ &= \nabla_{\boldsymbol{M}}(x)(s)+\nabla_{\boldsymbol{M}}(y)(s)+B(-x,sy)+B(-y,sx) \\ &= \nabla_{\boldsymbol{M}}(x)(s)+\nabla_{\boldsymbol{M}}(y)(s)-(B(x,sy)+B(y,sx)) \\ &= \nabla_{\boldsymbol{M}}(x)(s)+\nabla_{\boldsymbol{M}}(y)(s) \quad \text{in } R/2R.\end{aligned}$$

The formula (2) holds because

$$\begin{aligned}\nabla_{\boldsymbol{M}}(gx)(s) &= B(\Delta_\alpha(s)-gx, sgx) \\ &= B(g^{-1}\Delta_\alpha(s)-x, g^{-1}sgx) \\ &= B(\Delta_\alpha(g^{-1}sg)-x, g^{-1}sgx) \\ &= \nabla_{\boldsymbol{M}}(x)(g^{-1}sg) \quad \text{in } R/2R.\end{aligned}$$

The last assertion in the proposition is true since

$$\begin{aligned}\nabla_{\boldsymbol{M}}(rx)(s) &= B(\Delta_\alpha(s)-rx, srx) \\ &= B(\Delta_\alpha(s), srx)-B(rx,srx) \\ &= rB(\Delta_\alpha(s), sx)-r^2B(x,sx) \\ &= rB(\Delta_\alpha(s), sx)-rB(x,sx) \\ &= rB(\Delta_\alpha(s)-x, sx) \\ &= r\nabla_{\boldsymbol{M}}(x)(s) \quad \text{in } R/2R.\end{aligned}$$

We have established the proposition above. □

Let $\mathcal{SH}(R,G,S,\Theta)$, $\mathcal{SH}(R,G,S,\Theta)^{\text{t-iso}}$ and $\mathcal{SH}(R,G,S,\Theta)^{\text{triv}}$ denote the family consisting of objects $\boldsymbol{M}$ with $\nabla_{\boldsymbol{M}} = 0$ of $\mathcal{H}(R,G,\Theta)$, $\mathcal{H}(R,G,\Theta)^{\text{t-iso}}$ and $\mathcal{H}(R,G,\Theta)^{\text{triv}}$, respectively. We denote the Grothendieck groups of these under orthogonal sum by

$$\mathrm{KSH}_0(R,G,S,\Theta), \quad \mathrm{KSH}_0(R,G,S,\Theta)^{\text{t-iso}} \quad \text{and} \quad \mathrm{KSH}_0(R,G,S),$$

respectively. Moreover, we define the *special Grothendieck-Witt groups*

$$\mathrm{SGW}_0(R,G,S,\Theta), \quad \mathrm{SGW}_0(R,G,S,\Theta)^{\text{t-iso}}, \quad \mathrm{SGW}_0(R,G,S)$$

by

$$\begin{aligned}&\mathrm{SGW}_0(R,G,S,\Theta) = \mathrm{KSH}_0(R,G,S,\Theta)/\langle[\boldsymbol{M}]-[U^\perp/U, B^\perp, \mathrm{triv}]\rangle, \\ &\mathrm{SGW}_0(R,G,S,\Theta)^{\text{t-iso}} = \mathrm{KSH}_0(R,G,S,\Theta)^{\text{t-iso}}/\langle[\boldsymbol{M}]-[U^\perp/U, B^\perp, \mathrm{triv}]\rangle, \\ &\mathrm{SGW}_0(R,G,S) = \mathrm{KSH}_0(R,G,S)/\langle[\boldsymbol{M}]-[U^\perp/U, B^\perp, \mathrm{triv}]\rangle,\end{aligned}$$

where $(\boldsymbol{M}, U)$ ranges over all Quillen pairs in $\mathcal{SH}(R,G,S,\Theta)$, $\mathcal{SH}(R,G,S,\Theta)^{\text{t-iso}}$ and $\mathcal{SH}(R,G,S,\Theta)^{\text{triv}}$, respectively. Here we remark that if $\boldsymbol{M} \in \mathcal{SH}(R,G,S,\Theta)$ admits a Quillen submodule, then $\boldsymbol{M}$ belongs to $\mathcal{SH}(R,G,S,\Theta)^{\text{t-iso}}$. By definition, there are canonical homomorphisms

$$\mathrm{SGW}_0(R,G,S) \to \mathrm{SGW}_0(R,G,S,\Theta)^{\text{t-iso}}$$

and

$$\mathrm{SGW}_0(R,G,S,\Theta)^{\text{t-iso}} \to \mathrm{SGW}_0(R,G,S,\Theta).$$

Proposition 5.3. *The homomorphism* $\mathrm{SGW}_0(R,G,S) \to \mathrm{SGW}_0(R,G,S,\Theta)^{\text{t-iso}}$ *is surjective, and the homomorphism* $\mathrm{SGW}_0(R,G,S,\Theta)^{\text{t-iso}} \to \mathrm{SGW}_0(R,G,S,\Theta)$ *is injective.*

Proof. The proof of the surjectivity of $\mathrm{SGW}_0(R,G,S) \to \mathrm{SGW}_0(R,G,S,\Theta)^{\text{t-iso}}$ is the same as that of $\mathrm{GW}_0(R,G) \to \mathrm{GW}_0(R,G,\Theta)^{\text{t-iso}}$ (see Proposition 4.9).

Let $\boldsymbol{M}$ be an object of $\mathcal{SH}(R,G,S,\Theta)^{\text{t-iso}}$ such that $[\boldsymbol{M}]=0$ in $\mathrm{SGW}_0(R,G,S,\Theta)$. Then there exist objects $\boldsymbol{M}'=(M',B',\alpha')$, $\boldsymbol{M}_1=(M_1,B_1,\alpha_1)$ with a Quillen submodule U_1, and $\boldsymbol{M}_2=(M_2,B_2,\alpha_2)$ with a Quillen submodule U_2 of $\mathcal{SH}(R,G,S,\Theta)$ such that

$$\boldsymbol{M}\oplus\boldsymbol{M}'\oplus\boldsymbol{M}_1\oplus(U_2^{\perp}/U_2,B_2^{\perp},\mathrm{triv})\cong\boldsymbol{M}'\oplus\boldsymbol{M}_2\oplus(U_1^{\perp}/U_1,B_1^{\perp},\mathrm{triv}).$$

By definition, both $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ belong to $\mathcal{SH}(R,G,S,\Theta)^{\text{t-iso}}$. The object $\boldsymbol{M}'$ above may be replaced by

$$\boldsymbol{M}''=(M',B',\alpha')\oplus(M',-B',-\alpha').$$

Then $\boldsymbol{M}''$ has the Quillen submodule

$$U''=\{(x,x)\in M'\oplus M' \mid x\in M'\},$$

and hence belongs to $\mathcal{SH}(R,G,S,\Theta)^{\text{t-iso}}$, which lets us conclude that

$$[\boldsymbol{M}]=0 \ \text{ in } \mathrm{SGW}_0(R,G,S,\Theta)^{\text{t-iso}}.$$

□

Proposition 5.4. *If, for each $s\in S$, there is at most one element $t\in\Theta$ such that $\rho^{(2)}(t)\ni s$, then* $\mathrm{SGW}_0(R,G,S,\Theta)$*,* $\mathrm{SGW}_0(R,G,S,\Theta)^{\text{t-iso}}$ *and* $\mathrm{SGW}_0(R,G,S)$ *are commutative rings, possibly without unit. If R is square identical, and for each $s\in S$ there exists exactly one element $t\in\Theta$ such that $\rho^{(2)}(t)\ni s$, then* $\mathrm{SGW}_0(R,G,S,\Theta)$ *is a commutative ring with unit.*

Proof. Let $\boldsymbol{M}_1=(M_1,B_1,\alpha_1)$ and $\boldsymbol{M}_2=(M_2,B_2,\alpha_2)$ be objects of $\mathcal{H}(R,G,\Theta)$ and $\mathcal{SH}(R,G,S,\Theta)$, respectively. Then

$$\begin{aligned}
\nabla_{\boldsymbol{M}_1\otimes_R\boldsymbol{M}_2}(x_1\otimes x_2)(s) &= B_1\otimes_R B_2(\Delta_{\alpha_1\otimes_R\alpha_2}(s)-x_1\otimes x_2, s(x_1\otimes x_2))\\
&= B_1\otimes_R B_2(\Delta_{\alpha_1}(s)\otimes\Delta_{\alpha_2}(s)-x_1\otimes x_2, sx_1\otimes sx_2)\\
&= B_1(\Delta_{\alpha_1}(s),sx_1)B_2(\Delta_{\alpha_2}(s),sx_2)-B_1(x_1,sx_1)B_2(x_2,sx_2)\\
&= B_1(\Delta_{\alpha_1}(s)-x_1,sx_1)B_2(\Delta_{\alpha_2}(s),sx_2)\\
&\quad + B_1(x_1,sx_1)B_2(\Delta_{\alpha_2}(s)-x_2,sx_2)\\
&= \nabla_{\boldsymbol{M}_1}(x_1)(s)B_2(\Delta_{\alpha_2}(s),sx_2)+B_1(x_1,sx_1)\nabla_{\boldsymbol{M}_2}(x_2)(s)\\
&= \nabla_{\boldsymbol{M}_1}(x_1)(s)B_2(\Delta_{\alpha_2}(s),sx_2) \quad \text{in } R/2R.
\end{aligned}$$

By using this and Proposition 5.2 (1), we can show that the product $\boldsymbol{M}_1\otimes_R\boldsymbol{M}_2$ belongs to $\mathcal{SH}(R,G,S,\Theta)$ if $\boldsymbol{M}_1$ does. Therefore, the special Grothendieck-Witt groups are commutative rings.

Next we shall prove the last claim in the proposition. Let (R,B,α) denote the object in $\mathcal{H}(R,G,\Theta)$ such that G acts trivially on R, $B(r_1,r_2)=r_1r_2$ (r_1, $r_2\in R$) and $\alpha(t)=1$ ($t\in\Theta$). Then, the associated $\nabla: M\to\mathrm{Map}(S,R/2R)$ is trivial, since

$$\nabla(x)(s)=B(1-r,sr)=r-r^2=0 \quad \text{in } R/2R.$$

Thus, (R,B,α) belongs to $\mathcal{SH}(R,G,S,\Theta)$, and therefore we can now conclude that the ring $\mathrm{SGW}_0(R,G,S,\Theta)$ possesses a unit. □

Proposition 5.5. *The group* $\mathrm{SGW}_0(R,G,S)$ *is a module over the ring* $\mathrm{GW}_0(R,G)$.

Proof. Let $\boldsymbol{M}_1 = (M_1, B_1, \mathrm{triv})$ and $\boldsymbol{M}_2 = (M_2, B_2, \mathrm{triv})$ be arbitrary objects of $\mathcal{H}(R,G,\Theta)^{\mathrm{triv}}$ and $\mathcal{SH}(R,G,S,\Theta)^{\mathrm{triv}}$, respectively. Then, as in the proof of Proposition 5.4, we have

$$\nabla_{\boldsymbol{M}_1\otimes\boldsymbol{M}_2}(x_1\otimes x_2)(s) = \nabla_{\boldsymbol{M}_1}(x_1)(s)B_2(\Delta_{\alpha_2}(s), sx_2) + B_1(x_1, sx_1)\nabla_{\boldsymbol{M}_2}(x_2)(s).$$

Since $\Delta_{\alpha_2}(s) = 0$ and $\nabla_{\boldsymbol{M}_2}(x_2)(s) = 0$, $\nabla_{\boldsymbol{M}_1\otimes\boldsymbol{M}_2}$ vanishes. Thus $\boldsymbol{M}_1\otimes\boldsymbol{M}_2$ belongs to $\mathcal{SH}(R,G,S,\Theta)^{\mathrm{triv}}$. □

6. $R[G]$-valued λ-Hermitian forms

Let λ stand for 1 or -1 and let $w : G \to \{-1, 1\}$ be a homomorphism. The group ring $A = R[G]$ is equipped with the anti-involution $-$ defined by

$$\overline{\sum_{g\in G} r_g g} = \sum_{g\in G} w(g) r_g g^{-1} \quad (r_g \in R).$$

Definition 6.1. Let M be an $R[G]$-module. A map $B : M\times M \to R[G]$ is called an *$R[G]$-valued λ-Hermitian form* (or *λ-Hermitian form*) on M if the following conditions (1)–(3) are satisfied:

(1) B is R-bilinear,
(2) $B(ax, by) = bB(y,x)\overline{a}$,
(3) $B(x,y) = \lambda\overline{B(y,x)}$,

for all x, $y \in M$, a, $b \in R[G]$.

Let $B : M\times M \to R[G]$ be a λ-Hermitian form. For x, $y \in M$, $B(x,y)$ can be written as $\sum_{g\in G} B(x,y)_g g$ with $B(x,y)_g \in R$. Define the R-homomorphism $\varepsilon : R[G] \to R$ by

$$\varepsilon\left(\sum_{g\in G} r_g g\right) = r_e \quad (r_g \in R). \tag{6.1}$$

Lemma 6.2. $B(x,y)_g = \varepsilon(B(x, g^{-1}y))$ *for all* x, $y \in M$ *and* $g \in G$, *and consequently*

$$B(x,y) = \sum_{g\in G} \varepsilon(B(x, g^{-1}y))g.$$

Proof. By definition, we have $B(x,y)_e = \varepsilon(B(x,y))$. By observing the coefficients of g in $B(x,y)$ and

$$gB(x, g^{-1}y) = \sum_{h\in G} B(x, g^{-1}y)_h gh,$$

we have $B(x, g^{-1}y)_e = B(x,y)_g$. Thus, $B(x,y)_g = \varepsilon(B(x, g^{-1}y))$. □

Lemma 6.3. *Let* $\boldsymbol{M}$ *be as above. Then the composition* $\varepsilon\circ B : M\times M \to R$ *is a* λ*-symmetric,* (G,w)*-invariant,* R*-bilinear form on* M*. Namely, the following hold:*

(1) $\varepsilon(B(x+x', ry)) = r\varepsilon(B(x,y)) + r\varepsilon(B(x',y))$,
(2) $\varepsilon(B(x,y)) = \lambda\varepsilon(B(y,x))$,
(3) $\varepsilon(B(gx,gy)) = w(g)\varepsilon(B(x,y))$,

for any $r \in R$, x, x', $y \in M$ *and* $g \in G$.

Proof. (1) The proof is straightforward.

(2) The equality follows from $B(x,y) = \lambda\overline{B(y,x)}$.

(3) By comparing the coefficients of e in $B(x,gy)$ and $w(g)B(g^{-1}x,y)$:

$$B(x,gy) = \sum_{h\in G} B(x,gy)_h h,$$
$$w(g)B(g^{-1}x,y) = \sum_{h\in G} w(g)B(g^{-1}x,y)_h h,$$

we have $\varepsilon(B(x,gy)) = w(g)\varepsilon(B(g^{-1}x,y))$, which is equivalent to the equality (3). □

An $R[G]$-valued λ-Hermitian form B on an $R[G]$-projective module M is said to be *nonsingular* if the associated map

$$M \to \mathrm{Hom}_{R[G]}(M, R[G]);\ x \longmapsto B(x,-)$$

is bijective.

Lemma 6.4. *Let B be an $R[G]$-valued λ-Hermitian form on an $R[G]$-projective module M. Then B is nonsingular if and only if the induced R-bilinear form $\varepsilon\circ B : M\times M \to R$ is nonsingular.*

Let H and K be finite groups with homomorphisms $w_H : H \to \{-1,1\}$ and $w_K : K \to \{-1,1\}$, respectively. Let $\varphi : H \to K$ be a monomorphism such that $w_K \circ \varphi = w_H$. Let N be an $R[K]$-module and $B : N\times N \to R[K]$ a λ-Hermitian form. We define the map $\varphi^{\#}B : \varphi^{\#}N \times \varphi^{\#}N \to R[H]$ by

$$\varphi^{\#}B(x,y) = \sum_{h\in H} \varepsilon(B(x,\varphi(h)^{-1}y))h \quad (x,\ y \in \varphi^{\#}N). \tag{6.2}$$

It immediately follows that $\varphi^{\#}B$ is an $R[H]$-valued λ-Hermitian form on $\varphi^{\#}N$. If B is nonsingular, then so is $\varphi^{\#}B$. Next let M be a stably free $R[H]$-module. Then

$$\varphi_{\#}M = R[K] \otimes_{R[H],\varphi} M$$

is clearly a stably $R[K]$-free module. Let $B : M\times M \to R[H]$ be a λ-Hermitian form. We define the R-bilinear map $\varphi_{\#}B : \varphi_{\#}M \times \varphi_{\#}M \to R[K]$ so that

$$\varphi_{\#}B(a\otimes_\varphi x, b\otimes_\varphi y) = \sum_{k\in K} w_K(a)\delta_{a\varphi(H),k^{-1}b\varphi(H)}\varepsilon(B(x,\varphi^{-1}(a^{-1}k^{-1}b)y))k, \tag{6.3}$$

for $a,\ b \in K$, $x,\ y \in M$.

Lemma 6.5. *Let $\varphi_{\#}B$ be as above. Then*

$$\varphi_{\#}B(a\otimes_\varphi x, b\otimes_\varphi y) = b\varphi'(B(x,y))\overline{a},$$

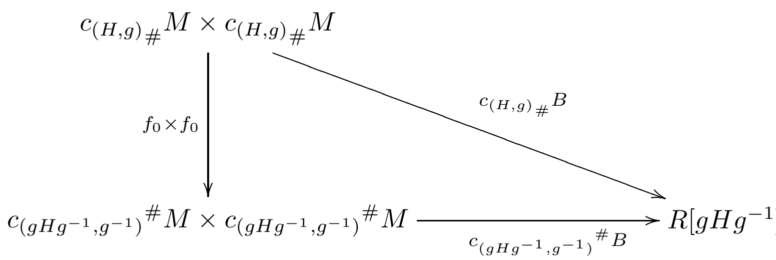
for $a,\ b \in K$, $x,\ y \in M$; and $\varphi_{\#}B$ is an $R[K]$-valued λ-Hermitian form on $\varphi_{\#}M$, where $\varphi' : R[H] \to R[K]$ is the ring homomorphism canonically induced by $\varphi : H \to K$. If B is nonsingular, then so is $\varphi_{\#}B$.

Proof. The formula in the lemma is true because

$$\begin{aligned}\varphi_{\#}B(a\otimes_\varphi x, b\otimes_\varphi y) &= \sum_{k\in K} w_K(a)\delta_{a\varphi(H),k^{-1}b\varphi(H)}\varepsilon(B(x,\varphi^{-1}(a^{-1}k^{-1}b)y))k \\ &= b(\sum_{k\in K}\delta_{\varphi(H),a^{-1}k^{-1}b\varphi(H)}\varepsilon(B(x,\varphi^{-1}(a^{-1}k^{-1}b)y))b^{-1}ka)\overline{a} \\ &= b(\sum_{k'\in K}\delta_{\varphi(H),k'^{-1}\varphi(H)}\varepsilon(B(x,\varphi^{-1}(k'^{-1})y))k')\overline{a} \\ &= b\varphi'(\sum_{k'\in K}\delta_{\varphi(H),k'^{-1}\varphi(H)}\varepsilon(B(x,\varphi^{-1}(k'^{-1})y))\varphi^{-1}(k'))\overline{a} \\ &= b\varphi'(B(x,y))\overline{a}.\end{aligned}$$

One can check the latter claim in the lemma by using this formula. □

Proposition 6.6. *Let H be a subgroup of G, B an $R[H]$-valued λ-Hermitian form on an $R[H]$-module M, and g an element of G. Provided $w_H = w_{gHg^{-1}}\circ c_{(H,g)}$, the diagram*



commutes, where f_0 is the canonical $R[gHg^{-1}]$-isomorphism (cf. Proposition 3.2).

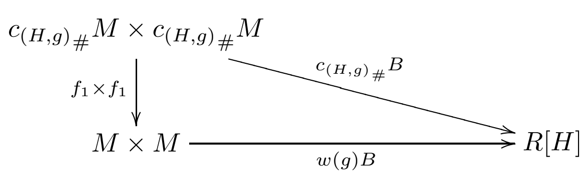
The proof of the proposition is straightforward.

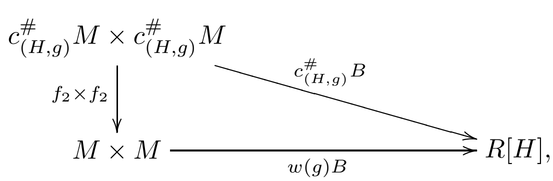
Given a datum $\mathcal{D} = (R, G, w, \lambda)$ as above, we obtain the datum

$$\mathcal{D}_H = (R, H, w|_H, \lambda)$$

for each subgroup H of G.

Proposition 6.7. *Let H be a subgroup of G and $B : M\times M \to R[H]$ a λ-Hermitian form on an $R[H]$-module M. Then for each $g \in H$, the following diagrams commute:*





where f_1 and f_2 are the canonical isomorphisms (cf. Proposition 3.4).

Proof. The commutability of the first diagram follows from

$$(c_{(H,g)_{\#}}B)(e\otimes x, e\otimes y) = \sum_{h\in H} \varepsilon(B(x, g^{-1}h^{-1}gy))h$$

and

$$\begin{aligned} B(f_1(e\otimes x), f_1(e\otimes y)) &= B(gx, gy) \\ &= \sum_{h\in H} \varepsilon(B(gx, h^{-1}gy))h \\ &= w(g)\sum_{h\in H} \varepsilon(B(x, g^{-1}h^{-1}gy))h. \end{aligned}$$

The commutability of the second diagram follows from

$$(c_{(H,g)}{}^{\#}B)(x,y) = \sum_{h\in H} \varepsilon(B(x, gh^{-1}g^{-1}y))h$$

and

$$\begin{aligned} B(f_2(x), f_2(y)) &= B(g^{-1}x, g^{-1}y) \\ &= \sum_{h\in H} \varepsilon(B(g^{-1}x, h^{-1}g^{-1}y))h \\ &= w(g)\sum_{h\in H} \varepsilon(B(x, g^{-1}h^{-1}gy))h. \end{aligned}$$

□

Proposition 6.8. *For any subgroups H and K of G, each $R[H]$-valued λ-Hermitian form $B : M\times M \to R[H]$ on an $R[H]$-module M satisfies the w-Mackey double coset formula. Namely,*

$$\begin{aligned} &(\mathrm{Res}^G_K \mathrm{Ind}^G_H B)\circ(\omega\times\omega) \\ &\qquad = \sum_{KgH\in K\backslash G/H} w(g)(\mathrm{Ind}^K_{K\cap gHg^{-1}} c_{(H\cap g^{-1}Kg, g)_{\#}} \mathrm{Res}^H_{H\cap g^{-1}Kg} B), \end{aligned}$$

where ω is the canonical isomorphism (cf. Proposition 3.5). Particularly, in the case $w(G) = \{1\}$, B satisfies the Mackey double coset formula.

Proof. It suffices to prove that

$$\begin{aligned} &(\mathrm{Res}^G_K \mathrm{Ind}^G_H B)(ag\otimes x, bg\otimes y) \\ &\qquad = w(g)(\mathrm{Ind}^K_{K\cap gHg^{-1}} c_{(H\cap g^{-1}Kg, g)_{\#}} \mathrm{Res}^H_{H\cap g^{-1}Kg} B)(a\otimes(e\otimes x), b\otimes(e\otimes y)) \end{aligned}$$

for any $g\in G$, a, $b\in K$, x, $y\in \mathrm{Res}^H_{H\cap g^{-1}Kg} M$. This equality holds because

$$\begin{aligned} &(\mathrm{Res}^G_K \mathrm{Ind}^G_H B)(ag\otimes x, bg\otimes y) \\ &\qquad = \sum_{k\in K} w(ag)\delta_{agH, k^{-1}bgH}\varepsilon(B(x, (ag)^{-1}k^{-1}bgy))k \\ &\qquad = w(g)\sum_{k\in K} w(a)\delta_{agH, k^{-1}bgH}\varepsilon(B(x, g^{-1}(a^{-1}k^{-1}b)gy))k \end{aligned}$$

and

$$\begin{aligned}
&(\mathrm{Ind}_{K\cap gHg^{-1}}^{K} c_{(H\cap g^{-1}Kg,g)_\#}\mathrm{Res}_{H\cap g^{-1}Kg}^{H} B)(a\otimes(e\otimes x), b\otimes(e\otimes y)) \\
&= \sum_{k\in K} w(a)\delta_{a(K\cap gHg^{-1}),k^{-1}b(K\cap gHg^{-1})} \\
&\qquad\qquad \cdot (c_{(H\cap g^{-1}Kg,g)_\#}\mathrm{Res}_{H\cap g^{-1}Kg}^{H} B)(e\otimes x, a^{-1}k^{-1}b(e\otimes y))k \\
&= \sum_{k\in K} w(a)\delta_{a(K\cap gHg^{-1}),k^{-1}b(K\cap gHg^{-1})} B(x, g^{-1}(a^{-1}k^{-1}b)gy)k.
\end{aligned}$$

□

7. Positioned quadratic $R[G]$-modules

In this paper λ stands for either 1 or -1. Let $w : G \to \{-1, 1\}$ be a group homomorphism. Set

$$G^{\lambda}(2) = \{g \in G(2) \mid w(g) = \lambda\},$$
$$G^{-\lambda}(2) = \{g \in G(2) \mid w(g) = -\lambda\}.$$

Clearly we have $g = \lambda\overline{g}$ for $g \in G^{\lambda}(2)$ and $g = -\lambda\overline{g}$ for $g \in G^{-\lambda}(2)$. Let S and Q be conjugation-invariant subsets of $G^{\lambda}(2)$ and $G^{-\lambda}(2)$, respectively. We shall define the Witt group of Θ-positioned quadratic $R[G]$-modules, which is the Wall group (cf. [27]) in the case where Q, S and Θ are the empty set, and the Bak group (cf. [1], [19]) in the case where S and Θ are the empty set. The datum

$$\boldsymbol{A} = (R, G, Q, S, \lambda, w)$$

is relevant to the group. Define R-submodules $A_s = A_s(G, S; R)$, $A_q = A_q(G, S; R)$ and $\Lambda = \Lambda(G, Q; R)$ of $A := R[G]$ as follows:

$$A_s = R[S] \ (= \langle s \mid s \in S\rangle_R),$$
$$A_q = R[G \smallsetminus S] \ (= \langle g \mid g \in G \smallsetminus S\rangle_R),$$
$$\Lambda = \langle x - \lambda\overline{x} \mid x \in A\rangle_R + \langle g \mid g \in Q\rangle_R.$$

This module Λ is called the *form parameter* generated by Q.

Definition 7.1. A map $q : M \to A_q/\Lambda$ is called an $\boldsymbol{A}$-*quadratic form* (or *quadratic form*) on M with respect to B if the following conditions (1)–(3) are fulfilled:
(1) $q(gx) = gq(x)\overline{g}$ and $q(rx) = r^2q(x)$ in $A_q/\Lambda = A/(\Lambda + A_s)$,
(2) $q(x+y) - q(x) - q(y) = B(x, y)$ in $A_q/\Lambda = A/(\Lambda + A_s)$,
(3) $\widetilde{q(x)} + \lambda\overline{\widetilde{q(x)}} = B(x, x)$ in $A_q = A/A_s$,
for all x, $y \in M$, $g \in G$, $r \in R$, where $\widetilde{q(x)} \in A_q$ is a lifting of $q(x)$.

A triple (M, B, q) consisting of an $R[G]$-module M, an $R[G]$-valued λ-Hermitian form B on M and an $\boldsymbol{A}$-quadratic form q on M with respect to B, is called an $\boldsymbol{A}$-*quadratic* $R[G]$-*module* (or λ-*quadratic* $R[G]$-*module*).

Let Θ be a finite G-set. A quadruple (M, B, q, α) consisting of an $\boldsymbol{A}$-quadratic $R[G]$-module (M, B, q) and a G-map $\alpha : \Theta \to M$ is called a Θ-*positioned* $\boldsymbol{A}$-*quadratic* $R[G]$-*module* (or Θ-*positioned* λ-*quadratic* $R[G]$-*module*).

Let $\mathcal{Q}(\boldsymbol{A}, \Theta)$ (or $\mathcal{Q}(R, G, Q, S, \Theta)$) denote the family of all Θ-positioned $\boldsymbol{A}$-quadratic $R[G]$-modules (M, B, q, α) such that M is a stably free $R[G]$-module and B is nonsingular.

Let $\boldsymbol{M} = (M, B, q, \alpha) \in \mathcal{Q}(\boldsymbol{A}, \Theta)$. The map α is said to be *totally isotropic* (resp. *trivial*) if $B(\mathrm{Im}(\alpha), \mathrm{Im}(\alpha)) = 0$ and $q(\mathrm{Im}(\alpha)) = 0$ (resp. $\mathrm{Im}(\alpha) = 0$). Set

$$\mathcal{Q}(\boldsymbol{A}, \Theta)^{\text{t-iso}} = \{(M, B, q, \alpha) \in \mathcal{Q}(\boldsymbol{A}, \Theta) \mid \alpha \text{ is totally isotropic}\},$$
$$\mathcal{Q}(\boldsymbol{A}, \Theta)^{\text{triv}} = \{(M, B, q, \alpha) \in \mathcal{Q}(\boldsymbol{A}, \Theta) \mid \alpha \text{ is trivial}\}.$$

Let $\mathrm{KQ}_0(\boldsymbol{A}, \Theta)$, $\mathrm{KQ}_0(\boldsymbol{A}, \Theta)^{\text{t-iso}}$ and $\mathrm{KQ}_0(\boldsymbol{A})$ denote the Grothendieck groups of $\mathcal{Q}(\boldsymbol{A}, \Theta)$, $\mathcal{Q}(\boldsymbol{A}, \Theta)^{\text{t-iso}}$ and $\mathcal{Q}(\boldsymbol{A}, \Theta)^{\text{triv}}$, respectively, under orthogonal sum.

A stably $R[G]$-free, $R[G]$-direct summand L of M is called a *Lagrangian submodule* of $\boldsymbol{M}$ if $B(L, L) = 0$, $q(L) = 0$, $L^\perp = L$ and $\mathrm{Im}(\alpha) \subset L$, where

$$L^\perp = \{x \in M \mid B(x, y) = 0 \ (\forall y \in L)\}.$$

If $\boldsymbol{M}$ has a Lagrangian submodule, then $\boldsymbol{M}$ is called a *null module*. The groups defined by

$$\mathrm{WQ}_0(\boldsymbol{A}, \Theta) = \mathrm{KQ}_0(\boldsymbol{A}, \Theta)/\langle \text{null modules in } \mathcal{Q}(\boldsymbol{A}, \Theta)\rangle,$$
$$\mathrm{WQ}_0(\boldsymbol{A}, \Theta)^{\text{t-iso}} = \mathrm{KQ}_0(\boldsymbol{A}, \Theta)^{\text{t-iso}}/\langle \text{null modules in } \mathcal{Q}(\boldsymbol{A}, \Theta)^{\text{t-iso}}\rangle,$$
$$\mathrm{WQ}_0(\boldsymbol{A}) = \mathrm{KQ}_0(\boldsymbol{A})/\langle \text{null modules in } \mathcal{Q}(\boldsymbol{A}, \Theta)^{\text{triv}}\rangle$$

are called the *Witt groups* of Θ-positioned $\boldsymbol{A}$-quadratic $R[G]$-modules. If the context is clear, those Witt groups are also denoted by

$$\mathrm{WQ}_0(R, G, Q, S, \Theta), \quad \mathrm{WQ}_0(R, G, Q, S, \Theta)^{\text{t-iso}}, \quad \mathrm{WQ}_0(R, G, Q, S),$$

respectively.

8. The special Witt groups

Let $\boldsymbol{A} = (R, G, Q, S, \lambda, w)$ be as in the previous section, Θ a finite G-set and $\rho^{(2)} : \Theta \to \mathfrak{P}(S)$ a G-map (cf. Section 5). Let $\boldsymbol{M} = (M, B, q, \alpha)$ be a Θ-positioned $\boldsymbol{A}$-quadratic $R[G]$-module, where $\alpha : \Theta \to M$. The associated map $\nabla_{\boldsymbol{M}} : M \to \mathrm{Map}(S, R/2R)$ is defined by

$$\nabla_{\boldsymbol{M}}(x)(s) = \varepsilon(B(\Delta_\alpha(s) - x, sx)), \tag{8.1}$$

for $x \in M$ and $s \in S$, where $\Delta_\alpha : S \to M$ is the map defined by (5.1).

If $\boldsymbol{M} \in \mathcal{Q}(\boldsymbol{A}, \Theta)$ satisfies $\nabla_{\boldsymbol{M}} = 0$, then we call $\boldsymbol{M}$ a *special* Θ*-positioned* $\boldsymbol{A}$*-quadratic* $R[G]$*-module* (or a *special* Θ*-positioned* λ*-quadratic* $R[G]$*-module*). Set

$$\mathcal{SQ}(\boldsymbol{A}, \Theta) = \{\boldsymbol{M} \in \mathcal{Q}(\boldsymbol{A}, \Theta) \mid \nabla_{\boldsymbol{M}} = 0\},$$
$$\mathcal{SQ}(\boldsymbol{A}, \Theta)^{\text{t-iso}} = \{\boldsymbol{M} \in \mathcal{Q}(\boldsymbol{A}, \Theta)^{\text{t-iso}} \mid \nabla_{\boldsymbol{M}} = 0\},$$
$$\mathcal{SQ}(\boldsymbol{A}, \Theta)^{\text{triv}} = \{\boldsymbol{M} \in \mathcal{Q}(\boldsymbol{A}, \Theta)^{\text{triv}} \mid \nabla_{\boldsymbol{M}} = 0\}.$$

The corresponding Grothendieck groups are denoted by

$$\mathrm{KSQ}_0(\boldsymbol{A}, \Theta), \quad \mathrm{KSQ}_0(\boldsymbol{A}, \Theta)^{\text{t-iso}}, \quad \mathrm{KSQ}_0(\boldsymbol{A})$$

respectively, or by

$$\mathrm{KSQ}_0(R, G, Q, S, \Theta), \quad \mathrm{KSQ}_0(R, G, Q, S, \Theta)^{\text{t-iso}}, \quad \mathrm{KSQ}_0(R, G, Q, S)$$

respectively. Further, define the *special Witt groups*

$$\mathrm{SWQ}_0(\boldsymbol{A}, \Theta) \ \ (= \mathrm{SWQ}_0(R, G, Q, S, \Theta)),$$
$$\mathrm{SWQ}_0(\boldsymbol{A}, \Theta)^{\text{t-iso}} \ \ (= \mathrm{SWQ}_0(R, G, Q, S, \Theta)^{\text{t-iso}}),$$
$$\mathrm{SWQ}_0(\boldsymbol{A}) \ \ (= \mathrm{SWQ}_0(R, G, Q, S))$$

by

$$\mathrm{SWQ}_0(\boldsymbol{A},\Theta) = \mathrm{KSQ}_0(\boldsymbol{A},\Theta)/\langle \text{null modules in } \mathcal{SQ}(\boldsymbol{A},\Theta)\rangle,$$
$$\mathrm{SWQ}_0(\boldsymbol{A},\Theta)^{\text{t-iso}} = \mathrm{KSQ}_0(\boldsymbol{A},\Theta)^{\text{t-iso}}/\langle \text{null modules in } \mathcal{SQ}(\boldsymbol{A},\Theta)^{\text{t-iso}}\rangle,$$
$$\mathrm{SWQ}_0(\boldsymbol{A}) = \mathrm{KSQ}_0(\boldsymbol{A})/\langle \text{null modules in } \mathcal{SQ}(\boldsymbol{A},\Theta)^{\text{triv}}\rangle,$$

respectively.

9. Tensor products of Hermitian modules and quadratic modules

Let $\boldsymbol{A} = (R, G, Q, S, \lambda, w)$ be as in Section 7, and Θ a finite G-set. Let $\boldsymbol{M} = (M, B, q)$ be an $\boldsymbol{A}$-quadratic $R[G]$-module. By definition, B is a map $M \times M \to R[G]$ and q is a map $M \to A_q/\Lambda$. We write G as a disjoint union of the form

$$G = \{e\} \amalg G(2) \amalg C \amalg C^{-1},$$

where C is a subset of G consisting of elements of order ≥ 3 and $C^{-1} = \{g^{-1} \mid g \in C\}$. Set

$$\mathcal{Q}(G) = \{e\} \cup (G^{\lambda}(2) \smallsetminus S) \cup (G^{-\lambda}(2) \smallsetminus Q) \cup C.$$

Let R_g stand for the R-module defined by

$$R_g = \begin{cases} R/(1-\lambda)R & (g = e), \\ R & (g \in G^{\lambda}(2)), \\ R/2R & (g \in G^{-\lambda}(2)), \\ R & (\text{otherwise}), \end{cases}$$

for each $g \in G$. Then $q(x)$, $x \in M$, can be regarded as the formal sum

$$\sum_{g \in \mathcal{Q}(G)} q(x)_g g$$

with $q(x)_g \in R_g$; namely, $q : M \to A_q/\Lambda$ can be regarded as the map

$$M \to \bigoplus_{g \in \mathcal{Q}(G)} R_g;\ x \longmapsto (q(x)_g).$$

We set $q(x)_g = \lambda w(g) q(x)_{g^{-1}}$ for $g \in G$ with $g^{-1} \in \mathcal{Q}(G)$. This definition is compatible with the ambiguity of choice of $\mathcal{Q}(G)$, because

$$\widetilde{q(x)_g} g = \lambda w(g) \widetilde{q(x)_g} g^{-1} \quad \mathrm{mod}\ \Lambda.$$

Let $\boldsymbol{M}_1 = (M_1, B_1, \alpha_1)$ and $\boldsymbol{M}_2 = (M_2, B_2, q_2, \alpha_2)$ be objects in $\mathcal{H}(R, G, S, \Theta)$ and $\mathcal{Q}(\boldsymbol{A}, \Theta)$, respectively. We define an object $\boldsymbol{M}_1 \cdot \boldsymbol{M}_2$ in $\mathcal{Q}(\boldsymbol{A}, \Theta)$ as the product of $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ as follows. For the sake of convenience, $\boldsymbol{M} = (M, B, q, \alpha)$ stands for $\boldsymbol{M}_1 \cdot \boldsymbol{M}_2$ for a while.

First, M is defined as the R-module $M_1 \otimes_R M_2$ with the G-action: $(g, x \otimes y) \mapsto (gx) \otimes (gy)$, where $g \in G$, $x \in M_1$ and $y \in M_2$. Since M_1 is R-free and M_2 is stably $R[G]$-free, M is stably $R[G]$-free.

Second, $B : M \times M \to R[G]$ is defined as the R-bilinear form such that

$$B(x \otimes y, x' \otimes y') = \sum_{g \in G} B_1(x, g^{-1}x') \varepsilon(B_2(y, g^{-1}y'))g.$$

The equality $B(u,v)=\lambda\overline{B(v,u)}$ $(u,\, v\in M)$ holds since

$$\begin{aligned}
B(x\otimes y, x'\otimes y') &= \sum_{g\in G} B_1(x, g^{-1}x')\varepsilon(B_2(y, g^{-1}y'))g\\
&= \sum_{g\in G} \lambda B_1(g^{-1}x', x)\varepsilon(B_2(g^{-1}y', y))g\\
&= \lambda\sum_{g\in G} w(g)B_1(x', gx)\varepsilon(B_2(y', gy))g\\
&= \lambda\sum_{g\in G} B_1(x', gx)\varepsilon(B_2(y', gy))\overline{g^{-1}}\\
&= \lambda\overline{\sum_{g\in G} B_1(x', gx)\varepsilon(B_2(y', gy))g^{-1}}\\
&= \lambda\overline{B(x'\otimes y', x\otimes y)}.
\end{aligned}$$

The equality $B(au,bv)=bB(u,v)\overline{a}$ $(a,\, b\in G,\ u,\, v\in M)$ holds because

$$\begin{aligned}
B(a(x\otimes y), b(x'\otimes y')) &= \sum_{g\in G} B_1(ax, g^{-1}bx')\varepsilon(B_2(ay, g^{-1}by'))g\\
&= b\sum_{h\in G} B_1(ax, h^{-1}x')\varepsilon(B_2(ay, h^{-1}y'))h\\
&= b\sum_{h\in G} w(a)B_1(x, a^{-1}h^{-1}x')\varepsilon(B_2(y, a^{-1}h^{-1}y'))h\\
&= b\sum_{h\in G} w(a)B_1(x, (ha)^{-1}x')\varepsilon(B_2(y, (ha)^{-1}y'))h\\
&= b\sum_{k\in G} w(a)B_1(x, k^{-1}x')\varepsilon(B_2(y, k^{-1}y'))ka^{-1}\\
&= b\ B(x\otimes y, x'\otimes y')\ \overline{a}.
\end{aligned}$$

Thus, B is an $R[G]$-valued λ-Hermitian form on M. Note that B_1 and $\varepsilon\circ B_2$ are both nonsingular. So, $B_1\otimes(\varepsilon\circ B_2)$ is nonsingular, which implies that B is nonsingular.

Third, we describe the definition of $q: M\to A_q/\Lambda$. Let $F(M_1\times M_2)$ denote the R-free module with basis $\{(x,y)\mid x\in M_1,\ y\in M_2\}$ (although it may not be finitely generated), T the subset of $F(M_1\times M_2)$ consisting of all elements of the form

$$r(x,y)-(rx,y),\quad r(x,y)-(x,ry),$$
$$(x+x',y)-(x,y)-(x',y),\quad\text{or}\quad (x,y+y')-(x,y)-(x,y'),$$

where r ranges over R, x and x' over M_1, y and y' over M_2; and let $[\ \ ]: F(M_1\times M_2)\to M_1\otimes M_2$ denote the canonical map.

Lemma 9.1. *Let f be a map from $F(M_1\times M_2)$ to $A_q/\Lambda = A/(A_s+\Lambda)$. If the following conditions* (1)–(3) *are fulfilled for all $r\in R$, u, $v\in F(M_1\times M_2)$ and $t\in T$:*

(1) $f(ru)=r^2f(u)$,
(2) $f(u+v)=f(u)+f(v)+B([u],[v])$,
(3) $f(t)=0$,

then f factors through $M_1\otimes M_2\to A_q/\Lambda$.

The proof is elementary, and we omit it.

Define a map $f : F(M_1 \times M_2) \to A_q/\Lambda = A/(A_s + \Lambda)$ by

$$f(\sum_i r_i(x_i, y_i)) = \sum_i \sum_{g \in \mathcal{Q}(G)} r_i^2 B_1(x_i, g^{-1}x_i) q_2(y_i)_g g + \sum_{i<j} r_i r_j B(x_i \otimes y_i, x_j \otimes y_j),$$

for finitely many distinct (x_i, y_i) with $x_i \in M_1$, $y_i \in M_2$, where $r_i \in R$.

By definition, we have $f(ru) = r^2 f(u)$ for all $r \in R$ and $u \in F(M_1 \times M_2)$.

Note that for $u = \sum_i r_i(x_i, y_i)$ and $v = \sum_i r_i'(x_i, y_i)$, we have

$$\begin{aligned} f(u+v) = &\sum_i \sum_{g \in \mathcal{Q}(G)} (r_i + r_i')^2 B_1(x_i, g^{-1}x_i) q_2(y_i)_g g \\ &+ \sum_{i<j} (r_i + r_i')(r_j + r_j') B(x_i \otimes y_i, x_j \otimes y_j). \end{aligned}$$

Thus, we have

$$\begin{aligned} &f(u+v) - f(u) - f(v) \\ &\quad = \sum_i \sum_{g \in \mathcal{Q}(G)} 2 r_i r_i' B_1(x_i, g^{-1}x_i) q_2(y_i)_g g + \sum_{i<j} (r_i r_j' + r_i' r_j) B(x_i \otimes y_i, x_j \otimes y_j). \end{aligned}$$

On the other hand, in A_q/Λ we have

$$\begin{aligned} B(\sum_i r_i x_i \otimes y_i, \sum_i r_i' x_i \otimes y_i) &= \sum_i r_i r_i' B(x_i \otimes y_i, x_i \otimes y_i) \\ &\quad + \sum_{i<j} (r_i r_j' B(x_i \otimes y_i, x_j \otimes y_j) + r_j r_i' B(x_j \otimes y_j, x_i \otimes y_i)) \\ &= \sum_i r_i r_i' B(x_i \otimes y_i, x_i \otimes y_i) \\ &\quad + \sum_{i<j} (r_i r_j' B(x_i \otimes y_i, x_j \otimes y_j) + r_i' r_j B(x_i \otimes y_i, x_j \otimes y_j)). \end{aligned}$$

Moreover, in $A/(A_s + \Lambda)$ we have

$$\begin{aligned} B(x_i \otimes y_i, x_i \otimes y_i) &= \sum_{g \in G} B_1(x_i, g^{-1}x_i) \varepsilon(B_2(y_i, g^{-1}y_i)) g \\ &= \sum_{g \in \mathcal{Q}(G)} B_1(x_i, g^{-1}x_i) 2 q_2(y_i)_g g. \end{aligned}$$

Thus we obtain $f(u+v) - f(u) - f(v) = B([u],[v])$ in A_s/Λ.

It is clear that $f(t) = 0$ for all $t \in T$.

Since the conditions (1)–(3) in Lemma 9.1 are satisfied, we obtain the map $q : M \to A_q/\Lambda$ by $q([u]) = f(u)$ for $u \in F(M_1 \times M_2)$. Immediately we have $q(r[u]) = r^2 q([u])$ and $q([u+v]) - q([u]) - q([v]) = B([u],[v])$ for $r \in R$ and u,

$v \in F(M_1 \times M_2)$. For $g \in G$ and $u = (x, y)$, we have

$$\begin{aligned} q(g[u]) &= f(gx, gy) \\ &= \sum_{h \in \mathcal{Q}(G)} B_1(gx, h^{-1}gx) q_2(gy)_h h \\ &= \sum_{h \in \mathcal{Q}(G)} w(g) B_1(x, g^{-1}h^{-1}gx) q_2(y)_{g^{-1}hg} h \\ &= \sum_{h \in \mathcal{Q}(G)} w(g) B_1(x, k^{-1}x) q_2(y)_k gkg^{-1} \\ &= g \sum_{h \in \mathcal{Q}(G)} B_1(x, k^{-1}x) q_2(y)_k k \overline{g} \\ &= g f(x \otimes y) \overline{g} \\ &= g q([u]) \overline{g}, \end{aligned}$$

where $k = g^{-1}hg$. Thus, $q(gz) = gq(z)\overline{g}$ for all $g \in G$ and $z \in M$.

Next we check the property (3) in Definition 7.1. For $u = (x, y)$ we have

$$\begin{aligned} \widetilde{q([u])} + \lambda \overline{\widetilde{q([u])}} &= \sum_{g \in \mathcal{Q}(G)} B_1(x, g^{-1}x)(\widetilde{q_2(y)}_g g + \lambda \widetilde{q_2(y)}_g \overline{g}) \\ &= \sum_{g \in G} B_1(x, g^{-1}x) B_2(y, y)_g g \\ &= B([u], [u]) \quad \text{in } A_q = A/A_s, \end{aligned}$$

which shows that $\widetilde{q(z)} + \lambda \overline{\widetilde{q(z)}} = B(z, z)$ for all $z \in M$.

Putting all together, we see that the current triple (M, B, q) is an $\boldsymbol{A}$-quadratic $R[G]$-module.

Defining $\alpha : \Theta \to M$ by $\alpha(t) = \alpha_1(t) \otimes \alpha_2(t)$ for $t \in \Theta$, we establish $\boldsymbol{M}_1 \cdot \boldsymbol{M}_2$ $(= \boldsymbol{M} = (M, B, q, \alpha))$ from $\boldsymbol{M}_1 = (M_1, B_1, \alpha_1)$ and $\boldsymbol{M}_2 = (M_2, B_2, q_2, \alpha_2)$.

Theorem 9.2. *Let $\boldsymbol{A} = (R, G, Q, S, \lambda, w)$ and Θ be as above. Then*

$$\mathrm{WQ}_0(\boldsymbol{A}, \Theta), \quad \mathrm{WQ}_0(\boldsymbol{A}, \Theta)^{\text{t-iso}} \quad \textit{and} \quad \mathrm{WQ}_0(\boldsymbol{A})$$

are modules over $\mathrm{GW}_0(R, G, S, \Theta)$, and $\mathrm{WQ}_0(\boldsymbol{A})$ is one over $\mathrm{GW}_0(R, G, S)$ by the pairing

$$(\boldsymbol{M}_1, \boldsymbol{M}_2) \longmapsto \boldsymbol{M}_1 \cdot \boldsymbol{M}_2.$$

10. Tensor products and ∇-invariants

In this section we invoke that R is square identical. Let Q, S, w, λ and Θ be as in Section 7, and let $\rho^{(2)} : \Theta \to \mathfrak{P}(S)$ be a G-map such that for every $s \in S$, there exists exactly one $t \in \Theta$ with $\rho^{(2)}(t) = s$. Hence, by Proposition 5.4, $\mathrm{SGW}_0(R, G, S, \Theta)$ is a commutative ring with unit.

Proposition 10.1. *Let $\boldsymbol{M}_1 = (M_1, B_1, \alpha_1)$ and $\boldsymbol{M}_2 = (M_2, B_2, q_2, \alpha_2)$ be objects in $\mathcal{SH}(R, G, S, \Theta)$ and $\mathcal{SQ}(\boldsymbol{A}, \Theta)$, respectively. Then $\boldsymbol{M} = \boldsymbol{M}_1 \cdot \boldsymbol{M}_2 = (M, B, q, \alpha)$ defined in the previous section lies in $\mathcal{SQ}(\boldsymbol{A}, \Theta)$.*

Proof. It was already shown that $\boldsymbol{M} = \boldsymbol{M}_1 \cdot \boldsymbol{M}_2$ belongs to $\mathcal{Q}(\boldsymbol{A}, \Theta)$. Therefore, it suffices to show that $\nabla_{\boldsymbol{M}} = 0$. By definition, we have

$$\begin{aligned}
\nabla_{\boldsymbol{M}}(x \otimes y)(s) &= \varepsilon(B(\Delta_\alpha(s) - x \otimes y, s(x \otimes y))) \\
&= \varepsilon(B(\Delta_{\alpha_1}(s) \otimes \Delta_{\alpha_2}(s) - x \otimes y, sx \otimes sy)) \\
&= \varepsilon(B(\Delta_{\alpha_1}(s) \otimes \Delta_{\alpha_2}(s), sx \otimes sy)) - \varepsilon(B(x \otimes y, sx \otimes sy)) \\
&= B_1(\Delta_{\alpha_1}(s), sx)\varepsilon(B_2(\Delta_{\alpha_2}(s), sy)) - B_1(x, sx)\varepsilon(B_2(y, sy)) \\
&= B_1(\Delta_{\alpha_2}(s) - x, sx)\varepsilon(B_2(\Delta_{\alpha_2}(s), sy)) \\
&\quad + B_1(x, sx)\varepsilon(B_2(\Delta_{\alpha_2}(s) - y, sy)) \\
&= \nabla_{\boldsymbol{M}_1}(x)(s)\varepsilon(B_2(\Delta_{\alpha_2}(s), sy)) + B_1(x, sx)\nabla_{\boldsymbol{M}_2}(y)(s) \\
&= 0 \quad \text{in } R/2R
\end{aligned}$$

for $x \in M_1$, $y \in M_2$, and $s \in S$. By using Proposition 5.2 (1), we have $\nabla_{\boldsymbol{M}} = 0$. □

The next theorem follows.

Theorem 10.2. *Let* $\boldsymbol{A} = (R, G, Q, S, \lambda, w)$ *and* Θ *be as above. Then*

$$\mathrm{SWQ}_0(\boldsymbol{A}, \Theta), \quad \mathrm{SWQ}_0(\boldsymbol{A}, \Theta)^{\text{t-iso}} \quad \textit{and} \quad \mathrm{SWQ}_0(\boldsymbol{A})$$

are modules over $\mathrm{SGW}_0(R, G, S, \Theta)$.

11. The Mackey and Green structures of GW and SGW

Let S be a conjugation-invariant subset of $G(2)$, and set

$$S_H = H \cap S$$

for each $H \in \mathcal{S}(G)$. Let $Z^{(0)}$ be a finite G-set and let $\mathfrak{P}(Z^{(0)})$ stand for the set of all subsets of $Z^{(0)}$. Let $\mathcal{S}(G) \to \mathfrak{P}(Z^{(0)})$; $H \mapsto Z^{(0)}_H$, be an intersection-preserving G-map (see (3.1)), where $\mathcal{S}(G)$ is the set of all subgroups of G on which G acts by conjugation.

Define Θ_H by

$$\Theta_H = S_H \amalg Z^{(0)}_H.$$

It immediately follows that the map $H \mapsto \Theta_H$ is intersection preserving. Define $\rho^{(2)}_H : \Theta_H \to \mathfrak{P}(S_H)$ by

$$\rho^{(2)}_H(t) = \begin{cases} \{t\} & (t \in S_H), \\ \emptyset & (t \in Z^{(0)}_H). \end{cases}$$

Then, obviously, for each $s \in S_H$, there exists exactly one $t \in \Theta_H$ with $s \in \rho^{(2)}_H(t)$. In this case, $\mathrm{GW}_0(R, H, \Theta_H)$ is a commutative ring with unit for each subgroup H of G, and so is $\mathrm{SGW}_0(R, H, S_H, \Theta_H)$ if R is square identical.

Now let $\varphi : H \to K$ be a morphism in $\mathcal{G}$, namely one of an inclusion map, a conjugation map, or a composition of such maps. Then we have the associated φ-equivariant map $\psi : \Theta_H \to \Theta_K$. Actually, if φ is the inclusion map $j_{H,K} : H \to K$, then $S_H \subset S_K$ and $Z^{(0)}_H \subset Z^{(0)}_K$, and therefore the associated $\psi : \Theta_H \to \Theta_K$ is the inclusion map; if φ is the conjugation map $c_{(H,g)} : H \to gHg^{-1}$, then the associated $\psi : \Theta_H \to \Theta_{gHg^{-1}} = g\Theta_H$ is the left translation $\ell_{(\Theta_H,g)}$ by g. Since the G-action on S is given by conjugation, $\ell_{(\Theta_H,g)}|_{S_H}$ is the conjugation $c_{(H,g)}|_{S_H}$ by g. Thus, there are canonical correspondences

$$\mathrm{GW}_0(R, H, \Theta_H) \to \mathrm{GW}_0(R, K, \Theta_K);\ [M, B, \alpha] \mapsto [\varphi_\# M, \varphi_\# B, \psi_\# \alpha]$$

and

$$\mathrm{GW}_0(R, K, \Theta_K) \to \mathrm{GW}_0(R, H, \Theta_H);\ [N, B, \alpha] \mapsto [\varphi^{\#}N, \varphi^{\#}B, \psi^{\#}\alpha].$$

Lemma 11.1. *$\nabla_{\boldsymbol{\varphi_{\#}M}} = 0$ for any morphism $\varphi : H \to K$ in $\mathcal{G}$ and any object $\boldsymbol{M} = (M, B, \alpha)$ in $\mathcal{SH}(R, H, \Theta_H)$.*

Proof. For the proof, we may suppose that $\varphi = j_{H,K}$ or $c_{(H,g)}$. For any $z = k \otimes_\varphi x \in \varphi_{\#}M$ with $k \in K$, $x \in M$ and $s \in S_K$, we have

$$\begin{aligned}\nabla_{\boldsymbol{\varphi_{\#}M}}(k \otimes_\varphi x)(s) &= \varphi_{\#}B(\Delta_{\psi_{\#}\alpha}(s) - k \otimes_\varphi x, s(k \otimes_\varphi x)) \\ &= \varphi_{\#}B(\Delta_{\psi_{\#}\alpha}(s), s(k \otimes_\varphi x)) - \varphi_{\#}B(k \otimes_\varphi x, s(k \otimes_\varphi x)) \quad \text{in } R/2R.\end{aligned}$$

By definition, we have

$$\begin{aligned}&\varphi_{\#}B(\Delta_{\psi_{\#}\alpha}(s), s(k \otimes_\varphi x)) = \varphi_{\#}B(\Delta_{\psi_{\#}\alpha}(s), k \otimes_\varphi x) \\ &= \varphi_{\#}B(\psi_{\#}\alpha(s), k \otimes_\varphi x) \\ &= \sum_{[a,s'] \in K \times_{H,\varphi} \Theta_H} \{\varphi_{\#}B(a \otimes_\varphi \alpha(s'), k \otimes_\varphi x) \mid s' \in S_H,\ a\varphi(s')a^{-1} = s\} \\ &= \sum_{[a,s'] \in K \times_{H,\varphi} \Theta_H} \{\delta_{a\varphi(H), k\varphi(H)} B(\alpha(s'), \varphi^{-1}(a^{-1}k)x) \mid s' \in S_H,\ \varphi(s') = a^{-1}sa\} \\ &= \sum_{[a,s'] \in K \times_{H,\varphi} \Theta_H} \{\delta_{\varphi(H), a^{-1}k\varphi(H)} B(\alpha(s'), \varphi^{-1}(a^{-1}k)x) \mid s' \in S_H,\ \varphi(s') = a^{-1}sa\} \\ &= \sum_{[k,s''] \in K \times_{H,\varphi} \Theta_H} \{B(\alpha(s''), x) \mid s'' \in S_H,\ \varphi(s'') = k^{-1}sk\} \\ &= \sum_{[k,s''] \in K \times_{H,\varphi} \Theta_H} \{B(\alpha(s''), s''x) \mid s'' \in S_H,\ \varphi(s'') = k^{-1}sk\} \\ &= \sum_{[k,s''] \in K \times_{H,\varphi} \Theta_H} \{B(x, s''x) \mid s'' \in S_H,\ \varphi(s'') = k^{-1}sk\} \\ &= \begin{cases} B(x, \varphi^{-1}(k^{-1}sk)x) & (\text{if } k^{-1}sk \in \varphi(H)), \\ 0 & (\text{otherwise}). \end{cases}\end{aligned}$$

On the other hand,

$$\begin{aligned}\varphi_{\#}B((k \otimes_\varphi x), s(k \otimes_\varphi x)) &= \varphi_{\#}B(k \otimes_\varphi x, sk \otimes_\varphi x) \\ &= \delta_{k\varphi(H), sk\varphi(H)} B(x, \varphi^{-1}(k^{-1}sk)x) \\ &= \begin{cases} B(x, \varphi^{-1}(k^{-1}sk)x) & (\text{if } k^{-1}sk \in \varphi(H)), \\ 0 & (\text{otherwise}). \end{cases}\end{aligned}$$

This gives us $\nabla_{\boldsymbol{\varphi_{\#}M}}(z)(s) = 0$ for all $z \in \varphi_{\#}M$ and $s \in \Theta_K$. □

Proposition 11.2. *Let S_H, Z_H and Θ_H be as above. Then, the Grothendieck-Witt ring functor $H \mapsto \mathrm{GW}_0(R, H, \Theta_H)$, $H \in \mathcal{S}(G)$, is a Mackey functor, and so is the special Grothendieck-Witt ring functor $\mathrm{SGW}_0(R, H, S_H, \Theta_H)$, $H \in \mathcal{S}(G)$.*

Proof. This follows from Propositions 3.2, 3.4, 3.5, 4.3, 4.5 and 4.6, and Lemma 11.1. □

Theorem 11.3. *Let S_H, Z_H and Θ_H be as above. Then, the Grothendieck-Witt ring functor $H \mapsto \mathrm{GW}_0(R, H, \Theta_H)$, $H \in \mathcal{S}(G)$, is a Green functor, and the special*

Grothendieck-Witt ring functor $H \mapsto \mathrm{SGW}_0(R, H, S_H, \Theta_H)$, $H \in \mathcal{S}(G)$, *is a Green functor, possibly without unit. If* R *is square identical, then the functor* $H \mapsto \mathrm{SGW}_0(R, H, S_H, \Theta_H)$, $H \in \mathcal{S}(G)$, *is a Green functor.*

Proof. The theorem follows from Propositions 3.1, 3.3, 4.2, 4.4, 4.10 and 5.4. □

Theorem 11.4. *The special Grothendieck-Witt group functor*

$$H \mapsto \mathrm{SGW}_0(R, H, S_H)$$

is a module over the Grothendieck-Witt ring functor $H \to \mathrm{GW}_0(R, H, S_H)$.

Proof. By Proposition 5.5, $\mathrm{SGW}_0(R, H, S_H)$ is a module over $\mathrm{GW}_0(R, H)$. The required properties for a Frobenius pairing follow from Propositions 3.1, 3.3, 4.2 and 4.4. □

12. The pairing $\mathrm{SGW}_0 \times \mathrm{SWQ}_0 \to \mathrm{SWQ}_0$

Let $S \subset G(2)$, S_H, $Z_H^{(0)}$, Θ_H, $\rho_H^{(2)}$ be as in Section 11, where $H \in \mathcal{S}(G)$. Let $w : G \to \{-1, 1\}$ be a homomorphism and let λ stand for either 1 or -1. In the current section we invoke

$$S \subset G^\lambda(2).$$

Let Q be a conjugation-invariant subset of $G^{-\lambda}(2)$. We set $Q_H = H \cap Q$, $A_H = R[H]$, and $\boldsymbol{A}_H = (R, H, Q_H, S_H, \lambda, w|_H)$ for $H \in \mathcal{S}(G)$.

Let $\varphi : H \to K$, where H, $K \in \mathcal{S}(G)$, be a monomorphism such that $w|_K \circ \varphi = w|_H$, $\varphi(Q_H) \subset Q_K$, and $\varphi(S_H) \subset S_K$.

Let $\boldsymbol{N} = (N, B, q)$ be an $\boldsymbol{A}_K$-quadratic $R[K]$-module. We can write $q(x)$ as $\sum_{g \in \mathcal{Q}(K)} q(x)_g g$, where $\mathcal{Q}(K) = K \cap \mathcal{Q}(G)$ and $q(x)_g \in R_g$. We define $\varphi^\# q : \varphi^\# M \to (A_H)_q/\Lambda_H = R[H]/(R[S_H] + \Lambda_H)$ by

$$\varphi^\# q(x) = \sum_{h \in \mathcal{Q}(H)} q(x)_{\varphi(h)} h$$

for $x \in \varphi^\# M$, where Λ_H is the smallest form parameter of $R[H]$ including Q_H.

Lemma 12.1. *The* $\varphi^\# q$ *above is an* $\boldsymbol{A}_H$*-quadratic form on* $\varphi^\# N$ *with respect to* $\varphi^\# B$.

Proof. The proof is straightforward, as follows: For $g \in H$ and $x \in \varphi^\# N$, we have

$$\begin{aligned}
\varphi^\# q(gx) &= \sum_{h \in \mathcal{Q}(H)} q(gx)_{\varphi(h)} h \\
&= \sum_{h \in \mathcal{Q}(H)} q(\varphi(g)x)_{\varphi(h)} h \\
&= \sum_{h \in \mathcal{Q}(H)} w(\varphi(g)) q(x)_{\varphi(g)^{-1}\varphi(h)\varphi(g)} h \\
&= g(\sum_{h \in \mathcal{Q}(H)} q(x)_{\varphi(g^{-1}hg)} g^{-1}hg)\overline{g} \\
&= g\varphi^\# q(x)\overline{g}.
\end{aligned}$$

For x, $y \in \varphi^{\#}N$, we have

$$\begin{aligned}\varphi^{\#}q(x+y) - \varphi^{\#}q(x) - \varphi^{\#}q(x) &= \sum_{h\in\mathcal{Q}(H)} (q(x+y)_{\varphi(h)} - q(x)_{\varphi(h)} - q(y)_{\varphi(h)})h \\ &= \sum_{h\in H} B(x,y)_{\varphi(h)}h \\ &= \sum_{h\in H} \varepsilon(B(x,\varphi(h)^{-1}y))h \\ &= \varphi^{\#}B(x,y)\end{aligned}$$

in $A_H/(\Lambda_H + (A_H)_s)$.

For $x \in \varphi^{\#}N$, we have

$$\begin{aligned}\widetilde{\varphi^{\#}q(x)} + \lambda\overline{\widetilde{\varphi^{\#}q(x)}} &= \sum_{h\in\mathcal{Q}(H)} (\widetilde{q(x)_{\varphi(h)}}h + \lambda\widetilde{q(x)_{\varphi(h)}}\overline{h}) \\ &= \sum_{h\in\mathcal{Q}(H)} (\widetilde{q(x)_{\varphi(h)}}h + \lambda w(h)\widetilde{q(x)_{\varphi(h)}}h^{-1}) \\ &= \sum_{h\in\mathcal{Q}(H)} (\widetilde{q(x)_{\varphi(h)}}h + \widetilde{q(x)_{\varphi(h)^{-1}}}h^{-1}) \\ &= \varphi^{\#}B(x,x) \quad \text{in } A_H/(A_H)_s.\end{aligned}$$

□

Proposition 12.2. *Let $\varphi : H \to K$, $\boldsymbol{A}_H$ and $\boldsymbol{A}_K$ be as above, and let $\boldsymbol{M}_1 = (M_1, B_1)$ and $\boldsymbol{M}_2 = (M_2, B_2, q_2)$ be a Hermitian $R[K]$-module and an $\boldsymbol{A}_K$-quadratic module, respectively. Then $(\varphi^{\#}\boldsymbol{M}_1)\cdot(\varphi^{\#}\boldsymbol{M}_2) = \varphi^{\#}(\boldsymbol{M}_1\cdot\boldsymbol{M}_2)$.*

Proof. Let x, $x' \in M_1$ and y, $y' \in M_2$. Then

$$\begin{aligned}B_{\varphi^{\#}\boldsymbol{M}_1\cdot\varphi^{\#}\boldsymbol{M}_2}(x\otimes y, x'\otimes y') &= \sum_{h\in H} B_1(x,\varphi(h)^{-1}x')\varepsilon(B_2(y,\varphi(h)^{-1}y'))h \\ &= B_{\varphi^{\#}(\boldsymbol{M}_1\cdot\boldsymbol{M}_2)}(x\otimes y, x'\otimes y').\end{aligned}$$

In addition,

$$\begin{aligned}q_{\varphi^{\#}\boldsymbol{M}_1\cdot\varphi^{\#}\boldsymbol{M}_2}(x\otimes y) &= \sum_{h\in\mathcal{Q}(H)} B_1(x,\varphi(h)^{-1}x)q_2(y)_{\varphi(h)}h \\ &= q_{\varphi^{\#}(\boldsymbol{M}_1\cdot\boldsymbol{M}_2)}(x\otimes y).\end{aligned}$$

We have established the proposition. □

Now let $\boldsymbol{M} = (M, B, q)$ be an $\boldsymbol{A}_H$-quadratic $R[H]$-module such that M is stably $R[H]$-free and B is nonsingular. Let $\{g_1, \ldots, g_\ell\}$ be a complete set of representatives of $K/\varphi(H)$, where g_i are elements in K. We define $\varphi_{\#}q : \varphi_{\#}M \to (A_K)_q/\Lambda = A_K/(\Lambda_K + (A_K)_s)$ by

$$\varphi_{\#}q\left(\sum_{i=1}^{\ell} g_i \otimes_\varphi x_i\right) = \sum_{i=1}^{\ell} g_i\varphi(q(x_i))\overline{g_i} + \sum_{1\le i<j\le\ell} g_j\varphi\left(B(x_i,x_j)\right)\overline{g_i}.$$

Lemma 12.3. *The* $\varphi_{\#}q$ *above is a quadratic form on* $\varphi_{\#}M$ *with respect to* $\varphi_{\#}B$. *Namely, the following hold:*

(1) $\varphi_{\#}q(ru) = r^2\varphi_{\#}q(u)$,
(2) $\varphi_{\#}q(u+v) - \varphi_{\#}q(u) - \varphi_{\#}q(v) = \varphi_{\#}B(u,v)$,
(3) $\widetilde{\varphi_{\#}q}(u) + \lambda\overline{\widetilde{\varphi_{\#}q}(u)} = \varphi_{\#}B(u,u)$ *in* $A_K/(A_K)_s$,
(4) $\varphi_{\#}q(ku) = k\varphi_{\#}q(u)\overline{k}$,

for all $r \in R$, u, $v \in \varphi_{\#}M$, $k \in K$.

Proof. The equality (1) holds clearly.

The proof of (2) runs as follows:

$$\begin{aligned}
\varphi_{\#}q&\left(\sum_i g_i \otimes_\varphi x_i + \sum_i g_i \otimes_\varphi y_i\right) - \varphi_{\#}q\left(\sum_i g_i \otimes_\varphi x_i\right) - \varphi_{\#}q\left(\sum_i g_i \otimes_\varphi y_i\right) \\
&= \sum_{i=1}^{\ell} g_i(\varphi(q(x_i+y_i)) - \varphi(q(x_i)) - \varphi(q(y_i)))\overline{g_i} \\
&\quad + \sum_{1\le i<j\le\ell} g_j\varphi\left(B(x_i+y_i, x_j+y_j) - B(x_i,x_j) - B(y_i,y_j)\right)\overline{g_i} \\
&= \sum_{i=1}^{\ell} g_i\varphi(B(x_i,y_i))\overline{g_i} + \sum_{1\le i\ne j\le\ell} g_j\varphi\left(B(x_i,y_j)\right)\overline{g_i} \\
&= \varphi_{\#}B\left(\sum_{i=1}^{\ell} g_i \otimes_\varphi x_i,\ \sum_{j=1}^{\ell} g_j \otimes_\varphi y_j\right).
\end{aligned}$$

The equality (3) holds because

$$\begin{aligned}
\varphi_{\#}\widetilde{q(g_i \otimes_\varphi} x) &+ \lambda\overline{\varphi_{\#}\widetilde{q(g_i \otimes_\varphi} x)} \\
&= g_i\varphi(\widetilde{q(x)})\overline{g_i} + \lambda g_i\varphi(\overline{\widetilde{q(x)}})\overline{g_i} \\
&= g_i\varphi(B(x,x))\overline{g_i} \\
&= \varphi_{\#}B(g_i \otimes_\varphi x, g_i \otimes_\varphi x).
\end{aligned}$$

For $k \in K$, we can write kg_i in the form $g_{\sigma(i)}\varphi(h_i)$ with $h_i \in H$. Then

$$\begin{aligned}
\varphi_{\#}q(k(g_i \otimes_\varphi x)) &= \varphi_{\#}q(g_{\sigma(i)} \otimes_\varphi h_i x) \\
&= g_{\sigma(i)}\varphi(q(h_i x))\overline{g_{\sigma(i)}} \\
&= g_{\sigma(i)}\varphi(h_i)\varphi(q(x))\overline{g_{\sigma(i)}\varphi(h_i)} \\
&= kg_i\varphi(q(x))\overline{g_i}\overline{k} \\
&= k\varphi_{\#}q(g_i \otimes_\varphi x)\overline{k}.
\end{aligned}$$

The equation (4) follows from this and (2) above. □

Proposition 12.4. *Let H be a subgroup of G, $q : M \to (A_H)_q/\Lambda_H$ an $\boldsymbol{A}_H$-quadratic form on M, and g an element of G. Then the diagram*

$$\begin{array}{ccc} {c_{(H,g)}}_{\#}M & \xrightarrow{\quad {c_{(H,g)}}_{\#}q \quad} & \\ \Big\downarrow f_0 & & \searrow \\ c_{(gHg^{-1},g^{-1})}{}^{\#}M & \xrightarrow[\;c_{(gHg^{-1},g^{-1})}{}^{\#}q\;]{} & (A_{gHg^{-1}})_q/\Lambda_{gHg^{-1}} \end{array}$$

commutes, where f_0 is the canonical $R[gHg^{-1}]$-isomorphism (cf. Proposition 3.2).

The proof of the proposition is straightforward.

Proposition 12.5. *Let H be a subgroup of G and $q : M \to (A_H)_q/\Lambda_H$ an $\boldsymbol{A}_H$-quadratic form on M. Then for each $g \in H$, the following diagrams commute:*

$$\begin{array}{ccc} {c_{(H,g)}}_{\#}M & \xrightarrow{\quad {c_{(H,g)}}_{\#}q \quad} & \\ \Big\downarrow f_1 & & \searrow \\ M & \xrightarrow[\;w(g)q\;]{} & (A_H)_q/\Lambda_H, \end{array}$$

$$\begin{array}{ccc} c^{\#}_{(H,g)}M & \xrightarrow{\quad c^{\#}_{(H,g)}q \quad} & \\ \Big\downarrow f_2 & & \searrow \\ M & \xrightarrow[\;w(g)q\;]{} & (A_H)_q/\Lambda_H, \end{array}$$

where f_1 and f_2 are the canonical isomorphisms (cf. Proposition 3.4).

The proposition follows straightforwardly from the definition.

Proposition 12.6. *For any subgroups H and K of G, each $\boldsymbol{A}_H$-quadratic form $q : M \to (A_H)_q/\Lambda_H$ satisfies the w-Mackey double coset formula. Namely,*

$$(\mathrm{Res}^G_K \mathrm{Ind}^G_H q) \circ \omega = \sum_{KgH \in K\backslash G/H} w(g)\mathrm{Ind}^K_{K\cap gHg^{-1}} {c_{(H\cap g^{-1}Kg,g)}}_{\#} \mathrm{Res}^H_{H\cap g^{-1}Kg} q,$$

where ω is the canonical isomorphism (cf. Proposition 3.5). Particularly, in the case $w(G) = \{1\}$, q satisfies the Mackey double coset formula.

Proof. It suffices to prove that

$$(\mathrm{Res}^G_K \mathrm{Ind}^G_H q)(ag \otimes x) = w(g)(\mathrm{Ind}^K_{K\cap gHg^{-1}} {c_{(H\cap g^{-1}Kg,g)}}_{\#} \mathrm{Res}^H_{H\cap g^{-1}Kg} q)(a \otimes (e \otimes x))$$

for any $g \in G$, $a \in K$, $x \in \mathrm{Res}^H_{H\cap g^{-1}Kg} M$. This is valid because

$$\begin{aligned}(\mathrm{Res}^G_K \mathrm{Ind}^G_H q)(ag \otimes x) &= \sum_{k \in \mathcal{Q}(K)} (\mathrm{Ind}^G_H q)(ag \otimes x)_k k \\ &= \sum_{k \in \mathcal{Q}(K)} (agq(x)\overline{ag})_k k\end{aligned}$$

and

$$\begin{aligned}&(\mathrm{Ind}^K_{K\cap gHg^{-1}} c_{(H\cap g^{-1}Kg,g)_\#} \mathrm{Res}^H_{H\cap g^{-1}Kg} q)(a \otimes (e \otimes x)) \\ &\quad = \sum_{k \in \mathcal{Q}(K)} (a(c_{(H\cap g^{-1}Kg,g)_\#} \mathrm{Res}^H_{H\cap g^{-1}Kg} q)(e \otimes x)\overline{a})_k k \\ &\quad = \sum_{k \in \mathcal{Q}(K)} (ag(\mathrm{Res}^H_{H\cap g^{-1}Kg} q)(x) g^{-1}\overline{a})_k k \\ &\quad = w(g) \sum_{k \in \mathcal{Q}(K)} (agq(x)\overline{ag})_k k.\end{aligned}$$

□

Proposition 12.7. *Let $\boldsymbol{A}_H$ and Θ_H be as above for each $H \in \mathcal{S}(G)$. Then the Witt group functor $H \mapsto \mathrm{WQ}_0(\boldsymbol{A}_H, \Theta_H)$, $H \in \mathcal{S}(G)$, and the special Witt group functor $H \mapsto \mathrm{SWQ}_0(\boldsymbol{A}_H, \Theta_H)$, $H \in \mathcal{S}(G)$, are both w-Mackey functors, and hence modules over the Burnside ring functor $H \mapsto \Omega(G)$, $H \in \mathcal{S}(G)$.*

Proof. The claim for the Witt group functor follows from Propositions 3.2, 3.4, 3.5, 6.6, 6.7, 6.8, 12.4, 12.5, and 12.6.

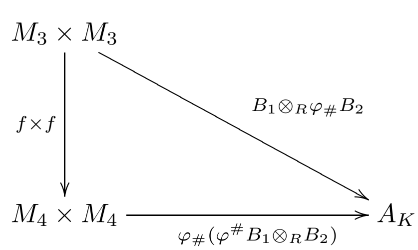
Let $\boldsymbol{M} = (M, B, q, \alpha)$ be a Θ_H-positioned $\boldsymbol{A}_H$-quadratic $R[H]$-module. By Lemma 6.3, $\varepsilon \circ B : M \times M \to R$ is a λ-symmetric, $(H, w|_H)$-invariant, R-bilinear form. For a morphism $\varphi : H \to K$ in $\mathcal{G}$, the same argument as the proof of Lemma 11.1 shows that if $\nabla_{\boldsymbol{M}} = 0$ (see (8.1)), then $\nabla_{\varphi_\# \boldsymbol{M}} = 0$. (In fact, consider the case where R is replaced by $R/2R$.) Thus, the claim for the special Witt group functor also follows. □

In the remainder of this section, let $\varphi : H \to K$ be a morphism in $\mathcal{G}$.

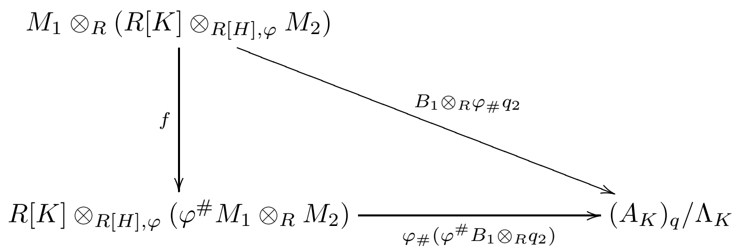
Proposition 12.8. *Let $\boldsymbol{M}_1 = (M_1, B_1, \alpha_1)$ and $\boldsymbol{M}_2 = (M_2, B_2, q_2, \alpha_2)$ be objects in $\mathcal{H}(R, K, \Theta_K)$ and $\mathcal{Q}(\boldsymbol{A}_H, \Theta_H)$, respectively. Let*

$$f : M_1 \otimes_R \varphi_\# M_2 \to \varphi_\#(\varphi^\# M_1 \otimes_R M_2)$$

denote the canonical isomorphism, namely $f(x \otimes (k \otimes_\varphi y)) = k \otimes_\varphi (k^{-1}x \otimes y)$ for $k \in K$, $x \in M_1$ and $y \in M_2$. Then the diagram



where $M_3 = (M_1 \otimes_R (R[K] \otimes_{R[H],\varphi} M_2))$ *and* $M_4 = R[K] \otimes_{R[H],\varphi} (\varphi^{\#} M_1 \otimes_R M_2)$, *and the diagram*



commute.

Proof. Let k, $k' \in K$, x, $x' \in M_1$, and y, $y' \in M_2$.

The commutability $B_1 \otimes (\varphi_{\#} B_2) = \varphi_{\#}((\varphi^{\#} B_1) \otimes B_2)$ via f holds because

$$\begin{aligned}
&B_1 \otimes (\varphi_{\#} B_2)(x \otimes (k \otimes_{\varphi} y), x' \otimes (k' \otimes_{\varphi} y')) \\
&\quad = \sum_{g \in K} B_1(x, g^{-1}x') \varepsilon(\varphi_{\#} B_2(k \otimes_{\varphi} y, g^{-1}(k' \otimes_{\varphi} y')))g \\
&\quad = \sum_{g \in K} w(k) \delta_{k\varphi(H), g^{-1}k'\varphi(H)} B_1(x, g^{-1}x') \varepsilon(B_2(y, \varphi^{-1}(k^{-1}g^{-1}k')y'))g
\end{aligned}$$

and

$$\begin{aligned}
&\varphi_{\#}((\varphi^{\#} B_1) \otimes B_2)(k \otimes_{\varphi} (k^{-1}x \otimes y), k' \otimes_{\varphi} ({k'}^{-1}x' \otimes y')) \\
&\quad = \sum_{g \in K} w(k) \delta_{k\varphi(H), g^{-1}k'\varphi(H)} \\
&\qquad\qquad \cdot \varepsilon \Big((\varphi^{\#} B_1 \otimes B_2)((k^{-1}x \otimes y), \varphi^{-1}(k^{-1}g^{-1}k')({k'}^{-1}x' \otimes y')) \Big) g \\
&\quad = \sum_{g \in K} w(k) \delta_{k\varphi(H), g^{-1}k'\varphi(H)} \\
&\qquad\qquad \cdot B_1(k^{-1}x, (k^{-1}g^{-1}k'){k'}^{-1}x') \varepsilon(B_2(y, \varphi^{-1}(k^{-1}g^{-1}k')y'))g \\
&\quad = \sum_{g \in K} w(k) \delta_{k\varphi(H), g^{-1}k'\varphi(H)} B_1(k^{-1}x, k^{-1}g^{-1}x') \varepsilon(B_2(y, \varphi^{-1}(k^{-1}g^{-1}k')y'))g \\
&\quad = \sum_{g \in K} w(k) \delta_{k\varphi(H), g^{-1}k'\varphi(H)} B_1(x, g^{-1}x') \varepsilon(B_2(y, \varphi^{-1}(k^{-1}g^{-1}k')y'))g.
\end{aligned}$$

The commutability $B_1 \otimes (\varphi_{\#} q_2) = \varphi_{\#}((\varphi^{\#} B_1) \otimes q_2)$ via f follows from

$$
\begin{aligned}
B_1 \otimes (\varphi_{\#} q_2)(x \otimes (k \otimes_\varphi y)) &= \sum_{g \in \mathcal{Q}(K)} B_1(x, g^{-1}x) \varphi_{\#} q_2(k \otimes_\varphi y)_g g \\
&= \sum_{g \in \mathcal{Q}(K)} B_1(x, g^{-1}x)(k\varphi(q_2(y))\overline{k})_g g \\
&= \sum_{g \in \mathcal{Q}(K)} B_1(x, g^{-1}x)\varphi(q_2(y))_{k^{-1}gk} w(k) g \\
&= \sum_{g \in \mathcal{Q}(K) \cap k\varphi(H)k^{-1}} B_1(x, g^{-1}x) q_2(y)_{\varphi^{-1}(k^{-1}gk)} w(k) g \\
&= k \left(\sum_{a \in k^{-1}\mathcal{Q}(K)k \cap \varphi(H)} B_1(x, ka^{-1}k^{-1}x) q_2(y)_{\varphi^{-1}(a)} a \right) \overline{k} \\
&= k \left(\sum_{b \in \mathcal{Q}(H)} B_1(x, k\varphi(b)^{-1}k^{-1}x) q_2(y)_b \varphi(b) \right) \overline{k}
\end{aligned}
$$

and

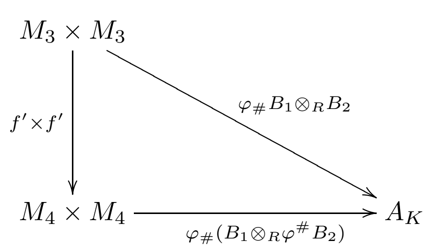
$$
\begin{aligned}
\varphi_{\#}((\varphi^{\#} B_1) \otimes q_2)(k \otimes_\varphi (k^{-1}x \otimes y)) &= k\varphi((\varphi^{\#} B_1) \otimes q_2(k^{-1}x \otimes y))\overline{k} \\
&= k\varphi \left(\sum_{h \in \mathcal{Q}(H)} \varphi^{\#} B_1(k^{-1}x, h^{-1}k^{-1}x) q_2(y)_h h \right) \overline{k} \\
&= k\varphi \left(\sum_{h \in \mathcal{Q}(H)} B_1(k^{-1}x, \varphi(h)^{-1}k^{-1}x) q_2(y)_h h \right) \overline{k} \\
&= k\varphi \left(\sum_{h \in \mathcal{Q}(H)} B_1(x, k\varphi(h)^{-1}k^{-1}x) q_2(y)_h h \right) \overline{k}.
\end{aligned}
$$

□

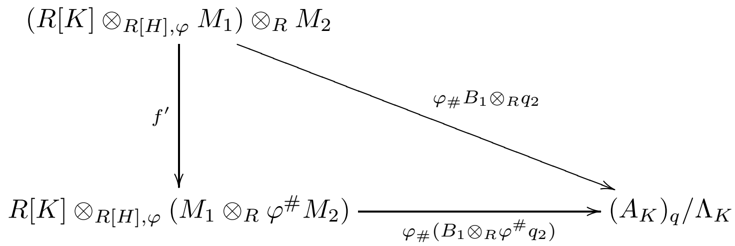
Proposition 12.9. *Let* $\boldsymbol{M}_1 = (M_1, B_1, \alpha_1)$ *and* $\boldsymbol{M}_2 = (M_2, B_2, q_2, \alpha_2)$ *be objects in* $\mathcal{H}(R, H, \Theta_H)$ *and* $\mathcal{Q}(\boldsymbol{A}_K, \Theta_K)$*, respectively. Let*

$$f' : (\varphi_{\#} M_1) \otimes_R M_2 \to \varphi_{\#}(M_1 \otimes_R \varphi^{\#} M_2)$$

denote the canonical isomorphism, namely $f'((k \otimes_\varphi x) \otimes y) = k \otimes_\varphi (x \otimes k^{-1}y)$ *for* $k \in K$*,* $x \in M_1$ *and* $y \in M_2$*. Then the diagram*



where $M_3 = (R[K] \otimes_{R[H],\varphi} M_1) \otimes_R M_2$ *and* $M_4 = R[K] \otimes_{R[H],\varphi} (M_1 \otimes_R \varphi^{\#} M_2)$, *and the diagram*



commute.

Proof. Let $k,\ k' \in K,\ x,\ x' \in M_1$, and $y,\ y' \in M_2$.

The commutability $(\varphi_{\#} B_1) \otimes B_2 = \varphi_{\#}(B_1 \otimes (\varphi^{\#} B_2))$ via f' holds because

$$\begin{aligned}
&(\varphi_{\#} B_1) \otimes B_2((k \otimes_\varphi x) \otimes y, (k' \otimes_\varphi x') \otimes y') \\
&\qquad = \sum_{g \in K} (\varphi_{\#} B_1)((k \otimes_\varphi x), g^{-1}(k' \otimes_\varphi x'))\varepsilon(B_2(y, g^{-1}y'))g \\
&\qquad = \sum_{g \in K} \delta_{k\varphi(H), g^{-1}k'\varphi(H)} B_1(x, \varphi^{-1}(k^{-1}g^{-1}k')x')\varepsilon(B_2(y, g^{-1}y'))g
\end{aligned}$$

and

$$\begin{aligned}
&\varphi_{\#}(B_1 \otimes (\varphi^{\#} B_2))(k \otimes_\varphi (x \otimes k^{-1}y), k' \otimes_\varphi (x' \otimes k'^{-1}y')) \\
&\quad = \sum_{g \in K} w(k)\delta_{k\varphi(H), g^{-1}k'\varphi(H)} \\
&\qquad\qquad \cdot \varepsilon\left(B_1 \otimes (\varphi^{\#} B_2)(x \otimes k^{-1}y, \varphi^{-1}(k^{-1}g^{-1}k')(x' \otimes k'^{-1}y'))\right) g \\
&\quad = \sum_{g \in K} w(k)\delta_{k\varphi(H), g^{-1}k'\varphi(H)} B_1(x, \varphi^{-1}(k^{-1}g^{-1}k')x')\varepsilon(B_2(k^{-1}y, k^{-1}g^{-1}y'))g \\
&\quad = \sum_{g \in K} \delta_{k\varphi(H), g^{-1}k'\varphi(H)} B_1(x, \varphi^{-1}(k^{-1}g^{-1}k')x')\varepsilon(B_2(y, g^{-1}y'))g.
\end{aligned}$$

The commutability $(\varphi_{\#} B_1) \otimes q_2 = \varphi_{\#}(B_1 \otimes (\varphi^{\#} q_2))$ via f' follows from

$$\begin{aligned}
(\varphi_{\#} B_1) \otimes q_2((k \otimes_\varphi x) \otimes y) &= \sum_{g \in \mathcal{Q}(K)} (\varphi_{\#} B_1)(k \otimes_\varphi x, g^{-1}(k \otimes_\varphi x))q_2(y)_g g \\
&= \sum_{g \in \mathcal{Q}(K)} \delta_{k\varphi(H), g^{-1}k\varphi(H)} B_1(x, \varphi^{-1}(k^{-1}g^{-1}k)x)q_2(y)_g g \\
&= \sum_{g \in \mathcal{Q}(k\varphi(H)k^{-1})} B_1(x, \varphi^{-1}(k^{-1}g^{-1}k)x)q_2(y)_g g \\
&= k \sum_{h \in \mathcal{Q}(H)} B_1(x, h^{-1}x)q_2(y)_{k\varphi(h)k^{-1}} \varphi(h)k^{-1} \\
&= k \sum_{h \in \mathcal{Q}(H)} w(k)B_1(x, h^{-1}x)q_2(k^{-1}y)_{\varphi(h)} \varphi(h)k^{-1} \\
&= k\varphi\left(\sum_{h \in \mathcal{Q}(H)} B_1(x, h^{-1}x)q_2(k^{-1}y)_{\varphi(h)} h\right)\overline{k}
\end{aligned}$$

and

$$\begin{aligned}\varphi_{\#}(B_1 \otimes (\varphi^{\#} q_2))(k \otimes_{\varphi} (x \otimes k^{-1}y)) &= k\varphi(B_1 \otimes (\varphi^{\#} q_2)(x \otimes k^{-1}y))\overline{k} \\ &= k\varphi\left(\sum_{h \in \mathcal{Q}(H)} B_1(x, h^{-1}x) q_2(k^{-1}y)_{\varphi(h)} h\right)\overline{k}.\end{aligned}$$

□

Let $\psi : \Theta_H \to \Theta_K$ denote the map associated with φ.

Theorem 12.10. *Let $\boldsymbol{A}_H$ and Θ_H be as above for each $H \in \mathcal{S}(G)$. Then the w-Mackey functor $H \mapsto \mathrm{WQ}_0(\boldsymbol{A}_H, \Theta_H)$, $H \in \mathcal{S}(G)$, is a module over the Green functor $H \mapsto \mathrm{GW}_0(R, H, \Theta_H)$, $H \in \mathcal{S}(G)$. If R is square identical, then the w-Mackey functor $H \mapsto \mathrm{SWQ}_0(\boldsymbol{A}_H, \Theta_H)$, $H \in \mathcal{S}(G)$, is a module over the Green functor $H \mapsto \mathrm{SGW}_0(R, H, S_H, \Theta_H)$, $H \in \mathcal{S}(G)$.*

Proof. By Proposition 3.3 we have $\alpha_1 \otimes (\psi_{\#}\alpha_2) = \psi_{\#}((\psi^{\#}\alpha_1) \otimes \alpha_2)$ via f in Proposition 12.8. By Proposition 3.3 we have $(\psi_{\#}\alpha_1) \otimes \alpha_2 = \psi_{\#}(\alpha_1 \otimes \psi^{\#}\alpha_2)$ via f' in Proposition 12.9. The theorem follows from Propositions 12.2, 12.8 and 12.9. □

13. Applications of induction and restriction

Let Z^0 be a finite G-set, and let $\mathcal{S}(G) \to \mathfrak{P}(Z^0)$; $H \mapsto Z_H^{(0)}$ be an intersection-preserving G-map. Let S be a conjugation-invariant subset of $G(2)$. We set $S_H = H \cap S$ and $\Theta_H = S_H \amalg Z_H^{(0)}$. Define $\rho_H^{(2)} : \Theta_H \to \mathfrak{P}(S_H)$ by

$$\rho_H^{(2)}(t) = \begin{cases} \{t\} & (t \in S_H), \\ \emptyset & (t \in Z_H^{(0)}). \end{cases}$$

Further, let $\mathcal{F}$ be a conjugation-invariant subset of $\mathcal{S}(G)$ such that

$$\Theta_G \times \Theta_G = \bigcup_{H \in \mathcal{F}} \Theta_H \times \Theta_H, \tag{13.1}$$

and let β be an element in the Burnside ring $\Omega(G)$ such that

$$\mathrm{Res}_H^G \beta = 1_{\Omega(H)} \quad \text{for any } H \in \mathcal{F}.$$

Theorem 13.1. *Let x be an arbitrary element in $\mathrm{SGW}_0(R, G, S, \Theta_G)$. If $\mathcal{F}$ contains all 2-hyperelementary (resp. cyclic) subgroups of G, then $(1_{\Omega(G)} - \beta)^2 x = 0$ (resp. $(1_{\Omega(G)} - \beta)^{2k+3} x = 0$, where k is the integer such that $|G| = 2^k m$ with an odd integer m).*

For the proof, we recall two lemmas.

Lemma 13.2 (A. Dress [11, Theorems 1 and 3])**.** *For a set $\mathcal{H}$ of subgroups of G, the restriction homomorphism*

$$\mathrm{Res} : \mathrm{GW}_0(\mathbb{Z}, G) \to \bigoplus_{H \in \mathcal{H}} \mathrm{GW}_0(\mathbb{Z}, H)$$

has the following properties.

(1) *If $\mathcal{H}$ contains all 2-hyperelementary subgroups of G, then* Res *is injective.*

(2) *If $\mathcal{H}$ contains all cyclic subgroups of G, then the kernel of* Res *is annihilated by 4.*

For a subgroup H of G, we denote by χ_H the homomorphism $\Omega(G) \to \mathbb{Z}$ such that $\chi_H([X]) = |X^H|$ for every finite G-set X.

Lemma 13.3 ([15, Proposition 6.3])**.** *Let x be an element of $\Omega(G)$ such that $\chi_H(x) \equiv 0 \mod 2$ for all $H \in \mathcal{S}(G)$. Then x^{k+1} lies in $2\Omega(G)$, where k is the integer such that $|G| = 2^k m$ with an odd integer m.*

Proof of Theorem 13.1. Let H be a 2-hyperelementary subgroup of G.

First consider the case where $\mathcal{F}$ contains all 2-hyperelementary subgroups of G. Then, it is obvious that $\mathrm{Res}^G_H(1_{\Omega(G)} - \beta) = 0$. Since the Green functor $\mathrm{GW}_0(\mathbb{Z}, -)$ is a module over the Green functor $\Omega(-)$, $\mathrm{Res}^G_H((1_{\Omega(G)} - \beta)\mathrm{GW}_0(\mathbb{Z}, G)) = 0$.

Next, consider the case where $\mathcal{F}$ contains all cyclic subgroups of G. Then

$$\chi_K(1_{\Omega(G)} - \beta) \equiv 0 \mod 2$$

for any subgroup K of H, and hence $\mathrm{Res}^G_H(1_{\Omega(G)} - \beta)^{2k+2}$ lies in $4\Omega(H)$. So, we can write $\mathrm{Res}^G_H(1_{\Omega(G)} - \beta)^{2k+2} = 4\gamma$ for some $\gamma \in \Omega(H)$. Clearly, $\mathrm{Res}^H_C \gamma = 0$ for all cyclic subgroups of H. Thus by (2) of Dress' Lemma, $\gamma\mathrm{GW}_0(\mathbb{Z}, H)$ is annihilated by 4, and hence $\mathrm{Res}^G_H((1_{\Omega(G)} - \beta)^{2k+2}\mathrm{GW}_0(\mathbb{Z}, G)) = 0$. By (1) of Dress' Lemma, we obtain

$$(1 - \beta)\mathrm{GW}_0(\mathbb{Z}, G) = 0 \quad \text{or} \quad (1_{\Omega(G)} - \beta)^{2k+2}\mathrm{GW}_0(\mathbb{Z}, G) = 0.$$

Since the canonical map $\mathrm{GW}_0(\mathbb{Z}, G) \to \mathrm{GW}_0(R, G)$ is an $\Omega(G)$-homomorphism of a ring with unit, it follows that

$$(1 - \beta)\mathrm{GW}_0(R, G) = 0 \quad \text{or} \quad (1_{\Omega(G)} - \beta)^{2k+2}\mathrm{GW}_0(R, G) = 0.$$

Noting that the Mackey functor $\mathrm{SGW}_0(R, -, S_-)$ is a module over the Green functor $\mathrm{GW}_0(R, -)$, we obtain

$$(1 - \beta)\mathrm{SGW}_0(R, G, S) = 0 \quad \text{or} \quad (1_{\Omega(G)} - \beta)^{2k+2}\mathrm{SGW}_0(R, G, S) = 0.$$

Recall Proposition 5.3, namely the fact that the canonical homomorphism

$$\mathrm{SGW}_0(R, G, S) \to \mathrm{SGW}_0(R, G, S, \Theta_G)^{\text{t-iso}}$$

is surjective. In addition, the homomorphism is an $\Omega(G)$-homomorphism. Hence, we conclude that

$$(1 - \beta)\mathrm{SGW}_0(R, G, S, \Theta_G)^{\text{t-iso}} = 0 \quad \text{or} \quad (1_{\Omega(G)} - \beta)^{2k+2}\mathrm{SGW}_0(R, G, S, \Theta_G)^{\text{t-iso}} = 0.$$

On the other hand, it is easy to check that $(1 - \beta)\mathrm{SGW}_0(R, G, S, \Theta_G)$ is contained in (the image by the canonical map from) $\mathrm{SGW}_0(R, G, S, \Theta_G)^{\text{t-iso}}$.

Putting all together, we establish that

$$(1 - \beta)^2\mathrm{SGW}_0(R, G, S, \Theta_G) = 0 \quad \text{or} \quad (1_{\Omega(G)} - \beta)^{2k+3}\mathrm{SGW}_0(R, G, S, \Theta_G) = 0.$$

□

Proof of Theorem 1.2. Here $Z^{(0)}$ is the empty set. Since $H \mapsto \mathrm{SGW}(R, H, S_H, S_H)$ is a Mackey functor, it is a module over the Burnside ring functor $H \mapsto \Omega(H)$ by [7, Proposition 6.2.3]. For each subgroup H of G we have

$$\Theta_H \times \Theta_H = (S \cap H) \times (S \cap H) = (S \times S) \cap (H \times H).$$

Thus (13.1) is fulfilled, and Theorem 1.2 follows from Theorem 13.1. □

Now let $w : G \to \{-1, 1\}$ be a homomorphism, $\lambda = 1$ or -1, and let Q be a conjugation-invariant subset of $G^{-\lambda}(2)$. Suppose $S \subset G^{\lambda}(2)$. For each $H \in \mathcal{S}(G)$, we set $A_H = R[H]$, $Q_H = H \cap Q$, and $\boldsymbol{A}_H = (R, H, Q_H, S_H, \lambda, w|_H)$.

Theorem 13.4. *Suppose R is square identical. Let x be an arbitrary element of the special Witt group $\mathrm{SWQ}_0(\boldsymbol{A}_G, \Theta_G)$. If $\mathcal{F}$ contains all 2-hyperelementary (resp. cyclic) subgroups of G, then $(1_{\Omega(G)} - \beta)^2 x = 0$ (resp. $(1_{\Omega(G)} - \beta)^{2k+3} x = 0$, where $|G| = 2^k m$ with m odd).*

Proof. The theorem follows from Proposition 12.10 and Theorem 13.1. □

Proof of Theorem 1.3. Theorem 1.3 follows from Theorem 13.4. □

Theorem 13.5. *Suppose that R is square identical, $\mathcal{F}$ contains any cyclic subgroup of G, and β has the form*

$$\beta = \sum_{H \in \widetilde{\mathcal{F}}} n_H [G/H],$$

with $n_H \in \mathbb{Z}$ for some lower closed subset $\widetilde{\mathcal{F}}$ of $\mathcal{S}(G)$; namely, any subgroup H of G lies in $\widetilde{\mathcal{F}}$ whenever $K \in \widetilde{\mathcal{F}}$ and $H \subset K$. Then

$$\mathrm{SWQ}_0(R, G, Q, S, \Theta_G) = \sum_{H \in \widetilde{\mathcal{F}}} \mathrm{Ind}_H^G \mathrm{SWQ}_0(R, H, Q_H, S_H, \Theta_H),$$

and the restriction homomorphism

$$\mathrm{Res} : \mathrm{SWQ}_0(R, G, Q, S, \Theta_G) \to \bigoplus_{H \in \widetilde{\mathcal{F}}} \mathrm{SWQ}_0(R, H, Q_H, S_H, \Theta_H)$$

is injective.

Proof. By hypothesis, we can write

$$(1_{\Omega(G)} - \beta)^{2|G|+3} = [G/G] - \sum_{H \in \widetilde{\mathcal{F}}} m_H [G/H]$$

with $m_H \in \mathbb{Z}$. For an arbitrary element $x \in \mathrm{SWQ}_0(R, G, Q, S, \Theta_G)$, Theorem 13.4 implies that

$$x = \sum_{H \in \widetilde{\mathcal{F}}} m_H [G/H] \cdot x = \sum_{H \in \widetilde{\mathcal{F}}} m_H \mathrm{Ind}_H^G (\mathrm{Res}_H^G x).$$

Moreover, if $\mathrm{Res}_H^G x = 0$ for every $H \in \widetilde{\mathcal{F}}$, then we conclude that $x = 0$. □

Proof of Theorem 1.4. Since G is a nonsolvable group, there exists an idempotent $\beta \in \Omega(G)$ such that $\chi_K(\beta) = 0$ for any nonsolvable subgroup K of G and $\chi_H(\beta) = 1$ for any solvable subgroup H of G. This element β has the form $\beta = \sum_H n_H [G/H]$ with $n_H \in \mathbb{Z}$, where H runs over the set of all solvable subgroups of G. Thus, Theorem 1.4 follows from Theorem 13.5. □

References

1. A. Bak, *K-Theory of Forms*, Annals of Mathematics Studies 98, Princeton Univ. Press, Princeton, 1981. MR **84m:**10012
2. A. Bak, *Induction for finite groups revisited*, J. Pure and Applied Algbera **104** (1995), pp. 235–241. MR **96m:**19002

3. A. Bak and M. Morimoto, *Equivariant surgery and applications*, in: Proceedings of Conference on Topology in Hawaii 1990 (ed. K. H. Dovermann), pp. 13–25, World Scientific Publ., Singapore, 1992. MR **93e:**57058
4. A. Bak and M. Morimoto, *K-theoretic groups with positioning map and equivariant surgery*, Proc. Japan Acad. **70** Ser. A (1994), 6–11. MR **95e:**19006
5. A. Bak and M. Morimoto, *Equivariant surgery with middle-dimensional singular sets. I*, Forum Math. **8** (1996), 267–302. MR **97b:**57031
6. A. Bak and W. Scharlau, *Grothendieck and Witt groups of orders and finite groups*, Invent. Math. **23** (1974), 207–240. MR **49:**5093
7. T. tom Dieck, *Transformation Groups and Representation Theory*, Lecture Notes in Mathematics 766, Springer-Verlag, Berlin–Heidlberg–New York, 1979. MR **82c:**57025
8. T. tom Dieck, *Transformation Groups*, de Gruyter Studies in Mathematics 8, Walter de Gruyter, Berlin, 1987. MR **89c:**57048
9. A. Dress, *A characterization of solvable groups*, Math. Zeit. **110** (1969), 213–217. MR **40:**1491
10. A. Dress, *Contributions to the theory of induced representations*, in: Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic, Proc. Conf., Battelle Memorial Inst., Seattle, 1972, Lecture Notes in Mathematics 342, pp. 183–240, Springer-Verlag, Berlin–Heidlberg–New York, 1973. MR **52:**5787
11. A. Dress, *Induction and structure theorems for Grothendieck and Witt rings of orthogonal representations of finite groups*, Bull. Amer. Math. Soc. **79** (1973), 741–745. MR **49:**7345
12. A. Dress, *Induction and structure theorems for orthogonal representations of finite groups*, Ann. of Math. **102** (1975), 291–325. MR **52:**8235
13. I. Hambleton and L. Taylor, *A guide to the calculation of the surgery obstruction groups for finite groups*, in: Surveys on Surgery Theory, vol. 1 (ed. S. Cappell, A. Ranicki and J. Rosenberg), Annals of Mathematics Studies 145, pp. 225–274, Princeton Univ. Press, Princeton, 2000. MR **2001e:**19007
14. I. Hambleton, L. Taylor and B. Williams, *An introduction to maps between surgery obstruction groups*, in: Algebraic Topology, Aarhus 1982, Lecture Notes in Mathematics 1051, pp. 29–127, Springer-Verlag, Berlin-Heidelberg-New York, 1984. MR **86b:**57017
15. E. Laitinen and M. Morimoto, *Finite groups with smooth one fixed point actions on spheres*, Forum Math. **10** (1998), 479–520. MR **99k:**57078
16. E. Laitinen, M. Morimoto and K. Pawałowski, *Deleting-inserting theorem for smooth actions of finite solvable groups on spheres*, Comment. Math. Helvetica **70** (1995), 10–38. MR **96b:**57043
17. M. Morimoto, *On one fixed point actions on spheres*, Proc. Japan Acad. **63** Ser. A Math. Sci. (1987), 95–97. MR **88j:**57039
18. M. Morimoto, *Most of the standard spheres have one fixed point actions of A_5*, in: Transformation Groups (ed. K. Kawakubo), Lecture Notes in Mathematics 1375, pp. 240–258, Springer-Verlag, Berlin–Heidlberg–New York, 1989. MR **90i:**57029
19. M. Morimoto, *Bak groups and equivariant surgery*, K-Theory **2** (1989), 465–483. MR **90d:**57037
20. M. Morimoto, *Most standard spheres have smooth one fixed point actions of A_5. II*, K-Theory **4** (1991), 289–302. MR **92h:**57055
21. M. Morimoto, *Equivariant surgery theory: Deleting–inserting theorems of fixed point manifolds on spheres and disks*, K-Theory **15** (1998), 13–32. MR **99i:**57056
22. M. Morimoto, *Equivariant surgery with middle dimensional singular sets. II: Equivariant framed cobordism invariance*, Trans. Amer. Math. Soc. **353** (2001), 2427–2440. MR **2001m:**57060
23. M. Morimoto, *The Burnside ring revisited*, in: Current Trends in Transformation Groups (eds. A. Bak, M. Morimoto and F. Ushitaki), K-Monographs in Mathematics 7, pp. 129–145, Kluwer Academic Publishers, Dordrecht-Boston-London, 2002.
24. M. Morimoto and K. Pawałowski, *Smooth actions of finite Oliver groups on spheres*, Topology **42** (2003), 395–421.
25. M. Morimoto and K. Uno, *Remarks on one fixed point A_5-actions on homotopy spheres*, in: Algebraic Topology Poznań 1989 (eds. S. Jackowski, B. Oliver and K. Pawałowski), Lecture Notes in Mathematics 1478, pp. 337–364, Springer-Verlag, Berlin–Heidlberg–New York, 1991. MR **92i:**57031

26. T. Petrie, *One fixed point actions on spheres I*, Advances in Mathematics **46** (1982), 3–14. MR **84b:**57027
27. C. T. C. Wall, *Surgery on Compact Manifolds*, London Mathematical Society Monographs, No. 1, Academic Press, London-New York, 1970. MR **55:**4217

Department of Environmental and Mathematical Sciences, Faculty of Environmental Science and Technology, Okayama University, Okayama, 700-8530 Japan
E-mail address: `morimoto@ems.okayama-u.ac.jp`