

## LEFT-DETERMINED MODEL CATEGORIES AND UNIVERSAL HOMOTOPY THEORIES

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**ABSTRACT.** We say that a model category is left-determined if the weak equivalences are generated (in a sense specified below) by the cofibrations. While the model category of simplicial sets is not left-determined, we show that its non-oriented variant, the category of symmetric simplicial sets (in the sense of Lawvere and Grandis) carries a natural left-determined model category structure. This is used to give another and, as we believe simpler, proof of a recent result of D. Dugger about universal homotopy theories.

### 1. INTRODUCTION

Recall that a *model category*  $\mathcal{K}$  is a complete and cocomplete category  $\mathcal{K}$  equipped with three classes of morphisms  $\mathcal{C}$ ,  $\mathcal{W}$  and  $\mathcal{F}$ , called *cofibrations*, *weak equivalences* and *fibrations*, such that

- (1)  $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$  and  $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$  are weak factorization systems and
- (2)  $\mathcal{W}$  is closed under retracts (in the category  $\mathcal{K}^{\rightarrow}$  of morphisms of  $\mathcal{K}$ ) and has the 2-out-of-3 property

(see [Q], [H], [Ho] or [AHRT2]). Model categories were introduced by D. Quillen to provide a foundation of homotopy theory. Here a weak factorization system is a pair  $(\mathcal{L}, \mathcal{R})$  of morphisms such that every morphism has a factorization as an  $\mathcal{L}$ -morphism followed by an  $\mathcal{R}$ -morphism, and  $\mathcal{R} = \mathcal{L}^{\square}$ ,  $\mathcal{L} = \square\mathcal{R}$  where  $\mathcal{L}^{\square}$  ( $\square\mathcal{R}$ ) consists of all morphisms having the right (left) lifting property w.r.t.  $\mathcal{L}$  ( $\mathcal{R}$ , respectively). The morphism  $l$  has a *left lifting property* with respect to a morphism  $r$  (or  $r$  has a *right lifting property* w.r.t.  $l$ ) if in every commutative square

$$\begin{array}{ccc}
 A & \xrightarrow{u} & C \\
 \downarrow l & & \downarrow r \\
 B & \xrightarrow{v} & D
 \end{array}$$

there exists a diagonal  $d : B \rightarrow C$ .

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A model category is determined by any two of the three classes above. Clearly,  $\mathcal{C}$  and  $\mathcal{W}$  determine  $\mathcal{F}$  because  $\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^\square$ , and from  $\mathcal{C}$  and  $\mathcal{F}$  one obtains the morphisms of  $\mathcal{W}$  as composites  $g \cdot f$  with  $f \in {}^\square\mathcal{F}$  and  $g \in \mathcal{C}^\square$ . In this paper we are interested in the model categories whose model structure is determined by its cofibrations only, and we therefore call them *left-determined*. For example, the model category **SComp** of simplicial complexes is left-determined while the model category **Simp** of simplicial sets is not left-determined.

**Simp** is, of course, the presheaf category  $\mathbf{Set}^{\mathbf{\Delta}^{\text{op}}}$  where  $\mathbf{\Delta}$  is the category of nonzero finite ordinals and order-preserving maps. F. W. Lawvere [L] and M. Grandis [G] introduced *symmetric simplicial sets* as functors  $\mathbf{F}^{\text{op}} \rightarrow \mathbf{Set}$  where  $\mathbf{F}$  is the category of nonzero finite cardinals (= finite sets) and arbitrary maps. We will show that the category **SSimp** =  $\mathbf{Set}^{\mathbf{F}^{\text{op}}}$  of symmetric simplicial sets is a left-determined model category. Moreover, the model categories **SSimp** and **Simp** are Quillen equivalent, i.e., they have equivalent homotopy categories.

D. Dugger [D] has recently shown that, for a small category  $\mathcal{X}$ ,  $\mathbf{Simp}^{\mathcal{X}^{\text{op}}}$  is a universal model category over  $\mathcal{X}$ . In particular, **Simp** is a universal model category over the (one-morphism) category  $\mathbf{1}$ . We will give another proof of his result, by showing that also **SSimp** $^{\mathcal{X}^{\text{op}}}$  serves as a universal model category over  $\mathcal{X}$ . Since **SSimp** is left-determined, our proof is simpler.

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After having completed this work we learned that the concept of a left-determined model category was independently developed by J. H. Smith [S], who used the term minimal model category instead. He also observed that the usual model structure on simplicial sets fails to be left-determined.

## 2. LEFT-DETERMINED MODEL CATEGORIES

**Definition 2.1.** A model category  $\mathcal{K}$  is *left-determined* if  $\mathcal{W}$  is the smallest class of morphisms satisfying the following conditions:

- (i)  $\mathcal{C}^\square \subseteq \mathcal{W}$ ,
- (ii)  $\mathcal{W}$  is closed under retracts and satisfies the 2-out-of-3 property,
- (iii)  $\mathcal{C} \cap \mathcal{W}$  is stable under pushout and closed under transfinite composition.

We will denote the smallest class of morphisms satisfying (i)–(iii) by  $\mathcal{W}_{\mathcal{C}}$ . It has the property that  $\mathcal{W}_{\mathcal{C}} \subseteq \mathcal{W}$  for each model category  $\mathcal{K}$  having  $\mathcal{C}$  as the class of cofibrations. Left-determined model categories are those for which  $\mathcal{W} = \mathcal{W}_{\mathcal{C}}$ . Recall that  $\mathcal{C}^\square$  denotes the class of morphisms having the right lifting property w.r.t.  $\mathcal{C}$ . Of course,  $\mathcal{C}^\square = \mathcal{F} \cap \mathcal{W}$  is the class of trivial fibrations.

In general, given  $\mathcal{C}$ , the first principal problem is whether  $\mathcal{C}$  and  $\mathcal{W}_{\mathcal{C}}$  yield a model category. The next theorem gives an affirmative answer under an additional set-theoretic hypothesis, the Vopěnka's Principle. (Subsequently, J. H. Smith informed us that he has been able to prove the theorem even absolutely, i.e., without any additional set-theoretic hypothesis.) Recall that Vopěnka's Principle is a set-theoretic axiom implying the existence of very large cardinals (see [AR]). We denote by  $\text{cof}(\mathcal{I})$  the smallest class of morphisms containing  $\mathcal{I}$ , closed under retracts in comma-categories  $A \backslash \mathcal{K}$  and satisfying (iii). The smallest class containing  $\mathcal{I}$  and satisfying (iii) is denoted by  $\text{cell}(\mathcal{I})$  (see [AHRT1]).

**Theorem 2.2.** *Let  $\mathcal{I}$  be a (small) set of morphisms in a locally presentable category  $\mathcal{K}$ . Under Vopěnka’s principle,  $\mathcal{C} = \text{cof}(\mathcal{I})$  and  $\mathcal{W} = \mathcal{W}_{\mathcal{C}}$  yield a model category structure on  $\mathcal{K}$ .*

*Proof.* According to the theorem of J. H. Smith (see [B, 1.7]), it suffices to show that  $\mathcal{W}_{\mathcal{C}}$  satisfies the solution set condition at  $\mathcal{I}$ . This means that for every  $f \in \mathcal{I}$  there is a subset  $\mathcal{X}_f$  of  $\mathcal{W}_{\mathcal{C}}$  such that every morphism  $f \rightarrow g$ ,  $g \in \mathcal{W}_{\mathcal{C}}$ , factorizes through some  $h \in \mathcal{X}_f$ . Since  $f \backslash \mathcal{W}_{\mathcal{C}}$  is a full subcategory of  $f \backslash \mathcal{K}$  and  $f \backslash \mathcal{K}$  is locally presentable (see [AR, 1.57]),  $f \backslash \mathcal{W}_{\mathcal{C}}$  has a small dense subcategory  $\mathcal{X}_f$  provided that we assume Vopěnka’s principle (see [AR, 6.6]). Without any loss of generality, we may assume that  $\mathcal{X}_f$  contains the initial object of  $f \backslash \mathcal{W}_{\mathcal{C}}$  provided that it exists. A morphism  $f \rightarrow g$  in  $f \backslash \mathcal{W}_{\mathcal{C}}$  is either initial in  $f \backslash \mathcal{W}_{\mathcal{C}}$  and thus belongs to  $\mathcal{X}_f$ , or it factorizes through some morphism  $f \rightarrow h$  from  $\mathcal{X}_f$ . Hence  $\mathcal{X}_f$ ,  $f \in \mathcal{I}$ , provide a solution set condition at  $\mathcal{I}$ .  $\square$

A model category is called *cofibrantly generated* if  $\mathcal{C} = \text{cof}(\mathcal{I})$  and  $\mathcal{C} \cap \mathcal{W} = \text{cof}(\mathcal{J})$  for sets  $\mathcal{I}$  and  $\mathcal{J}$ . Following J. H. Smith, a model category  $\mathcal{K}$  is called *combinatorial* if it is cofibrantly generated and the category  $\mathcal{K}$  is locally presentable. The model categories from Theorem 2.2 are combinatorial.

Left-determined model categories are, in some sense, related to left Bousfield localizations. Recall that, having model categories  $\mathcal{K}$  and  $\mathcal{L}$ , a *left Quillen functor*  $H : \mathcal{K} \rightarrow \mathcal{L}$  is a left adjoint functor preserving cofibrations and trivial cofibrations (i.e., elements of  $\mathcal{C} \cap \mathcal{W}$ ). Every left Quillen functor preserves weak equivalences between cofibrant objects (see [Ho]). An object  $A$  of a model category  $\mathcal{K}$  is *cofibrant* if  $0 \rightarrow A$  is a cofibration.

A model category  $\mathcal{K}$  is called *functorial* if both weak factorization systems  $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$  and  $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$  are functorial. This means that, for a weak factorization system  $(\mathcal{L}, \mathcal{R})$ , there is a functor  $F : \mathcal{K}^{\rightarrow} \rightarrow \mathcal{K}$  and natural transformations  $\lambda : \text{dom} \rightarrow F$  and  $\varrho : F \rightarrow \text{cod}$  such that  $f = \varrho_f \cdot \lambda_f$  is an  $(\mathcal{L}, \mathcal{R})$ -factorization of a morphism  $f : A \rightarrow B$ ; of course,  $\text{dom}(f) = A$  and  $\text{cod}(f) = B$ . This definition of a functorial weak factorization system is given in [RT] where its relation to functoriality in the sense of Hovey [Ho] is explained. Each combinatorial model category is functorial. In a functorial model category  $\mathcal{K}$  we have a *cofibrant replacement functor*  $Q : \mathcal{K} \rightarrow \mathcal{K}$  where

$$0 \longrightarrow Q(A) \xrightarrow{q_A} A$$

is a functorial weak factorization in  $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ . Then  $q : Q \rightarrow \text{Id}_{\mathcal{K}}$  is a natural transformation.

Let  $\mathcal{K}$  be a model category and  $\mathcal{Z}$  a class of morphisms of  $\mathcal{K}$ . A *left Bousfield localization* of  $\mathcal{K}$  w.r.t.  $\mathcal{Z}$  is a model category structure  $\mathcal{K} \backslash \mathcal{Z}$  on the category  $\mathcal{K}$  such that

- (a)  $\mathcal{K} \backslash \mathcal{Z}$  has the same cofibrations as  $\mathcal{K}$ ,
- (b) weak equivalences of  $\mathcal{K} \backslash \mathcal{Z}$  contain both the weak equivalences of  $\mathcal{K}$  and the morphisms of  $\mathcal{Z}$  and
- (c) each left Quillen functor  $H : \mathcal{K} \rightarrow \mathcal{L}$  such that  $H \cdot Q$  sends  $\mathcal{Z}$ -morphisms to weak equivalences is a left Quillen functor  $\mathcal{K} \backslash \mathcal{Z} \rightarrow \mathcal{L}$

(see [H, 3.3.1]). J. H. Smith proved that if  $\mathcal{K}$  is a left proper combinatorial model category and  $\mathcal{Z}$  is a set of morphisms, then a left Bousfield localization  $\mathcal{K} \backslash \mathcal{Z}$  exists (see [S]). (The model category is called *left proper* if every pushout of a weak

equivalence along a cofibration is a weak equivalence.) As a consequence, we get the following result.

**Theorem 2.3.** *Let  $\mathcal{K}$  be a left proper, combinatorial model category and  $\mathcal{Z}$  a class of morphisms of  $\mathcal{K}$ . Under Vopěnka’s principle, a left Bousfield localization  $\mathcal{K}\backslash\mathcal{Z}$  exists.*

*Proof.* We can express  $\mathcal{Z}$  as a union of an increasing chain of (small) subsets  $\mathcal{Z}_i$  indexed by ordinals. Let  $\mathcal{W}_i$  denote the class of weak equivalences in the model category  $\mathcal{K}\backslash\mathcal{Z}_i$  (which exists by the result of J. Smith). Then we have  $\mathcal{W}_i \subseteq \mathcal{W}_j$  for  $i \leq j$ ; this follows from  $Id_{\mathcal{K}} : \mathcal{K} \rightarrow \mathcal{K}\backslash\mathcal{Z}_j$  being a left Quillen functor sending  $\mathcal{Z}_i$  morphisms to weak equivalences. Hence  $Q = Id_{\mathcal{K}} \cdot Q : \mathcal{K}\backslash\mathcal{Z}_i \rightarrow \mathcal{K}\backslash\mathcal{Z}_j$  is a left Quillen functor. Let  $f : A \rightarrow B$  be a weak equivalence from  $\mathcal{W}_i$  and consider

$$\begin{array}{ccc}
 QA & \xrightarrow{Qf} & QB \\
 \downarrow r_A & & \downarrow r_B \\
 A & \xrightarrow{f} & B
 \end{array}$$

Since  $r_A, r_B \in \mathcal{C}^\square$ , we have  $r_A, r_B, Qf \in \mathcal{W}_j$ . Hence  $f \in \mathcal{W}_j$ . Put  $\mathcal{W}_* = \bigcup_{i \in \text{Ord}} \mathcal{W}_i$ .

Then  $\mathcal{W}_*$  is closed under retracts and satisfies the 2-out-of-3 property. Analogously as in Theorem 2.2, Vopěnka’s principle guarantees that  $\mathcal{K}, \mathcal{C}$  and  $\mathcal{W}_*$  form a model category  $\mathcal{K}\backslash\mathcal{Z}$ .

Let  $H : \mathcal{K} \rightarrow \mathcal{L}$  be a left Quillen functor such that  $H \cdot Q$  sends  $\mathcal{Z}$ -morphisms to weak equivalences. Since  $H : \mathcal{K}\backslash\mathcal{Z} \rightarrow \mathcal{L}$  is a left Quillen functor for each  $i$ ,  $H : \mathcal{K}\backslash\mathcal{Z}_i \rightarrow \mathcal{L}$  is a left Quillen functor. Hence  $\mathcal{K}\backslash\mathcal{Z}$  is a left Bousfield localization of  $\mathcal{K}$  w.r.t.  $\mathcal{Z}$ . □

Using [AR] as well, C. Casacuberta, D. Sceveneles and J. H. Smith [CSS] proved a related result saying that cohomological localizations of simplicial sets exist under Vopěnka’s Principle.

*Remark 2.4.* In analogy with the definition of a left-determined model category, we define  $\mathcal{W}_{\mathcal{X}}$ , where  $\mathcal{X}$  is a class of morphisms in a model category  $\mathcal{K}$ , as the smallest class of morphisms satisfying

- (i)  $\mathcal{W} \cup \mathcal{X} \subseteq \mathcal{W}_{\mathcal{X}}$ ,
- (ii)  $\mathcal{W}_{\mathcal{X}}$  is closed under retracts and satisfies the 2-out-of-3 property,
- (iii)  $\mathcal{C} \cap \mathcal{W}_{\mathcal{X}}$  is stable under pushout and closed under transfinite composition.

Under Vopěnka’s principle,  $\mathcal{K}, \mathcal{C}$  and  $\mathcal{W}_{\mathcal{X}}$  form a model category structure for each combinatorial model category  $\mathcal{K}$ . This model category structure is evidently  $\mathcal{K}\backslash\mathcal{X}$ , provided that  $\mathcal{K}\backslash\mathcal{X}$  exists (because  $\mathcal{W}_{\mathcal{X}}$  is contained in the class of weak equivalences of  $\mathcal{K}\backslash\mathcal{X}$ ).

Let  $\mathcal{K}$  be a cofibrantly generated model category and  $\mathcal{X}$  a small category. Then there is a cofibrantly generated model category structure on the functor category  $\mathcal{K}^{\mathcal{X}^{\text{op}}}$  (see [H, 14.2.1]); this structure is called the *Bousfield-Kan structure*. To recall it we denote by

$$ev_{\mathcal{X}} : \mathcal{K}^{\mathcal{X}^{\text{op}}} \rightarrow \mathcal{K}$$

the evaluation functor given by  $ev_X(A) = A(X)$  and by

$$F_X : \mathcal{K} \rightarrow \mathcal{K}^{\mathcal{X}^{op}}$$

its left adjoint given by

$$F_X(K)(Y) = \coprod_{\mathcal{X}^{op}(X,Y)} K.$$

If  $\mathcal{I}$  (resp.,  $\mathcal{J}$ ) is the set of generating (resp., trivial) cofibrations in  $\mathcal{K}$ , then the Bousfield-Kan model structure has  $\overline{\mathcal{I}} = \bigcup_{X \in \text{ob}(\mathcal{X})} F_X(\mathcal{I})$  as generating cofibrations

and  $\overline{\mathcal{J}} = \bigcup_{X \in \text{ob}(\mathcal{X})} F_X(\mathcal{J})$  as generating trivial cofibrations. Then

- (a)  $\varphi : A \rightarrow B$  is a weak equivalence in  $\mathcal{K}^{\mathcal{X}^{op}}$  iff  $\varphi_X : A(X) \rightarrow B(X)$  is a weak equivalence in  $\mathcal{K}$  for each  $X$  in  $\mathcal{X}$ ,
- (b)  $\varphi : A \rightarrow B$  is a fibration in  $\mathcal{K}^{\mathcal{X}^{op}}$  iff  $\varphi_X : A(X) \rightarrow B(X)$  is a fibration in  $\mathcal{K}$  for each  $X$  in  $\mathcal{X}$ .

Consequently, trivial fibrations are also morphisms in  $\mathcal{K}^{\mathcal{X}^{op}}$ , which are pointwise trivial fibrations in  $\mathcal{K}$ .

**Proposition 2.5.** *Let  $\mathcal{K}$  be a cofibrantly generated, left-determined model category and  $\mathcal{X}$  a small category. Then  $\mathcal{K}^{\mathcal{X}^{op}}$  is a left-determined model category.*

*Proof.* Let  $\mathcal{I}$  (resp.,  $\mathcal{J}$ ) be the set of generating (resp., trivial) cofibrations in  $\mathcal{K}$ , and let  $\overline{\mathcal{W}}$  be the set of weak equivalences in  $\mathcal{K}^{\mathcal{X}^{op}}$ . We have  $\mathcal{W}_{\text{cof}(\overline{\mathcal{I}})} \subseteq \overline{\mathcal{W}}$ . Let  $w \in \overline{\mathcal{W}}$ . Then  $w = f \cdot g$  where  $f \in \overline{\mathcal{I}}^{\square}$  and  $g \in \text{cof}(\overline{\mathcal{J}})$ . We have  $f \in \mathcal{W}_{\text{cof}(\overline{\mathcal{I}})}$ . To prove that  $g \in \mathcal{W}_{\text{cof}(\overline{\mathcal{I}})}$ , that is,  $w \in \mathcal{W}_{\text{cof}(\overline{\mathcal{I}})}$ , it suffices to show that  $\overline{\mathcal{J}} \subseteq \mathcal{W}_{\text{cof}(\overline{\mathcal{I}})}$ . But this follows from  $\mathcal{J} \subseteq \mathcal{W}_{\text{cof}(\mathcal{I})}$  and the fact that  $F_X$  preserve colimits.  $\square$

### 3. SYMMETRIC SIMPLICIAL SETS

A trivial example of a left-determined model category is the category **Set** of sets, with  $\mathcal{C}$  the class of all monomorphisms, and with  $\mathcal{W}$  the class of the morphisms between nonempty sets and the identity morphism on  $\emptyset$ . To give a nontrivial example we recall that a *simplicial complex* is a set  $X$  equipped with a set  $\mathcal{X}$  of nonempty finite subsets of  $X$  such that

- (a)  $\{x\} \in \mathcal{X}$  for each  $x \in X$ ,
- (b)  $A \in \mathcal{X}, \emptyset \neq B \subseteq A \Rightarrow B \in \mathcal{X}$ .

Elements  $A \in \mathcal{X}$  with  $|A| = n + 1$  are called (non-degenerated) *n-simplices*. For  $n = 0, 1$  and  $2$  we speak about *vertices*, *edges* and *triangles*, respectively. If  $|A| \leq 2$  for each  $A \in \mathcal{X}$ , then  $(X, \mathcal{X})$  is called a (non-oriented) *graph* (with loops). Morphisms of complexes  $(X, \mathcal{X}) \rightarrow (Y, \mathcal{Y})$  are maps  $h : X \rightarrow Y$  with  $h(\mathcal{X}) \subseteq \mathcal{Y}$ . We denote the category of simplicial complexes by **SComp**. We will show that  $\mathcal{C} = \text{Mono}$  yields a left-determined model category structure on **SComp**. But the disadvantage of simplicial complexes is that each simplex is uniquely determined by its vertices, which makes colimits in **SComp** bad. This led S. Eilenberg and B. Zilber [EZ] to introduce complete semisimplicial complexes, which later were renamed as simplicial sets, and which are oriented. Surprisingly, non-oriented simplicial sets were introduced only recently by F. W. Lawvere [L] and M. Grandis [G]; they are called symmetric simplicial sets. Their position to simplicial complexes

is the same as the position of multigraphs to graphs in graph theory (one admits multiple edges).

**Definition 3.1.** Let  $\mathbf{F}$  denote the category of nonzero finite cardinals (and all maps). A *symmetric simplicial set* is by definition a functor  $\mathbf{F}^{\text{op}} \rightarrow \mathbf{Set}$ . The category  $\mathbf{Set}^{\mathbf{F}^{\text{op}}}$  of symmetric simplicial sets will be denoted by  $\mathbf{SSimp}$ .

We will recall the basic properties of symmetric simplicial sets (see [G]). We have the Yoneda embedding

$$Y : \mathbf{F} \rightarrow \mathbf{SSimp}.$$

Its values  $\Delta_{n-1} = Y(n)$  are in fact simplicial complexes, which yields the functor

$$\mathbf{F} \rightarrow \mathbf{SComp}.$$

The Yoneda embedding  $Y$  extends along this functor to the full embedding

$$G : \mathbf{SComp} \rightarrow \mathbf{SSimp}$$

sending a simplicial complex  $(X, \mathcal{X})$  to the functor  $A : \mathbf{F}^{\text{op}} \rightarrow \mathbf{Set}$  given by  $A(n) = \{S \in \mathcal{X} \mid |S| \leq n-1\}$ . In what follows we will identify a simplicial complex  $(X, \mathcal{X})$  with its image under  $G$ . Hence  $\mathbf{SComp}$  will be considered as a full subcategory of  $\mathbf{SSimp}$ .

We have the functor  $U : \mathbf{SSimp} \rightarrow \mathbf{Set}$  given by precomposition with  $\mathbf{1} \rightarrow \mathbf{F}^{\text{op}}$  (sending the object of  $\mathbf{1}$  to 1). We can view  $U(A)$  as the set of vertices of a symmetric simplicial set  $A$  and the whole  $A$  as the set  $U(A)$  equipped with  $n$ -simplices corresponding to morphisms  $\Delta_n \rightarrow A$ . We will use the notation  $A = (UA, \mathcal{A})$  where  $\mathcal{A}$  is the set of simplices of  $A$ . For instance,  $\Delta_n$  has all nonempty subsets of  $\{0, 1, \dots, n\}$  as simplices. But note that the functor  $U$  is not faithful.

The embedding  $\Delta \rightarrow \mathbf{F}$  induces the faithful functor

$$H : \mathbf{SSimp} \rightarrow \mathbf{Simp}.$$

It has a left adjoint

$$L : \mathbf{Simp} \rightarrow \mathbf{SSimp}$$

sending each simplicial set to its symmetrization. There is also a right adjoint

$$R : \mathbf{Simp} \rightarrow \mathbf{SSimp}.$$

Let  $\partial\Delta_n$  be the boundary of  $\Delta_n$  for  $n > 0$ , i.e.,  $U(\partial\Delta_n) = n+1$ , and simplices of  $\partial\Delta_n$  are all nonempty subsets of  $\{0, 1, \dots, n\}$  distinct from  $\{0, 1, \dots, n\}$ . Let  $i_n : \partial\Delta_n \rightarrow \Delta_n$ ,  $n > 0$ , be the embeddings. Let  $\mathcal{I} = \{i_n \mid n \geq 0\}$  where

$$i_0 : 0 \rightarrow \Delta_0.$$

In what follows, the class of all monomorphisms of  $\mathbf{SSimp}$  is denoted by  $Mono$ .

**Lemma 3.2.**  $\text{cof}(\mathcal{I}) = Mono$ .

The proof is the same as for simplicial sets.

Given  $0 \leq k \leq n$ , the *k-horn*  $\Delta_n^k$  is the simplicial complex whose simplices are all subsets  $\emptyset \neq S \subsetneq \{0, 1, \dots, n\}$  distinct from  $\{0, \dots, k-1, k+1, \dots, n\}$ . Let  $\mathcal{J}$  be the set of inclusions

$$j_n : \Delta_n^0 \rightarrow \Delta_n, \quad n > 0.$$

**Lemma 3.3.**  $j_n \in \mathcal{W}_{Mono} \cap Mono$  for each  $n > 0$ .

*Proof.* Evidently, each morphism  $s_n : \Delta_n \rightarrow \Delta_0$ ,  $n \geq 0$ , belongs to  $Mono^\square$ . Therefore, by 2.1 (i) and (ii), each morphism  $u : \Delta_0 \rightarrow \Delta_n$  belongs to  $\mathcal{W}_{Mono}$  and, consequently, to  $\mathcal{W}_{Mono} \cap Mono$ . Hence  $j_1 \in \mathcal{W}_{Mono} \cap Mono$ .

Assume that  $j_1, \dots, j_n \in \mathcal{W}_{Mono} \cap Mono$ . Consider the pushout

$$\begin{array}{ccc} \Delta_n + \Delta_0 & \xrightarrow{p_n} & \Delta_n \\ \text{id}_{\Delta_n} + u_1^0 \downarrow & & \downarrow g_n \\ \Delta_n + \Delta_1 & \longrightarrow & P_n \end{array}$$

where  $u_1^0(0) = 0$  and  $p_n$  is induced by  $\text{id}_{\Delta_n}$  and  $u_n^0 : \Delta_0 \rightarrow \Delta_n$  (again  $u_n^0(0) = 0$ ). Since  $\text{id}_{\Delta_n} + u_1^0 \in \mathcal{W}_{Mono} \cap Mono$ , we have  $g_n \in \mathcal{W}_{Mono} \cap Mono$  (by 2.1 (iii)).  $P_n$  is the simplicial complex given by attaching an edge at the vertex 0. By successive use of  $j_2, \dots, j_n$ , we fill horns to simplices. This is done via pushouts, starting with

$$\begin{array}{ccc} \Delta_2^0 & \xrightarrow{h} & P_n \\ j_2 \downarrow & & \downarrow \\ \Delta_2 & \longrightarrow & P'_n \end{array}$$

where  $h$  sends one edge of  $\Delta_2^0$  to the attached edge and the other edge to an edge of  $\Delta_n$ . Doing this for all edges of  $\Delta_n$  containing 0, we start to fill by using  $j_3$ , etc. At the end we obtain  $\Delta_{n+1}^0$  and a morphism

$$q_n : \Delta_n \xrightarrow{g_n} P_n \longrightarrow P'_n \longrightarrow \dots \longrightarrow \Delta_{n+1}^0$$

which belongs to  $\mathcal{W}_{Mono} \cap Mono$ . Since, in the diagram

$$\begin{array}{ccc} \Delta_n & \xrightarrow{q_n} & \Delta_{n+1}^0 \\ & \searrow s_n & \swarrow t_n \\ & \Delta_0 & \end{array}$$

we have  $s_n \in \mathcal{W}_{Mono}$ , we get  $t_n \in \mathcal{W}_{Mono}$ . Since, in the diagram

$$\begin{array}{ccc} \Delta_{n+1}^0 & \xrightarrow{j_{n+1}} & \Delta_{n+1} \\ & \searrow t_n & \swarrow s_{n+1} \\ & \Delta_0 & \end{array}$$

we have  $t_n, s_{n+1} \in \mathcal{W}_{Mono}$ , we get  $j_{n+1} \in \mathcal{W}_{Mono}$ . Hence  $j_{n+1} \in \mathcal{W}_{Mono} \cap Mono$ .  $\square$

**Theorem 3.4.** **SSimp** is a left-determined model category with  $\mathcal{C} = \text{Mono}$  and  $\mathcal{F} = \mathcal{J}^\square$ .

*Proof.*  $(\text{Mono}, \text{Mono}^\square)$  is a weak factorization system (see [B] or [AHRT2]); analogously for  $(\text{cof}(\mathcal{J}), \mathcal{J}^\square)$ . To prove the result it suffices to show that

$$\mathcal{W}_{\text{Mono}} \cap \text{Mono} = \text{cof}(\mathcal{J})$$

(cf. [B]). Following Lemma 3.3, we have

$$\text{cof}(\mathcal{J}) \subseteq \mathcal{W}_{\text{Mono}} \cap \text{Mono}.$$

The opposite inclusion will follow from properties of the adjunction  $L \dashv H$  between symmetric simplicial sets and simplicial sets.

Since  $L$  preserves monomorphisms,  $H$  preserves trivial fibrations, i.e.,

$$H(\text{Mono}^\square) \subseteq \mathcal{W},$$

where  $\mathcal{W}$  denotes the class of weak equivalences of simplicial sets. Since  $H$  preserves monomorphisms as well, we have

$$H(\mathcal{W}_{\text{Mono}} \cap \text{Mono}) \subseteq \mathcal{W} \cap \text{Mono}^* = \text{cof}(\mathcal{J}^*)$$

where  $\text{Mono}^*$  denotes the monomorphisms in **Simp** and  $\mathcal{J}^*$  is the generating set of horns in simplicial sets (cf. [Ho]). Since  $L(\mathcal{J}^*) = \mathcal{J}$ , we have  $L(\text{cof}(\mathcal{J}^*)) \subseteq \text{cof}(\mathcal{J})$ . Consequently,

$$LH(\mathcal{W}_{\text{Mono}} \cap \text{Mono}) \subseteq \text{cof}(\mathcal{J}).$$

The functor  $H$  sends a symmetric simplicial set  $A = (UA, \mathcal{A})$  to the simplicial set  $HA$  having as (oriented) simplices all possible orientations of simplices from  $\mathcal{A}$ . The functor  $L$  then produces from each orientation a non-oriented simplex in  $LHA$ . Hence  $LH$  multiplies each non-degenerated  $n$ -simplex in  $A$   $n!$  times. By sending each simplex in  $\mathcal{A}$  to its standard orientation, we get a natural transformation  $\varrho : Id \rightarrow LH$  that splits the adjunction counit  $\varepsilon : LH \rightarrow Id$ . Hence each morphism  $f$  in **SSimp** is a retract of  $LH(f)$ . Consequently,

$$\mathcal{W}_{\text{Mono}} \cap \text{Mono} \subseteq \text{cof}(\mathcal{J}).$$

□

*Remark 3.5.* Both  $L$  and  $H$  are left Quillen functors. Moreover,  $L \dashv H$  is a Quillen equivalence. Following [Ho, 1.3.13], this amounts to showing that

$$X \xrightarrow{\eta_X} H LX \xrightarrow{Hr_{LX}} H(LX)_f$$

is a weak equivalence for each simplicial set  $X$  (where  $\eta$  is the adjunction unit and  $r_{LX} : LX \rightarrow (LX)_f$  is a fibrant replacement) and that

$$\varepsilon_Y : LHY \rightarrow Y$$

is a weak equivalence for each fibrant symmetric simplicial set  $Y$ . But  $\eta_X$  is a trivial cofibration because it is given by completing horns to simplices, and  $Hr_{LX}$  is a trivial cofibration too, because  $H$  is a left Quillen functor. That  $\varepsilon_Y$  is a trivial fibration follows from its description given in the proof above.

As a consequence we obtain that **Simp** and **SSimp** have equivalent homotopy categories.

*Remark 3.6.* The model category **Simp** is not left-determined. To prove this we consider the class  $\mathcal{X}$  of morphisms  $f : A \rightarrow B$  such that one of the following possibilities occurs ( $U^*$  denotes the underlying functor **Simp**  $\rightarrow$  **Set**):



- (a) there are vertices  $b_1 \in U^*B$ ,  $b_2 \in U^*B - (U^*f)(U^*A)$ , an edge  $e$  in  $B$  from  $b_1$  to  $b_2$ , but no edge in  $B$  from  $b_2$  to  $b_1$ ;
- (b) there are vertices  $a_1, a_2 \in U^*A$  and an edge  $e$  in  $A$  from  $a_1$  to  $a_2$  such that there is no edge in  $A$  from  $a_2$  to  $a_1$ , but there is an edge in  $B$  from  $U^*(f)(a_2)$  to  $U^*(f)(a_1)$ ;
- (c) there are vertices  $a_1, a_2 \in U^*A$  and an edge  $e$  in  $B$  from  $U^*(f)(a_1)$  to  $U^*(f)(a_2)$ , but there is no edge in  $B$  from  $U^*(f)(a_2)$  to  $U^*(f)(a_1)$  and no edge in  $A$  from  $a_1$  to  $a_2$ .

Since (the oriented) horn  $j_1^* : \Delta_1^0 \rightarrow \Delta_1$  belongs to  $\mathcal{X}$ , it suffices to show that  $\mathcal{X} \cap \mathcal{W}_{Mono^*} = \emptyset$ . But  $\mathcal{X} \cap Mono^\square = \emptyset$  and no element of  $\mathcal{X}$  can arise by operations 2.1 (ii) and (iii) from morphisms not belonging to  $\mathcal{X}$ .

*Remark 3.7.* **SComp** is a left-determined model category with  $\mathcal{C} = Mono$  and  $\mathcal{F} = \mathcal{J}^\square$ . In fact, both  $i_n$ ,  $n \geq 0$  and  $j_n$ ,  $n > 0$  are morphisms of simplicial complexes. Hence the result follows from Theorem 3.4.

**Corollary 3.8.** *For each small category  $\mathcal{X}$  the functor category  $\mathbf{SSimp}^{\mathcal{X}}$  is a left-determined model category (with the Bousfield-Kan model category structure).*

The proof follows from Theorem 3.4 and Proposition 2.5.

#### 4. UNIVERSAL MODEL CATEGORIES

We will show that  $\mathbf{SSimp}^{\mathcal{X}^{op}}$  is a universal model category over  $\mathcal{X}$  in the sense of D. Dugger [D] for each small category  $\mathcal{X}$ . In particular, **SSimp** is a universal model category over the one-morphism category. We will denote by

$$Y^* : \mathcal{X} \longrightarrow \mathbf{SSimp}^{\mathcal{X}^{op}}$$

the composition

$$\mathcal{X} \xrightarrow{Y_{\mathcal{X}}} \mathbf{Set}^{\mathcal{X}^{op}} \xrightarrow{D_{\mathcal{X}}} (\mathbf{Set}^{\mathcal{X}^{op}})^{\mathbf{F}^{op}}$$

where  $D_{\mathcal{X}}$  is a left adjoint to the underlying functor

$$U_{\mathcal{X}} : (\mathbf{Set}^{\mathcal{X}^{op}})^{\mathbf{F}^{op}} \longrightarrow \mathbf{Set}^{\mathcal{X}^{op}}$$

given by evaluation at 1, i.e.,  $U_{\mathcal{X}} = ev_1$ . Of course, we use the identifications

$$(\mathbf{Set}^{\mathbf{F}^{op}})^{\mathcal{X}^{op}} \cong \mathbf{Set}^{(\mathbf{F} \times \mathcal{X})^{op}} \cong (\mathbf{Set}^{\mathcal{X}^{op}})^{\mathbf{F}^{op}}.$$

Objects of  $\mathbf{SSimp}^{\mathcal{X}^{op}}$  may be called symmetric simplicial presheaves; then  $D_{\mathcal{X}}(A)$  is the discrete symmetric simplicial presheaf over  $A$ . We also have the Yoneda embedding

$$\bar{Y} : \mathbf{F} \times \mathcal{X} \longrightarrow \mathbf{SSimp}^{\mathcal{X}^{op}},$$

and we will use the notation

$$\Delta_{n,X} = \bar{Y}(n+1, X)$$

for  $n \geq 0$  and  $X \in \mathcal{X}$ . It is easy to see that

$$\Delta_{n,X} = F_X(\Delta_n)$$

where  $F_X : \mathbf{SSimp} \longrightarrow \mathbf{SSimp}^{\mathcal{X}^{op}}$  is a left adjoint to the evaluation functor  $ev_X$ . We will also denote

$$\partial\Delta_{n,X} = F_X(\partial\Delta_n).$$

**Theorem 4.1.** *Let  $\mathcal{K}$  be a functorial model category,  $\mathcal{X}$  a small category and  $H : \mathcal{X} \rightarrow \mathcal{K}$  a functor such that all objects  $HX, X \in \mathcal{X}$  are cofibrant. Then there is a left Quillen functor  $H^* : \mathbf{SSimp}^{\mathcal{X}^{\text{op}}} \rightarrow \mathcal{K}$  such that  $H^* \cdot Y^* = H$ .*

*Proof.* Let  $\mathbf{F}_n$  be the full subcategory of  $\mathbf{F}$  consisting of cardinals  $0 < k \leq n + 1$ . We get the induced inclusions

$$\mathbf{SSimp}_n^{\mathcal{X}^{\text{op}}} \subseteq \mathbf{SSimp}^{\mathcal{X}^{\text{op}}}$$

where  $\mathbf{SSimp}_n = \mathbf{Set}^{\mathbf{F}_n^{\text{op}}} (\mathbf{SSimp}_n^{\mathcal{X}^{\text{op}}} \hookrightarrow \mathbf{SSimp}^{\mathcal{X}^{\text{op}}}$  is given by  $\mathbf{SSimp}_n \hookrightarrow \mathbf{SSimp}$ , which is induced by the functor  $\mathbf{F}_n \hookrightarrow \mathbf{F} \xrightarrow{Y} \mathbf{SSimp}$ ). In particular,  $\mathbf{SSimp}_0^{\mathcal{X}^{\text{op}}} \cong \mathbf{Set}^{\mathcal{X}^{\text{op}}}$  is the category of discrete symmetric simplicial presheaves. Since  $\mathcal{K}$  is cocomplete and  $\mathbf{Set}^{\mathcal{X}^{\text{op}}}$  is a free cocompletion of  $\mathcal{X}$  (see [AR, 1.45]),  $H$  extends to a colimit preserving functor

$$H_0^* : \mathbf{SSimp}_0^{\mathcal{X}^{\text{op}}} \rightarrow \mathcal{K}$$

such that  $H_0^* Y = H$ .

Assume that we have the functor

$$H_n^* : \mathbf{SSimp}_n^{\mathcal{X}^{\text{op}}} \rightarrow \mathcal{K}$$

extending  $H_{n-1}^*$ . Since  $\partial\Delta_{n+1,X}, \Delta_{0,X}$  belong to  $\mathbf{SSimp}_n^{\mathcal{X}^{\text{op}}}$  for  $X \in \mathcal{X}$ , we can define  $H_{n+1}^*(\Delta_{n+1,X})$  by the functorial (cofibration, trivial fibration) factorization

$$H_n^*(\partial\Delta_{n+1,X}) \xrightarrow{c_{n+1,X}} H_{n+1}^*(\Delta_{n+1,X}) \xrightarrow{r_{n+1,X}} H_n^*(\Delta_{0,X})$$

of the morphism  $H_n^*(F_X(p_n)) : H_n^*(\partial\Delta_{n+1,X}) \rightarrow H_n^*(\Delta_{0,X})$  where  $p_{n+1} : \partial\Delta_{n+1} \rightarrow \Delta_0$ . To get an extension  $H_{n+1}^*$  of  $H_n^*$ , below we will define

- (a)  $H_{n+1}^*(f)$  for  $f = \overline{Y}(f_1, f_2) : \Delta_{n+1,X} \rightarrow \Delta_{n+1,Y}$  where  $f_1 : \{0, 1, \dots, n + 1\} \rightarrow \{0, 1, \dots, n + 1\}$  is a bijection,
- (b)  $H_{n+1}^*(t)$  for  $t = \overline{Y}(t_1, t_2) : \Delta_{m,X} \rightarrow \Delta_{n+1,Y}$  where  $m \leq n$ ,

and

- (c)  $H_{n+1}^*$  for  $u = \overline{Y}(u_1, u_2) : \Delta_{n+1,X} \rightarrow \Delta_{m,Y}$  where  $m \leq n$ .

(a)  $f_1$  induces the isomorphisms  $\overline{f}_1 : \Delta_{n+1} \rightarrow \Delta_{n+1}, \partial\overline{f}_1 : \partial\Delta_{n+1} \rightarrow \partial\Delta_{n+1}$  and  $f_2$  induces a natural transformation  $\varphi_{f_2} : F_X \rightarrow F_Y$ . Hence  $f$  induces the homomorphism  $\partial f : \partial\Delta_{n+1,X} \rightarrow \partial\Delta_{n+1,Y}$ . We define  $H_{n+1}^*(f)$  by the functorial filling

$$\begin{array}{ccccc} H_n^*(\partial\Delta_{n+1,X}) & \xrightarrow{c_{n+1,X}} & H_{n+1}^*(\Delta_{n+1,X}) & \xrightarrow{r_{n+1,X}} & H_n^*(\Delta_{0,X}) \\ \downarrow H_n^*(\partial f) & & \downarrow H_{n+1}^*(f) & & \downarrow H_n^*(\varphi_{f_2})_{\Delta_0} \\ H_n^*(\partial\Delta_{n+1,Y}) & \xrightarrow{c_{n+1,Y}} & H_{n+1}^*(\Delta_{n+1,Y}) & \xrightarrow{r_{n+1,Y}} & H_n^*(\Delta_{0,Y}) \end{array}$$

- (b) Since  $t$  factorizes through  $i_{n+1,Y}$ , i.e.,

$$t : \Delta_{m,X} \xrightarrow{t'} \partial\Delta_{n+1,Y} \xrightarrow{i_{n+1,Y}} \Delta_{n+1,Y},$$

we put  $H_{n+1}^*(t) = c_{n+1,Y} \cdot H_n^*(t')$ .

(c) To define  $H_{n+1}^*(u)$  for each  $u$ , it suffices to do this for the retraction  $u^0$  of  $\Delta_{n+1,X}$  to one of its faces  $\Delta_{n,X}$ . In this case, we take  $H_{n+1}^*(u^0)$  given by the lifting property

$$\begin{array}{ccc}
 H_n^*(\partial\Delta_{n+1,X}) & \xrightarrow{c_{n+1,X}} & H_{n+1}^*(\Delta_{n+1,X}) \\
 \downarrow H_n^*(u^0 \cdot i_{n+1,X}) & \nearrow H_{n+1}^*(u^0) & \downarrow r_{n+1,X} \\
 H_n^*(\Delta_{n,X}) & \xrightarrow{H_n^*(s_{n,X})} & H_n^*(\Delta_{0,X})
 \end{array}$$

where  $s_{n,X} = \overline{Y}(s, \text{id}_X)$  and  $s : n + 1 \rightarrow 1$ .

To prove that  $H_{n+1}^*$  is a functor, it suffices to consider the following cases:

(1) In

$$\begin{array}{ccccc}
 & & H_n^*(\Delta_{m,X}) & & \\
 & & \downarrow H_{n+1}^*(t) & & \\
 & H_n^*(t') & \swarrow & & \\
 H_n^*(\partial\Delta_{n+1,Y}) & \xrightarrow{c_{n+1,Y}} & H_{n+1}^*(\Delta_{n+1,Y}) & \xrightarrow{r_{n+1,Y}} & H_n^*(\Delta_{0,Y}) \\
 \downarrow H_n^*(\partial f) & & \downarrow H_{n+1}^*(f) & & \downarrow H_n^*((\varphi_{f_2})_{\Delta_0}) \\
 H_n^*(\partial\Delta_{n+1,Z}) & \xrightarrow{c_{n+1,Z}} & H_{n+1}^*(\Delta_{n+1,Z}) & \xrightarrow{r_{n+1,Z}} & H_n^*(\Delta_{0,Z})
 \end{array}$$

we have  $(ft)' = \partial f \cdot t'$  and thus  $H_{n+1}^*(f) \cdot H_{n+1}^*(t) = H_{n+1}^*(ft)$ .

(2) In

$$\begin{array}{ccc}
 & & H_n^*(\Delta_{m,X}) \\
 & & \downarrow H_{n+1}^*(t) \\
 & H_n^*(t') & \swarrow \\
 H_n^*(\partial\Delta_{n+1,Y}) & \xrightarrow{c_{n+1,Y}} & H_{n+1}^*(\Delta_{n+1,Y}) \\
 \downarrow H_n^*(u^0 \cdot i_{n+1,Y}) & \nearrow H_{n+1}^*(u^0) & \downarrow r_{n+1,Y} \\
 H_n^*(\Delta_{n,Y}) & \xrightarrow{H_n^*(s_{n,Y})} & H_n^*(\Delta_{0,Y})
 \end{array}$$

we have  $u^0 \cdot t = u^0 \cdot i_{n+1,Y} \cdot t'$  and thus  $H_{n+1}^*(u^0) \cdot H_{n+1}^*(t) = H_{n+1}^*(u^0 \cdot t)$ .

We have defined  $H_{n+1}^*$  on the image of the Yoneda embedding

$$\bar{Y}_n : \mathbf{F}_n \times \mathcal{X} \rightarrow \mathbf{Set}^{(\mathbf{F}_n \times \mathcal{X})^{op}} \cong \mathbf{SSimp}_n^{\mathcal{X}^{op}}.$$

Since this is a free cocompletion of  $\mathbf{F}_n \times \mathcal{X}$ , we obtain a colimit-preserving functor  $H_{n+1}^* : \mathbf{SSimp}_n^{\mathcal{X}^{op}} \rightarrow \mathcal{K}$  extending  $H_n^*$ .

We have constructed an increasing chain of colimit-preserving functors  $H_n^* : \mathbf{SSimp}_n^{\mathcal{X}^{op}} \rightarrow \mathcal{K}$  for  $n = 0, 1, \dots$ , which yields a colimit-preserving functor

$$H^* : \mathbf{SSimp}^{\mathcal{X}^{op}} \rightarrow \mathcal{K}$$

with the restriction  $H_n^*$  on  $\mathbf{SSimp}_n^{\mathcal{X}^{op}}$ . Moreover,  $H^*$  is a left adjoint functor (see (a) in the proof of 1.45 in [AR]). In particular,  $H^*Y^* = H_0^*Y^* = H$ . It remains to be proved that  $H^*$  preserves cofibrations and trivial cofibrations. Since  $\partial\Delta_{n+1,X}$  is a colimit of  $\partial_{m,X}$ ,  $m \leq n$ , by (b) of the construction we get that  $H^*(i_{n+1,X}) = c_{n+1,X}$ . Hence  $H^*$  preserves cofibrations (using Lemma 3.2 and the fact that  $H^*$  preserves colimits). Since  $s_{n+1,X} = s_{n,X} \cdot u^0$ , by (c) of the construction we get that  $H^*(s_{n+1,X}) = r_{n+1,X}$ . Hence  $H^*(s_{n+1,X})$  is a weak equivalence in  $\mathcal{K}$  for  $n > 0$  and  $X \in \mathcal{X}$ . Since  $s_{m,X} \cdot f = s_{n,Y}$  for each morphism  $f : \Delta_{m,X} \rightarrow \Delta_{m,Y}$ ,  $H^*(f)$  is a weak equivalence in  $\mathcal{K}$  as well. In particular,  $H^*F_X(j_1) : H^*F_X(\Delta_1^0) = H^*\Delta_{0,X} \rightarrow H^*\Delta_{1,X}$  is a weak equivalence. Since it is also a cofibration, it is a trivial cofibration. Assume that  $H^*F_X(j_1), \dots, H^*F_X(j_n)$  are trivial cofibrations in  $\mathcal{K}$ . In the notation of Lemma 3.3, we get that  $H^*F_X(g_n)$  is a trivial cofibration in  $\mathcal{K}$  because  $u_1^0 = j_1$ . Since by assumption  $H^*F_X(j_1), \dots, H^*F_X(j_n)$  are trivial cofibrations,  $H^*F_X(q_n)$  is a trivial cofibration too. Therefore  $H^*F_X(t_n)$  is a weak equivalence in  $\mathcal{K}$  and thus  $H^*F_X(j_{n+1})$  is a weak equivalence too. Since it is a cofibration, it is a trivial cofibration.  $\square$

*Remark 4.2.* Let  $\mathcal{K}$  be a functorial model category,  $\mathcal{X}$  a small category and  $H : \mathcal{X} \rightarrow \mathcal{K}$  an arbitrary functor. Then there is a left Quillen functor  $H^* : \mathbf{SSimp}^{\mathcal{X}^{op}} \rightarrow \mathcal{K}$  and a natural transformation  $\gamma : H^*Y^* \rightarrow H$  which is a pointwise trivial fibration in  $\mathcal{K}$ .

In fact, let  $H_0 : \mathcal{X} \rightarrow \mathcal{K}$  be the composition  $Q \cdot H$  where  $Q : \mathcal{K} \rightarrow \mathcal{K}$  is the (functorial) cofibrant replacement functor and  $\gamma = qH : H_0 \rightarrow H$  the corresponding natural transformation. Then  $\gamma$  is a pointwise trivial fibration in  $\mathcal{K}$  (because  $q : Q \rightarrow Id_{\mathcal{K}}$  is). If we start the construction of the functor  $H^*$  with  $H_0$  (i.e.,  $H_0^* \cdot Y^* = H_0$ ), we get the result.

**Corollary 4.3.** *Let  $\mathcal{K}$  be a functorial model category and  $K$  an object in  $\mathcal{K}$ . Then there is a left Quillen functor  $H^* : \mathbf{SSimp} \rightarrow \mathcal{K}$  and a trivial fibration  $\gamma : H^*(\Delta_0) \rightarrow K$ .*

This is a special case of Theorem 4.1 (for  $\mathcal{X}$  a one-morphism category).

*Remark 4.4.* Again if  $K$  is cofibrant, then  $H^*(\Delta_0) = K$ .

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