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SINGULARITIES OF THE HYPERGEOMETRIC SYSTEM ASSOCIATED WITH A MONOMIAL CURVE

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Abstract. We compute, using $\mathcal{D}$-module restrictions, the slopes of the irregular hypergeometric system associated with a monomial curve. We also study rational solutions and reducibility of such systems.

1. Introduction

Let $A_n = \mathbf{C}\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n\rangle$ be the Weyl algebra of order n over the complex numbers $\mathbf{C}$ and let $\mathbf{C}[\partial] = \mathbf{C}[\partial_1, \dots, \partial_n]$ be the subring of A_n of linear differential operators with constant coefficients.

Let $A = (a_{ij})$ be an integer $(d \times n)$-matrix of rank d. We denote by $I_A \subset \mathbf{C}[\partial]$ the *toric ideal* associated to A: i.e., I_A is the ideal generated by the set

$$\{\partial^u - \partial^v \mid u, v \in \mathbf{N}^n,\ Au^T = Av^T\}$$

where $()^T$ means "transpose".

We denote by θ the vector $(\theta_1, \dots, \theta_n)^T$ with $\theta_i = x_i\partial_i$. For a given $\beta = (\beta_1, \dots, \beta_d)^T \in \mathbf{C}^d$ we consider the column vector (in A_n^d) $A\theta - \beta$ and we denote by $\langle A\theta - \beta\rangle$ the left ideal of A_n generated by the entries of $A\theta - \beta$.

Following Gel'fand, Zelevinskiĭ and Kapranov [6], we denote by $H_A(\beta)$ the left ideal of A_n generated by $I_A \cup \langle A\theta - \beta\rangle$. It is called the GKZ-hypergeometric system associated to the pair (A, β). The quotient $\mathcal{H}_A(\beta) = A_n/H_A(\beta)$ is a holonomic A_n-module (see e.g. [17]).

If the toric ideal I_A is homogeneous, i.e., if the $\mathbf{Q}$-row span of A contains $(1, \dots, 1)$, it is known ([8]; see also [17]) that $\mathcal{H}_A(\beta)$ is regular holonomic and the book [17] is devoted to an algorithmic study of such systems. Especially, the book gives an algorithmic method to construct series solutions around singular points of the system.

In this article, we start a study of singularities of GKZ-hypergeometric systems for non-homogeneous toric ideals I_A by treating the "first" case when $d = 1$, $A = (a_1, a_2, \dots, a_n) \in \mathbf{Z}^n$, $a_1 = 1$. We evaluate the geometric slopes of $\mathcal{H}_A(\beta)$ by successive restrictions of the number of variables. The slopes characterize Gevrey class solutions around a singular locus. Let us remark on an analytic meaning of slopes. Let $X = \mathbf{C}^n$ and $Y = \{x \mid x_n = 0\} \subset X$. We denote by $\mathcal{O}_{\widehat{X|Y}}$ the

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formal completion of $\mathcal{O}_X$ along Y (i.e., series that are formal in x_n and convergent in $x_1, \dots, x_{n-1}$). To each real number $s \in [1, +\infty)$ we denote by $\mathcal{O}_{X|Y}(s)$ the subsheaf of $\mathcal{O}_{\widehat{X|Y}}$ of Gevrey functions of order s (along Y). The sheaf $\mathcal{O}_{X|Y}(1)$ is the restriction $\mathcal{O}_{X|Y}$ and, by definition, we write $\mathcal{O}_{X|Y}(+\infty) = \mathcal{O}_{\widehat{X|Y}}$. For any holonomic $\mathcal{D}_X$-module M, Z. Mebkhout [14] associates the sheaf $\mathrm{Irr}_Y(s)(M)$ as the solution sheaf $\mathbf{R}Hom_{\mathcal{D}}(M, \mathcal{O}_{X|Y}(s)/\mathcal{O}_{X|Y})$. One fundamental result in the irregularity of $\mathcal{D}$-modules is the fact that $\mathrm{Irr}_Y(s)(M)$ is a perverse sheaf, for any s (see [14]). These sheaves define a filtration of the irregularity of M along Y, i.e., $\mathrm{Irr}_Y(M) := \mathrm{Irr}_Y(+\infty)(M)$. The main result of [11] is that $1/(1-s)$ is a slope of M with respect to Y if and only if s is a gap of the graduation defined by the filtration on the irregularity. In other words, $1/(1-s)$ is a slope if and only if $\mathrm{Irr}_Y(s)(M)/\mathrm{Irr}_Y(<s)(M) \neq 0$.

Our evaluation of the slopes is done as follows: (1) We translate Laurent and Mebkhout's theorem [12] on restrictions and slopes of $\mathcal{D}$-modules into an algorithm to evaluate the slopes by utilizing the results of [2] and [15]. (2) Apply our general algorithm to the hypergeometric system associated to $A = (1, a_2, \dots, a_n)$. This system has many nice properties and our algorithm outputs the slopes without computation on computers.

In the last section we study rational solutions and reducibility of our systems.

2. Micro-characteristic varieties

In this section, following Laurent [10], we describe micro-characteristic varieties for a given $\mathcal{D}$-module. We will state a result of Laurent and Mebkhout [12, Corollaire 2.2.9] (see also [14, p. 125] and [11, p. 42]), allowing to reduce our general problem of evaluating the slopes to fewer variables.

In this section $X = \mathbf{C}^n$ and $\mathcal{D}_X = \mathcal{D}$ is the sheaf of linear differential operators with holomorphic function coefficients. Let M be a coherent $\mathcal{D}$-module. Recall that the characteristic variety of M (denoted by $\mathrm{Ch}(M)$) is an analytic subvariety of the cotangent bundle T^*X.

Suppose that $Y \subset X$ is a smooth hypersurface. We say that Y is *non-characteristic* for M if $T^*_Y X \cap \mathrm{Ch}(M) \subset T^*_X X$. Here $T^*_Y X$ is the conormal bundle to Y in X and $T^*_X X$ is the zero section of T^*X.

Now, following Laurent [9], [10], we shall define the notion of *non-micro-characteristic variety* for M. To simplify the presentation we will assume that $(x_1, x_2, \dots, x_n)$ are local coordinates in X and that Y is defined by $x_n = 0$. We denote by $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ local coordinates in T^*X. Sometimes it will be useful to write $x_1 = y_1, \dots, x_{n-1} = y_{n-1}$, $x_n = t$, $\xi_1 = \eta_1, \dots, \xi_{n-1} = \eta_{n-1}$ and $\xi_n = \tau$.

Let us denote by Λ the conormal bundle of Y in T^*X (i.e., $\Lambda = T^*_Y X$). So, in local coordinates, $\Lambda = \{(y, t, \eta, \tau) \in T^*X \mid t = \eta = 0\}$. Here, $\eta = (\eta_1, \dots, \eta_{n-1})$ and $y = (y_1, \dots, y_{n-1})$. We denote by (y, τ, y^*, τ^*) local coordinates on the cotangent bundle $T^*\Lambda$.

We denote by $V_\bullet(\mathcal{D})$ (or simply by V) the Malgrange-Kashiwara filtration associated to Y on $\mathcal{D}$ and by $F_\bullet(\mathcal{D})$ (or simply by F) the order filtration on $\mathcal{D}$. For a given rational number $p/q \geq 0$ we denote by $L_{p/q}$ the filtration on $\mathcal{D}$ defined by $pF + qV$. The $L_{p/q}$-order of a monomial $y^\alpha t^l \partial_y^\beta \partial_t^k$ is equal to $p(|\beta| + k) + q(k - l)$, where $|\beta| = \beta_1 + \cdots + \beta_{n-1}$. We will simply write $L = L_{p/q}$ if no confusion arises.

For $p > 0$ the associated graded ring $\mathrm{gr}^L(\mathcal{D})$ is canonically isomorphic to $\pi_* \mathcal{O}_{[T^*\Lambda]}$ [10, p. 407], where $\pi : T^*\Lambda \to \Lambda$ is the canonical projection and $\mathcal{O}_{[T^*\Lambda]}$ denotes

holomorphic functions on $T^*\Lambda$ that are polynomials on the fibers of π. In local coordinates, $\mathrm{gr}^L(\mathcal{D}_0)$ is expressed as $\mathbf{C}\{y\}[\tau, y^*, \tau^*]$ where $\mathcal{D}_0$ is the stalk of $\mathcal{D}$ at the origin.

Given a differential operator

$$P = \sum_{\alpha l \beta k} p_{\alpha l \beta k} y^\alpha t^l \partial_y^\beta \partial_t^k,$$

the L-order of P is the maximun value of $p(|\beta| + k) + q(k - l)$ over the monomials of P. For $p > 0$ we define the L-principal symbol of P by

$$\sigma^L(P) = \sum p_{\alpha l \beta k} y^\alpha (\tau^*)^l (y^*)^\beta (-\tau)^k$$

where the sum is taken over monomials with maximal L-order. The L-principal symbol of P is an element of $\mathrm{gr}^L(\mathcal{D})$ and then is a function on $T^*\Lambda$. In the classical case, i.e., for $L = F$, $\mathrm{gr}^F(\mathcal{D})$ is identified with $\mathbf{C}\{x\}[\xi_1, \ldots, \xi_n] = \mathbf{C}\{y, t\}[\eta, \tau]$ and the F-principal symbol of P is simply denoted by $\sigma^F(P) = \sum p_{\alpha l \beta k} y^\alpha t^l \eta^\beta \tau^k$ where the sum is taken for $|\beta| + k$ maximum.

To each left ideal $I \subset \mathcal{D}$ we denote by $\sigma^L(I)$ the ideal of $\mathrm{gr}^L(\mathcal{D})$ generated by the set of $\sigma^L(P)$ for $P \in I$.

For each L-filtration on $\mathcal{D}$ we associate a "good" L-filtration on M, by means of a finite presentation. The associated $\mathrm{gr}^L(\mathcal{D})$-module $\mathrm{gr}^L(M)$ is coherent (see [10, 3.2.2]). The radical of the annihilating ideal $\mathrm{Ann}_{\mathrm{gr}^L(\mathcal{D}_X)}(\mathrm{gr}^L(M))$, which is independent of the "good" filtration on M, defines an analytic subvariety of $T^*\Lambda$. This variety is called the *L-characteristic variety* of M and it is denoted by $\mathrm{Ch}^L(M)$.

Suppose now that $Z \subset X$ is a smooth hypersurface transverse to Y. Suppose for simplicity that Z is defined in local coordinates by $y_1 = 0$. The conormal space $\Lambda' := T^*_{Y \cap Z} Z$ is a smooth subvariety of $\Lambda = T^*_Y X$ defined in local coordinates by $y_1 = 0$. So $T^*_{\Lambda'}\Lambda$ is the subvariety of $T^*\Lambda$ defined in local coordinates by $y_1 = y'^* = \tau^* = 0$, where $y' = (y_2, \ldots, y_{n-1})$.

Definition 2.1 ([9], [12])**.** We say that Z is *non-micro-characteristic of type L* for M if $T^*_{\Lambda'}\Lambda \cap \mathrm{Ch}^L(M)$ is contained in $T^*_\Lambda \Lambda$. Sometimes we will say that, if this condition holds, Z is *non-L-micro-characteristic* for M.

The sheaf of rings $\mathrm{gr}^L(\mathcal{D})$ is endowed with two graduations: The first is induced by the F-filtration and the second is induced by the V-filtration. Recall Laurent's definition of slope of a coherent $\mathcal{D}$-module M.

Definition 2.2 ([10])**.** The rational number $-p/q$ is said to be a *slope* of M with respect to Y at the origin if and only if the radical of the ideal $\mathrm{Ann}_{\mathrm{gr}^L(\mathcal{D}_X)}(\mathrm{gr}^L(M))$ is not bihomogeneous for F- nor V-graduations.

An important consequence of the work [11] (see Théorème 2.4.2) is what follows: A holonomic $\mathcal{D}_X$-module M is regular with respect to Y at the origin if and only if M has no slope with respect to Y at the origin. We will use this fact freely in the text.

Remark. In [12], ∞ and 0 (F and V) were included in the set of the slopes. We do not include them in the set of the slopes in this paper.

Finally, the following result by Laurent and Mebkhout allows induction on the number of variables to calculate slopes.

Theorem 2.3 ([12, Corollaire 2.2.9])**.** *Let M be a holonomic $\mathcal{D}_X$-module. Let Z and Y be transverse smooth hypersurfaces on X such that Z is non-L-micro-characteristic (for all $p > 0, q > 0$) for M. Then the slopes of M with respect to Y equal the slopes of M' with respect to $Y' = Z \cap Y$, where M' is the restriction of M to Z.*

This is a deep result in $\mathcal{D}$-module theory. Its proof uses the algebraic-analytic comparison theorem ([11, Theorem 2.4.2]) and a Cauchy-Kowalewska theorem for Gevrey functions with respect to Z ([12, Corollaire 2.2.4]; see also [14, Théorème 6.3.4]).

3. Computing slopes by reducing the number of variables

We have introduced the notion of the slopes and the invariance of them under restrictions satisfying a condition on L-characteristic varieties. In the sequel, we assume that our ideal is that of the Weyl algebra A_n. Constructions in sheaves such as restrictions and L-characteristic varieties in the previous section can be done via constructions in the Weyl algebra as we usually see in the computational $\mathcal{D}$-module theory.

We are interested in computation of the slopes. The slopes of A_n/I (at the origin) along $x_n = 0$ can be computed by the ACG algorithm introduced in [2]. In this section, translating Laurent and Mebkhout's result into computer algebra algorithms, we will give a preprocessing method for the ACG algorithm to accelerate the original. The preprocessing is useful for a class of inputs including GKZ hypergeometric ideals as we will see in Section 4. Let us first recall the ACG algorithm.

A weight vector is an element $W = (u_1, \dots, u_n, v_1, \dots, v_n) \in \mathbf{R}^{2n}$ such that $u_i + v_i \geq 0$ for all i. This weight vector $W = (u, v)$ induces a natural filtration on A_n, and it is called the W-filtration. The associated graded ring is denoted by $\mathrm{gr}^W(A_n)$, and for each left ideal $I \subset A_n$ the associated graded ideal is denoted by $\mathrm{in}_W(I)$ or $\mathrm{in}_{(u,v)}(I)$. Here, the *initial ideal* $\mathrm{in}_{(u,v)}(I)$ is the ideal generated by $\mathrm{in}_{(u,v)}(f), f \in I$ in $\mathrm{gr}^W(A_n)$. When $u_i + v_i > 0$, it is an ideal in the polynomial ring of $2n$ variables: $\mathrm{gr}^W(A_n) = \mathbf{C}[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$. The initial ideal of I with respect to the weight (u, v) is generated by the (u, v)-initial terms of a Gröbner basis of I by an order that refines the partial order defined by (u, v). See, e.g., [17, Theorem 1.1.6].

Consider the filtration $L = pF + qV, p > 0, q > 0$ introduced in the previous section. The ideal $\sigma^L(I)$, which gives the L-characteristic variety, can be expressed in terms of the initial ideal as follows:

$$\sigma^L(I) = \mathrm{gr}^L(\mathcal{D}) \cdot \mathrm{in}_\ell(I)|_{x_1 \mapsto y_1, \dots, x_{n-1} \mapsto y_{n-1}, x_n \mapsto \tau^*, \xi_1 \mapsto y_1^*, \dots, \xi_{n-1} \mapsto y_{n-1}^*, \xi_n \mapsto -\tau},$$

$$\ell = p(\overbrace{0, \dots, 0}^{n}, \overbrace{1, \dots, 1}^{n}) + q(\overbrace{0, \dots, 0, -1}^{n}, \overbrace{0, \dots, 0, 1}^{n}).$$

For two weight vectors W and W' and a term order $<$, we denote by $<_{W,W'}$ the order

$$\begin{aligned} & x^\alpha \partial^\beta <_{W,W'} x^a \partial^b \\ \Leftrightarrow\ & W \cdot (\alpha, \beta) < W \cdot (a, b) \\ & \text{or } W \cdot (\alpha, \beta) = W \cdot (a, b) \ \text{ and } W' \cdot (\alpha, \beta) < W' \cdot (a, b) \\ & \text{or } W^* \cdot (\alpha, \beta) = W^* \cdot (a, b) \ \text{ for both } \ W^* = W, W' \ \text{ and } x^\alpha \partial^\beta < x^a \partial^b. \end{aligned}$$

To each differential operator $P = \sum p_{\alpha\beta} x^\alpha \partial^\beta \in A_n$ we associate the Newton polygon $N(P)$ of P (with respect to $x_n = 0$) defined as the convex hull of the subset of $\mathbf{Z}^2$,

$$\bigcup_{p_{\alpha\beta} \neq 0} (|\beta|, \beta_n - \alpha_n) + (-\mathbf{N})^2.$$

Let I be a left ideal in A_n. As we said in Definition 2.2, the notion of slope of a differential system was introduced by Y. Laurent [10]. Let us give here a slightly different but equivalent definition: the number r, $-\infty < r < 0$, is a *geometric* slope of I (or of A_n/I) with respect to $x_n = 0$ if and only if $\sqrt{\sigma^{(-r)F+V}(I)}$ is not bihomogeneous with respect to the weight vectors $F = (0, \ldots, 0, 1, \ldots, 1)$ and $V = (0, \ldots, 0, -1, 0, \ldots, 0, 1)$. Following [2], we say that the number r, $-\infty < r < 0$ is an *algebraic* slope of I (or of A_n/I) if and only if $\sigma^{(-r)F+V}(I)$ is not bihomogeneous with respect to the weight vectors F and V. The geometric slope is simply called the slope in this paper if confusion does not arise. Note that we may consider $\mathrm{in}_L(I)$ instead of $\sigma^L(I)$ so long as we are concerned about homogeneity. For algebraic or geometric slope r, the weight vector $L = (-r)F + V$ lies on a face of the Gröbner fan of I ([3], [17]), which yields the following algorithm.

Algorithm 3.1 ([2], ACG algorithm)**.**
Input: $G = \{P_1, \ldots, P_m\}$ (generators of an ideal I).
Output: All algebraic and geometric slopes of A_n/I with respect to $x_n = 0$ at the origin.

```
geometric_slope = ∅; algebraic_slope = ∅;
F = (0,...,0,1,...,1); V = (0,...,0,−1,0,...,0,1);
p = 1; q = 0; slope = −∞; previous_slope = slope;
while (slope! = 0) {
    L = pF + qV;
    G = a Gröbner basis of I with respect to the order <_{L,V};
    slope = the minimum of 0 and
        {the slopes r of the Newton polygon N(P) | P ∈ G, r > previous_slope}
    if slope = 0, then return (algebraic_slope and geometric_slope).
    if σ^L(G) is not homogeneous for F nor V then {
      algebraic_slope = algebraic_slope ∪ {slope}
    }
    if √(σ^L(G)) is not homogeneous for F nor V then {
      geometric_slope = geometric_slope ∪ {slope}
    }
    p = numerator(|slope|); q = denominator(|slope|);
    previous_slope = slope;
}
return(algebraic_slope and geometric_slope)
```

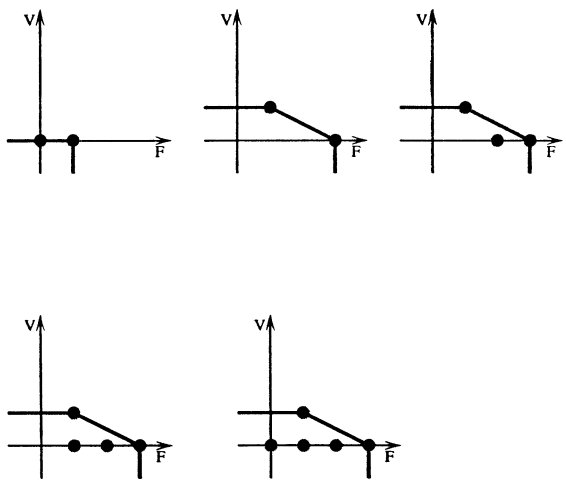

FIGURE 1.

Here, we use the convention $F := \infty F + V$. Note that the ideal $J = \langle f_1, \ldots, f_m \rangle \subset \mathbf{C}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$ is homogeneous for the weight $(u, v) \in \mathbf{Z}^{2n}$ if and only if all (u, v)-homogeneous subsums of f_i belong to the ideal J.

For ordinary differential equations, the ACG algorithm is nothing but the well-known Newton polygon method; see two examples below.

Example 3.2. Put $G = \{x_1^{p+1}\partial_1 + p\}$, $n = 1$, $p \in \mathbf{N}$. Then, the ACG algorithm returns $\{-p\}$. ($\exp(x_1^{-p})$ is a classical solution of G.)

Example 3.3. Put $G = \{2x_1(x_1\partial_1)^2 + x_1^2\partial_1 + 1\}$, $n = 1$. Then, the ACG algorithm returns $\{-1/2\}$. ($\exp(x_1^{-1/2})$ is a classical solution of G.)

The next example of two variables is generated by the computer algebra system `kan/k0` [18].

Example 3.4. We consider the GKZ hypergeometric ideal I associated to the matrix $A = (1, 3)$ and $\beta = -3$. We will compute the slopes of A_2/I at the origin along $x_2 = 0$ by the ACG algorithm. The Gröbner basis of I for the weight $(\overbrace{0}^{x_1}, \overbrace{0}^{x_2}, \overbrace{1}^{\partial_1}, \overbrace{1}^{\partial_2})$ is

```
[ x1*Dx1+3*x2*Dx2+3, -Dx1^3+Dx2, -3*x2*Dx1^2*Dx2-5*Dx1^2-x1*Dx2,
 -9*x2^2*Dx1*Dx2^2-36*x2*Dx1*Dx2+x1^2*Dx2-20*Dx1,
 -27*x2^3*Dx2^3-189*x2^2*Dx2^2-x1^3*Dx2-276*x2*Dx2-60 ]
```

Here, `Dx`i and `x`i stand for ∂_i and x_i, respectively. The Newton polygons $N(P)$'s are given in Figure 1.

From the Newton polygons, $-1/2$ is the candidate of the first slope. Next, we compute the Gröbner basis for the weight $(0, -2, 1, 3) = (0, 0, 1, 1) + 2(0, -1, 0, 1)$. The Gröbner basis is

```
[ x1*Dx1+3*x2*Dx2+3, Dx2-Dx1^3 ]
```

The radical is generated by

```
 [ -x1*Dx1-3*x2*Dx2, -Dx1^3+Dx2, 3*x2*Dx1^2*Dx2+x1*Dx2,
 -9*x2^2*Dx1*Dx2^2+x1^2*Dx2, 27*x2^3*Dx2^3+x1^3*Dx2 ]
```

Here, `x1` $= y_1$, `x2` $= \tau^*$, `Dx1` $= y_1^*$, `Dx2` $= -\tau$. It is not bihomogeneous, and then $-1/2$ is a geometric and algebraic slope.

By looking at the two Newton polygons of `x1*Dx1+3*x2*Dx2+3`, `Dx2-Dx1^3`, we see that there are no slopes larger than $-1/2$. Then, the ACG algorithm terminates here.

The ACG algorithm requires a repetition of Gröbner basis computations in the Weyl algebra of $2n$ variables to evaluate the slopes of A_n/I. However, if $x_i = 0$ is non-micro-characteristic of type $L = (-r)F + V$ for all $-\infty < r < 0$ and the restriction of A_n/I to $x_i = 0$ is singly generated, we can preprocess the input so that the input ideal for the ACG algorithm lies in the Weyl algebra of $2(n-1)$ variables. The correctness of the following algorithm can be shown by Laurent and Mebkhout's Theorem 2.3.

Algorithm 3.5 (Computing slopes with a preprocessing)**.**

Step 1: Check if $x_i = 0$ is non-micro-characteristic of A_n/I for all types L by calling Algorithm 3.6.

Step 2: Compute the restriction of A_n/I to $x_i = 0$ and check if it is expressed as D'/I' where D' is the Weyl algebra of $2(n-1)$ variables.

Step 3: If we failed either in Step 1 or in Step 2, then apply the ACG algorithm for I.

If we succeeded both in Step 1 and in Step 2, then try to reduce more variables or apply the ACG algortihm for I'.

We can compute restrictions of a given $\mathcal{D}$-module by using Oaku's algorithm [15]. This algorithm is implemented in computer algebraic systems Macaulay2 and Kan [7], [18], [13]. Therefore, the remaining algorithmic question for the preprocessing is to determine the range of type $L = (-r)F + V$ for which $x_i = 0$ $(i \leq n-1)$ is non-micro-characteristic. It follows from the definition of non-micro-characteristic that the question is nothing but to find the segment $(-\infty, r_1)$ such that

$$\mathcal{V}(\sigma^{(-r)F+V}(I), y_1^*, \ldots, y_{i-1}^*, y_i, y_{i+1}^*, \ldots, y_{n-1}^*, \tau^*) \subseteq \mathcal{V}(y_1^*, \ldots, y_{n-1}^*, \tau^*) = T_\Lambda^*\Lambda$$

for $r \in (-\infty, r_1)$. Here, $\mathcal{V}(f_1, \ldots, f_m)$ is the affine variety defined by the polynomials $f_1, \ldots, f_m$. The inclusion condition can be algebraically rephrased as

$$\sqrt{\mathrm{in}_{(-r)F+V}(I), \xi_1, \ldots, \xi_{i-1}, x_i, \xi_{i+1}, \ldots, \xi_n} \ni \xi_i.$$

Since the Gröbner fan is a finite union of Gröbner cones [2], the range can be determined by a similar method with the ACG algorithm.

Algorithm 3.6.

`range_of_nonMC`(H, r_0)
Input: H is a finite set in A_n, r_0 is a negative number or $-\infty$.
Output: r_1 such that $x_1 = 0$ is non-micro-characteristic of type $(-r)F + V$ for $A_n/A_n \cdot \{H\}$, for $r \in [r_0, r_1)$.
`previous_slope` = `slope` $= r_0$;
`F` $= (0, \ldots, 0, 1, \ldots, 1)$; `V` $= (0, \ldots, 0, -1, 0, \ldots, 0, 1)$;
`while` (`slope`! $= 0$) {
    `p` = numerator(|`slope`|); `q` = denominator(|`slope`|);
    `L` = `pF` + `qV`;
    `G` = a Gröbner basis of H with respect to $<_L$;

```
    if √⟨σ^L(H), y_1, y_2^*, ..., y_{n-1}^*, τ^*⟩ ∋ y_1^* and
      √⟨σ^{pF+(q+ε)V}(H), y_1, y_2^*, ..., y_{n-1}^*, τ^*⟩ ∋ y_1^*
    then {
      previous_slope = slope;
      slope = the minimum of 0 and
        {the slopes r of the Newton polygon N(P) | P∈G, r>previous_slope}
    } else {
    return(slope);
    }
}
return(0);
```

Here, ε is a sufficiently small positive rational number so that $pF + (q+\varepsilon)V$ lies in the interior of a Gröbner cone.

In the case when $r_0 = -\infty$, we use the convention numerator$(|r_0|) = 1$ and denominator$(|r_0|) = 0$, and F-non-micro-characteristic means that it is non-characteristic in the classical sense.

Example 3.7. Suppose $n \geq 2$. Then `range_of_nonMC`$(\{\partial_1^2 - \partial_2\}, -\infty)$ returns 0.

If the function `range_of_nonMC`$(I, -\infty)$ returns 0, then $x_1 = 0$ is non-micro-characteristic of type $pL + qV$ ($p > 0$, $q \geq 0$) for A_n/I. We note that it is not always a clever strategy to call the function with the full set of generators. In fact, if $x_1 = 0$ is non-micro-characteristic of type L for A_n/J, then it is non-micro-characteristic of type L for A_n/I for any $I \supseteq J$. Therefore, for Step 1, it is sometimes more efficient to call the function `range_of_nonMC` for a subset of the generators of the input ideal as we will see in the case of GKZ hypergeometric ideals in the next section.

4. Computing slopes of $\mathcal{H}_{(1,a_2,\ldots,a_n)}(\beta)$

Put $A = (1, a_2, \ldots, a_n)$, $a_1 = 1 < a_2 < \cdots < a_n$. We will evaluate the slopes of the GKZ hypergeometric ideal $H_A(\beta)$ associated to the $1 \times n$ matrix A and $\beta \in \mathbf{C}$ by using the general algorithm given in Section 3. To apply this algorithm, we need to find non-micro-characteristic varieties and compute the restrictions of $\mathcal{H}_A(\beta)$ to these varieties. When $f_1, \ldots, f_m$ are polynomials in $\mathbf{C}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$, we denote by $\mathcal{V}(f_1, \ldots, f_m)$ the affine subvariety in $\mathbf{C}^{2n}$ defined by the f_i.

The following theorem can be shown by a standard method of Koszul complex ([1], [6]).

Theorem 4.1. *The characteristic variety of $\mathcal{H}_A(\beta)$ is $\mathcal{V}(\xi_1, \ldots, \xi_{n-1}, x_n\xi_n)$. In particular, the singular locus of $\mathcal{H}_A(\beta)$ is $x_n = 0$.*

Note that there is no slope along $x_i = 0$, $1 \leq i \leq n-1$, which can be shown easily.

Recall that $F = (0, \ldots, 0, 1, \ldots, 1)$ and $V = (0, \ldots, 0, -1, 0, \ldots, 0, 1)$. For a positive number p and a non-negative number q, we define the weight vector $L = pF + qV$. In Section 2, we explained the notion of non-micro-characteristic. When the variety is $y_i = x_i = 0$, this notion is rephrased as follows: for a given left

A_n-module A_n/I, the hyperplane $y_i = 0$ $(1 \le i \le n-1)$ is called non-micro-characteristic of type L when

$$\sqrt{\langle \sigma^L(I), y_i, y_j^*, (j \neq i), \tau^* \rangle} \ni y_i^*.$$

Proposition 4.2. *For the hypergeometric A_n-module $\mathcal{H}_A(\beta)$, the variety $y_i = 0$ $(1 \le i \le n-2)$ is non-micro-characteristic of type L for all $L = pF + qV$, $p > 0$.*

Proof. Consider $\partial_i^{a_j} - \partial_j^{a_i} \in H_A(\beta)$. For all L and for $i < j \le n-1$, we have $\sigma^L(\partial_i^{a_j} - \partial_j^{a_i}) = (y_i^*)^{a_j}$, which implies that $y_i = 0$ is non-micro-characteristic of type L. □

Now, let us apply the second step of the algorithm to evaluate the slopes, i.e., we will compute the restriction of $\mathcal{H}_A(\beta)$ to $y_i = x_i = 0$ $(1 < i \le n-2)$. Let s be an indeterminate. Consider the ideal $H_A[s]$ in $A_n[s]$ generated by $A\theta - s$ and I_A.

Theorem 4.3. *We have a left $D'[s]$-module isomorphism*

$$A_n[s]/(A_n[s]H_A[s] + x_i A_n[s]) \simeq D'[s]/D'[s]H_{A'}[s], \quad i \neq 1. \tag{4.1}$$

Here, $D' = \mathbf{C}\langle x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n, \partial_1, \dots, \partial_{i-1}, \partial_{i+1}, \dots, \partial_n \rangle$ and $A' = (1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$.

Proof. Fix an order $\prec$ such that $\partial_i \succ \partial_n \succ \partial_{n-1} \succ \cdots \succ \partial_1$. Then,

$$\partial_n - \partial_1^{a_n},\ \partial_{n-1} - \partial_1^{a_{n-1}},\ \dots, \partial_2 - \partial_1^{a_2}$$

is the reduced Gröbner basis of I_A with respect to $\succ$.

For each i $(2 \le i \le n)$, by applying the same method as in the proof of [17, Theorem 3.1.3], we can prove that $\mathrm{in}_{(-e_i, e_i)}(H_A(s))$ is generated by $A\theta - s$ and $\mathrm{in}_{e_i}(\partial_j - \partial_1^{a_j})$ $(j = 2, \dots, n)$.

Define the V_i-filtration $F_k[s]$ of $A_n[s]$ by

$$F_k[s] = \left\{ \sum a_{\alpha\beta\gamma} x^\alpha \partial^\beta s^\gamma \mid (\beta - \alpha) \cdot e_i \le k \right\}.$$

We compute the restriction of $A_n[s]/H_A(s)$ to $x_i = 0$ by Oaku's algorithm (see, e.g., [17, Theorem 5.2.6, Algorithm 5.2.8]). Since $\mathrm{in}_{(-e_i, e_i)}(x_i\partial_i - x_i\partial_1^{a_i}) = x_i\partial_i$, the b-function of $H_A(s)$ along $x_i = 0$ is $b(p) = p$. Therefore, by [17, Theorem 5.2.6], we have the following isomorphism of left $D'[s]$-modules:

$$\begin{aligned}
& A_n[s]/(H_A(s) + x_i A_n[s]) \\
\simeq\ & F_0[s]/\left(F_0[s](A\theta - s) + \sum_{j=2, j\neq i}^{n} F_0[s](\partial_j - \partial_1^{a_j}) + F_{-1}[s](\partial_i - \partial_1^{a_i}) + x_i F_1[s] \right) \\
\simeq\ & D'[s]/\left(D'[s](A\theta - s) + \sum_{j=2, j\neq i}^{n} D'[s](\partial_j - \partial_1^{a_j}) \right).
\end{aligned}$$

□

We can specialize s to any complex number β. In fact, we have the following theorem.

Theorem 4.4.

$$A_n/(A_n H_A(\beta) + x_i A_n) \simeq D'/D' H_{A'}(\beta), \quad i \neq 1.$$

Proof. Put $L = x_1\partial_1 + a_2x_2\partial_2 + \cdots + a_nx_n\partial_n - \beta$. We note that $[\partial_k - \partial_1^{a_k}, L] = a_k(\partial_k - \partial_1^{a_k})$. Fix a lexicographic order $\prec$ such that $x_1 \succ \cdots \succ x_n \succ \partial_i \succ \partial_n \succ \partial_{n-1} \succ \cdots \succ \partial_1$. Then, we can show that

$$\{L, \partial_n - \partial_1^{a_n},\ \partial_{n-1} - \partial_1^{a_{n-1}},\ \ldots, \partial_2 - \partial_1^{a_2}\}$$

is a Gröbner basis of $H_A(\beta)$ with respect to the order $\succ_{(-e_i,e_i)}$ by Buchberger's S-pair criterion. In fact, we have

$$\mathrm{sp}(\underline{\partial_k} - \partial_1^{a_k}, \underline{x_1\partial_1} + a_2x_2\partial_2 + \cdots + a_nx_n\partial_n - \beta) = L(\partial_k - \partial_1^{a_k}) - (\partial_k - \partial_1^{a_k})L \to 0.$$

The remaining checks are easy. Therefore, we conclude that $\mathrm{in}_{(-e_i,e_i)}(H_A(\beta))$ is generated by $A\theta - \beta$ and $\mathrm{in}_{e_i}(\partial_j - \partial_1^{a_j})$ $(j = 2, \ldots, n)$. The rest of the proof is the same as that of Theorem 4.3. □

The theorem means that the restriction of $\mathcal{H}_A(\beta)$ to $x_i = 0$ can be exactly expressed in terms of the GKZ system for smaller A. By applying our Algorithm 3.5 for computing slopes by reduction of the number of the variables to the variables $x_2, \ldots, x_{n-2}$, we obtain the following theorem from Proposition 4.2 and Theorem 4.4.

Theorem 4.5. *The geometric slopes of $\mathcal{H}_A(\beta)$ along $x_n = 0$ at the origin and $\mathcal{H}_{(1,a_{n-1},a_n)}(\beta)$ along $x_3 = 0$ at the origin coincide.*

Example 4.6. We note that (the algebraic slopes) $\neq$ (the geometric slopes) in general. For example, let us apply the ACG algorithm to get the algebraic slopes of $H_A(-30)$ for $A = (1, 3, 7)$. This ideal is generated by

$$x_1\partial_1 + 3x_2\partial_2 + 7x_3\partial_3 + 30, \partial_1^3 - \partial_2, -\partial_1^2\partial_2 + \partial_3, \partial_2^3 - \partial_1^2\partial_3.$$

The output is

$$\{-1, -3/4, -1/2\}.$$

On the other hand, if we apply the ACG algorithm to get the geometric slopes, the output is $\{-3/4\}$.

Example 4.7. (the slopes of $\mathcal{H}_{(1,a_{n-1},a_n)}(\beta)$) $\neq$ (the slopes of $\mathcal{H}_{(1,a_n)}(\beta)$) in general.

Let us take the example: $A = (1, 3, 7)$. Consider the hypergeometric A_3-module A_3/I where $I = H_{(1,3,7)}(-30)$. As we have seen, the slope of this system along $x_3 = 0$ is $\{-3/4\}$.

Consider $\mathrm{in}_L(I)$ for $L = F + 4V$. By computing the Gröbner basis with respect to L, we can see that

$$\mathcal{V}(\mathrm{in}_L(I)) = \mathcal{V}(x_2, \xi_1, \xi_3) \cup \mathcal{V}(\xi_1, \xi_2, \xi_3).$$

It is not included in $\mathcal{V}(\xi_2)$. Hence $x_2 = 0$ is micro-characteristic of type L, and we cannot apply the restriction criterion.

The condition "non-microcharacteristic for all the filtration $pF + qV$" cannot be taken as a way to evaluate the slopes by the restriction. In fact, it can be easily checked by the ACG algorithm that the set of the geometric slopes of $\mathcal{H}_{(1,a_n)}(\beta)$ is equal to $\{1/(1 - a_n)\}$. Hence, the set of the slopes of $\mathcal{H}_{(1,7)}(\beta)$ is $\{-1/6\}$, which is not equal to $\{-3/4\}$.

We have shown that the computation of the slopes of $\mathcal{H}_A(\beta)$ is reduced to the case of three variables. The slopes in this case are as follows.

Theorem 4.8. (the slopes of $\mathcal{H}_{(1,a_{n-1},a_n)}(\beta)$) $= \{a_{n-1}/(a_{n-1} - a_n)\}$.

Proof. We fix some notation:

(1) $A = (1, a, b) \in \mathbf{Z}^3$ and $1 < a < b$.
(2) $P_1 = \partial_1^a - \partial_2$, $P_2 = \partial_1^b - \partial_3$, $P_3 = \partial_2^b - \partial_3^a$, $P_4 = x_1\partial_1 + ax_2\partial_2 + bx_3\partial_3 - \beta$.
(3) Let Λ be the linear form with slope $-a/(b-a)$ (i.e., $\Lambda = aF + (b-a)V$).
(4) Let L, L' be linear forms. We say that $L > L'$ if slope$(L) >$ slope(L').
(5) We will write $y_1 = x_1$, $y_2 = x_2$, $t = x_3$.

The operators P_1, P_2, P_3, P_4 are in $H = H_A(\beta)$. Then, we have the following claims.

(1) For all linear forms L we have $\sigma^L(P_1) = (\eta_1^*)^a$ and so $\eta_1^* \in \sqrt{\sigma^L(H)}$ for all L.
(2) For all linear forms L we have $\sigma^L(P_4) = y_1\eta_1^* + ay_2\eta_2^* + b\tau^*(-\tau)$.
(3) For all linear forms $L > \Lambda$ we have $\sigma^L(P_3) = (\tau)^a$ and so $\tau \in \sqrt{\sigma^L(H)}$ for all $L > \Lambda$.
(4) Thus, for all $L > \Lambda$ we have $\mathrm{Ch}^L(\mathcal{H}) \subset T^*_{y_2=0}\mathbf{C}^3 \cup T^*_{\mathbf{C}^3}\mathbf{C}^3$, and then $\sqrt{\sigma^L(H)}$ is bihomogeneous and L is not a geometric slope of $\mathcal{H}$.
(5) On the other hand, for $L < \Lambda$, we have $\sigma^L(P_3) = (\eta_2^*)^b$. Then $\eta_2^* \in \sqrt{\sigma^L(H)}$ and $\mathrm{Ch}^L(\mathcal{H}) \subset T^*_{t=0}\mathbf{C}^3 \cup T^*_{\mathbf{C}^3}\mathbf{C}^3$. So L is not a geometric slope of $\mathcal{H}$.
(6) Therefore, the only possible geometric slope of $\mathcal{H}$ is Λ.

Now, suppose that Λ is not a slope. Then, there is no slope, which implies that the L-characteristic variety $\mathrm{Ch}^L(\mathcal{H}_A(\beta))$ is the same for all $L = pF + qV$, $p, q > 0$ by [3] and [10, Théorème 3.4.1]. It follows from

$$[P_1, P_2] = 0,\ [P_1, P_4] = aP_1,\ [P_2, P_4] = bP_2$$

and the Buchberger algorithm that $\{P_1, P_2, P_4\}$ is a Gröbner basis for the order defined by the weight vector $L = (0, 0, -N, 1, 1, N+1)$, $N \geq b$ and a tie-breaking term order such that $x_2 \succ \partial_3 \succ \partial_1 \succ \partial_2$. Therefore, the initial ideal $\mathrm{in}_L(H_A(\beta))$ is generated by ξ_1^a, ξ_3, $x_1\xi_1 + ax_2\xi_2 + bx_3\xi_3$ and hence the L-characteristic variety is equal to $T^*_{y_2=0}\mathbf{C}^3 \cup T^*_{\mathbf{C}^3}\mathbf{C}^3$. This fact contradicts that the L-characteristic variety is the same for all L. □

5. Rational solutions and reducibility

Our ultimate aim in studying the slopes of $H_A(\beta)$ is to get a better understanding of solutions of this system. We are far from the goal, but to this end, it will be useful to present some facts on classical solutions and a relation to generalized confluent hypergeometric functions.

In the case of the hypergeometric ideal associated to homogeneous monomial curves, Cattani, D'Andrea, and Dickenstein [5] studied rational solutions and reducibility of the system. We will study rational solutions and reducibility of our system.

Theorem 5.1. *Any rational solution of the hypergeometric system $H_{(1,a_2,\ldots,a_n)}(\beta)$ is a polynomial. It has a polynomial solution if and only if $\beta \in \mathbf{N} = \{0, 1, \ldots\}$. The polynomial solution is the residue of $\exp\left(\sum x_i t^{a_i}\right) t^{-\beta}$ at the origin $t = 0$:*

$$\int_C \exp\left(\sum x_i t^{a_i}\right) t^{-\beta} \frac{dt}{t}.$$

Here, C is a circle that encircles the origin in the positive direction.

Proof. Since the singular locus of $H_A(\beta)$ ($A = (a_1, a_2, \dots, a_n)$, $a_1 = 1$) is $x_n = 0$, any rational solution f is a Laurent polynomial with poles on $x_n = 0$. Take a weight vector $w = (0, 1, 1, \dots, 1)$. Then, we have $\mathrm{in}_w(I_A) = \langle \partial_2, \dots, \partial_n \rangle$. The initial term $\mathrm{in}_w(f)$ is annihilated by

$$\sum a_i \theta_i - \beta, \quad \partial_2, \dots, \partial_n.$$

Therefore, $x_1^\beta = \mathrm{in}_w(f)$. This implies $\beta \in \mathbf{N}$, because f has a pole only on $x_n = 0$.

Take a $\mathbf{Z}$-basis of $\mathrm{Ker}\,(\mathbf{Z}^n \xrightarrow{A} \mathbf{Z})$ as

$$(-a_2, 1, 0, \dots, 0), (-a_3, 0, 1, 0, \dots, 0), \dots, (-a_n, 0, \dots, 0, 1)$$

to construct series solutions. Since [17, Proposition 3.4.1] holds for non-homogeneous A as well, a formal series solution g of $H_A(\beta)$ satisfying $\mathrm{in}_w(g) = x_1^\beta$ can be uniquely expressed as

$$\sum_{m \in \mathbf{N}^{n-1}} \frac{\beta(\beta-1)\cdots(\beta - \sum m_k a_k + 1)}{m_2! \cdots m_n!} \left(\frac{x_2}{x_1^{a_2}}\right)^{m_2} \cdots \left(\frac{x_n}{x_1^{a_n}}\right)^{m_n} x_1^\beta. \tag{5.1}$$

When $\beta \in \mathbf{N}$, it is a polynomial. The rest of the theorem is easy to show. □

Let R be the ring of differential operators of n variables with rational function coefficients over $\mathbf{k} = \mathbf{C}$. A left ideal J of R is called *irreducible* when J is a maximal ideal in R. We will study the reducibility of $R \cdot H_A(\beta)$.

We assume that J is zero-dimensional, i.e., $r = \dim_{\mathbf{C}(x)} R/J < +\infty$. Let $V = V(J)$ be the vector space of holomorphic solutions of J on a simply connected open set contained in the non-singular domain of J. It is known that $\dim_{\mathbf{C}} V = r$. Define $I(V)$ by $R \cdot \{\ell \in R \,|\, \ell \bullet f = 0 \text{ for all } f \in V\}$. If $J \subset I(V)$, $J \neq I(V)$, then we have $\dim_{\mathbf{C}(x)} R/J < \dim_{\mathbf{C}} V = r$ because of the zero-dimensionality of J. Therefore, we have

$$J = I(V(J)).$$

Under this correspondence of ideals and solutions, a zero-dimensional ideal J of R is reducible if and only if there exists a proper subspace W of the solution space of $V(J)$ such that $0 < \dim_{\mathbf{C}(x)} R/I(W) < \dim_{\mathbf{C}(x)} R/J$. In the case of one variable, the reducibility is equivalent to saying that the generator of the ideal can be factored in R.

Theorem 5.2. *The system of differential equations $R \cdot H_A(\beta)$ is reducible if and only if $\beta \in \mathbf{Z}$.*

Proof. Any curve is Cohen-Macaulay. By applying the theorem of Adolphson [1], the holonomic rank of $H_A(\beta)$ is a_n for all β.

Put $M(\beta) = A_n/H_A(\beta)$. Consider the left A_n-morphism

$$\partial_1 \ : \ M(\beta) \to M(\beta + 1). \tag{5.2}$$

It has the inverse when $\beta \neq -1$. Therefore, $M(-1) \simeq M(-2) \simeq M(-3) \simeq \cdots$ and $M(0) \simeq M(1) \simeq M(2) \simeq \cdots$.

When $\beta \in \mathbf{N}$, the system admits a polynomial solution; then it is reducible. It is also easy to see that when $\beta \in \mathbf{Z}_{<0}$, the equation is reducible. In fact, consider the left $\mathcal{D}$-morphism

$$\partial_1 \ : \ M(-1) \to M(0).$$

It induces a morphism to the solutions by

$$f \to \partial_1 \bullet f.$$

The solution $f = 1$ of $M(0)$ is sent to zero. So the image of ∂_1 gives a proper subspace of solutions in the solution space of $M(-1)$. To find differential equations for the subspace, take all ℓ such that $\ell\partial_1 \in H_A(0)$. Then, $\{\ell\} \subset H_A(-1)$. By the isomorphism (5.2), we conclude that when $\beta \in \mathbf{Z}_{<0}$, the system is reducible.

Let us prove that the system is irreducible when $\beta \notin \mathbf{Z}$ by applying the result of Beukers, Brownawell, and Heckman [4]. For this purpose, we first construct a convergent series solution of $H_A(\beta)$. Take $w = (1, 1, \ldots, 1, 0)$. Then the degree of $\mathrm{in}_w(I_A)$ is equal to a_n. Since I_A contains the elements of the form $\underline{\partial_i^{a_n}} - \partial_n^{a_i}$, the radical of $\mathrm{in}_w(I_A)$ is $\langle \partial_1, \ldots, \partial_{n-1}\rangle$. Therefore, the top-dimensional standard pairs have the form $(\partial^b, \{n\})$.

Let v be the zero of the indicial ideal associated to $(\partial^b, \{n\})$:

$$v_1 = b_1, \ldots, v_{n-1} = b_{n-1}, v_n = \frac{\beta - \sum a_i b_i}{a_n}.$$

Assume $\beta \notin \mathbf{Z}$ or $\beta \gg 0$. Taking the lattice basis

$$(a_2, -1, 0, \ldots, 0), \ldots, (a_n, 0, \ldots, 0, -1),$$

we have the following a_n linearly independent convergent series solutions:

$$\sum_{m \in \mathbf{N}^{n-1}} \frac{v_2(v_2-1)\cdots(v_2-m_2+1)\cdots v_n(v_n-1)\cdots(v_n-m_n+1)}{(v_1+1)(v_1+2)\cdots(v_1+\sum a_k m_k)} \cdot \left(\frac{x_1^{a_2}}{x_2}\right)^{m_2} \cdots \left(\frac{x_1^{a_n}}{x_n}\right)^{m_n} x^v. \tag{5.3}$$

We consider the change of variables:

$$y_1 = x_1, y_2 = x_1^{a_2}/x_2, \ldots, y_n = x_1^{a_n}/x_n.$$

The inverse of this change of variables is also rational, and the change of variables induces that of ∂_i. We denote by Φ the operation of this change of variables of x_i and ∂_i. Since the irreducibility is invariant under any birational change of variables, we will prove the irreducibility of the ideal $J = R \cdot \Phi(x^{-v} H_A(\beta) x^v)$ where $R = \mathbf{C}(y)\langle \partial_{y_1}, \ldots, \partial_{y_n}\rangle$.

Let V be the solution space of J spanned by the series $\Phi((5.3) \cdot x^{-v})$ near $y = 0$. If $J = I(V)$ is reducible, then there exists a proper subspace W of J such that $0 < \dim_{\mathbf{C}(y)} R/I(W) < a_n$. We consider the vector space $W' = \{\, f(0, \ldots, 0, y_n) \mid f \in W\}$. It is easy to see that $\dim_{\mathbf{C}(y_n)} R'/I(W') \le \dim_{\mathbf{C}} W$ where $R' = \mathbf{C}(y_n)\langle \partial_{y_n}\rangle$. Let us prove that $\dim_{\mathbf{C}} W' = a_n$ when $\beta \notin \mathbf{Z}$, which implies the irreduciblity of J by a contradiction.

We restrict the series $\Phi((5.3) \cdot x^{-v})$ to $y_1 = x_1 = 0, y_2 = x_1^{a_2}/x_2 = 0, \ldots, y_{n-1} = x_1^{a_{n-1}}/x_n = 0$ and replace $y_n = x_1^{a_n}/x_n$ by z. Without loss of generality, we may assume $v_1 = 0$. Then the restricted series has the form

$$\sum_{m=0}^{\infty} \frac{v_n(v_n-1)\cdots(v_n-m+1)}{(a_n m)!}(-z)^m. \tag{5.4}$$

It is annihilated by the ordinary differential operator

$$(a_n\theta_z)(a_n\theta_z - 1)\cdots(a_n\theta_z - a_n + 1) - z(\theta_z + v_n).$$

By replacing $z/a_n^{a_n}$ by x, we obtain the generalized hypergeometric ordinary differential equation

$$\theta_x(\theta_x - 1/a_n)\cdots(\theta_x - (a_n - 1)/a_n) - x(\theta_x + v_n). \tag{5.5}$$

By [4], this ordinary differential equation of rank a_n is reducible if and only if $v_n - k/a_n \notin \mathbf{Z}$ for all $k = 0, 1, \ldots, a_n - 1$. If one of them is an integer, β becomes an integer. Therefore, the ideal $I(W')$ contains the principal ideal generated by (5.5), which is maximal when $\beta \notin \mathbf{Z}$. We conclude that $I(W')$ is generated by (5.5) and hence $\dim_{\mathbb{C}} W' = a_n$. □

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