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PSEUDO-HOLOMORPHIC CURVES IN COMPLEX GRASSMANN MANIFOLDS

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ABSTRACT. It is proved that the Kähler angle of the pseudo-holomorphic sphere of constant curvature in complex Grassmannians is constant. At the same time we also prove several pinching theorems for the curvature and the Kähler angle of the pseudo-holomorphic spheres in complex Grassmannians with non-degenerate associated harmonic sequence.

1. INTRODUCTION

In this paper we study conformal minimal two-spheres in complex Grassmann manifolds by using the harmonic sequence. Given a harmonic map φ of surfaces M into the complex Grassmannian $\mathbf{G}_{k,n}$, by using the ∂'-transform Chern and Wolfson ([3], [10]) obtained the following harmonic sequence associated to φ:

$$\varphi = \varphi_0 \xrightarrow{\partial'} \varphi_1 \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \varphi_j \xrightarrow{\partial'} \cdots ,$$

where $\varphi_{j+1} = \partial' \varphi_j$, $j = 0, 1, \cdots$, and $\varphi_j : M \to \mathbf{G}_{k_j,n}$ are harmonic maps, $k_j = \operatorname{rank}(\varphi_j)$. If φ_j is anti-holomorphic, then $k_{j+1} = 0$. When φ is holomorphic we call φ_j a pseudo-holomorphic curve generated by φ. Such curves with the induced metrics from the associated complex Grassmann manifolds form a class of minimal immersions. When $k_j = k_{j+1}$ we say that φ_j is *non-degenerate*. When $k_j = k_{j+1}$ for all j we say that the harmonic sequence associated to the map φ is *non-degenerate*.

When specialized to $\mathbf{G}_{1,n} = \mathbf{CP}^{n-1}$, any pseudo-holomorphic curve is obtained from a holomorphic curve projected into $\mathbf{CP}^{n-1}$. Calabi ([2]) showed that any simply connected holomorphic curve in $\mathbf{CP}^{n-1}$ is completely determined, up to holomorphic isometries of $\mathbf{CP}^{n-1}$, by its induced metric. Calabi also showed that a simply connected holomorphic curve of constant curvature in $\mathbf{CP}^{n-1}$ is the Veronese curve, up to unitary equivalence. For a pseudo-holomorphic curve in $\mathbf{CP}^{n-1}$, Bolton, Jensen, Rigoli and Woodward ([1]) showed that, up to a holomorphic isometry of $\mathbf{CP}^{n-1}$, the harmonic sequence determined by any linearly full conformal minimal immersion of constant curvature in $\mathbf{CP}^{n-1}$ is the Veronese sequence, in which each map is a minimal immersion with constant curvature and constant Kähler angle.

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It is well known that the rigidity fails for pseudo-holomorphic curves or holomorphic curves generalized to $\mathbf{G}_{k,n}$ ([5], [14]). For example, Chi and Zheng ([5]) classified the holomorphic curves of the Riemann sphere into $\mathbf{G}_{2,4}$ with the induced constant curvature 2 into two classes, up to unitary equivalence, in which none of the curves are congruent. Let $\varphi : \mathbf{S}^2 \to \mathbf{G}_{k,n}$ be a pseudo-holomorphic curve in a complex Grassmannian $\mathbf{G}_{k,n}$. Problem: *Is the Kähler angle $\theta(\varphi)$ of φ constant when its Gauss curvature $K(\varphi)$ is constant? What are the relationships between the Kähler angle and the Gauss curvature of φ and its ramification index?* In this paper we will investigate these questions.

In the second and third sections of this paper we obtain some fundamental formulas for pseudo-holomorphic curves in complex Grassmann manifolds.

In the fourth section, by using these formulas we prove that the curvatures of pseudo-holomorphic curves are equal to $4/N$ (N is a positive integer) if these curvatures are constant (this result was proved by Chi and Zheng in [5])(Theorem 4.1), and prove that Kähler angles of pseudo-holomorphic curves of constant curvature are constant (Theorem 4.2). In this section, we also give a harmonic sequence, in which each map is a minimal immersion with constant curvature and constant Kähler angle.

In the final section, we give some pinching theorems for pseudo-holomorphic curves with the associated non-degenerate harmonic sequence for curvatures and Kähler angles (Theorems 5.2, 5.6 and 5.7). At the same time we also show that the Kähler angle of a pseudo-holomorphic curve is independent of its ramification index under the assumption of Theorem 5.2.

2. Minimal Immersions and Harmonic Sequences

Let $U(n)$ be the unitary group. Let M be a simply connected domain in the unit sphere $\mathbf{S}^2$ and let $(z, \overline{z})$ be a complex coordinate on M. We take the metric $ds_M^2 = dzd\overline{z}$ on M. Denote

$$\partial = \frac{\partial}{\partial z}, \quad \overline{\partial} = \frac{\partial}{\partial \overline{z}}, \quad A_z = \frac{1}{2}s^{-1}\partial s, \quad A_{\overline{z}} = \frac{1}{2}s^{-1}\overline{\partial} s.$$

Let $s : M \to U(n)$ be a smooth map; then s is a harmonic map if and only if it satisfies the following equation ([9]):

$$\overline{\partial} A_z = [A_z, A_{\overline{z}}]. \tag{1}$$

If $s : \mathbf{S}^2 \to U(n)$ is a harmonic map, then s is a conformal map; so s is a minimal immersion. Let $\omega = g^{-1}dg$ be a Maurer-Cartan form on $U(n)$, and let $ds^2_{U(n)} = \frac{1}{8}\operatorname{tr}\omega\omega^*$ be the metric on $U(n)$. Then the metric induced by s on $\mathbf{S}^2$ is given by

$$ds^2 = -\mathrm{tr} A_z A_{\overline{z}} dz d\overline{z}. \tag{2}$$

Let $\mathbf{G}_{k,n}$ be the complex Grassmann manifold consisting of all complex k-dimensional subspaces in $\mathbf{C}^n$. Here we consider $\mathbf{G}_{k,n}$ as the set of Hermitian orthogonal projections onto a k-dimensional subspace in $\mathbf{C}^n$, i.e., $\mathbf{G}_{k,n} = \{\varphi$ is the Hermitian orthogonal projection onto a k-dimensional subspace in $\mathbf{C}^n\}$. Then $\varphi : \mathbf{S}^2 \to \mathbf{G}_{k,n}$ is a Hermitian orthogonal projection onto a k-dimensional subbundle $\eta \subset \mathbf{S}^2 \times \mathbf{C}^n$, and $s = \varphi - \varphi^\perp$ is a map from $\mathbf{S}^2$ into $U(n)$. It is well known that φ is harmonic if and only if s is harmonic. If $\varphi^\perp \overline{\partial} \varphi = 0$ or $\varphi^\perp \partial \varphi = 0$, we call φ a *holomorphic curve* or an *anti-holomorphic curve* in $\mathbf{G}_{k,n}$.

Using φ, the harmonic sequences (see [3], [10]) are given by

$$\varphi = \varphi_0 \xrightarrow{\partial'} \varphi_1 \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \varphi_\alpha \xrightarrow{\partial'} \cdots , \tag{3}$$

$$\varphi = \varphi_0 \xrightarrow{\partial''} \varphi_{-1} \xrightarrow{\partial''} \cdots \xrightarrow{\partial''} \varphi_{-\alpha} \xrightarrow{\partial''} \cdots , \tag{4}$$

where $\varphi_\alpha : \mathbf{S}^2 \times \mathbf{C}^n \to \mathrm{Im}(\varphi^\perp_{\alpha-1}\partial\varphi_{\alpha-1})$ and $\varphi_{-\alpha} : \mathbf{S}^2 \times \mathbf{C}^n \to \mathrm{Im}(\varphi^\perp_{-\alpha+1}\overline{\partial}\varphi_{-\alpha+1})$ are Hermitian orthogonal projections, $\alpha = 1, 2, \cdots$.

Proposition 2.1 ([7]). *For* (3) *and* (4)*, we have*

$$\varphi_\alpha\partial\varphi_\alpha = -\varphi^\perp_{\alpha-1}\partial\varphi_{\alpha-1}, \quad \varphi^\perp_\alpha\overline{\partial}\varphi_\alpha = -\varphi_{\alpha-1}\overline{\partial}\varphi_{\alpha-1},$$

where $\alpha = \pm1, \pm2, \cdots$.

If φ_0 is a holomorphic curve in (3) or an anti-holomorphic curve in (4), then elements in (3) or (4) are finite and are mutually orthogonal. If there exists a holomorphic curve φ_0 in $\mathbf{G}_{k,n}$ such that φ is an element in the harmonic sequence (3), i.e., $\varphi = \varphi_\alpha : \mathbf{S}^2 \to \mathbf{G}_{k_\alpha,n}$ belongs to the harmonic sequence

$$0 \xrightarrow{\partial'} \varphi_0 \xrightarrow{\partial'} \varphi_1 \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \varphi = \varphi_\alpha \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \varphi_{\alpha_0} \xrightarrow{\partial'} 0, \tag{5}$$

then we call φ a *pseudo-holomorphic curve* in complex Grassmann manifolds, and α_0 is called the *length* of the harmonic sequence (5).

Now we assume that $\varphi = \varphi_\alpha : \mathbf{S}^2 \to \mathbf{G}_{k_\alpha,n}$ is a pseudo-holomorphic curve. Then we may choose the local unitary frame $e_1, e_2, \cdots, e_n$ on $\mathbf{S}^2 \times \mathbf{C}^n$ such that $e_{k_{\alpha-1}+1}, \cdots, e_{k_\alpha}$ span $\mathrm{Im}(\varphi^\perp_{\alpha-1}\partial\varphi_{\alpha-1})$, where $k_\alpha = \mathrm{rank}(\varphi^\perp_{\alpha-1}\partial\varphi_{\alpha-1})$, $\alpha = 1, 2, \cdots$, $k_0 = \mathrm{rank}(\varphi_0)$.

Let $W_\alpha = (e_{k_{\alpha-1}+1}, e_{k_\alpha}, \cdots, e_{k_\alpha})$ be an $(n \times k_\alpha)$-matrix. Then we have

$$\varphi_\alpha = W_\alpha W^*_\alpha, \tag{6}$$

$$W^*_\alpha W_\alpha = I_{k_\alpha \times k_\alpha}, \quad W^*_\alpha W_{\alpha+1} = 0, \quad W^*_\alpha W_{\alpha-1} = 0. \tag{7}$$

By (7) and a straightforward computation we obtain

$$\begin{cases} \partial W_\alpha = W_{\alpha+1}\Omega_\alpha + W_\alpha\Psi_\alpha, \\ \overline{\partial} W_\alpha = -W_{\alpha-1}\Omega^*_{\alpha-1} - W_\alpha\Psi^*_\alpha, \end{cases} \tag{8}$$

where Ω_α is a $(k_{\alpha+1} \times k_\alpha)$-matrix and Ψ_α is a $(k_\alpha \times k_\alpha)$-matrix, $\alpha = 0, 1, 2, \cdots$.

It is well known that $\Omega_\alpha = 0$ or $\Omega_{\alpha-1} = 0$ in (8) if and only if φ_α is anti-holomorphic or holomorphic. It is very evident that integrability conditions for (8) are

$$\overline{\partial}\Omega_\alpha = \Psi^*_{\alpha+1}\Omega_\alpha - \Omega_\alpha\Psi^*_\alpha, \tag{9}$$

$$\overline{\partial}\Psi_\alpha + \partial\Psi^*_\alpha = \Omega^*_\alpha\Omega_\alpha + \Psi^*_\alpha\Psi_\alpha - \Omega_{\alpha-1}\Omega^*_{\alpha-1} - \Psi_\alpha\Psi^*_\alpha. \tag{10}$$

By (8), $A^{(\alpha)}_z$ and $A^{(\alpha)}_{\overline{z}}$ for φ_α are given by

$$A^{(\alpha)}_z = -W_\alpha\Omega_{\alpha-1}W^*_{\alpha-1} - W_{\alpha+1}\Omega_\alpha W^*_\alpha, \tag{11}$$

$$A^{(\alpha)}_{\overline{z}} = W_\alpha\Omega^*_\alpha W^*_{\alpha+1} + W_{\alpha-1}\Omega^*_\alpha W^*_\alpha. \tag{12}$$

It can easily be checked that (9) is equivalent to (1). An immediate consequence of (8) is

Proposition 2.2. *Let $\varphi = \varphi_\alpha : \mathbf{S}^2 \to \mathbf{G}_{k_\alpha,n}$ be a pseudo-holomorphic curve, with Ω_α and Ψ_α determined by equations* (8). *Then Ω_α and Ψ_α satisfy equations* (9) *and* (10).

Let $\varphi^{(\alpha)} = \varphi_0 \oplus \cdots \oplus \varphi_\alpha$ for (5) and $k_{(\alpha)} = k_0 + \cdots + k_\alpha$. Then by Proposition 2.1 we have

$$\partial \varphi^{(\alpha)} = \varphi_\alpha^\perp \partial \varphi_\alpha. \tag{13}$$

Hence $\varphi^{(\alpha)} : \mathbf{S}^2 \to \mathbf{G}_{k_{(\alpha)},n}$ is a holomorphic map, and the harmonic map sequence (5) becomes

$$0 \xrightarrow{\partial'} \varphi^{(\alpha)} \xrightarrow{\partial'} \varphi_{\alpha+1} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \varphi_{\alpha_0} \xrightarrow{\partial'} 0. \tag{14}$$

If $k_\alpha = k_{\alpha+1}$, i.e., $\operatorname{rank}(\varphi_\alpha) = \operatorname{rank}(\varphi_{\alpha+1})$, then φ_α is called *non-degenerate*. If φ_α is non-degenerate for $\alpha = 0, 1, \cdots, \alpha_0 - 1$ in (5), i.e., $k_0 = k_1 = \cdots = k_{\alpha_0}$, then the harmonic sequence (5) is called the *non-degenerate harmonic sequence* associated to the harmonic map $\varphi = \varphi_\alpha$. Now we assume that φ_α is non-degenerate; then $\det(\Omega_\alpha)$ is a well-defined invariant on $\mathbf{S}^2$ and has only isolated zeros. Let

$$l_\alpha = \operatorname{tr}(\Omega_\alpha \Omega_\alpha^*). \tag{15}$$

Then

$$l_\alpha = \operatorname{tr}(\varphi_\alpha^\perp \partial \varphi_\alpha \overline{\partial} \varphi_\alpha) = \operatorname{tr}(\partial \varphi^{(\alpha)} \overline{\partial} \varphi^{(\alpha)}), \quad l_{\alpha-1} + l_\alpha = -\operatorname{tr}(A_z^{(\alpha)} A_{\overline{z}}^{(\alpha)}),$$

and we have

Proposition 2.3. *If $\varphi = \varphi_\alpha : \mathbf{S}^2 \to \mathbf{G}_{k_\alpha,n}$ is a non-degenerate pseudo-holomorphic curve, then*

$$2\partial\overline{\partial} \log |\det(\Omega_\alpha)| = l_{\alpha-1} - 2l_\alpha + l_{\alpha+1}. \tag{16}$$

Proof. By (9) and the rule of differentiating a determinant, we get

$$\overline{\partial} \log \det(\Omega_\alpha) = \operatorname{tr}(\Omega_\alpha^{-1} \overline{\partial} \Omega_\alpha) = \operatorname{tr} \Psi_{\alpha+1}^* - \operatorname{tr} \Psi_\alpha^*,$$
$$\partial \log \det(\Omega_\alpha^*) = \operatorname{tr}((\Omega_\alpha^*)^{-1} \partial \Omega_\alpha^*) = \operatorname{tr} \Psi_{\alpha+1} - \operatorname{tr} \Psi_\alpha.$$

It is not difficult to obtain (16) by (10). ☐

Remark. If φ_α is non-degenerate for all α in (5), then

$$2\partial\overline{\partial} \log |\det(\Omega_\alpha)| = l_{\alpha-1} - 2l_\alpha + l_{\alpha+1} \tag{17}$$

for $\alpha = 0, 1, \cdots, \alpha_0 - 1$, where $l_{-1} = l_{\alpha_0} = 0$. When $k_\alpha = 1$ for all α, then $l_\alpha = |\det(\Omega_\alpha)|^2$, and (17) is just the *unintegrated Plücker formulae* for l_α derived by Bolton, Jensen, Rigoli and Woodward in [1].

3. Kähler Angles and Gauss Curvatures

If $\varphi : M \to \mathbf{G}_{k,n}$ is a conformal immersion of a Riemann surface M, we define the Kähler angle of φ to be the function $\theta : M \to [0, \pi]$ given in terms of a complex coordinate z on M by

$$\tan \frac{\theta(p)}{2} = \frac{|d\varphi(\partial/\partial \overline{z})|}{|d\varphi(\partial/\partial z)|}, \quad p \in M. \tag{18}$$

It is clear that θ is globally defined and is smooth at p unless $\theta(p) = 0$ or π. Let $z = x + \sqrt{-1}y$, and let J denote the complex structure on $\mathbf{G}_{k,n}$; then θ is the angle between $Jd\varphi(\partial/\partial x)$ and $d\varphi(\partial/\partial y)$. The importance of the Kähler angle in the theory of minimal immersions of surfaces into Kähler manifolds was pointed out by Chern and Wolfson [4]. Indeed, φ is holomorphic if and only if $\theta(p) = 0$ for all $p \in M$, while φ is anti-holomorphic if and only if $\theta(p) = \pi$ for all $p \in M$.

Now suppose that $\varphi : \mathbf{S}^2 \to \mathbf{G}_{k,n}$ is a conformal minimal immersion in the harmonic sequence (5). Then each $\varphi_\alpha : \mathbf{S}^2 \to \mathbf{G}_{k_\alpha,n}$ is a conformal minimal immersion. So there exists a finite set X_α (see [1]) such that the Kähler angle

$$\theta_\alpha : \mathbf{S}^2 \setminus X_\alpha \to [0, \pi]$$

is well defined, and is smooth on $\mathbf{S}^2 \setminus X_\alpha$.

Let $t_\alpha = \left(\tan \frac{\theta_\alpha}{2}\right)^2$. Then, in terms of a local complex coordinate z,

$$t_\alpha = \frac{|d\varphi_\alpha(\partial/\partial\overline{z})|^2}{|d\varphi_\alpha(\partial/\partial z)|^2} = \frac{l_{\alpha-1}}{l_\alpha}. \tag{19}$$

Let ds^2_α and $ds^2_{(\alpha)}$ be the metrics on $\mathbf{S}^2 \setminus X_\alpha$ induced by φ_α and $\varphi^{(\alpha)}$ respectively. Then by (11), (12) and (13) we have

$$ds^2_\alpha = (l_{\alpha-1} + l_\alpha) dz d\overline{z}, \quad ds^2_{(\alpha)} = l_\alpha dz d\overline{z}. \tag{20}$$

The Laplacians $\triangle_\alpha$ and $\triangle_{(\alpha)}$ for ds^2_α and $ds^2_{(\alpha)}$ are given by

$$\triangle_\alpha = \frac{4}{l_{\alpha-1} + l_\alpha} \partial\overline{\partial}, \quad \triangle_{(\alpha)} = \frac{4}{l_\alpha} \partial\overline{\partial}, \tag{21}$$

and the curvatures K_α, $K_{(\alpha)}$ of φ_α and $\varphi^{(\alpha)}$ by

$$K_\alpha = -\frac{2}{l_{\alpha-1} + l_\alpha} \partial\overline{\partial} \log(l_{\alpha-1} + l_\alpha), \quad K_{(\alpha)} = -\frac{2}{l_\alpha} \partial\overline{\partial} \log l_\alpha, \tag{22}$$

the area forms dv_α and $dv_{(\alpha)}$ by

$$dv_\alpha = (l_{\alpha-1} + l_\alpha) \frac{d\overline{z} \wedge dz}{2\sqrt{-1}}, \quad dv_{(\alpha)} = l_\alpha \frac{d\overline{z} \wedge dz}{2\sqrt{-1}}. \tag{23}$$

Choose holomorphic sections $f_1, \cdots, f_{k_{(\alpha)}}$ in $\Gamma(\mathbf{S}^2 \times \mathbf{C}^n)$ so that they span $\mathrm{Im}(\varphi^{(\alpha)})$ and

$$f_1 \wedge \cdots \wedge f_{k_{(\alpha)}} : \mathbf{S}^2 \to \mathbf{C}^{\binom{n}{k_{(\alpha)}}} \tag{23}$$

is a nowhere zero holomorphic curve.

Let $F^{(\alpha)} = f_1 \wedge \cdots \wedge f_{k_{(\alpha)}}$. Now consider the Plücker embedding (see [12], [13])

$$[F^{(\alpha)}] : \mathbf{S}^2 \to \mathbf{CP}^{\binom{n}{k_{(\alpha)}}-1}, \tag{24}$$

which is a holomorphic isometry, and

$$[F^{(\alpha)}]^* ds^2_{\mathbf{CP}^{\binom{n}{k_{(\alpha)}}-1}} = l_\alpha dz d\overline{z}. \tag{25}$$

By [1], we have

$$\partial\overline{\partial} \log |F^{(\alpha)}|^2 = l_\alpha, \tag{26}$$

and the degree δ_α of $F^{(\alpha)}$ is given by

$$\delta_\alpha = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbf{S}^2} \partial\overline{\partial} \log |F^{(\alpha)}|^2 d\overline{z} \wedge dz = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbf{S}^2} l_\alpha d\overline{z} \wedge dz, \tag{27}$$

which is equal to the degree of the polynomial function $F^{(\alpha)}$ in z. We call δ_α the *degree* of the holomorphic curve $\varphi^{(\alpha)}$. Thus from (17) and (27) we get

Proposition 3.1. *If $\varphi = \varphi_\alpha : \mathbf{S}^2 \to \mathbf{G}_{k_\alpha,n}$ is a non-degenerate pseudo-holomorphic curve, then*

$$-\sharp_\alpha = \delta_{\alpha-1} - 2\delta_\alpha + \delta_{\alpha+1}, \tag{28}$$

where $\sharp_\alpha = -\frac{1}{\pi\sqrt{-1}}\int_{\mathbf{S}^2} \partial\overline{\partial}\log|\det\Omega_\alpha| d\overline{z}\wedge dz$ *is the number of singular points of* Ω_α*, i.e., the number of zeros of* $\det\Omega_\alpha$.

Remark. If φ_α is non-degenerate for $\alpha = 0, 1, \cdots, \alpha_0 - 1$, then $-\sharp_\alpha = \delta_{\alpha-1} - 2\delta_\alpha + \delta_{\alpha+1}$ for all α, and $\delta_{-1} = \delta_{\alpha_0} = 0$; in particular, when $k_0 = \cdots = k_{\alpha_0} = 1$, (28) is the global Plücker formula (see [6]).

Let $ds^2 = |\det\Omega_\alpha|^2 dz d\overline{z} = \psi_\alpha\overline{\psi}_\alpha$, where ψ_α is a type (1, 0) analytic 1-form. Then $ds^2 = \psi_\alpha \oplus \overline{\psi}_\alpha$ is a singular Hermitian metric. Let $D_S = \sum_{p\in\mathbf{S}^2} \mathrm{ord}_p(\psi_\alpha)p$ be the singular divisor of $(\mathbf{S}^2, ds^2)$, i.e., the zero divisor of ψ_α. By the Gauss-Bonnet-Chern theorem we have

$$\sharp_\alpha = \tau_\alpha + 2,$$

where $\tau_\alpha = \deg D_S$.

We say that τ_α is the *ramification index* of φ_α. Evidently, τ_α is a non-negative integer. If $\tau_\alpha = 0$, φ_α is called *unramified* by Bolton et al. ([1]).

Let (5) be the non-degenerate harmonic sequence; if $\tau_\alpha = 0$ for $\alpha = 0, 1, \cdots, \alpha_0 - 1$, the harmonic sequence (5) is called *totally unramified*. Let $\varphi = \varphi_\alpha : \mathbf{S}^2 \to \mathbf{G}_{k_\alpha,n}$ be the pseudo-holomorphic conformal immersion with the non-degenerate associated harmonic sequence (5); we say that φ is a *totally unramified pseudo-holomorphic conformal immersion* if $\varphi_0, \cdots, \varphi_{\alpha_0}$ is totally unramified.

If $\varphi_\alpha : \mathbf{S}^2 \to \mathbf{G}_{k_\alpha,n}$ is a conformal minimal immersion with constant Kähler angle, then we have

$$t_\alpha = \frac{\delta_{\alpha-1}}{\delta_\alpha}, \tag{29}$$

and from (19) and (22) it follows that

$$K_\alpha = -\frac{2}{l_{\alpha-1}+l_\alpha}\partial\overline{\partial}\log l_{\alpha-1} = -\frac{2}{l_{\alpha-1}+l_\alpha}\partial\overline{\partial}\log l_\alpha. \tag{30}$$

4. Conformal Minimal Immersions with Constant Curvatures

It is well known that any complex submanifold of a (simply-connected, complete) space of constant holomorphic curvature is completely determined, up to holomorphic isometries of the ambient space, by its induced metric (see [2], [8]). The Veronese sequence is the harmonic sequence

$$0 \xrightarrow{\partial'} \varphi_0 \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \varphi_n \xrightarrow{\partial'} 0, \tag{31}$$

where $n = \deg(\varphi_0)$, and each $\varphi_\alpha = [g_{\alpha,0}, \cdots, g_{\alpha,n}] : \mathbf{S}^2 \to \mathbf{CP}^n$ is given by

$$g_{\alpha,j} = \frac{\alpha!}{(1+z\overline{z})^\alpha}\sqrt{\binom{n}{j}}z^{j-\alpha}\sum_k(-1)^k\binom{j}{\alpha-k}\binom{n-j}{k}(z\overline{z})^k, \quad \alpha, j = 0, 1, \ldots, n.$$

Each map φ_α in the Veronese sequence (31) is a conformal minimal immersion with constant curvature

$$K(\varphi_\alpha) = \frac{4}{n+2\alpha(n-\alpha)} \tag{32}$$

and constant Kähler angle θ_α given by

$$\left(\tan\frac{\theta_\alpha}{2}\right)^2 = \frac{\alpha(n-\alpha+1)}{(\alpha+1)(n-\alpha)}. \tag{33}$$

Bolton, Jensen, Rigoli and Woodward ([1]) showed that, up to a holomorphic isometry of $\mathbf{CP}^n$, the harmonic sequence determined by $\varphi : \mathbf{S}^2 \to \mathbf{CP}^n$, which is a linearly full conformal minimal immersion of constant curvature, is the Veronese sequence. It is very complicated for pseudo-holomorphic curves in complex Grassmann manifolds; for example, rigidity fails, but we still believe that there are some good geometric properties. In this section we discuss pseudo-holomorphic curves of constant curvature in complex Grassmann manifolds, and Kähler angles.

Let $\varphi_\alpha : \mathbf{S}^2 \to \mathbf{G}_{k_\alpha,n}$ be a pseudo-holomorphic curve with constant curvature. Then we know that

$$[F^{(\alpha-1)}] : \mathbf{S}^2 \to \mathbf{CP}^{\binom{n}{k_{(\alpha-1)}}-1}, \quad [F^{(\alpha)}] : \mathbf{S}^2 \to \mathbf{CP}^{\binom{n}{k_{(\alpha)}}-1}$$

are two holomorphic curves with degrees $\delta_{\alpha-1}$ and δ_α respectively. Consider the tensor product of $[F^{(\alpha-1)}]$ and $[F^{(\alpha)}]$,

$$T^{(\alpha)} = F^{(\alpha-1)} \otimes F^{(\alpha)}. \tag{34}$$

Then

$$[T^{(\alpha)}] : \mathbf{S}^2 \to \mathbf{CP}^{\binom{n}{k_{(\alpha-1)}}\binom{n}{k_{(\alpha-1)}}-1}$$

is a well-defined holomorphic curve, and from (25) the metric induced by $[T^{(\alpha)}]$ is given by

$$[T^{(\alpha)}]^* ds^2_{\mathbf{CP}^{\binom{n}{k_{(\alpha-1)}}\binom{n}{k_{(\alpha-1)}}-1}} = [F^{(\alpha-1)}]^* ds^2_{\mathbf{CP}^{\binom{n}{k_{(\alpha-1)}}-1}} + [F^{(\alpha)}]^* ds^2_{\mathbf{CP}^{\binom{n}{k_{(\alpha)}}-1}},$$

i.e.,

$$[T^{(\alpha)}]^* ds^2_{\mathbf{CP}^{\binom{n}{k_{(\alpha-1)}}\binom{n}{k_{(\alpha-1)}}-1}} = (l_{\alpha-1}+l_\alpha)dzd\overline{z}. \tag{35}$$

Hence the curvature K_α of φ_α is equal to the curvature of $[T^{(\alpha)}]$. From [1], an immediate consequence is

Theorem 4.1. *If $\varphi : \mathbf{S}^2 \to \mathbf{G}_{k,n}$ is a pseudo-holomorphic curve with constant curvature $K(\varphi)$, then $K(\varphi) = 4/N$, where N is a positive integer.*

This theorem was proved by Chi and Zheng ([5]) by the method of the moving frame. In the following we will prove

Theorem 4.2. *If $\varphi_\alpha : \mathbf{S}^2 \to \mathbf{G}_{k_\alpha,n}$ is a pseudo-holomorphic curve with constant curvature K_α, then the Kähler angle θ_α of φ_α is constant.*

Proof. From (22) we have

$$K_\alpha(l_{\alpha-1}+l_\alpha) = -2\partial\overline{\partial}\log(l_{\alpha-1}+l_\alpha). \tag{36}$$

When K_α is constant, from (22), (23), (27) and the Gauss-Bonnet theorem it follows that

$$K_\alpha = \frac{4}{\delta_{\alpha-1}+\delta_\alpha}. \tag{37}$$

Hence from (22) we obtain

$$-\frac{2}{l_{\alpha-1}+l_\alpha}\partial\overline{\partial}\log(l_{\alpha-1}+l_\alpha) = \frac{4}{\delta_{\alpha-1}+\delta_\alpha}. \tag{38}$$

Choose a complex coordinate z on $\mathbf{S}^2 \setminus \{pt\}$ so that

$$l_{\alpha-1}+l_\alpha = \frac{\delta_{\alpha-1}+\delta_\alpha}{(1+z\overline{z})^2}. \tag{39}$$

From (38), (39) and (26) we obtain

$$\partial\overline{\partial}\log\frac{|F^{(\alpha-1)}|^2|F^{(\alpha)}|^2}{(1+z\overline{z})^{\delta_{\alpha-1}+\delta_\alpha}} = 0. \tag{40}$$

Since we can choose holomorphic sections $f_1, \cdots, f_{k_{(\alpha)}}$ in $\Gamma(\mathbf{S}^2 \times \mathbf{C}^n)$ such that the maps $F^{(\alpha-1)}$ and $F^{(\alpha)}$ are polynomial functions on $\mathbf{C}$ of degrees $\delta_{\alpha-1}$ and δ_α respectively, it folows that $\frac{|F^{(\alpha-1)}|^2|F^{(\alpha)}|^2}{(1+z\overline{z})^{\delta_{\alpha-1}+\delta_\alpha}}$ is globally defined on $\mathbf{C}$ and has a non-zero constant limit c, as $z \to \infty$. So from (40) we get

$$\frac{|F^{(\alpha-1)}|^2|F^{(\alpha)}|^2}{(1+z\overline{z})^{\delta_{\alpha-1}+\delta_\alpha}} = c.$$

Then we have

$$|F^{(\alpha-1)}|^2 = c_{\alpha-1}(1+z\overline{z})^{\delta_{\alpha-1}}, \quad |F^{(\alpha)}|^2 = c_\alpha(1+z\overline{z})^{\delta_\alpha},$$

where $c_{\alpha-1}$ and c_α are constants.

Hence, $l_{\alpha-1} = \frac{\delta_{\alpha-1}}{(1+z\overline{z})^2}$ and $l_\alpha = \frac{\delta_\alpha}{(1+z\overline{z})^2}$, namely, φ_α is of constant curvature and constant Kähler angle. □

From (19) and (22) we know that if $\varphi_\alpha : \mathbf{S}^2 \to \mathbf{G}_{k_\alpha,n}$ is a pseudo-holomorphic curve with constant Kähler angle θ_α, then $K_\alpha = \frac{1}{1+t_\alpha}K_{(\alpha)}$.

Remark. We do not need to assume that $\varphi_\alpha : \mathbf{S}^2 \to \mathbf{G}_{k_\alpha,n}$ is non-degenerate in Theorem 4.2.

To conclude this section, we give an example. This example is a harmonic sequence, in which the Gauss curvature and the Kähler angle of each element are constant.

Let $f_0(z) = (1,\ 0,\ \sqrt{2}z,\ 0,\ z^2)$ and $g_0(z) = (0,\ 1,\ 0,\ z,\ 0)$; then

$$\varphi_0 = \frac{1}{(1+z\overline{z})^2}\begin{pmatrix} 1 & 0 & \sqrt{2}z & 0 & z^2 \\ 0 & 1+z\overline{z} & 0 & z(1+z\overline{z}) & 0 \\ \sqrt{2}\overline{z} & 0 & 2z\overline{z} & 0 & \sqrt{2}z^2\overline{z} \\ 0 & \overline{z}(1+z\overline{z}) & 0 & z\overline{z}(1+z\overline{z}) & 0 \\ \overline{z}^2 & 0 & \sqrt{2}z\overline{z}^2 & 0 & z^2\overline{z}^2 \end{pmatrix} : \mathbf{S}^2 \to \mathbf{G}_{2,5}$$

determined by $f_0(z)$ and $g_0(z)$ is a holomorphic map.

An immediate computation shows that

$$f_1(z,\overline{z}) = \varphi_0^{\perp}(\partial f_0(z)) = \left(-\frac{2\overline{z}}{1+z\overline{z}},\ 0,\ \frac{\sqrt{2}(1-z\overline{z})}{1+z\overline{z}},\ 0,\ \frac{2z}{1+z\overline{z}}\right),$$

$$g_1(z,\overline{z}) = \varphi_0^{\perp}(\partial g_0(z)) = \left(0,\ -\frac{\overline{z}}{1+z\overline{z}},\ 0,\ \frac{1}{1+z\overline{z}},\ 0\right),$$

and $\varphi_1(z,\overline{z})$ determined by f_1 and g_1 is given by

$$\varphi_1 = \frac{1}{(1+z\overline{z})^2}\begin{pmatrix} 2z\overline{z} & 0 & \sqrt{2}z(z\overline{z}-1) & 0 & -2z^2 \\ 0 & z\overline{z}(1+z\overline{z}) & 0 & -z(1+z\overline{z}) & 0 \\ \sqrt{2}\overline{z}(z\overline{z}-1) & 0 & (z\overline{z}-1)^2 & 0 & \sqrt{2}z(z\overline{z}-1) \\ 0 & -\overline{z}(1+z\overline{z}) & 0 & 1+z\overline{z} & 0 \\ -2\overline{z}^2 & 0 & \sqrt{2}\overline{z}(z\overline{z}-1) & 0 & 2z\overline{z} \end{pmatrix},$$

which is obviously a pseudo-holomorphic curve into $\mathbf{G}_{2,5}$. Similarly, we have

$$f_2 = \varphi_1^{\perp}(\partial f_1) = \left(\frac{2\overline{z}^2}{(1+z\overline{z})^2},\ 0,\ -\frac{2\sqrt{2}\overline{z}}{(1+z\overline{z})^2},\ 0,\ \frac{2}{(1+z\overline{z})^2}\right),$$

$$g_2 = \varphi_1^{\perp}(\partial g_1) = (0,\ 0,\ 0,\ 0,\ 0),$$

and φ_2, determined by f_2 and g_2, is given by

$$\varphi_2 = \frac{1}{(1+z\overline{z})^2}\begin{pmatrix} z^2\overline{z}^2 & 0 & -\sqrt{2}z^2\overline{z} & 0 & z^2 \\ 0 & 0 & 0 & 0 & 0 \\ -\sqrt{2}z\overline{z}^2 & 0 & 2z\overline{z} & 0 & -\sqrt{2}z \\ 0 & 0 & 0 & 0 & 0 \\ \overline{z}^2 & 0 & -\sqrt{2}\overline{z} & 0 & 1 \end{pmatrix}.$$

φ_2 is an anti-holomorphic curve, which is isomorphic to the Veronese curve, in $\mathbf{CP}^2$.

Hence we obtain a harmonic sequence from φ_0:

$$0 \xrightarrow{\partial'} \varphi = \varphi_0 \xrightarrow{\partial'} \varphi_1 \xrightarrow{\partial'} \varphi_2 \xrightarrow{\partial'} 0.$$

By a straightforward computation we obtain

$$l_0 = \frac{3}{(1+z\overline{z})^2}, \quad l_1 = \frac{2}{(1+z\overline{z})^2}, \quad l_2 = 0.$$

It is very easy to see that $K(\varphi_0) = 4/3$, $K(\varphi_1) = 4/5$, $K(\varphi_2) = 2$ and $t_1 = 3/2$.

It is well known that the rigidity of holomorphic curves in Grassmannians fails; so this example is a special harmonic sequence.

5. Pinching Theorem for Curvature and Kähler Angle

In this section we will discuss curvature pinching and Kähler angle pinching of non-degenerate pseudo-holomorphic spheres in complex Grassmann manifolds.

Let $\varphi = \varphi_\alpha : \mathbf{S}^2 \to \mathbf{G}_{k_\alpha,n}$ be a pseudo-holomorphic curve with the non-degenerate associated harmonic sequence (5), and let α_0 be the length of its associated harmonic sequence. Then from (28) we have

$$\delta_\alpha = -\delta_{\alpha-2} + 2\delta_{\alpha-1} - \tau_{\alpha-1} - 2 \tag{41}$$

for $\alpha = 1, \cdots, \alpha_0$, and

$$\tau_\alpha = (\delta_\alpha - \delta_{\alpha+1}) - (\delta_{\alpha-1} - \delta_\alpha) - 2 \tag{42}$$

for $\alpha = 0, 1, \cdots, \alpha_0 - 1$.

It is an immediate consequence of (41) and (42) that

$$\delta_\alpha = (\alpha+1)(\delta_0-\alpha) - \sum_{\beta=0}^{\alpha-1}(\alpha-\beta)\tau_\beta \tag{43}$$

for $\alpha = 1, \cdots, \alpha_0$, and

$$\tau_0 + \cdots + \tau_\alpha = (\delta_\alpha - \delta_{\alpha+1}) + \delta_0 - 2(\alpha+1) \tag{44}$$

for $\alpha = 0, 1, \cdots, \alpha_0 - 1$, where δ_0 is the degree of the holomorphic map φ_0 in (5).

From (43) and (44) we have also

$$\sum_{\alpha=0}^{\alpha_0-1}(\alpha_0-\alpha)\tau_\alpha = (\alpha_0+1)(\delta_0-\alpha_0) \tag{45}$$

and

(46)

$$\delta_\alpha = (\alpha+1)(\alpha_0-\alpha) + \frac{\alpha_0-\alpha}{\alpha_0+1}\sum_{\beta=0}^{\alpha-1}(\beta+1)\tau_\beta + \frac{\alpha+1}{\alpha_0+1}\sum_{\beta=\alpha}^{\alpha_0-1}(\alpha_0-\beta)\tau_\beta.$$

Denoting $\tau = \min\{\tau_0, \cdots, \tau_{\alpha_0-1}\}$ (≥ 0), we immediately obtain

$$\delta_0 \geq \alpha_0(1+\tfrac{1}{2}\tau), \quad \delta_\alpha \geq (\alpha+1)(\alpha_0-\alpha)(1+\tfrac{1}{2}\tau), \tag{47}$$

and "=" holds if and only if $\tau_0 = \cdots = \tau_{\alpha_0-1}$, where $\alpha = 0, 1, \cdots, \alpha_0 - 1$.

Obviously, φ is a totally unramified non-degenerate pseudo-holomorphic minimal immersion, i.e., the harmonic sequence $\varphi_0, \cdots, \varphi_{\alpha_0} : \mathbf{S}^2 \to \mathbf{G}_{k,n}$ is non-degenerate and totally unramified if and only if the degree δ_0 of φ_0 is α_0. For a totally unramified non-degenerate harmonic sequence $\varphi_0, \cdots, \varphi_{\alpha_0} : \mathbf{S}^2 \to \mathbf{G}_{k,n}$ we have

$$\delta_\alpha = (\alpha+1)(\alpha_0-\alpha). \tag{48}$$

At first, by using the Gauss-Bonnet theorem we have

Lemma 5.1. *Suppose that the curvature K_α of φ_α satisfies either $K_\alpha \geq \dfrac{4}{\delta_{\alpha-1}+\delta_\alpha}$ or $K_\alpha \leq \dfrac{4}{\delta_{\alpha-1}+\delta_\alpha}$. Then $K_\alpha = \dfrac{4}{\delta_{\alpha-1}+\delta_\alpha}$.*

Remark. In Lemma 5.1 we do not need to assume that φ_α is non-degenerate.

Theorem 5.2. *Let $\varphi : \mathbf{S}^2 \to \mathbf{G}_{k,n}$ be a pseudo-holomorphic curve with non-degenerate associated harmonic sequence, and suppose that φ is the α-th element φ_α of its non-degenerate associated harmonic sequence.*

(i) If $K(\varphi) \geq \dfrac{4}{(\alpha_0+2\alpha(\alpha_0-\alpha))(1+\frac{1}{2}\tau)}$, then

$$K(\varphi) = \frac{4}{(\alpha_0+2\alpha(\alpha_0-\alpha))(1+\frac{1}{2}\tau)},$$

and $\tau_\beta = \tau$ for all β.

(ii) If $K(\varphi) \leq \dfrac{4}{(\alpha_0+2\alpha(\alpha_0-\alpha))(1+\frac{1}{2}\tau)}$ and if $\tau_\beta = \tau$ for all β, then

$$K(\varphi) = \frac{4}{(\alpha_0+2\alpha(\alpha_0-\alpha))(1+\frac{1}{2}\tau)}.$$

Proof. From (47) we see that

$$\delta_{\alpha-1} + \delta_\alpha \geq (\alpha_0 + 2\alpha(\alpha_0 - \alpha))(1 + \tfrac{1}{2}\tau),$$

with equality if and only if $\tau_\beta = \tau$ for all β. The result is now immediate from Lemma 5.1. □

Remark. We have $t_\alpha = \dfrac{\alpha(\alpha_0 - \alpha + 1)}{(\alpha+1)(\alpha_0 - \alpha)}$ under the assumption of Theorem 5.2. This shows that the Kähler angle θ_α is independent of τ.

The following is an immediate consequence of Theorem 5.2.

Corollary 5.3. *Let $\varphi : \mathbf{S}^2 \to \mathbf{G}_{k,n}$ be a holomorphic curve with non-degenerate associated harmonic sequence. Suppose $K(\varphi) \leq \dfrac{4}{\alpha_0(1 + \frac{1}{2}\tau)}$ and $\tau_\beta = \tau$ for all β. Then $K(\varphi) = \dfrac{4}{\alpha_0(1 + \frac{1}{2}\tau)}$.*

Similarly, the following theorem is also an immediate consequence of Theorem 5.2.

Corollary 5.4. *Let $\varphi : \mathbf{S}^2 \to \mathbf{G}_{k,n}$ be a holomorphic curve with non-degenerate associated harmonic sequence, and suppose $K(\varphi) \geq \dfrac{4}{\alpha_0(1 + \frac{1}{2}\tau)}$. Then $K(\varphi) = \dfrac{4}{\alpha_0(1 + \frac{1}{2}\tau)}$, and $\tau_\beta = \tau$ for all β.*

We now prove a pinching theorem for the Kähler angle. Let $\varphi : \mathbf{S}^2 \to \mathbf{G}_{k,n}$ be a pseudo-holomorphic sphere and let $\varphi_0, \cdots, \varphi_{\alpha_0}$ be the associated harmonic sequence. We assume that $\varphi = \varphi_\alpha$.

Lemma 5.5 ([1]). *If the Kähler angle t_α of φ_α satisfies either $t_\alpha \geq \dfrac{\delta_{\alpha-1}}{\delta_\alpha}$ or $t_\alpha \leq \dfrac{\delta_{\alpha-1}}{\delta_\alpha}$, then $t_\alpha = \dfrac{\delta_{\alpha-1}}{\delta_\alpha}$.*

Lemma 5.6. *Let φ_α be a pseudo-holomorphic curve with non-degenerate associated harmonic sequence. If $\tau_\beta = \tau$ for all β, and t_α satisfies either $t_\alpha \geq \dfrac{\alpha(\alpha_0 - \alpha + 1)}{(\alpha+1)(\alpha_0 - \alpha)}$ or $t_\alpha \leq \dfrac{\alpha(\alpha_0 - \alpha + 1)}{(\alpha+1)(\alpha_0 - \alpha)}$, then $t_\alpha = \dfrac{\alpha(\alpha_0 - \alpha + 1)}{(\alpha+1)(\alpha_0 - \alpha)}$.*

The proof of the above theorem is immediate from Lemma 5.5 and (47).

Using (46), we can also prove the following.

Theorem 5.7. *Let $\varphi : \mathbf{S}^2 \to \mathbf{G}_{k,n}$ be a pseudo-holomorphic curve with non-degenerate associated harmonic sequence, and suppose that φ is the α-th element φ_α of its non-degenerate associated harmonic sequence. If $t_\alpha \leq \frac{1}{2}$ (resp. $t_\alpha \geq 2$), then $t_\alpha = 0$ (resp. $t_\alpha = \infty$), i.e., φ is a holomorphic (resp. anti-holomorphic) curve.*

Proof. When $\alpha \neq 0$ and α_0, by (46) an immediate computation shows that

$$\frac{1}{2} < \frac{\delta_{\alpha-1}}{\delta_\alpha} < 2.$$

Hence, by Lemma 5.5, if $t_\alpha \leq \frac{1}{2}$ (resp. $t_\alpha \geq 2$), then $\alpha = 0$ (resp. $\alpha = \alpha_0$), i.e., φ is a holomorphic (resp. anti-holomorphic) curve. □

We believe that $\tau \neq 0$ for the non-degenerate harmonic sequence associated to the holomorphic curve of constant curvature, except for the Veronese sequence.

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