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INVERSE FUNCTIONS OF POLYNOMIALS AND ORTHOGONAL POLYNOMIALS AS OPERATOR MONOTONE FUNCTIONS

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ABSTRACT. We study the operator monotonicity of the inverse of every polynomial with a positive leading coefficient. Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of orthonormal polynomials and p_{n+} the restriction of p_n to $[a_n, \infty)$, where a_n is the maximum zero of p_n. Then p_{n+}^{-1} and the composite $p_{n-1} \circ p_{n+}^{-1}$ are operator monotone on $[0, \infty)$. Furthermore, for every polynomial p with a positive leading coefficient there is a real number a so that the inverse function of $p(t+a) - p(a)$ defined on $[0, \infty)$ is semi-operator monotone, that is, for matrices $A, B \geq 0$, $(p(A+a) - p(a))^2 \leq ((p(B+a) - p(a))^2$ implies $A^2 \leq B^2$.

1. INTRODUCTION

Let A, B be bounded selfadjoint operators on a Hilbert space. Then $A \leq B$ means that $B - A$ is positive semi-definite by definition. A real-valued continuous function $f(t)$ defined on a finite or infinite interval I in $\mathbf{R}$ is called an *operator monotone function* on I if for every pair A, B whose spectra lie in the interval I, $A \leq B$ implies $f(A) \leq f(B)$. Likewise, a continuous function h on I is called an *operator concave function* on I if $h(sA + (1-s)B) \geq sh(A) + (1-s)h(B)$ for all A, B whose spectra lie in the interval I and for every $s : 0 \leq s \leq 1$. A nonnegative continuous function $f(t)$ on $[0, \infty)$ is operator monotone if and only if it is operator concave [4]. If a sequence of operator monotone functions converges to f pointwise on I, then f is also operator monotone. The sum of operator monotone functions is also an operator monotone function. By the Löwner theorem [7] (see also [4], [6]), f is operator monotone if and only if f has an analytic extension $f(z)$ to the open upper half plane Π_+ so that $f(z)$ maps Π_+ into itself. Thus if $f(t) \geq 0$ and $g(t) \geq 0$ are operator monotone, so is $f(t)^\mu g(t)^\lambda$ for $0 \leq \mu, \lambda \leq 1,\ \mu + \lambda \leq 1$. If $f(t)$ is operator monotone on (a, b) and continuous on $[a, b)$, then $f(t)$ is clearly operator monotone on $[a, b)$. It is well known that t^α $(0 < \alpha \leq 1)$, $\log t$ and $\frac{t}{t+\lambda}$ $(\lambda > 0)$ are operator monotone on $(0, \infty)$.

By Herglotz's theorem, an operator monotone function $f(t)$ on $(0, \infty)$ is represented as follows:

$$f(t) = a + bt + \int_0^\infty \left(-\frac{1}{x+t} + \frac{x}{x^2+1} \right) d\nu(x), \tag{1}$$

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where a, b are real constants with $b \geq 0$ and $d\nu$ is a nonnegative Borel measure on $[0, \infty)$ satisfying

$$\int_0^\infty \frac{d\nu(x)}{x^2+1} < \infty. \tag{2}$$

It is known that for the Gamma function Γ, $\frac{\Gamma'(t)}{\Gamma(t)}$ is operator monotone on $(0, \infty)$ (see p. 30 of [2]). Thanks to (1), we can see that if $f(t) \geq 0$ is operator monotone on $(0, \infty)$, so is $f(t^\alpha)^{1/\alpha}$ for $0 < \alpha < 1$. For further details on the operator monotone function we refer the reader to [1], [2], [5], [8].

The function $f(t)$ on $[0, \infty)$ is called a *semi-operator monotone function* if $f(t) \geq 0$ and $f(t^{1/2})^2$ is operator monotone on $[0, \infty)$. If $f(t)$ is semi-operator monotone, so is $f(t) + a$ for a scalar $a > 0$. By the Lőwner theorem, $f(t)$ is semi-operator monotone if and only if $f(t)$ has an analytic extension $f(z)$ to the open first quadrant Q and $f(z)$ maps Q into itself. It is also equivalent to each of the following:

$$\begin{aligned} 0 \leq A, B, \quad A^2 \leq B^2 \qquad &\Rightarrow \quad f(A)^2 \leq f(B)^2, \\ 0 \leq A, B, \quad ||A\mathbf{x}|| \leq ||B\mathbf{x}|| \qquad &\Rightarrow \quad ||f(A)\mathbf{x}|| \leq ||f(B)\mathbf{x}||. \end{aligned}$$

Operator monotone functions are useful in the study of electrical networks and in quantum physics. However, not many concrete examples of operator monotone functions are known despite the formula (1). We tried to systematically seek operator monotone functions in our previous papers [9], [10]. This paper is continued from them and concerned with polynomials: throughout this paper we assume that a polynomial means a *real polynomial whose leading coefficient is positive*. Our objective is to study the operator monotonicity of the inverse of every polynomial. First, we will deal with a sequence of orthonormal polynomials $\{p_n\}_{n=0}^\infty$ and show that p_{n+}^{-1} and the composite $p_{n-1} \circ p_{n+}^{-1}$ are both operator monotone on $[0, \infty)$, where a_n is the maximum zero of p_n and p_{n+} is the restriction of p_n to $[a_n, \infty)$. This deduces an interesting inequality. Second, we will study a more general function and show an extention of [10] which will be needed in the last section. Last, we will see that for every polynomial p there is a real number a so that the inverse function of $p(t+a) - p(a)$ defined on $[0, \infty)$ is semi-operator monotone, that is, for $A, B \geq a$, $(p(A) - p(a))^2 \leq ((p(B) - p(a))^2$ implies $(A - a)^2 \leq (B - a)^2$.

2. Orthogonal polynomials

In this section we mainly deal with real polynomials whose zeros are all real, especially orthgonal polynomials. First of all we recall the following:

Theorem A ([9]). *Let us define functions $u(t)$ on $[-a_1, \infty)$ and $v(t)$ on $[-b_1, \infty)$ by*

$$u(t) = \prod_{i=1}^k (t + a_i)^{\gamma_i}, \quad v(t) = \prod_{j=1}^l (t + b_j)^{\lambda_j},$$

where $a_1 < a_2 < \cdots < a_k$, $0 < \gamma_i$, and $b_1 < b_2 < \cdots < b_l$, $0 < \lambda_j$. If $\gamma_1 \geq 1$, then the inverse function $u^{-1}(s)$ is operator monotone on $[0, \infty)$. Moreover, if

$$\sum_{b_j < t} \lambda_j \leq \sum_{a_i < t} \gamma_i \quad \textit{for every} \;\; t \in \mathbf{R}, \tag{3}$$

then $v \circ u^{-1}(s)$ is operator monotone on $[0, \infty)$*, that is,*

$$u(A) \leq u(B) \quad (A, B \geq 0) \Rightarrow v(A) \leq v(B).$$

Condition (3) implies that $a_1 \leq b_1$; hence $v \circ u^{-1}(s)$ is well defined. Also, (3) is equivalent to

$$\sum_{b_j < a_{i+1}} \lambda_j \leq \gamma_1 + \cdots + \gamma_i \quad (i = 1, \cdots, k).$$

It is clear that for such a $u(t)$, $v(t) := u(t+c)$ with $c > 0$ satisfies (3). Namely, for $A, B \geq -a_1$ and for a scalar $c > 0$,

$$u(A) \leq u(B) \Rightarrow u(A+c) \leq u(B+c).$$

In case u and v are both polynomials, (3) means that the number of zeros of u in any interval $(-a_{i+1}, \infty)$ is not less than that of v, where zeros are counted according to their multiplicities. The following says that v may have imaginary zeros:

Proposition 2.1. *Define the function g by*

$$g(t) = \prod_{j=1}^{l} \{(t+b_j)^2 + c_j^2\}^{\lambda_j} \quad (c_j \geq 0).$$

For $u(t)$ given in Theorem A*, if*

$$\sum_{b_j < a_{i+1}} 2\lambda_j \leq \gamma_1 + \cdots + \gamma_i \quad (i = 1, \cdots, k),$$

then $g \circ u^{-1}$ is operator monotone.

Proof. Define a function $v(t)$ by

$$v(t) = \prod_{j=1}^{l} (t+b_j)^{2\lambda_j}.$$

By Theorem A, u^{-1}, $v \circ u^{-1}$ and $(t+b_j)^{2\lambda_j} \circ u^{-1}$ are all operator monotone. Therefore, letting $u^{-1}(z)$ be the analytic extension of u^{-1} to Π_+ and putting $w = u^{-1}(z)$ for each $z \in \Pi_+$, $(w+b_j)^{2\lambda_j}$ and $v(w)$ are all in Π_+. Since $0 < \arg\{(w+b_j)^2 + c_j^2\} \leq \arg(w+b_j)^2$, we have $0 < \arg g(w) \leq \arg v(w)$. This implies that the analytic extension $g \circ u^{-1}(z)$ maps Π_+ into itself. This completes the proof. $\square$

Let $d\mu$ be a positive Borel measure on $\mathbf{R}$ such that

$$\int_{-\infty}^{\infty} |t|^n d\mu(t) < \infty \text{ for } n = 0, 1, 2, \cdots.$$

Then there is a sequence of real polynomials $\{p_n\}_{n=0}^{\infty}$ with the following properties:

$$p_n(t) = c_n t^n + \cdots + c_0, \quad c_n > 0,$$
$$\int_{-\infty}^{\infty} p_n(t) p_m(t) d\mu(t) = \delta_{nm}.$$

This is called a *sequence of orthonormal polynomials associated with $d\mu$*. For instance, the sequence of Legendre polynomials is associated with $d\mu(t) = \chi_{[-1,1]} dt$ and that of Chebyshev polynomials with $2dt/(\pi\sqrt{1-t^2})$.

Proposition 2.2. *Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of orthonormal polynomials and p_{n+}^{-1} the inverse of the restriction p_{n+} of p_n to $[a_n, \infty)$, where a_n is the maximum zero of p_n. Then p_{n+}^{-1} and the composite $p_i \circ p_{n+}^{-1}$ are operator monotone on $[0, \infty)$ for $i = 1, \cdots, n-1$. Namely, for $a_n \leq A, B$,*

$$p_n(A) \leq p_n(B) \Rightarrow p_i(A) \leq p_i(B) \quad (i = 1, \cdots, n-1). \tag{4}$$

In particular, if the support of $d\mu$ is in $(-\infty, a]$, then (4) holds for every n and $A, B \geq a$.

Proof. It is known that each p_n has n simple zeros and there is one zero of p_{n-1} between any two consecutive zeros of p_n (see p. 61 of [3]). Thus, from Theorem A it follows that $p_{n+}^{-1}(s)$ and $p_{n-1}(p_{n+}^{-1}(s))$ are operator monotone. We can see that $p_i(p_{n+}^{-1}(s))$ is operator monotone as well. This gives (4). If the support of $d\mu$ is contained in $(-\infty, a]$, every zero of p_n is in this interval. Therefore, (4) holds for $A, B \geq a$ and for every n. □

Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of orthonormal polynomials associated with a measure with the support in $[-1, 1]$ like Legendre polynomials or Chebyshev polynomials, and let $q(t)$ be a polynomial whose degree is not larger than n. Suppose the real part of every zero of $q(t)$ is not larger than -1. Then, by applying Proposition 2.1 we have:

$$p_n(A) \leq p_n(B) \quad (A, B \geq 1) \quad \Rightarrow \quad q(A) \leq q(B).$$

Now we give an interesting inequality. Recall the definition of $P_2(a, b)$. A function $f(t)$ on (a, b) is said to be of class $P_2(a, b)$ if $f(A) \leq f(B)$ whenever $A \leq B$ for 2×2 matrices A, B with spectra in (a, b). An operator monotone function on (a, b) is evidently of class $P_2(a, b)$. It is well known that a C^1 function f is of class $P_2(a, b)$ if and only if

$$\left\{ \frac{f(t_1) - f(t_2)}{t_1 - t_2} \right\}^2 \leq f'(t_1) \cdot f'(t_2) \tag{5}$$

for every $t_1 \neq t_2$ in (a, b) (see pp. 75, 80 of [2]). Therefore, we have:
For a C^1 function $f(t)$ on (a, ∞) with $f'(t) > 0$, f^{-1} is of class $P_2(f(a), f(\infty))$ if and only if

$$\left\{ \frac{f(t_1) - f(t_2)}{t_1 - t_2} \right\}^2 \geq f'(t_1) \cdot f'(t_2). \tag{6}$$

Proposition 2.2 says that (6) holds for $f = p_{n+}$. Moreover, by putting $f = p_{i-1} \circ p_{i+}^{-1}$ in (5), we get

Corollary 2.3. *Under the same situation as* Proposition 2.2*,*

$$(t_1 - t_2)^2 \leq \frac{(p_2(t_1) - p_2(t_2))^2}{p_2'(t_1) p_2'(t_2)} \leq \cdots \leq \frac{(p_n(t_1) - p_n(t_2))^2}{p_n'(t_1) p_n'(t_2)}$$

for every $t_1 \neq t_2$ in (a_n, ∞).

At first sight, (6) seems to hold on $(0, \infty)$ for all polynomials with positive coefficients, but there is a counterexample:
Consider $f(t) = t^3 + t$ on $(0, \infty)$;then (6) is equivalent to $h(x, y) := x^4 + 2yx^3 - (5y^2 + 1)x^2 + (2y^3 + 2y)x + y^4 - y^2 \geq 0$, but $h(\frac{2}{5}, \frac{1}{3}) = -\frac{44}{3375} \cdot \frac{1}{15} < 0$.

This counterexample says that the inverse of t^3+t is not in $P_2(0,\infty)$. Moreover, we have:

For all $c,d>0$, the inverse function of $u_+(t)=ct+dt^3$ defined on $0\le t<\infty$ is not in $P_2(0,\infty)$.

We show this fact by giving a counterexample of a pair of 2×2 matrices $A,B\ge 0$ so that $dA^3+cA\ge dB^3+cB$ but $A\not\ge B$.

Example 2.1. We first consider the case $c=1/9, d=8/9$ for a technical reason. Put

$$A=\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad P=\begin{pmatrix} \frac{11}{32} & \alpha \\ \alpha & \frac{21}{32} \end{pmatrix},$$

where $(\frac{11}{32})^2+\alpha^2=\frac{11}{32}$. Then $P^2=P$ and

$$K:=\frac{8}{9}A+\frac{1}{9}A^3\ge P=\frac{8}{9}P+\frac{1}{9}P^3, \quad A\not\ge P,$$

because $\det(K-P)=\frac{2123}{9216}-\frac{231}{1024}=\frac{11}{2304}>0$ and $\det(A-P)=\frac{215}{1024}-\frac{231}{1024}<0$. Hence $A\not\ge P$. Next, define b by $b^2c=8d$; then for A,P given above,

$$\frac{c}{b}A+(\frac{d}{b^3})A^3=\frac{d}{b^3}(8A+A^3)\ge\frac{d}{b^3}(8P+P^3)=\frac{c}{b}P+(\frac{d}{b^3})P^3.$$

It will be shown in the last section that the above u_+^{-1} is semi-operator monotone. We end this section with a proposition on the special case where the larger side consists of a projection. This is simple but useful; in fact, taking account of it we constructed the above example.

Proposition 2.4. *Suppose that for $A,B\ge 0$ and for $c_n>0, c_i\ge 0$,*

$$c_1A+c_2A^2+\cdots+c_nA^n\ge c_1B+c_2B^2+\cdots+c_nB^n. \tag{7}$$

If $A=aP$ for a projection P and for a scalar $a>0$, then $A\ge B$.

Proof. We may assume $a=1$; in fact, when $a\ne 1$ we only need to rewrite (7) as

$$(c_1a)P+\cdots+(c_na^n)P^n\ge(c_1a)(B/a)+\cdots+(c_na^n)(B/a)^n$$

to get $A\ge B$. Put $b_i=c_i/(c_1+\cdots+c_n)$. Then (7) gives $P\ge b_1B+b_2B^2+\cdots+b_nB^n$. By the Jensen inequality or the Hölder inequality, $(B^j\mathbf{x},\mathbf{x})\ge(B\mathbf{x},\mathbf{x})^j$ for every unit vector $\mathbf{x}$. So we get

$$(P\mathbf{x},\mathbf{x})\ge\sum b_j(B\mathbf{x},\mathbf{x})^j, \quad \sum_{j=1}^n b_j=1,$$

from which it follows that if $(P\mathbf{x},\mathbf{x})=1$, then $(B\mathbf{x},\mathbf{x})\le 1$ and that if $(P\mathbf{x},\mathbf{x})=0$, then $(B\mathbf{x},\mathbf{x})=0$. Therefore $PBP\le P$ and $(1-P)B(1-P)=0$, which implies $B(1-P)=0$. Thus we obtain $A=P\ge PBP=B$. $\square$

As we saw in Example 2.1, in the case where B, not A, is a projection, (7) does not necessarily imply $A\ge B$.

3. General theorems

Before proceeding to the study of a polynomial with imaginary zeros, we need to extend our previous work [10]. We recall some results of it for the convenience of the reader and later reference.

Lemma B ([10]). *For $f(t)$ defined by* (1) *and* (2)*, suppose that*

$$f(\infty) = \lim_{t\to+\infty} f(t) < \infty.$$

Then, $b = 0$ and the function $x(x^2+1)^{-1}$ is integrable with respect to ν. Hence we have a representation

$$f(t) = f(\infty) - \int_0^\infty \frac{1}{x+t}\, d\nu(x) \quad (t > 0).$$

If we suppose moreover that $f(+0) = \lim_{t\to 0} f(t) > -\infty$, then the function x^{-1} is integrable with respect to ν.

Theorem C ([10]). *Let $h(t)$ be a real-valued differentiable function on (a,∞) and define*

$$u(t) = (t-a)^\gamma e^{h(t)} \qquad (a < t < \infty).$$

If $\gamma \geq 1$ and $-h'(t)$ is non-positive and operator monotone on (a,∞), then the inverse function $u^{-1}(s)$ is operator monotone on $(0,\infty)$.

We first state Theorem C in a different form.

Proposition 3.1. *Let $g(t)$ be a nonnegative and continuous function on $[a,\infty)$ that is differentiable on (a,∞) with $g'(t) > 0$. Put $u(t) = (t-a)^\gamma g(t)$ for $\gamma > 0$. If $-g'(t)/g(t)$ is operator monotone on (a,∞), then $u^{-1}(s^\gamma)$ is operator monotone on $0 \leq s < \infty$.*

Proof. Put $v(t) = (t-a)g(t)^{1/\gamma} = u(t)^{1/\gamma}$. Since

$$\frac{d}{dt}\log g(t)^{1/\gamma} = -\frac{\frac{d}{dt}g(t)^{1/\gamma}}{g(t)^{1/\gamma}} = -\frac{\frac{d}{dt}g(t)}{\gamma g(t)},$$

by Theorem C, $v^{-1}(s)$ is operator monotone on $(0,\infty)$ and hence on $[0,\infty)$, because it is continuous on $[0,\infty)$. The operator monotonicity of $u^{-1}(s^\gamma)$ follows from $u^{-1}(s^\gamma) = v^{-1}(s)$. $\square$

Since $u^{-1}(s^\gamma) = (u^{1/\gamma})^{-1}(s)$, $u^{-1}(s^\gamma)$ is operator monotone if and only if

$$u(A)^{1/\gamma} \leq u(B)^{1/\gamma} \quad (A, B \geq a) \Rightarrow A \leq B.$$

We remark that if $u^{-1}(s^\gamma)$ is operator monotone for $\gamma \geq 1$, so is $u^{-1}(s)$.

We consider a simple example: for $u(t)$ defined by $u(t) = t^{1/2}(t+1) \quad (t \geq 0)$, $u^{-1}(s)$ is then not operator monotone on $[0,\infty)$ as shown in [9]; however, the above proposition says that $u^{-1}(s^{1/2})$ is operator monotone there.

The following is the main theorem of this section, and it will be helpful in the study of polynomials in the last section.

Theorem 3.2. *Let $u(t)$ be a continuous function on $[a, \infty)$ that is differentiable on (a, ∞) with $u'(t) > 0$. Suppose the range of u is $[0, \infty)$. If $-u'(t)/u(t)$ is operator monotone on (a, ∞) and if*

$$\lim_{t \to a+0} (t-a)\frac{u'(t)}{u(t)} = \gamma > 0,$$

then the function $u^{-1}(s^\gamma)$ is operator monotone on $0 \leq s < \infty$. In particular, if $\gamma \geq 1$, then $u'(u^{-1}(s))$ is operator monotone on $0 < s < \infty$.

Proof. Assume first that $a = 0$. Then, by Lemma B, we have

$$-\frac{u'(t)}{u(t)} = -c - \int_0^\infty \frac{1}{x+t}\, d\nu(x), \tag{8}$$

where $c \geq 0$ and $x(x^2+1)^{-1}$ is integrable with respect to ν. Thus, by the assumption,

$$\gamma = \lim_{t \to +0} \int_0^\infty \frac{t}{x+t}\, d\nu(x).$$

Since ν is finite on each finite Borel set,

$$k(x) := \begin{cases} 1 & \text{if } 0 \leq x \leq 1, \\ \frac{1}{x} & \text{if } 1 \leq x \end{cases}$$

is integrable with respect to ν. Since $t/(x+t) \leq k(x)$ for $0 < t < 1$, we have

$$\gamma = \int_0^\infty \lim_{t \to +0} \frac{t}{x+t}\, d\nu(x) = \nu(\{0\}).$$

Denote the Dirac measure by δ and put $\mu = \nu - \gamma\delta$. Then μ is a positive Borel measure on $[0, \infty)$ and $x/(1+x^2)$ is integrable with respect to μ. Hence $\frac{1}{x+t}$ is integrable with respect to μ for each $t > 0$. By (8),

$$-c - \int_0^\infty \frac{1}{x+t}\, d\mu(x) = -\frac{u'(t)}{u(t)} + \frac{\gamma}{t}.$$

Putting $g(t) = u(t)/t^\gamma$, the right-hand side of the above equals $-g'(t)/g(t)$. Since the left-hand side is an operator monotone function on $0 < t < \infty$, so is $-g'(t)/g(t)$. By Proposition 3.1, $u^{-1}(s^\gamma)$ is hence operator monotone on $0 \leq s < \infty$. Assume next that $a \neq 0$. Putting $\tilde{u}(t) = u(t+a)$, we have

$$-\frac{\tilde{u}'(t)}{\tilde{u}(t)} = -\frac{u'(t+a)}{u(t+a)} \quad \text{and} \quad \lim_{t \to +0} t\frac{\tilde{u}'(t)}{\tilde{u}(t)} = \gamma.$$

$\tilde{u}^{-1}(s^\gamma)$ is therefore operator monotone, and hence so is $u^{-1}(s^\gamma) = \tilde{u}^{-1}(s^\gamma) + a$.

Suppose $\gamma \geq 1$. To prove the last statement of the theorem we may assume $a = 0$, for $\tilde{u}'(\tilde{u}^{-1}(s)) = u'(u^{-1}(s))$ with $\tilde{u}$ given above. By replacing t by $u^{-1}(s)$ in (8) and by multiplying both sides by s, we get

$$u'(u^{-1}(s)) = cs + \int_0^\infty \frac{s}{x + u^{-1}(s)} d\nu(x) \quad (s > 0). \tag{9}$$

Since $u^{-1}(s)$ is an operator monotone function defined on $0 < s < \infty$ with the range $(0, \infty)$, $u^{-1}(s^\alpha)^{1/\alpha}$ is operator monotone for every α in $(0, 1)$; this implies that $0 \leq \arg u^{-1}(z) \leq \arg z$ for $z \in \Pi_+$. Hence for each $x \geq 0$ and for all $z \in \Pi_+$,

$$0 \leq \arg z - \arg u^{-1}(z) \leq \arg z - \arg(x + u^{-1}(z)) \leq \arg z.$$

Thus, for each x the integrand of (9) has an analytic extension to Π_+ that maps Π_+ into itself; hence it is operator monotone on $0 < s < \infty$. This implies that $u'(u^{-1}(s))$ is operator monotone on $0 < s < \infty$. $\square$

Owing to Theorem 3.2 we are able to construct new operator monotone functions. Let us next consider the case of $a = -\infty$ in Theorem 3.2. Then, roughly speaking, $u(t)$ is an exponential function. We precisely have:

Proposition 3.3. *Let $u(t)$ be a positive, differentiable and increasing function on $(-\infty, \infty)$, and let $-u'(t)/u(t)$ be operator monotone on $(-\infty, \infty)$. Then $u(t) = c_1 e^{c_2 t}$ with $c_1, c_2 > 0$.*

Proof. By the Lőwner theorem, $-u'(t)/u(t)$ has an analytic extension to the whole space and maps the open upper (lower) half plane into itself; the range of the extension therefore does not contain $(0, \infty)$. By the Little Picard Theorem, $-u'(t)/u(t)$ is a constant. Thus we get the desired formula. $\square$

Proposition 3.4. *For $u(t)$ given in* Theorem 3.2*, suppose $\gamma \geq 1$. Let $v(t)$ be a nonnegative increasing function on $a < t < \infty$ and have an analytic extension $v(z)$ to Π_+. If the continuous branch of $\arg v(z)$ with $\arg v(t) = 0$ is nonnegative for $z \in \Pi_+$ and $\arg v(z) \leq \arg u(z)$ for z in Π_+, then $v \circ u^{-1}$ and $\log u(t) - \log v(t)$ are both operator monotone on $a < t < \infty$.*

Proof. $\log u(t) - \log v(t)$ has an analytic extension to Π_+ whose imaginary part is $\arg u(z) - \arg v(z) \geq 0$. By the open mapping theorem, the extended holomorphic function on Π_+ is constant or maps Π_+ into itself. Hence $\log u(t) - \log v(t)$ is operator monotone. Since $u^{-1}(s)$ is operator monotone, $u^{-1}(\Pi_+) \subset \Pi_+$. $v \circ u^{-1}(\Pi_+) \subset \Pi_+$ evidently follows from $0 \leq \arg v(z) \leq \arg u(z)$ for z in Π_+. Thus $v \circ u^{-1}$ is operator monotone. $\square$

As we mentioned in the first section, if $f(t) \geq 0$ is continuous on $[0, \infty)$, then f is operator monotone if and only if f is operator concave [4]. Now we slightly extend it to see that $u^{-1}(s^\gamma)$ in Theorem 3.2 is operator concave.

Proposition 3.5. *Let $f(t)$ be a continuous function on $[0, \infty)$, and let $f(\infty) > -\infty$. Then f is operator monotone if and only if f is operator concave.*

Proof. If $f(t)$ is operator monotone, then $f(t) - f(0)$ is nonnegative and operator monotone, because $f(t)$ is increasing. Thus it is operator concave; hence so is $f(t)$. Conversely, if $f(t)$ is operator concave, then $f(t)$ is naturally a concave function; hence $f(t)$ is increasing because of $f(\infty) > -\infty$. Since $0 \leq f(t) - f(0)$ is operator concave, one can see the operator monotonicity of $f(t)$. $\square$

4. Polynomials

This main section is devoted to the study of a polynomial with imaginary zeros. However, in the following theorem we treat a more general function $u(t)$, that is, the exponents γ and γ_i are not necessarily integers.

Theorem 4.1. *Let $u(t)$ be the function on $-a \leq t < \infty$ defined by*

$$u(t) = (t+a)^\gamma \prod_{i=1}^{k} (t+a_i)^{\gamma_i} \prod_{i=k+1}^{m} \{(t+a_i)^2 + b_i^2\}^{\gamma_i}, \tag{10}$$

where $a < a_i\,(i = 1, \cdots, k)$, $a \le a_i$, $0 < b_i, (i = k+1, \cdots, m)$, $0 < \gamma$, *and* $0 \le \gamma_i$. *Then the function* $u^{-1}(s^\gamma) + a$ *is semi-operator monotone on* $0 \le s < \infty$. *Furthermore, if* $a \le 0$, *then* $u^{-1}(s^\gamma)$ *is semi-operator monotone on* $0 \le s < \infty$.

Proof. Define a function $h(t)$ on $t \ge 0$ by $h(t) = u(\sqrt{t} - a)$. Then

$$h(t) = t^{\gamma/2} \prod_{i=1}^{k} (\sqrt{t} + c_i)^{\gamma_i} \prod_{i=k+1}^{m} \{(\sqrt{t} + c_i)^2 + b_i^2\}^{\gamma_i},$$

where $c_i = a_i - a \ge 0$. Since

$$t + c_i\sqrt{t} + b_i^2 \frac{\sqrt{t}}{\sqrt{t} + c_i}$$

is operator monotone on $0 \le t < \infty$, it is not difficult to see that $-h'(t)/h(t)$ is operator monotone on $0 \le t < \infty$ and that

$$\lim_{t \to +0} t \frac{h'(t)}{h(t)} = \gamma/2.$$

Therefore, by Theorem 3.2, $h^{-1}(s^{\gamma/2}) = (u^{-1}(s^{\gamma/2}) + a)^2$ is operator monotone on $0 \le s < \infty$. This implies that $u^{-1}(s^\gamma) + a$ is semi-operator monotone. Therefore, for $a \le 0$, $u^{-1}(s^\gamma) = (u^{-1}(s^\gamma) + a) - a$ is also semi-operator monotone. $\square$

By applying real polynomials to the above theorem we can easily obtain the following:

Theorem 4.2. *If a polynomial* p *has all zeros in* $\{z : \Re(z) \le -a\}$ *and* $-a$ *is a real zero with order* γ, *then* $p_+^{-1}(s^\gamma) + a$ *and* $p_+^{-1}(s) + a$ *are both semi-operator monotone on* $0 \le s < \infty$, *where* p_+ *is the restriction of* $p(t)$ *to* $[-a, \infty)$. *Furthermore, if* $a \le 0$, *then* $p_+^{-1}(s^\gamma)$ *and* $p_+^{-1}(s)$ *are also semi-operator monotone on* $0 \le s < \infty$.

In the above theorem we assumed that all zeros of p are in $\{z : \Re(z) \le -a\}$. The following example shows that we cannot remove this condition from the theorem.

Example 4.1. Put $p(t) = t^3 + 1$. Then $p(-1) = 0$, $p'(t) > 0$ on $(-1, \infty)$. We show that $(p_+^{-1}(s^{1/2}) + 1)^2$ is not operator monotone on $0 \le s < \infty$. To do it, we give a pair of operators A, B so that $-1 \le A, B$, $p(A)^2 \le p(B)^2$ but $(A+1)^2 \not\le (B+1)^2$. Set

$$A = -\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}.$$

Then $A \ge -1$, $B \ge -1$ and $B - A \not\ge 0$; hence $(A+1)^2 \not\le (B+1)^2$. However,

$$\begin{aligned} p(B)^2 - p(A)^2 &= \begin{pmatrix} (\frac{1}{3})^3 + 1 & 0 \\ 0 & -(\frac{1}{4})^3 + 1 \end{pmatrix}^2 - \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}^2 \\ &= \begin{pmatrix} (\frac{28}{27})^2 & 0 \\ 0 & (\frac{63}{64})^2 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{839}{1458} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1921}{4096} \end{pmatrix} \ge 0. \end{aligned}$$

$\square$

From now on, we deal with a polynomial p whose zeros are all in $\{z : \Re(z) \le -a\}$ but $p(-a) \ne 0$. Then p' has all zeros in $\{z : \Re(z) \le -a\}$ as well; so $p(t)$ is increasing on $(-a, \infty)$.

Lemma 4.3. *Let p be a real polynomial with degree less than 5, and let the zeros of p all be in $\{z : \Re(z) \leq -a\}$. Then the zeros of $p(t)-p(-a)$ are all in $\{z : \Re(z) \leq -a\}$ as well.*

Proof. The case where the degree is less than 4 is trivial; so we assume that the degree is 4. We may also assume $a = 0$. It is clear that a quadratic real polynomial $t^2 + \alpha t + \beta$ has both zeros in $\{z : \Re z \leq 0\}$ if and only if $\alpha \geq 0$, $\beta \geq 0$. Thus we can write $p(t)$ as $p(t) = (t^2 + bt + c)(t^2 + dt + e)$ with $b, c, d, e \geq 0$. Hence

$$p(t) - p(0) = t\{t^3 + (b+d)t^2 + (c+e+bd)t + cd + be\}. \tag{11}$$

Since all the zeros of $p'(t)$ are also in $\{z : \Re z \leq 0\}$, we have $p'(t) > 0$ for $t > 0$, which implies that $p(t) > 0$ for $t > 0$. Therefore, all real zeros of $p(t) - p(0)$ are non-positive. We can write the factorization of $p(t) - p(0)$ as follows:

$$p(t) - p(0) = t(t + \lambda)(t^2 + \alpha t + \beta), \tag{12}$$

where $\lambda \geq 0$, and α and β are real. By (11) and (12), we get $b + d = \lambda + \alpha$, $c + e + bd = \lambda\alpha + \beta$ and $cd + be = \lambda\beta$, from which it follows that $\beta \geq 0$, $\alpha \geq 0$. Hence the zeros of $p(t) - p(0)$ are all in $\{z : \Re z \leq 0\}$. □

By virtue of this lemma and Theorem 4.2, we obtain

Corollary 4.4. *Let p be a real polynomial whose leading coefficient is positive. If the zeros of p are all in $\{z : \Re z \leq -a\}$, and if the degree of p is less than 5, then the inverse of $s = p(t) - p(-a)$ defined on $-a \leq t < \infty$ is semi-operator monotone.*

The condition on the degree in Lemma 4.3 is necessary: indeed, our computer says that $p(t) = t^5 + t^4 + 4t^3 + 3t^2 + \frac{15}{4}t + 2$ has all zeros in $\{z : \Re z < 0\}$, but $p(t) - p(0) = t(t^4 + t^3 + 4t^2 + 3t + \frac{15}{4})$ has two zeros in $\{z : \Re z > 0\}$. Therefore, we cannot extend Corollary 4.4 to higher-degree polynomials in the same way. However, by parallel translation we can estimate the semi-operator monotonicity of inverse of every polynomial. We precisely have

Theorem 4.5. *Let p be a real polynomial whose leading coefficient is positive. Let $\alpha_k = a_k + ib_k \quad (k = 1, 2, \cdots, n)$ be the zeros of p. Then for a real number a such that $a - a_k > |b_k|$ for every k, the inverse of $s = p(t+a) - p(a)$ defined on $0 \leq t < \infty$ is semi-operator monotone on $0 \leq s < \infty$; that is, for $A, B \geq 0$,*

$$(p(A + a) - p(a))^2 \leq (p(B + a) - p(a))^2 \quad \Rightarrow A^2 \leq B^2.$$

Proof. Put $q(t) = p(t + a) - p(a)$. Then

$$q(z) = \prod_{k=1}^{n}(z + a - \alpha_k) - \prod_{k=1}^{n}(a - \alpha_k).$$

We will show that $|q(z)| > 0$ for $\Re z > 0$. Suppose $\Re z > 0$, and set $z = x + iy$. If α_k is real, then $|z + a - \alpha_k| = |x + iy + a - \alpha_k| \geq |x + a - \alpha_k| > |a - \alpha_k|$, because $a - \alpha_k > 0$. If α_k is an imaginary number, then $\overline{\alpha}_k$ is also a zero of p and by $a - a_k > |b_k|$,

$$\begin{aligned} |(z \quad + \quad & a - a_k)(z + a - \overline{a_k})| = |x + a - a_k + i(y - b_k)||x + a - a_k + i(y + b_k)| \\ > \quad & \{(a - a_k)^2 + (y - b_k)^2\}^{1/2}\{(a - a_k)^2 + (y + b_k)^2\}^{1/2} \\ \geq \quad & (a - a_k)^2 + b_k^2 = |(a - a_k)(a - \overline{a_k})|, \end{aligned}$$

where the second inequality can be shown by straightforward computation. By taking the products of two cases, we have

$$|g(z)| \geq \prod |z+a-\alpha_k| - \prod |a-\alpha_k| > 0 \quad (\Re z > 0),$$

which implies that $g(z) \neq 0$ for $\Re z > 0$. Since $g(0) = 0$, by Theorem 4.2, $g^{-1}(s)$ is semi-operator monotone on $0 < s < \infty$. □

In general, it is not easy to find the real number a in the theorem. So, the following corollary might be helpful.

Corollary 4.6. *Suppose $p(t) = a_n t^n + a_{n-1} + \cdots + a_0$ with $a_k > 0$ for each k. Set $\gamma = \max\{\frac{a_k}{a_{k+1}} : k = 0, 1, \cdots, n-1\}$. Then for any a so that $a > \sqrt{2}\gamma$, the inverse of $s = p(t+a) - p(a)$ defined on $0 \leq t < \infty$ is semi-operator monotone.*

Proof. By the Eneström-Kakeya theorem, all the zeros of p lie in $\{|z : |z| \leq \gamma\}$ (see p. 13 of [3]). Since $\sqrt{2}\gamma \geq |x| + |y|$ for $x + iy$ in $\{z : |z| \leq \gamma\}$, this corollary follows from the theorem. □

Now we are in a position to consider a composite function $q \circ p_+^{-1}$.

Lemma 4.7. *Let a, b, c and d be nonnegative real numbers. Then, in order that for every z in Q,*

$$\arg((z+a)^2 + b^2) \geq \arg((z+c)^2 + d^2), \tag{13}$$

it is necessary and sufficient that

$$c \geq a > 0, \quad a(c^2 + d^2) - c(a^2 + b^2) \geq 0,$$

or

$$c = a = 0, \quad d \geq b.$$

Proof. Put $z = x + iy$.

$$\begin{aligned} (13) \quad &\Leftrightarrow \quad 0 \leq \arg \frac{(z+a)^2 + b^2}{(z+c)^2 + d^2} < \pi \\ &\Leftrightarrow \quad 0 \leq \Im\{(z+a)^2 + b^2\}\{(\overline{z}+c)^2 + d^2\} \\ &\Leftrightarrow \quad 0 \leq x^2(c-a) + x(c^2 + d^2 - a^2 - b^2) + y^2(c-a) \\ &\qquad\quad +a(c^2 + d^2) - c(a^2 + b^2) \\ &\Leftrightarrow \quad c > a \text{ and } a(c^2+d^2) - c(a^2+b^2) \geq 0, \quad \text{or} \quad c = a \text{ and } d^2 \geq b^2. \end{aligned}$$

We consequently obtain the desired condition. □

For $\alpha = (a, b)$ and $\beta = (c, d)$ in the closure of Q, we write $\alpha \preceq \beta$ if (13) is satisfied, and $\alpha \prec \beta$ if $\alpha \preceq \beta,\ \alpha \neq \beta$.

"$\preceq$" is clearly an order and $0 = (0, 0) \preceq \beta$ for any $\beta = (c, d)$ in the closure of Q. The following is a simple example of p and q so that $q \circ p_+^{-1}$ is semi-operator monotone:

Consider $p(t) = t(t^2 + t + 1)(t^2 + at + b)$ and $q(t) = (t+1)^2(t^2 + ct + d)$, where a, b, c and d are nonnegative. Also, denote the restriction of p to $0 \leq t < \infty$ by p_+. If $a \leq c$ and $ad \geq bc$ or if $a = c = 0$ and $b \leq d$, then $q \circ p_+^{-1}$ is semi-operator monotone.

Proof. By Theorem 4.2, p_+^{-1} is semi-operator monotone. For each z in Q set $w = p_+^{-1}(z)$. Since w is in Q, $\arg(w+1) \leq \arg w$ and $\arg(w+1) \leq \arg(w^2+w+1)$. The above lemma says that $\arg(w^2+cw+d) \leq \arg(w^2+aw+b)$. Thus we have $0 < \arg q(w) \leq \arg p(w) = \arg z < \pi$. $q \circ p_+^{-1}$ is consequently semi-operator monotone. □

We end this paper with the following theorem:

Theorem 4.8. *Let $p(t)$ and $q(t)$ be functions defined on $[0,\infty)$ by*

$$p(t) = \Pi_{i=1}^{m}\{(t+a_i)^2+b_i^2\}^{\gamma_i}, \tag{14}$$

$$q(t) = \Pi_{j=1}^{n}\{(t+c_j)^2+d_j^2\}^{\lambda_j}, \tag{15}$$

where $a_1 = b_1 = 0, a_i, b_i, c_i, \gamma_i$ and λ_i are all nonnegative. Denote the points (a_i,b_i) and (c_i,d_i) by α_i and β_i, respectively. Let $\{z_i\}_{i=1}^{l}$ be a totally ordered set in the closure of Q such that $0 = z_1 \prec z_2 \prec \cdots \prec z_l$ and define Γ_k, Λ_k $(k=1,\cdots,l)$ by

$$\Gamma_k = \sum_{\alpha_i \preceq z_k} \gamma_i, \quad (k=1,\cdots,l),$$

$$\Lambda_1 = \sum_{z_2 \npreceq \beta_j} \lambda_j, \cdots, \Lambda_{l-1} = \sum_{z_l \npreceq \beta_j} \lambda_j, \Lambda_l = \sum_{j=1}^{n} \lambda_j.$$

If $\gamma_1 \geq \frac{1}{2}$ and $\Gamma_k \geq \Lambda_k$ $(k=1,2,\cdots,l)$, then $q \circ p_+^{-1}$ is semi-operator monotone.

Proof. Since p^{-1} is semi-operator monotone, its analytic extension $p^{-1}(z)$ maps Q into itself. It is clear that $q(t)$ has an analytic extension $q(z)$ to Q and its argument is not negative. So we have only to show that $\arg q(p^{-1}(z)) \leq \arg z$ for every $z \in Q$. Put $w = p^{-1}(z)$ and denote $\arg((w+a)^2+b^2)$ by $\arg\langle w;\alpha\rangle$ with $\alpha = (a,b)$ for convenience. Note that $\arg\langle w;\alpha\rangle \geq \arg\langle w;\beta\rangle$ for $w \in Q$ whenever $\alpha \preceq \beta$. Then

$$\begin{aligned}
\arg q(p^{-1}(z)) &= \sum_{j=1}^{n} \lambda_j \arg\langle w;\beta_j\rangle \\
&= \sum_{k=1}^{l-1} \sum_{\substack{z_k \preceq \beta_j \\ z_{k+1} \npreceq B_j}} \lambda_j \arg\langle w;\beta_j\rangle + \sum_{z_l \preceq \beta_j} \lambda_j \arg\langle w;\beta_j\rangle \\
&\leq \sum_{k=1}^{l-1} \sum_{\substack{z_k \preceq \beta_j \\ z_{k+1} \npreceq \beta_j}} \lambda_j \arg\langle w;z_k\rangle + \sum_{z_l \preceq \beta_j} \lambda_j \arg\langle w;z_l\rangle \\
&\leq \Lambda_1 \arg\langle w,z_1\rangle + \sum_{k=2}^{l} (\Lambda_k - \Lambda_{k-1}) \arg\langle w;z_k\rangle \\
&= \sum_{k=1}^{l-1} \Lambda_k(\arg\langle w;z_k\rangle - \arg\langle w;z_{k+1}\rangle) + \Lambda_l \arg\langle w;z_l\rangle \\
&\leq \sum_{k=1}^{l-1} \Gamma_k(\arg\langle w;z_k\rangle - \arg\langle w;z_{k+1}\rangle) + \Gamma_l \arg\langle w;z_l\rangle \\
&= \Gamma_1 \arg\langle w;z_1\rangle + \sum_{k=2}^{l} (\Gamma_k - \Gamma_{k-1}) \arg\langle w;z_k\rangle
\end{aligned}$$

$$\begin{aligned}
&= \Big(\sum_{\alpha_i \preceq z_1} \gamma_i\Big) \arg\langle w; z_1\rangle + \sum_{k=2}^{l}\Big(\sum_{\substack{\alpha_i \preceq z_k \\ \alpha_i \not\preceq z_{k-1}}} \gamma_i\Big) \arg\langle w; z_k\rangle \\
&= \sum_{\alpha_i \preceq z_1} \gamma_i \arg\langle w; z_1\rangle + \sum_{k=2}^{l} \sum_{\substack{\alpha_i \preceq z_k \\ \alpha_i \not\preceq z_{k-1}}} \gamma_i \arg\langle w; z_k\rangle \\
&\leq \sum_{\alpha_i \preceq z_1} \gamma_i \arg\langle w; \alpha_i\rangle + \sum_{k=2}^{l} \sum_{\substack{\alpha_i \preceq z_k \\ \alpha_i \not\preceq z_{k-1}}} \gamma_i \arg\langle w; \alpha_i\rangle \\
&\leq \sum_{i=1}^{m} \gamma_i \arg\langle w; \alpha_i\rangle = \arg p(w) = \arg z.
\end{aligned}$$

Thus the proof is complete. □

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