TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 355, Number 9, Pages 3825–3838
S 0002-9947(03)03345-2
Article electronically published on May 29, 2003

SELF-INTERSECTION CLASS FOR SINGULARITIES AND ITS APPLICATION TO FOLD MAPS

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Dedicated to Professor Takuo Fukuda on the occasion of his 60th birthday

ABSTRACT. Let $f : M \to N$ be a generic smooth map with corank one singularities between manifolds, and let $S(f)$ be the singular point set of f. We define the self-intersection class $I(S(f)) \in H^*(M; \mathbf{Z})$ of $S(f)$ using an incident class introduced by Rimányi but with twisted coefficients, and give a formula for $I(S(f))$ in terms of characteristic classes of the manifolds. We then apply the formula to the existence problem of fold maps.

1. INTRODUCTION

Given a smooth map $g : M \to N$ between smooth manifolds, does there exist a smooth map homotopic to g that has at most "nice" singularities? If not, then what is the obstruction?

Embeddings and immersions, which have no singularities at all, have been particularly well studied since the concept of a manifold was established around 1930. For example, Whitney proved that every n-dimensional manifold can be embedded in $\mathbf{R}^{2n}$ and immersed in $\mathbf{R}^{2n-1}$. Furthermore, the Smale–Hirsch theory [18] gives a satisfactory answer to the existence problem of immersions in terms of homotopy theory. Note that these results make sense only when $n = \dim M < \dim N = p$.

On the other hand, similar problems for the case $n \geq p$ seem to have been hardly studied except for maps of open manifolds (for example, see [22]). If M is closed and $n \geq p$, then every smooth map $f : M \to N$ into an open manifold N must have singularities. In this paper, we consider *fold maps* — smooth maps that have at most fold singularities (for details, see §4) — for such cases, and discuss their (non)existence problem. Note that a fold singularity is the simplest of the generic singularities: for example, a fold map for $p = 1$ and $N = \mathbf{R}$ is nothing but a Morse function.

When $p = 2$, the existence problem of fold maps has been solved: a smooth map of a closed connected n-dimensional manifold M with $n \geq 2$ into an orientable

Received by the editors September 12, 2002 and, in revised form, March 24, 2003.
2000 *Mathematics Subject Classification.* Primary 57R45; Secondary 57R42.
Key words and phrases. Self-intersection class, incident class, Thom polynomial, Pontrjagin class, twisted coefficient, fold map.
The first author has been partially supported by Grant-in-Aid for Scientific Research (No. 12740046), the Ministry of Education, Science and Culture, Japan. The second and the third authors have been partially supported by Grant-in-Aid for Scientific Research (No. 13640076), the Ministry of Education, Science and Culture, Japan. The third author has also been partially supported by Grant for Encouragement of Young Researchers, Kinki Univ. (G008).

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surface is homotopic to a fold map if and only if the Euler characteristic of M is even [31], [20], [9]. Recall that generic maps into surfaces have fold and cusp singularities, and that the number of cusp singularities has the same parity as the Euler characteristic (or equivalently, the top Stiefel-Whitney class $w_n(M) \in H^n(M;\mathbf{Z}_2) = \mathbf{Z}_2$) of the source manifold. The above result shows that this is the only obstruction for the existence of fold maps into orientable surfaces. In fact, our problem is closely related to the concept of the Thom polynomial of a given singularity type, as explained below.

Let Σ be a singularity type, which means a certain union of orbits of the group $\mathrm{Diff}(\mathbf{R}^n, 0) \times \mathrm{Diff}(\mathbf{R}^p, 0)$ with respect to the right-left action on some jet space $J^s(n,p)$. For a generic map $f : M \to N$ between smooth manifolds, $\Sigma(f) \subset M$ denotes the singular point set of f of type Σ. It is known that for Σ good enough, the closure $\overline{\Sigma(f)}$ of $\Sigma(f)$ represents a homology class and that its Poincaré dual, denoted by $\mathrm{Tp}(\Sigma)(f)$, can be described as a polynomial in characteristic classes of TM and f^*TN. This is called the *Thom polynomial* for Σ and does not depend on a particular choice of f (for example, see [17], [3], [24], [12]). Clearly, if $\mathrm{Tp}(\Sigma)(f)$ or $\mathrm{Tp}(\Sigma')(f)$ for some Σ' adjacent to Σ (i.e., $\Sigma' \subset \overline{\Sigma} \setminus \Sigma$) does not vanish, then there is no generic map $\widetilde{f}$ homotopic to f such that $\Sigma(\widetilde{f}) = \emptyset$. However, in general, even if $\mathrm{Tp}(\Sigma)(f)$ vanishes, we cannot conclude that the corresponding singularity can be eliminated by homotopy. An example in low dimensions was discovered by the second author, as follows.

Theorem 1.1 (Saeki [25], [26]). *Let M be a smooth 4-manifold with $H_*(M;\mathbf{Z}) \cong H_*(\mathbf{C}P^2;\mathbf{Z})$. Then there exists no fold map of M into any orientable 3-manifold.*

Namely, for M as above, every generic map $f : M \to N$ into an orientable 3-manifold N must have cusp singularities. Note that the Thom polynomials for cusp singularities (or A_2-type singularities) and its adjacent swallowtail singularities (or A_3-type singularities) both vanish in our case.

In the proof of Theorem 1.1, the following congruence, which the third author [29] proved by using a Rohlin type theorem, played an essential role: for a generic map $f : M \to N$ of a closed oriented 4-manifold M with $H_1(M;\mathbf{Z}) = 0$ into an orientable 3-manifold N, we have

$$S(f) \cdot S(f) \equiv 3\sigma(M) \pmod 4, \tag{1.1}$$

where $S(f) \subset M$ is the set of all singular points of f, called the *singular set* of f, $S(f) \cdot S(f)$ is the self-intersection number of $S(f)$ in M, and $\sigma(M)$ is the signature of M. In a sense, the congruence (1.1) gives us more information than the usual Thom polynomials.

The purpose of this paper is to describe what is the essential point behind the congruence (1.1) and to give its integral lift for general dimensions. For a certain generic map $f : M \to N$ between smooth manifolds whose singular point set $S(f)$ is a smooth submanifold, we define the *self-intersection class* of $S(f)$, denoted by $I(S(f))$, as the cohomology class in $H^*(M;\mathbf{Z})$ Poincaré dual to the homology class represented by the transverse intersection of $S(f)$ and its small perturbation in M. This class coincides with the Gysin map image of a special kind of the *incident class* introduced by Rimányi in his theory of Thom polynomials [24] but refined by using twisted coefficients (for more details, see §3). By using the desingularization method, we obtain a formula for $I(S(f))$ as follows, where Σ^ℓ denotes the submanifold of the 1-jet bundle $J^1(M,N)$ consisting of the 1-jets of kernel dimension ℓ.

Theorem 1.2. *Let M and N be manifolds of dimensions n and p respectively, where M is closed and $n-p+1=2k$, $n \geq p \geq 1$, $k \geq 1$. Furthermore, let $f: M \to N$ be a smooth map such that $j^1 f$ is transverse to Σ^{2k} and $\Sigma^\ell(f) = \emptyset$ for all $\ell \geq 2k+1$. Then we have*

$$I(S(f)) \equiv p_k(TM - f^*TN) \in H^{4k}(M; \mathbf{Z}) \quad (\textit{modulo } 2\textit{-torsion}), \tag{1.2}$$

where $p_k(TM - f^*TN)$ is the k-th Pontrjagin class of the difference bundle $TM - f^*TN$.

In the terminology of [15, Chapter VI, §1], the above condition on $j^1 f$ is equivalent to f being 1-generic and having corank at most one everywhere (the *corank* of f at x is defined to be $\min(n,p) - \operatorname{rank} df_x$). In this case, we say simply that f is a generic smooth map with corank one singularities. Note that then the singular set $S(f) = (j^1 f)^{-1}(\Sigma^{2k})$ is a smooth regular submanifold of M. Note also that this condition is generic provided that $n > 2p - 4$.

The above theorem implies, in particular, that for a generic map $f: M \to N$ of a closed oriented 4-manifold M into an orientable 3-manifold N, we have

$$S(f) \cdot S(f) = p_1[M] = 3\sigma(M) \in \mathbf{Z}, \tag{1.3}$$

by the Hirzebruch signature formula, where $p_1[M]$ denotes the first Pontrjagin number of M. This is nothing but an integral lift of the congruence (1.1). Clearly, Theorem 1.1 can be proved by using (1.3). As another application of the formula (1.2), we have the following necessary condition for the existence of fold maps.

Theorem 1.3. *Let $g: M \to N$ be a smooth map between smooth manifolds, where M is closed. We assume that $n = \dim M$ and $p = \dim N$ satisfy $n - p + 1 = 2k$ for some positive odd integer k. If there exists a fold map homotopic to g, then there exists a cohomology class $x \in H^{2k}(M; \mathcal{O}_{TM - g^*TN})$ such that*

$$x \smile x \equiv p_k(TM - g^*TN) \in H^{4k}(M; \mathbf{Z}) \quad (\textit{modulo } 4\textit{-torsion}),$$

where $\smile$ denotes the cup product and $\mathcal{O}_{TM-g^*TN}$ is the orientation local system associated with the difference bundle $TM - g^*TN$.

The above theorem is very useful for obtaining nonexistence results for fold maps. For example, we will show that for certain dimension pairs (n,p), every n-dimensional oriented cobordism class contains a connected manifold that admits no fold maps into certain p-dimensional manifolds (for details, see §4).

Throughout the paper, we work in the smooth category; that is, all manifolds and maps are differentiable of class C^∞ unless otherwise stated.

The authors would like to express their sincere gratitude to Julius Korbaš and Peter Zvengrowski for their useful suggestions.

2. Gysin map with twisted coefficients and self-intersection class

In this section, we recall the definition and some properties of the Gysin map with twisted coefficients induced by a smooth map between manifolds, and define the self-intersection class of a submanifold.

In the following, for a manifold X (or a vector bundle ξ), $\mathcal{O}_X$ (resp. $\mathcal{O}_\xi$) will denote the orientation local system associated with X (resp. ξ).

Let $f : M \to N$ be a smooth map between smooth manifolds of dimensions n and p respectively, and $\mathcal{L}$ an arbitrary local system over N. Take an embedding $\varepsilon : M \to \operatorname{Int} D^r$ for some r and identify M with the image of the embedding $(f, \varepsilon) : M \to N \times D^r$. Let U be a tubular neighbourhood of M in $N \times D^r$ with projection $p_U : U \to M$, where U is identified with the total space of the normal bundle ν of M in $N \times D^r$. Note that the orientation local system $\mathcal{O}_\nu$ is isomorphic to that associated with the difference bundle $f^*TN - TM$, i.e., $\mathcal{O}_\nu \cong \mathcal{O}_{f^*TN-TM} \cong \mathcal{O}_M \otimes f^*\mathcal{O}_N$.

Then, we define the (twisted) *Gysin map* (or the *Umkehr map*)

$$f_! : H^*(M; f^*\mathcal{L} \otimes \mathcal{O}_{f^*TN-TM}) \to H^{*+(p-n)}(N; \mathcal{L})$$

induced by f by the composition

$$\begin{aligned} H^*(M; f^*\mathcal{L} \otimes \mathcal{O}_{f^*TN-TM}) &\xrightarrow[\cong]{(p_U)^*} H^*(U; (p_U)^* f^*\mathcal{L} \otimes (p_U)^*\mathcal{O}_{f^*TN-TM}) \\ &\xrightarrow[\cong]{\smile u} H^{*+(p+r-n)}(U, U \smallsetminus M; (p_U)^* f^*\mathcal{L}) \\ &\xrightarrow[\cong]{ex} H^{*+(p+r-n)}(N \times D^r, N \times D^r \smallsetminus M; (p_N)^*\mathcal{L}) \\ &\xrightarrow{(\bar{i}_M)^*} H^{*+(p+r-n)}(N \times D^r, N \times \partial D^r; (p_N)^*\mathcal{L}) \xrightarrow[\cong]{((p_N)^*)^{-1}} H^{*+(p-n)}(N; \mathcal{L}), \end{aligned}$$

where $u \in H^{p+r-n}(U, U \smallsetminus M; (p_U)^*\mathcal{O}_{f^*TN-TM})$ is the Thom class of the normal bundle ν, ex denotes the excision isomorphism, $p_N : N \times D^r \to N$ is the projection to the first factor, $\bar{i}_M : (N \times D^r, N \times \partial D^r) \to (N \times D^r, N \times D^r \smallsetminus M)$ is the inclusion, and the final isomorphism comes from the Künneth theorem. It is known that the above definition does not depend on the choice of a particular embedding ε or r (for details, see [7], [8], [6], for example).

The following properties are known.

Lemma 2.1. (1) *If f is a proper map, then the twisted Gysin map $f_!$ coincides with the composition*

$$\begin{aligned} H^*(M; f^*\mathcal{L} \otimes \mathcal{O}_{f^*TN-TM}) &\xrightarrow{\frown [M]} H^c_{n-*}(M; f^*(\mathcal{L} \otimes \mathcal{O}_N)) \\ &\xrightarrow{f_*} H^c_{n-*}(N; \mathcal{L} \otimes \mathcal{O}_N) \xrightarrow{(\frown [N])^{-1}} H^{*+(p-n)}(N; \mathcal{L}), \end{aligned}$$

where H^c_* denotes the homology of closed support, and $[M] \in H^c_n(M; \mathcal{O}_M)$ and $[N] \in H^c_p(N; \mathcal{O}_N)$ are the fundamental classes of M and N respectively.

(2) *We have*

$$f_!(f^*x \smile y) = x \smile f_!(y)$$

for all $x \in H^*(N; \mathcal{L})$ and all $y \in H^*(M; f^*\mathcal{L}' \otimes \mathcal{O}_{f^*TN-TM})$, where $\mathcal{L}$ and $\mathcal{L}'$ are arbitrary local systems over N.

(3) *For smooth maps $f : M \to N$ and $g : N \to L$ with $\dim M = n, \dim N = p$ and $\dim L = \ell$, we have $(g \circ f)_! = g_! \circ f_! : H^*(M; (g \circ f)^*\mathcal{L} \otimes \mathcal{O}_{(g\circ f)^*TL-TM}) \to H^{*+(\ell-n)}(L; \mathcal{L})$ for any local system $\mathcal{L}$ over L.*

Using the twisted Gysin map, we define the self-intersection class of a submanifold as follows.

Definition 2.2. Let Y be a smooth submanifold of a smooth manifold X of codimension κ. Let U be a tubular neighbourhood of Y in X with projection $p_U : U \to Y$, where U is identified with the total space of the normal bundle ν_Y of

Y in X. Consider the Thom class $u \in H^\kappa(U, U \smallsetminus Y; (p_U)^*\mathcal{O}_{\nu_Y})$ of ν_Y. The *twisted Euler class* $e(\nu_Y) \in H^\kappa(Y; \mathcal{O}_{\nu_Y})$ of ν_Y is the image of u under the composition

$$H^\kappa(U, U \smallsetminus Y; (p_U)^*\mathcal{O}_{\nu_Y}) \xrightarrow{(\bar{i}_Y)^*} H^\kappa(U; (p_U)^*\mathcal{O}_{\nu_Y}) \underset{\cong}{\xrightarrow{((p_U)^*)^{-1}}} H^\kappa(Y; \mathcal{O}_{\nu_Y}),$$

where $\bar{i}_Y : U \to (U, U \smallsetminus Y)$ is the inclusion. Then the *self-intersection class* $I(Y) \in H^{2\kappa}(X; \mathbf{Z})$ of Y is defined by $I(Y) = i_!(e(\nu_Y))$, where $i_! : H^\kappa(Y; \mathcal{O}_{\nu_Y}) \to H^{2\kappa}(X; \mathbf{Z})$ is the Gysin map induced by the inclusion $i : Y \to X$.

It is clear that when X is oriented, the homology class Poincaré dual to the self-intersection class $I(Y)$ is represented by the transverse intersection of Y and its small perturbation in X as a $\mathbf{Z}$-cycle (even if Y is non-orientable). Note that this integral cycle is well-defined as a homology class, which is denoted by $Y \cdot Y \in H_{n-2\kappa}(X; \mathbf{Z})$ with $n = \dim X$ (see [4, p. 583]). Note that $Y \cdot Y$ depends on the choice of an orientation for X, while $I(Y)$ does not.

3. Proof of Theorem 1.2

In this section, we consider the self-intersection class of the singular set of a generic smooth map with corank one singularities and prove the formula (1.2).

Proof of Theorem 1.2. Let us first assume that $N = \mathbf{R}^p$. The general case will follow from this special case.

Let $\pi : G \to M$ be the Grassmannian bundle of unoriented $2k$-planes in TM and ϕ the tautological $2k$-plane bundle over G. Let us consider the vector bundle $\mathrm{Hom}(\phi, \varepsilon^p)$ over G, where $\varepsilon^p = \pi^* f^* T\mathbf{R}^p$ is the trivial p-plane bundle. There is a natural section s of this vector bundle associated with f, which is defined by $s(x, H) = df_x|_H : H \to \mathbf{R}^p$ for $x \in M$ and $H \subset TM_x$. Our assumption on $j^1 f$ implies that s is transverse to the zero section (for details, see [23]). We set $\widetilde{S}(f) = s^{-1}(0)$ and $\widetilde{\pi} = \pi|_{\widetilde{S}(f)}$. Furthermore, we denote by $j : \widetilde{S}(f) \hookrightarrow G$ the inclusion. Note that $\widetilde{\pi} : \widetilde{S}(f) \to S(f)$ is a diffeomorphism and that the normal bundle ν_j of the embedding j is isomorphic to $j^* \mathrm{Hom}(\phi, \varepsilon^p) \cong \mathrm{Hom}(j^*\phi, j^*\varepsilon^p)$. Note also that $\mathcal{O}_{\mathrm{Hom}(\phi,\varepsilon^p)} \cong (\mathcal{O}_\phi)^{\otimes p}$ and hence that $\mathcal{O}_{\nu_j} \cong j^*(\mathcal{O}_\phi)^{\otimes p}$, where for a local system $\mathcal{L}$, $\mathcal{L}^{\otimes p}$ denotes the p-fold tensor product of $\mathcal{L}$.

Let $\widetilde{U}$ be a tubular neighbourhood of $\widetilde{S}(f)$ in G with projection $p_{\widetilde{U}} : \widetilde{U} \to \widetilde{S}(f)$. Furthermore, let E denote the total space of the vector bundle $\mathrm{Hom}(\phi, \varepsilon^p)$ and $\pi_E : E \to G$ the projection, where we consider G to be embedded in E as the zero section. Note that $\widetilde{U}$ can be identified with the total space of $j^* \mathrm{Hom}(\phi, \varepsilon^p)$. Then, by considering the commutative diagram

$$\begin{array}{ccc}
H^{2kp}(\widetilde{U}, \widetilde{U} \smallsetminus \widetilde{S}(f); (p_{\widetilde{U}})^*\mathcal{O}_{\nu_j}) & \xrightarrow{ex} & H^{2kp}(G, G \smallsetminus \widetilde{S}(f); (\mathcal{O}_\phi)^{\otimes p}) \\
 & {\scriptstyle (s|_{\widetilde{U}})^*} \nwarrow & \uparrow {\scriptstyle s^*} \\
 & & H^{2kp}(E, E \smallsetminus G; (\pi_E)^*(\mathcal{O}_\phi)^{\otimes p}) \\
 & \xrightarrow{(\bar{i}_{\widetilde{S}(f)})^*} & H^{2kp}(G; (\mathcal{O}_\phi)^{\otimes p}) \\
 & & \cong \uparrow {\scriptstyle s^* = ((\pi_E)^*)^{-1}} \\
 & \xrightarrow{(\bar{i}_G)^*} & H^{2kp}(E; (\pi_E)^*(\mathcal{O}_\phi)^{\otimes p}),
\end{array}$$

we see that

$$j_!(1) = e(\mathrm{Hom}(\phi, \varepsilon^p)) = e(\phi)^p \in H^{2kp}(G; (\mathcal{O}_\phi)^{\otimes p}), \tag{3.1}$$

where $\bar{i}_{\widetilde{S}(f)} : G \to (G, G \smallsetminus \widetilde{S}(f))$ and $\bar{i}_G : E \to (E, E \smallsetminus G)$ are the inclusion maps.

Let $K = \ker df$ and $Q = \operatorname{coker} df$ be the kernel bundle of rank $2k$ and the cokernel bundle of rank 1 defined over $S(f)$, respectively. Note that $\mathrm{Hom}(K, Q)$ is isomorphic to the normal bundle ν of $i : S(f) \hookrightarrow M$ (for example, see [5], [15]). Over $\widetilde{S}(f)$, we have $\widetilde{\pi}^*\nu \cong \mathrm{Hom}(\widetilde{\pi}^*K, \widetilde{\pi}^*Q)$ and $\widetilde{\pi}^*K \cong j^*\phi$. Hence, we have $\widetilde{\pi}^*\mathcal{O}_\nu \cong \widetilde{\pi}^*\mathcal{O}_K \cong j^*\mathcal{O}_\phi$, since the rank of K is even.

Lemma 3.1. *We have*

$$e(\widetilde{\pi}^*\nu) \equiv e(j^*\phi) \in H^{2k}(\widetilde{S}(f); j^*\mathcal{O}_\phi) \quad (\textit{modulo } 2\textit{-torsion}). \tag{3.2}$$

Proof. When $\widetilde{\pi}^*Q$ is trivial, (3.2) obviously holds even without taking modulo 2-torsion, since $\widetilde{\pi}^*\nu \cong \widetilde{\pi}^*K \cong j^*\phi$. When $\widetilde{\pi}^*Q$ is not trivial, let $\widehat{\pi} : \widehat{S}(f) \to \widetilde{S}(f)$ be the double covering corresponding to $w_1(\widetilde{\pi}^*Q) \in H^1(\widetilde{S}(f); \mathbf{Z}_2)$, the first Stiefel-Whitney class. Then, since $\widehat{\pi}^*\widetilde{\pi}^*Q$ is trivial, we have $e(\widehat{\pi}^*\widetilde{\pi}^*\nu) = e(\widehat{\pi}^*j^*\phi) \in H^{2k}(\widehat{S}(f); \widehat{\pi}^*j^*\mathcal{O}_\phi)$. Then by Lemma 2.1 (2), we have

$$2e(\widetilde{\pi}^*\nu) = \widehat{\pi}_!(e(\widehat{\pi}^*\widetilde{\pi}^*\nu)) = \widehat{\pi}_!(e(\widehat{\pi}^*j^*\phi)) = 2e(j^*\phi),$$

since $\widehat{\pi}_!(1) = 2$. Thus (3.2) holds. □

Lemma 3.2. *We have*

$$(\mathcal{O}_\phi)^{\otimes n-2k} \otimes \mathcal{O}_G \cong \pi^*\mathcal{O}_M.$$

Proof. Recall that the Grassmann manifold $G_{2k}(\mathbf{R}^n)$ consisting of $2k$-planes in $\mathbf{R}^n$ is orientable if and only if $n - 2k$ is even. Furthermore, its orientation remains invariant under the orientation reversal of $\mathbf{R}^n$. Hence, when $n-2k$ is even, we have

$$(\mathcal{O}_\phi)^{\otimes n-2k} \otimes \mathcal{O}_G \cong \mathcal{O}_G \cong \pi^*\mathcal{O}_M.$$

When $n - 2k$ is odd, let γ be the tautological $2k$-plane bundle over $G_{2k}(\mathbf{R}^n)$. Then the orientation local system $\mathcal{O}_\gamma$ is isomorphic to $\mathcal{O}_{G_{2k}(\mathbf{R}^n)}$, since $TG_{2k}(\mathbf{R}^n)$ is isomorphic to $\mathrm{Hom}(\gamma, \gamma^\perp)$, where $\gamma^\perp$ is the orthogonal complement of γ. Hence, we have $(\mathcal{O}_\phi)^{\otimes n-2k} \otimes \mathcal{O}_G \cong \mathcal{O}_\phi \otimes \mathcal{O}_G \cong \pi^*\mathcal{O}_M$. □

By the above lemma, $\pi : G \to M$ induces the Gysin map

$$\pi_! : H^*(G; (\mathcal{O}_\phi)^{\otimes n-2k}) \cong H^*(G; \mathcal{O}_G \otimes \pi^*\mathcal{O}_M) \to H^{*-2k(n-2k)}(M; \mathbf{Z}).$$

Lemma 3.3. *We have*

$$\pi_!(e(\phi)^{n-2k}) = 1,$$

where $\pi_! : H^{2k(n-2k)}(G; (\mathcal{O}_\phi)^{\otimes n-2k}) \to H^0(M; \mathbf{Z})$ *is the Gysin map induced by* $\pi : G \to M$.

Proof. The proof is similar to that in Porteous' paper [23, Proposition 0.3], as follows.

We have only to consider the equality over a fibre $G(TM_x) = \pi^{-1}(x)$, that is, over the Grassmannian of TM_x, since the diagram

$$\begin{array}{ccc} H^{2k(n-2k)}(G; (\mathcal{O}_\phi)^{\otimes n-2k}) & \xrightarrow{\pi_!} & H^0(M; \mathbf{Z}) \\ {\scriptstyle (i_{G(TM_x)})^*}\downarrow & & \downarrow{\scriptstyle (i_x)^*} \\ H^{2k(n-2k)}(G(TM_x); (\mathcal{O}_{\phi(x)})^{\otimes n-2k}) & \xrightarrow{(\pi_x)_!} & H^0(x; \mathbf{Z}) \end{array}$$

is commutative, where $i_{G(TM_x)} : G(TM_x) \to G$ and $i_x : x \to M$ are the inclusions, $\pi_x = \pi|_{G(TM_x)} : G(TM_x) \to x$, and $\phi(x)$ is the restriction of ϕ to $G(TM_x)$.

Fix a nonzero vector $v \in TM_x$. Then it induces a section s_v of $\phi(x)$ defined by

$$s_v(H) = [v] \in TM_x/H^\perp \equiv H \quad \text{for } H \in G(TM_x),$$

where $H^{\perp}$ is the orthogonal complement of H with respect to a fixed inner product on TM_x. The zero set $s_v^{-1}(0)$ $(= \{H : v \in H^{\perp}\})$ represents the homology class in $H_{2k(n-2k)-2k}(G(TM_x); \mathcal{O}_{G(TM_x)} \otimes \mathcal{O}_{\phi(x)})$ Poincaré dual to the Euler class $e(\phi(x))$. If we take a set of linearly independent $(n-2k)$-vectors of TM_x, then we get $n-2k$ generically linearly independent sections s_i, $i = 1, 2, \ldots, n-2k$. The intersection of all $s_i^{-1}(0)$ consists of a unique point, which corresponds to the $2k$-plane orthogonal to the space spanned by the $(n-2k)$-vectors. Thus we have that the homology class in $H_0(G(TM_x); \mathcal{O}_{G(TM_x)} \otimes (\mathcal{O}_{\phi(x)})^{\otimes n-2k}) \cong H_0(G(TM_x); \mathbf{Z})$ Poincaré dual to $e(\phi(x))^{n-2k}$ is represented by a point. Hence, the result follows by Lemma 2.1 (1). $\square$

Let us go back to the proof of Theorem 1.2. We fix a Riemannian metric on M and consider the orthogonal decomposition $\pi^*TM = \phi \oplus \phi^{\perp}$. Note that $e(\phi)^2 = p_k(\phi)$ (for example, see [21, §15]). By using the product formula

$$p(\pi^*TM) \equiv p(\phi)p(\phi^{\perp}) \quad \text{(modulo 2-torsion)}$$

for Pontrjagin classes together with Lemma 2.1 and (3.1), we have

$$\begin{aligned}
& j_!(e(j^*\phi)) = e(\phi)j_!(1) = e(\phi)^{p+1} = p_k(\phi)e(\phi)^{n-2k} \\
\equiv\ & \left(p_k(\pi^*TM) - \sum_{\ell=0}^{k-1} p_\ell(\phi)p_{k-\ell}(\phi^{\perp})\right) e(\phi)^{n-2k} \\
\equiv\ & p_k(\pi^*TM)e(\phi)^{n-2k} - \left(\sum_{\ell=0}^{k-1} p_\ell(\phi)(p_{k-\ell}(\phi^{\perp})p_k(\phi))\right) e(\phi)^{n-2k-2} \\
\equiv\ & p_k(\pi^*TM)e(\phi)^{n-2k} - \left(\sum_{\ell_1=0}^{k-1} p_{\ell_1}(\phi)\right. \\
& \qquad \left.\left(p_{2k-\ell_1}(\pi^*TM) - \sum_{\ell_2=0}^{k-1} p_{\ell_2}(\phi)p_{2k-\ell_1-\ell_2}(\phi^{\perp})\right)\right) e(\phi)^{n-2k-2} \\
\equiv\ & \cdots \\
\equiv\ & p_k(\pi^*TM)e(\phi)^{n-2k} + T_1 + T_2 \quad \text{(modulo 2-torsion)}.
\end{aligned}$$

Here we set

$$T_1 = (-1)^s \sum_{I_s} p_{j_s}(\pi^*TM)p_{I_s}(\phi)e(\phi)^{n-2k-2s}, \tag{3.3}$$

$$T_2 = (-1)^{n_0+1} \sum_{I_{n_0+1}} p_{I_{n_0+1}}(\phi)p_{k_0}(\phi^{\perp})e(\phi)^{n-2k-2n_0}, \tag{3.4}$$

where $n_0 = [(n-2k)/2]$ is the greatest integer not exceeding $(n-2k)/2$, the sum in (3.3) runs over all multi-indices

$$I_s = (\ell_1, \ldots, \ell_s) \quad \text{with} \quad 0 \le \ell_1, \ldots, \ell_s \le k-1 \quad (1 \le s \le [(n-2k)/2] = n_0),$$

the sum in (3.4) runs over all multi-indices

$$I_{n_0+1} = (\ell_1, \ldots, \ell_{n_0+1}) \quad \text{with} \quad 0 \le \ell_1, \ldots, \ell_{n_0+1} \le k-1,$$

$p_{I_t} = p_{\ell_1}p_{\ell_2}\cdots p_{\ell_t}$, $j_s = (s+1)k - (\ell_1 + \cdots + \ell_s)$, and $k_0 = (n_0+1)k - (\ell_1 + \cdots + \ell_{n_0+1})$.

Since $\ell_t \le k-1$ for all $1 \le t \le n_0+1$, we have $k_0 \ge n_0+1 > (n-2k)/2$. Therefore, $p_{k_0}(\phi^{\perp})$ and hence T_2 vanishes. Furthermore, the degree of $p_{I_s}(\phi)e(\phi)^{n-2k-2s}$ is strictly smaller than $2k(n-2k)$. Therefore, its image by the Gysin map $\pi_!$ vanishes, and hence we have $\pi_! T_1 = 0$ by Lemma 2.1 (2). Thus, by Lemmas 3.3 and 2.1, we have

$$\pi_! j_!(e(j^*\phi)) \equiv p_k(TM)\pi_!(e(\phi)^{n-2k}) = p_k(TM) \quad \text{(modulo 2-torsion)}.$$

On the other hand, by (3.2) and Lemma 2.1 we have

$$\begin{aligned} \pi_! j_!(e(j^*\phi)) &= i_!\widetilde{\pi}_!(e(j^*\phi)) \equiv i_!\widetilde{\pi}_!(e(\widetilde{\pi}^*\nu)) \quad \text{(modulo 2-torsion)} \\ &= i_!(e(\nu)\widetilde{\pi}_!(1)) = i_!(e(\nu)). \end{aligned}$$

Thus, by definition of the self-intersection class, we get

$$I(S(f)) = i_!(e(\nu)) \equiv p_k(TM) \quad \text{(modulo 2-torsion)}.$$

This completes the proof for the case $N = \mathbf{R}^p$.

In the case of general N, we can show the assertion similarly in a standard way, which is sketched as follows (for more details, see [12] for example). Take a vector bundle ξ over M such that $f^*TN \oplus \xi$ is trivial (for instance, taking an embedding of N into a Euclidean space of sufficiently high dimension, set ξ to be the pull-back of the normal bundle of the embedding). Note that ξ defines the element $-f^*TN$ in the K-group. In the above proof, TM and df (i.e., $j^1 f$) should be replaced by $TM \oplus \xi$ and $df \oplus \mathrm{id}_\xi : TM \oplus \xi \to f^*TN \oplus \xi$, respectively. Finally, we get

$$i_!(e(\nu)) \equiv p_k(TM \oplus \xi) \equiv p_k(TM - f^*TN) \quad \text{(modulo 2-torsion)}.$$

This completes the proof of Theorem 1.2. □

Remark 3.4. When the normal bundle ν of the embedding $i : S(f) \hookrightarrow M$ is orientable, the Euler class $e(\nu) \in H^{2k}(S(f); \mathbf{Z})$ is a special case of the incident class defined by Rimányi [24], i.e., $e(\nu) = I(\Sigma^{2k}, \Sigma^{2k})(f)$. Note that our definition of the twisted Euler class uses the orientation local system so that everything works even if ν is non-orientable.

Now the formula (1.3) follows directly from Theorem 1.2, since $\langle I(S(f)), [M]\rangle = S(f) \cdot S(f)$ by the definition of $I(S(f))$ and every orientable 3-manifold is parallelizable, where $[M] \in H_4(M; \mathbf{Z})$ is the fundamental class of M, and $\langle \, , \, \rangle$ is the Kronecker product.

Remark 3.5. When $n - p + 1$ is odd, $e(\nu) \in H^{n-p+1}(S(f); \mathcal{O}_\nu)$ is of order at most two, since the dimension of a fibre of ν is odd. Hence $I(S(f)) = i_!(e(\nu))$ is an element of order at most two or, in other words, it vanishes modulo 2-torsion.

Remark 3.6. Fehér and Rimányi in [13] observed that if a singularity set $\eta(f)$ of type η is an orientable closed submanifold of an orientable manifold, then the self-intersection class of $\eta(f)$ is equal to the Thom polynomial of the complexified singularity $\eta_{\mathbf{C}}(f_{\mathbf{C}})$ multiplied by $(-1)^{m(m-1)/2}$, where m is the (real) codimension of η. It seems possible to generalize this to the non-orientable case, as we have done here for the singularity type A_1. In fact, for complex analytic maps $M^n \to N^{n-2k+1}$, the Thom polynomial for an A_1-type singularity coincides with $c_{2k}(TM - f^*TN)$. We conjecture that our formula would be true without taking modulo 2-torsion, and also for the case $\Sigma^\ell(f) \neq \emptyset$ for some $\ell \geq 2k+1$.

Remark 3.7. For a generic map $f : M \to N$ of a closed n-dimensional manifold into a p-dimensional manifold with $n - p + 1 = 2k$, $k \geq 1$, the Thom polynomial

$\mathrm{Tp}(A_1(f))$ coincides with the $2k$-th Stiefel-Whitney class $w_{2k}(TM - f^*TN) \in H^{2k}(M; \mathbf{Z}_2)$ by Thom [31]. Hence, we have

$$I(S(f)) \equiv w_{2k}(TM - f^*TN)^2 \pmod 2.$$

On the other hand, we have the basic congruence

$$w_{2k}^2 \equiv p_k \pmod 2$$

(see [21, Problem 15-A], for example). This means that the difference $I(S(f)) - p_k(TM - f^*TN)$ is a multiple of two.

Remark 3.8. In Theorem 1.2, we have assumed that the source manifold M is compact. However, this assumption is not necessary, as long as the map $f : M \to N$ is proper and we use the homology of closed support instead of the usual homology.

4. Nonexistence of fold maps

In this section, we give some necessary conditions for the existence of fold maps as an application of our formula (1.2). Let us first recall the following.

Definition 4.1. For a smooth map $f : M \to N$ with $n = \dim M \geq \dim N = p$, a point $q \in M$ is a *fold singularity* (or an A_1*-type singularity*) of f if f is of the form

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{p-1}, \pm x_p^2 \pm \cdots \pm x_n^2) \tag{4.1}$$

with respect to appropriate coordinates around q and $f(q)$. Let λ' be the number of negative signs appearing in the p-th component of the right-hand side of (4.1). Then, $\max\{\lambda', n-p+1-\lambda'\}$ is called the *reduced index* of q, which does not depend on a particular choice of coordinates (for details, see [20]). If the singular set $S(f)$ of f is empty or consists only of fold singularities, f is called a *fold map.*[1]

Now let us prove Theorem 1.3.

Proof of Theorem 1.3. Suppose that there is a fold map $f : M \to N$ homotopic to g. For $\lambda = k, k+1, \dots, n-p+1$, let $S_\lambda(f)$ be the set of fold singularities of f of reduced index λ. Note that $S(f)$ is the disjoint union of $S_k(f), S_{k+1}(f), \dots, S_{n-p+1}(f)$ and that each $S_\lambda(f)$ consists of some connected components of $S(f)$. As is easily observed, each $S_\lambda(f)$ is a $(p-1)$-dimensional regular submanifold of M, and $f|_{S_\lambda(f)}$ is an immersion for each λ. Furthermore, the normal bundle of the immersion is trivial for $\lambda \neq k$ (for example, see [25]). This implies that $\mathcal{O}_{S_\lambda(f)} \cong (i_\lambda)^* f^* \mathcal{O}_N$ for $\lambda \neq k$, where $i_\lambda : S_\lambda(f) \to M$ is the inclusion. Let $x_\lambda \in H^{2k}(M; \mathcal{O}_{f^*TN-TM}) = H^{2k}(M; \mathcal{O}_{g^*TN-TM})$ be the cohomology class Poincaré dual to the homology class in $H_{p-1}(M; f^*\mathcal{O}_N)$ represented by $S_\lambda(f)$, $\lambda \neq k$. Then, the self-intersection homology class $S_\lambda(f) \cdot S_\lambda(f) \in H_{n-4k}(M; \mathcal{O}_M)$ and the cohomology class $x_\lambda \smile x_\lambda \in H^{4k}(M; \mathbf{Z})$ are Poincaré dual to each other, and the self-intersection class $I(S_\lambda(f))$ coincides with $x_\lambda \smile x_\lambda$.

For $\lambda = k$, let ν_k be the normal bundle of $S_k(f)$ in M. By [25], its structure group can be reduced to the semi-direct product $G = (O(k) \times O(k)) \rtimes \mathbf{Z}_2 \subset O(2k)$. Since $O(k) \times O(k)$ is a subgroup of G of index two, the structure group of the pull-back $\pi_k^* \nu_k$ of ν_k by an appropriate double covering $\pi_k : \widetilde{S}_k(f) \to S_k(f)$ can be reduced to $O(k) \times O(k)$. This implies that $\pi_k^* \nu_k$ splits into the Whitney sum $\xi_1 \oplus \xi_2$ for some k-plane bundles ξ_1 and ξ_2 over $\widetilde{S}_k(f)$.

[1] In [15], such a map is called a *submersion with folds.*

Since k is odd by our assumption, the twisted Euler classes of ξ_1 and ξ_2 are elements of order at most two, and hence so is the Euler class $e(\pi_k^*\nu_k)$ of $\pi_k^*\nu_k$. Since $\pi_{k!}(e(\pi_k^*\nu_k)) = e(\nu_k)\pi_{k!}(1) = 2e(\nu_k)$, we see that $4e(\nu_k)$ vanishes. This implies that the self-intersection class $I(S_k(f))$ vanishes modulo 4-torsion.

Since $x_\lambda \smile x_{\lambda'} = 0$ for $\lambda \neq \lambda'$ and

$$I(S(f)) = \sum_{\lambda=k}^{n-p+1} I(S_\lambda(f)),$$

we conclude that the cohomology class

$$x = \sum_{\lambda=k+1}^{n-p+1} x_\lambda \in H^{2k}(M; \mathcal{O}_{g^*TN-TM})$$

satisfies

$$x \smile x \equiv I(S(f)) \quad \text{(modulo 4-torsion)}.$$

Hence, by Theorem 1.2, the result follows. This completes the proof of Theorem 1.3. □

As an important corollary, we have the following, which shows the effectiveness of the formula (1.3).

Corollary 4.2. *Let M be a closed oriented 4-manifold whose intersection form is isomorphic either to $\pm\langle 1\rangle$ or to $\pm(\langle 1\rangle \oplus \langle 1\rangle)$. Then, there exists no fold map $f : M \to N$ for any orientable 3-manifold N. In other words, every generic map $f : M \to N$ necessarily has cusp singularities.*

Proof. Suppose that there is a fold map $f : M \to N$. Then by Theorem 1.3, there exists an element $x \in H^2(M; \mathbf{Z})$ such that

$$x \smile x = p_1(TM - f^*TN) = p_1(M),$$

since every orientable 3-manifold is parallelizable. When the intersection form of M is isomorphic to $\pm\langle 1\rangle$, this implies that there exists an integer ℓ such that $\ell^2 = 3$, since $p_1[M] = 3\sigma(M)$. This is a contradiction. Similarly, when the intersection form of M is isomorphic to $\pm(\langle 1\rangle \oplus \langle 1\rangle)$, there must exist integers ℓ_1 and ℓ_2 such that $\ell_1^2 + \ell_2^2 = 6$, which is a contradiction again. Hence, there exists no fold map $f : M \to N$. □

Remark 4.3. In [27], the second author obtained the special case of Theorem 1.3 for $(n, p) = (4, 3)$ by using a different method. Corollary 4.2 was also obtained there. (In fact, in [27], it was proved that the sufficient condition for the nonexistence of fold maps mentioned in Corollary 4.2 is also necessary.) Note that Corollary 4.2 generalizes Theorem 1.1 and the main theorem of [30]. Note also that Akhmetiev and Sadykov [1] recently gave another proof of Theorem 1.1 from a slightly different point of view.

By using Theorem 1.3, we also have the following result for general dimensions.

Corollary 4.4. *For every dimension pair (n, p) such that $n - p + 1 = 2k$ for a positive odd integer k with $4k \leq n$, there exists a closed connected orientable manifold of dimension n that admits no fold map into any p-dimensional manifold N such that $p_i(N) = 0$ for all $1 \leq i \leq k$.*

Proof. Set

$$M = \begin{cases} \mathbf{C}P^{2k} \times S^{n-4k}, & n-4k \geq 2, \\ \mathbf{C}P^{2k-1} \times S^{n-4k+2}, & n-4k = 1, \\ \mathbf{C}P^{2k}, & n = 4k. \end{cases}$$

Note that its cohomology ring satisfies

$$H^*(M;\mathbf{Z}) \cong \begin{cases} \mathbf{Z}[a,b]/(a^{2k+1}, b^2), & n-4k \geq 2, \\ \mathbf{Z}[a,b]/(a^{2k}, b^2), & n-4k = 1, \\ \mathbf{Z}[a]/(a^{2k+1}), & n = 4k, \end{cases}$$

where a corresponds to a generator of $H^2(\mathbf{C}P^{2k};\mathbf{Z}) \cong \mathbf{Z}$ for $n-4k \neq 1$ (or a generator of $H^2(\mathbf{C}P^{2k-1};\mathbf{Z}) \cong \mathbf{Z}$ for $n-4k=1$), and b corresponds to a generator of $H^{n-4k}(S^{n-4k};\mathbf{Z}) \cong \mathbf{Z}$ for $n-4k \geq 2$ (or a generator of $H^{n-4k+2}(S^{n-4k+2};\mathbf{Z}) \cong \mathbf{Z}$ for $n-4k=1$). Note also that the total Pontrjagin class $p(M)$ of M satisfies

$$p(M) = \begin{cases} (1+a^2)^{2k+1}, & n-4k \neq 1, \\ (1+a^2)^{2k}, & n-4k = 1 \end{cases}$$

(for example, see [21]).

Suppose that there exists a fold map $f : M \to N$ for some p-dimensional manifold N such that $p_i(N) = 0$ for $1 \leq i \leq k$. Since M is simply connected, any local system over M is trivial. Then by Theorem 1.3, there exists an element $x \in H^{2k}(M;\mathbf{Z})$ such that

$$x \smile x = p_k(TM - f^*TN) = p_k(M) = \begin{cases} \binom{2k+1}{k} a^{2k}, & n-4k \neq 1, \\ \\ \binom{2k}{k} a^{2k}, & n-4k = 1, \end{cases}$$

since $H^*(M;\mathbf{Z})$ is torsion free. This implies that the integer

$$\binom{2k+1}{k} = \frac{(2k+1)!}{(k+1)!k!} \quad \text{or} \quad \binom{2k}{k} = \frac{(2k)!}{k!k!}$$

must be a square, which is a contradiction by a result of Erdös [11] (see also [16]). Hence, M is a desired manifold. This completes the proof. □

Remark 4.5. In the above corollary, if n is even and $k \geq 3$, then we can prove that $\mathbf{C}P^{n/2}$ is also a desired manifold except possibly for $(n,p) = (98,93)$, since the binomial coefficient

$$\binom{n/2+1}{k}$$

is a square if and only if $n = 98$ and $k = 3$ (see [16]).

Corollary 4.6. *Let (n,p) be a dimension pair such that $n-p+1 = 2k$ for a positive odd integer k with $4k < n$. Then, for every closed oriented n-dimensional manifold M, there exists a closed connected oriented n-dimensional manifold M' oriented cobordant to M such that M' admits no fold map into any p-dimensional manifold N with $p_i(N) = 0$ for all $1 \leq i \leq k$.*

Proof. Let M_0 be the n-dimensional manifold given by Corollary 4.4. Furthermore, let M_1 be a connected and simply connected manifold oriented cobordant to M and set $M' = M_1 \sharp M_0 \sharp \overline{M_0}$, where $\overline{M_0}$ denotes the manifold M_0 with orientation reversed.

Then, since we have $n > 4k$ by our assumption, by an argument similar to that in the proof of the above corollary, we see that there exists no element $x \in H^{2k}(M'; \mathbf{Z})$ such that $x \smile x \equiv p_k(M')$ modulo 4-torsion. Hence, by Theorem 1.3, M' admits no fold map into N. Since M' is oriented cobordant to M, the result follows. $\square$

Remark 4.7. Surprisingly enough, for $(n, p) = (4, 3)$, the conclusion of the above corollary does not hold in general. In fact, if a closed oriented 4-manifold M has signature $\sigma(M) \neq \pm 1, \pm 2$, then M always admits a fold map into $\mathbf{R}^3$ (see [27]). Note that, in this case, $k = 1$ and $4k = n$.

Remark 4.8. In [19], it has been shown that if M is a closed manifold of odd Euler characteristic, then M cannot admit any fold map into $\mathbf{R}^p$ for $p \neq 1, 3, 7$ (see also [28]). We can use this to obtain a result similar to Corollary 4.4 for other dimension pairs as well. However, such a result is not useful for the proof of Corollary 4.6.

Example 4.9. Let us consider the 4-dimensional complex projective space $\mathbf{C}P^4$. By [19], if $\mathbf{C}P^4$ admits a fold map into $\mathbf{R}^p$, then p must be equal to $1, 3$ or 7, since $\mathbf{C}P^4$ has odd Euler characteristic. Clearly, it admits a fold map into $\mathbf{R}$. However, it cannot admit a fold map into $\mathbf{R}^7$, since

$$\binom{5}{1} = 5$$

is not a square (for details, see the proof of Corollary 4.4 or Remark 4.5). We do not know if $\mathbf{C}P^4$ admits a fold map into $\mathbf{R}^3$ or not.

Similar observations hold also for $\mathbf{C}P^6, \mathbf{C}P^2 \times \mathbf{C}P^2, \mathbf{H}P^2$, etc. (refer to [21] for the description of their Pontrjagin classes). Details are left to the reader.

Remark 4.10. Let $f : M \to N$ be a fold map of a closed orientable n-dimensional manifold into an orientable p-dimensional manifold such that $n - p + 1 = 2k$. Let us suppose that the singular set $S(f)$ of f is orientable. We give an arbitrary orientation to $S(f)$ and let $v \in H^{2k}(M; \mathbf{Z})$ be the cohomology class Poincaré dual to the homology class represented by $S(f)$. Then by [31], the modulo two reduction of v coincides with the Stiefel-Whitney class $w_{2k}(TM - f^*TN) \in H^{2k}(M; \mathbf{Z}_2)$. Furthermore, by Theorem 1.2, $v \smile v \equiv p_k(TM - f^*TN)$ modulo 2-torsion. Summarizing, we see that there exists an element $v \in H^{2k}(M; \mathbf{Z})$ whose modulo two reduction coincides with $w_{2k}(TM - f^*TN)$ such that $v \smile v \equiv p_k(TM - f^*TN)$ modulo 2-torsion.

The above result can also be proved as follows. By Ando [2], there exists a fold map $f : M \to N$ homotopic to a given smooth map $g : M \to N$ such that $S(f)$ is orientable if and only if there exists a fibrewise epimorphism $\varphi : TM \oplus \varepsilon^1 \to g^*TN$, where ε^1 is the trivial line bundle over M. Set $\eta = \ker \varphi$, which is an orientable vector bundle of rank $n - p + 1 = 2k$, and let v denote the Euler class of η. Then its modulo two reduction coincides with $w_{2k}(TM - f^*TN)$, and $v \smile v = p_k(TM - f^*TN)$ (for details, see [21], for example).

The above observation suggests that the formula (1.2) should be true without taking modulo 2-torsion.

Remark 4.11. So far, we have obtained several nonexistence results for fold maps. For the existence results, refer to the works of Levine [20], Èliašberg [9], [10] and Ando [2]. For the dimension pair $(n, p) = (4, 3)$, see [27].

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