TWISTED SUMS WITH $C(K)$ SPACES

F. CABELLO SÁNCHEZ, J. M. F. CASTILLO, N. J. KALTON, AND D. T. YOST

Abstract. If $X$ is a separable Banach space, we consider the existence of non-trivial twisted sums $0 \to C(K) \to Y \to X \to 0$, where $K = [0,1]$ or $\omega^\omega$. For the case $K = [0,1]$ we show that there exists a twisted sum whose quotient map is strictly singular if and only if $X$ contains no copy of $\ell_1$. If $K = \omega^\omega$ we prove an analogue of a theorem of Johnson and Zippin (for $K = [0,1]$) by showing that all such twisted sums are trivial if $X$ is the dual of a space with summable Szlenk index (e.g., $X$ could be Tsirelson’s space); a converse is established under the assumption that $X$ has an unconditional finite-dimensional decomposition. We also give conditions for the existence of a twisted sum with $C(\omega^\omega)$ with strictly singular quotient map.

1. Introduction and preliminary remarks

Let $X$ and $Y$ be real Banach spaces. Then we say $\text{Ext}(X,Y) = \{0\}$ if every short exact sequence $0 \to Y \to Z \to X \to 0$ splits; informally this means that if $Z$ is a Banach space containing $Y$ and so that $Z/Y \sim X$, then there is a bounded projection of $Z$ onto $Y$. A space $Z$ with a subspace isomorphic to $Y$ so that $Z/Y$ is isomorphic to $X$ is often called a twisted sum of $Y$ and $X$ (order is important!). Thus $\text{Ext}(X,Y) = \{0\}$ if and only if every twisted sum of $Y$ and $X$ is trivial (i.e. reduces to $Y \oplus X$).

Fundamental tools for us are the pushout and pullback constructions. These are well-known to algebraists and topologists, but less so to analysts. So we will describe them briefly in the Banach space setting. If $T : E \to X$ and $S : E \to Y$ are two operators defined on the same Banach space, then their pushout $Z$ is defined as the quotient of $X \oplus Y$ by the closure of $\{(T e, -S e) : e \in E\}$, together with the natural mappings $X \to Z$ and $Y \to Z$ (i.e., the restrictions of the quotient mapping). In case one of the mappings, say $S$, is the inclusion mapping from a short exact sequence, then completing the diagram gives a second short exact sequence with the same quotient space $F$:

$$
\begin{array}{ccccccc}
0 & \to & E & \xrightarrow{S} & Y & \to & F & \to & 0 \\
0 & \to & X & \to & Z & \to & F & \to & 0.
\end{array}
$$

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Conversely, if we are given any commutative diagram as above, then $Z$ must be isomorphic to the pushout of $S$ and $T$; this observation will be used several times in the sequel. Note also that the operator $Y \to Z$ is an isomorphic embedding (respectively a quotient mapping) if and only if $T$ is. Furthermore, the lower sequence splits if and only if $T$ can be extended to $Y$. These well-known exercises follow from standard diagram-chasing arguments.

Dually, if $S : X \to E$ and $T : Y \to E$ are two operators into the same Banach space, then their pullback $Z$ is defined as the subspace of all $(x, y) \in X \oplus Y$ for which $Sx = Ty$, together with the natural mappings $Z \to Y$ and $Z \to X$. In case one of the original mappings, say $S$, is the quotient mapping from a short exact sequence, then completing the diagram gives a second short exact sequence with the same subspace $F$:

$$
\begin{array}{ccc}
0 & \to & F & \to & Z & \to & Y & \to & 0 \\
\end{array}
$$

Conversely, if we are given any commutative diagram as above, then $Z$ must be isomorphic to the pullback of $S$ and $T$. Note again that the operator $Z \to X$ is an isomorphic embedding (respectively a quotient mapping) if and only if $T$ is. For further information, see [16, Chap. 1] and the references therein.

Let $X$ be any separable Banach space and let $Q_X : \ell_1 \to X$ be any quotient map. We will keep the notation $\hat{X}$ for the kernel of $Q_X$ (which is unique up to automorphism provided it is infinite dimensional, see [35], [36, p. 108] or [15, p. 382]). The following theorem is well known:

**Theorem 1.1.** Suppose $X$ and $Y$ are separable Banach spaces. Then the following are equivalent:

1. $\text{Ext}(X, Y) = \{0\}$.
2. If $T : \hat{X} \to Y$ is a bounded operator, then there is a bounded extension $\hat{T} : \ell_1 \to Y$.
3. If $Z$ is a separable Banach space containing a subspace $E$ so that $Z/E \sim X$ and $T : E \to Y$ is a bounded operator, then there is an extension $\hat{T} : Z \to Y$.

**Proof.** It is trivial that (3) implies (1). For (1) implies (3) we use the pushout construction:

$$
\begin{array}{ccc}
0 & \to & E & \to & Z & \to & X & \to & 0 \\
\end{array}
$$

Now (1) implies the existence of a projection $P : W \to Y$, and then $PS$ extends $T$.

That (2) is equivalent to (3) is clear from the proof of Corollary 1.1 of [20]. Alternatively, [30, Prop. 3.1] proves directly the equivalence of (1) and (2). □

**Remark.** Of course all separability assumptions can be removed if we simply replace $\ell_1$ by $\ell_1(I)$ for a suitable index set.

There is an immediate corollary, which essentially says that $\text{Ext}(X, Y) = \{0\}$ is a three-space property of $X$:

**Corollary 1.2.** Suppose $Y$ is a Banach space and $X$ is a Banach space with a subspace $E$ so that $\text{Ext}(E, Y) = \{0\}$, and $\text{Ext}(X/E, Y) = \{0\}$. Then $\text{Ext}(X, Y) = \{0\}$. 
Proof. Let \( \widetilde{X} \) and \( Q_X \) be defined as above. Given \( T : \widetilde{X} \to Y \), we need to find an extension to all of \( \ell_1 \). We will apply Theorem 1.1.

If \( Q : X \to X/E \) is the obvious mapping, we may choose \( \widetilde{X}/E \) to be the kernel of \( Q \circ Q_X \). Then \( y \mapsto Q_X y \) is a quotient mapping from \( \widetilde{X}/E \) onto \( E \) with kernel \( \widetilde{X} \).

The implication (1) \( \Rightarrow \) (3) then gives us an extension \( \widetilde{T} : \widetilde{X}/E \to Y \) of \( T \), which by the implication (1) \( \Rightarrow \) (2) admits a further extension \( \widetilde{T} : \ell_1 \to Y \). \( \square \)

In this paper, we consider the case when the subspace of our twisted sum is \( C(K) \) for some compact metric space \( K \). If \( K \) is uncountable, then the theorem of Milutin [40, Theorem 8.5] implies we may consider \( K = [0, 1] \).

The following result is due to Johnson and Zippin [26], in view of Theorem 1.1:

Theorem 1.3. If \( X \) is isomorphic to the dual of a subspace of \( c_0 \) (so that \( \widetilde{X} \) can be assumed weak*-closed), then \( \text{Ext}(X, C(K)) = \{0\} \) for every compact \( K \).

In [28] the following converses were found. Throughout this paper, we will use (FDD) to indicate a finite-dimensional Schauder decomposition and (UFDD) to indicate an unconditional finite-dimensional Schauder decomposition. Recall also that \( X \) is said to have the strong Schur property if there is a constant \( c > 0 \) so that for any normalized sequence \((x_n)\) with \( \|x_m - x_n\| \geq \delta > 0 \) for any \( m \neq n \), there exists a subsequence \((x_{n_k})_{k \in \mathbb{N}}\) such that

\[
\left\| \sum_{k \in \mathbb{N}} \alpha_k x_k \right\| \geq c \delta \sum_{k \in \mathbb{N}} |\alpha_k|
\]

for any finitely supported sequence \((\alpha_k)_{k \in \mathbb{N}}\).

Theorem 1.4. If \( X \) is separable and \( \text{Ext}(X, C[0, 1]) = \{0\} \), then \( X \) has the strong Schur property. If \( X \) also has a (UFDD), then \( X \) is isomorphic to the dual of a subspace of \( c_0 \).

Let us remark at this point that Bourgain and Pisier [9] (cf. [16, §1.8]) showed that for any separable Banach space \( X \) that is not an \( L_\infty \)-space there is a space \( Y \) that is an \( L_\infty \)-space so that \( Y \) contains \( X \) as an uncomplemented subspace and \( Y/X \) has the Schur property and the Radon-Nikodým property.

Recall that an operator is called strictly singular if its restriction to an infinite-dimensional subspace of its domain is never an isomorphic embedding. In Section 2 we consider the problem of characterizing those separable Banach spaces \( X \) for which there is a short exact sequence \( 0 \to C[0, 1] \to Z \to X \to 0 \) so that the quotient map is strictly singular. We show in Theorem 2.3 that this is equivalent to the requirement that \( X \) contains no copy of \( \ell_1 \).

In Section 3 we consider quantitative results for the case \( K = \omega^N \). In this case \( C(K) \) is isomorphic to \( c_0 \), so that \( \text{Ext}(X, C(K)) = \{0\} \) for every separable \( X \) by Sobczyk’s theorem [43], but it is still worthwhile to consider projection constants. We need the following elementary result; we recall that \( Z \) is said to be separably injective if it is complemented in every separable superspace. As usual, \( I_X \) indicates the identity on a given Banach space \( X \).

Proposition 1.5. Let \( X \) be any separable Banach space, let \( Z \) be a separably injective Banach space and let \( k \) be a constant. Then the following are equivalent:

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(1) If $Y$ is a separable Banach space and $E$ is a closed subspace with $Y/E$ isometric to $X$, then for any bounded linear operator $T : E \to Z$ and any $\varepsilon > 0$, there is an extension $\tilde{T} : Y \to Z$ with $\|\tilde{T}\| < k\|T\| + \varepsilon$.

(2) If $0 \to Z \xrightarrow{j} Y \xrightarrow{q} X \to 0$ is an (isometric) exact sequence and any $\varepsilon > 0$, then there is a linear operator $P : Y \to Z$ with $Pj = I_Z$ and $\|P\| \leq k + \varepsilon$.

Proof. It is clear from the definition that if the short exact sequence is given, then we may find such a $P$ with $\|P\| \leq k + \varepsilon$. Conversely, suppose $Y$ is a separable Banach space and $E$ is a closed subspace with $Y/E$ isometric to $X$. If $T : E \to Z$ is an operator with $\|T\| \leq 1$, we form the pushout:

\[
\begin{array}{cccccc}
0 & \to & E & \xrightarrow{j} & Y & \xrightarrow{\tilde{T}} & X & \xrightarrow{S} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \| & & \\
0 & \to & Z & \xrightarrow{j'} & PO & \to & X & \to & 0
\end{array}
\]

Then, if $P : PO \to Z$ satisfies $Pj' = I_Z$, we see that $PS = \tilde{T}$ extends $T$ and $\|PS\| \leq \|P\|$.

Our results build on earlier work of Amir and Baker, who showed that the separable projection constant of $C(\omega^N)$ is $2N + 1$, [2], [3] and [4]. In particular, we show that, given any $\varepsilon > 0$, there is a space $Z$ containing $C(\omega^N)$ isometrically so that $X/C(\omega^N)$ is isometric to $c_0$ and the norm of any projection is at least $2N + 1 - \varepsilon$. However, our main motivation in Section 3 is to provide the necessary groundwork to study the case $K = \omega^\omega$, which is done in Section 4. Here we show results parallel to Theorems 1.3 and 1.4 above. We show that if $X$ is the dual of a space with summable Szlenk index [31], [23 §2], then Ext$(X, C(\omega^\omega)) = \{0\}$, and this condition is necessary if $X$ has a (UFDD). An example of such an $X$ is Tsirelson’s space [31].

We also consider the possibility of Ext$(X, C(\omega^\omega))$ being large in the sense that there is a twisted sum $0 \to C(\omega^\omega) \to Z \to X \to 0$ for which the quotient map is strictly singular. We show that a sufficient condition for the construction of such a short exact sequence is that $X$ has a shrinking (UFDD) and contains no subspace that is the dual of a space with summable Szlenk index. This leads to new counterexamples for several old problems.

We refer to [16] and [29] for a discussion of twisted sums in general. Let us note that in Section 4 it is important to consider twisted sums in the isometric category rather than the isomorphic category; hence the standard pushout and pullback constructions were defined above isometrically. Of course any isomorphic twisted sum can be equivalently renormed to an isometric twisted sum.

2. A universal twisted sum

**Theorem 2.1.** Suppose $X$ is a separable Banach space. Then there is a universal short exact sequence $0 \to C[0,1] \to Y \to X \to 0$ such that every short exact sequence $0 \to C[0,1] \to Z \to X \to 0$ can be identified with a pushout, i.e., there exist linear operators $S : C[0,1] \to C[0,1]$ and $S_1 : Y \to Z$ so that the following
Since it is unique up to automorphism, we may choose $e$.

We need the well-known result that there is a non-trivial twisted sum of $C[0,1]$ and $c_0$. The first published reference we know is [22, Theorem 6]. In [1] a stronger statement about the non-existence of Lipschitz liftings is proved; a non-separable version is to be found in [18]. The example, also studied in [27], can be described as follows. Let $Q = (q_n)$ be any dense sequence in $[0,1]$. We could for example order the rationals in $(0,1)$ into a sequence $(q_n)$, but we prefer not to be specific. Denote by $B$ the set of all functions from $[0,1]$ into $\mathbb{R}$ that are continuous at every $t \notin Q$ and left continuous with right limits at every $t \in Q$. Routine arguments show that all such functions are bounded and that the sup-norm makes $B$ into a Banach space. Clearly $C = C[0,1]$ is a closed subspace and $D/C$ is isometric to $c_0$. More precisely, let us denote by $J : D \to c_0$ the “jump function” $Jf = 1/2(f(q_n+) - f(q_n))$. Then $J$ maps $D$ onto $c_0$, and $d(f,C) = ||Jf||$ for all $f \in D$. We denote by $e_n$ the usual basis in $c_0$. It is well known [6, p. 33], [27, p. 20] that $D$ is isometric to the space of continuous functions on the Cantor set, but we do not need this representation.

**Lemma 2.2.** Let $(f_n)$ be any sequence of functions in $D$ for which $J(f_n) = e_n$ for all $n$. Then the sequence $(f_n)$ is not weakly Cauchy.
Proof. The assumption $J(f_n) = e_n$ means that $f_n(q_n) - f_n(q_m) = 2$ for all $n$. Let us assume $(f_n)$ is weakly Cauchy and hence bounded. We first note that if $I$ is any nonempty open interval in $(0,1)$, $\alpha \in \mathbb{R}$ and $m \in \mathbb{N}$, then there exist $n > m$ and a nonempty open interval $J$ with $\mathcal{J} \subset I$ such that for some $\beta$ with $|\beta - \alpha| \geq 1$, we have $|f_n(t) - \beta| \leq \frac{1}{4}$ for $t \in J$. Indeed, we just pick $n > m$ so that $q_m \in I$, and then let $\beta$ be either $f_n(q_n)$ or $f_n(q_m)$. The interval $J$ can then be chosen using the left- or right-hand limit condition.

Now we can use this inductively to create a subsequence $(f_{n_k})$ of $(f_n)$, a sequence of nonempty intervals $(I_k)$ with $I_{k+1} \subset I_k$, and a sequence of reals $(\alpha_k)$ with $|\alpha_{k+1} - \alpha_k| \geq 1$ so that $|f_{n_k}(t) - \alpha_k| \leq \frac{1}{4}$ for $t \in I_k$. If we pick $t_0 \in \bigcap_{k=1}^{\infty} I_k$ (which is nonempty by compactness), it is clear that $|f_{n_k}(t_0) - f_{n_{k+1}}(t_0)| \geq \frac{1}{2}$ for all $k$, and this gives us a contradiction.

**Theorem 2.3.** Suppose $X$ is a separable Banach space. Then there is a twisted sum

$$0 \to C[0,1] \to Y \overset{Q}{\to} X \to 0$$

with $Q$ strictly singular if and only if $X$ contains no copy of $\ell_1$.

Proof. If $\ell_1$ embeds into $X$, then, by the well-known lifting property of $\ell_1$ [36, p. 107], $Q$ cannot be strictly singular.

Conversely, suppose $\ell_1$ does not embed into $X$. We will argue that the universal twisted sum $Y$ given by Theorem 2.1 has a strictly singular quotient map $Q : Y \to X$. First we show that whenever $E$ is an infinite-dimensional closed subspace of $X$, there is a twisted sum $0 \to C[0,1] \to Z \to X \to 0$ so that the pullback by the inclusion $E \to X$ does not split. Since $X$ does not contain $\ell_1$, any such subspace $E$ contains a weakly null basic sequence $(x_n)_{n=1}^{\infty}$ [36, p. 5, Remark] spanning a subspace $E_0$. By considering the basis expansion we thus obtain a map $T_0 : E_0 \to c_0$ so that $T_0 (x_n) = e_n$, the $n$th-basis vector in $c_0$. Since $c_0$ is separably injective, we can extend $T_0$ to a bounded operator $T : X \to c_0$.

We now use the twisted sum of $C[0,1]$ and $c_0$ constructed above and form the pullback using $T$:}

$$0 \to C[0,1] \to D \overset{J}{\to} c_0 \to 0$$

We now need only show that the further pullback via the inclusion $E \to X$ does not split. Thus we consider

$$0 \to C[0,1] \to Z \overset{T|_E}{\to} E \to 0.$$
does the intermediate sequence \(0 \to C[0, 1] \to Q^{-1}(E) \to E \to 0\). Since \(E\) was arbitrary, we conclude that \(Q\) is strictly singular. \(\square\)

A simplification of this argument shows that if \(X\) is separable but fails the Schur property, then \(\text{Ext}(X, C[0, 1]) \neq \{0\}\). Of course Theorem 1.4 is stronger.

This essentially formal construction gives an interesting corollary:

**Corollary 2.4.** There is a twisted sum \(Y\) of \(C[0, 1]\) and \(c_0\) that is necessarily an \(L_\infty\)-space but is not isomorphic to a quotient of \(C(K)\) for any compact \(K\).

**Proof.** Taking \(X = c_0\) in Theorem 2.3 gives us an example with \(Q : Y \to c_0\) strictly singular. Since \(c_0\) is not reflexive, \(Q\) cannot be weakly compact. By a well-known result of Pelczyński [39, Theorem 1], \(Y\) cannot be isomorphic to a quotient of any \(C(K)\) space. \(\square\)

Note here that \(Y^*\) is isomorphic to an \(L_1(\mu)\)-space, but \(Y\) cannot be renormed so that \(Y^*\) is isometric to an \(L_1(\mu)\) by a result of Johnson and Zippin [25]. This easily gives a counterexample to the old problems 3c and 3e of Lindenstrauss and Rosenthal [35], although other much more sophisticated counterexamples have been known for some time [5], [8]. For a stronger example, see the end of §4.

3. Twisted sums with \(C(\omega^N)\)

If \(N \in \mathbb{N}\), then the space \(C(\omega^N)\) is isomorphic to \(c_0\), and so for any separable Banach space \(X\), we have \(\text{Ext}(X, C(\omega^N)) = \{0\}\). In this case it is natural to introduce the extension constant \(\pi_N(X)\), which we define to be the least constant so that if \(Y\) is a separable Banach space and \(E\) is a closed subspace with \(Y/E\) isometric to \(X\), then for any bounded linear operator \(T : E \to C(\omega^N)\) and \(\varepsilon > 0\), there is an extension \(\bar{T} : Y \to C(\omega^N)\) with \(\|T\| < \pi_N(X)\|\bar{T}\| + \varepsilon\). In view of Proposition 1.3, \(\pi_N(X)\) is also the least constant such that

\[
0 \to C(\omega^N) \overset{0}{\to} Y \overset{q}{\to} X \to 0
\]

is an (isometric) exact sequence and \(\varepsilon > 0\), then there is a linear operator \(P : Y \to C(\omega^N)\) with \(Pj = I_{C(\omega^N)}\) and \(\|P\| \leq \pi_N(X) + \varepsilon\).

The following theorem is due to Amir [2], [3] and Baker [4]:

**Theorem 3.1.** For any separable Banach space \(X\) we have \(\pi_N(X) \leq 2N + 1\), and there is a separable Banach space \(X\) such that \(\pi_N(X) = 2N + 1\).

In fact, it follows from the arguments in [3] that we may take \(X = C(\omega^{N-1})\). The main purpose of this section is to show that \(X\) may be chosen independently of \(N\), more precisely that \(\pi_N(c_0) = 2N + 1\). This will be needed in the next section, where it will also be useful to introduce an alternative constant \(\rho_N(X)\), defined as the least constant such that if \(T : X \to \ell_\infty(\omega^N)\) is a bounded operator satisfying \(d(Tx, C(\omega^N)) \leq \|x\|\) for \(x \in X\), and \(\varepsilon > 0\), there is a linear operator \(L : X \to C(\omega^N)\) with \(\|T - L\| \leq \rho_N(X) + \varepsilon\).

**Lemma 3.2.** For any separable Banach space \(X\) we have \(\rho_N(X) \leq \pi_N(X) \leq \rho_N(X) + 1\).

**Proof.** First suppose \(Y\) is a Banach space containing \(C(\omega^N)\) and such that \(Y/C(\omega^N)\) is isometric to \(X\). Then there is a bounded projection \(P_0 : Y \to C(\omega^N)\). (We may suppose \(\|P_0\| \leq 2N + 1\), but this is not necessary.) We can also find a linear operator
Suppose Lemma 3.3.

For some \( k \) and \( a \) either \( a \) is not, or if \( a \) is not, then \( a \) is not.

**Proof.** Define \( P \) homeomorphic to \( P \) routine to check that \( P \) satisfies \( d(Tx, C(\omega^N)) \leq \|x\| \). Hence, for \( \varepsilon > 0 \), we can find a linear operator \( L : X \to C(\omega^N) \) with \( \|T - L\| \leq \rho_N(X) + \varepsilon \). Now \( P = P_0 - Lg \) is a projection onto \( C(\omega^N) \). If \( y \in Y \), then \( Py = P_0y - Ty + (T - L)qy = Sy + (T - L)qy \), so that \( \|P\| \leq 1 + \rho_N(X) + \varepsilon \). Hence \( \pi_N(X) \leq 1 + \rho_N(X) \).

Conversely, suppose \( T : X \to \ell_\infty(\omega^N) \) is a bounded operator with

\[
d(Tx, C(\omega^N)) \leq \|x\|
\]

for \( x \in X \). Let \( Z \) be the space \( X \oplus C(\omega^N) \) normed by

\[
\|(x, h)\| = \max(\|x\|, \|h - Tx\|)
\]

Then the map \((x, h) \to x\) defines a quotient mapping of \( Y \) onto \( X \) (since \( d(Tx, C(\omega^N)) \leq \|x\| \) with \( \|x\| \)). Hence, if \( \varepsilon > 0 \), there is a projection \( P : Y \to E \) with \( \|P\| \leq \pi_N(X) + \varepsilon \). Then \( P \) takes the form \( P(x, h) = (0, h - Lx) \), where \( L : X \to C(\omega^N) \) is bounded. Now if \( x \in X \), we have \( P(x, Tx) = (0, Tx - Lx) \), so that \( \|Tx - Lx\| \leq \|P\|\|x\| \). Hence \( \rho_N(X) \leq \pi_N(X) \).

**Lemma 3.3.** Suppose \( K \) is a compact Hausdorff space and \( h \in \ell_\infty(K) \). Then

\[
d(h, C(K)) = \frac{1}{2} \sup_{s \in K} (\limsup_{t \to s} h(t) - \liminf_{t \to s} h(t)).
\]

**Proof.** Define \( f(s) = \liminf_{t \to s} h(t) \) and \( g(s) = \limsup_{t \to s} h(t) \) for \( s \in K \). It is routine to check that \( f \) is upper semicontinuous and that \( g \) is lower semicontinuous. If \( R = \frac{1}{2} \sup_{s \in K} (g(s) - f(s)) \), then a classical interpolation theorem gives us a continuous function \( h \) satisfying \( g - R \leq h \leq f + R \). Clearly \( f \leq h \leq g \), and so \( -R \leq h - h \leq R \), as required.

We now need a representation of \( \omega^N \). To this end we consider the power set of \( \mathbb{N} \), i.e., \( 2^\mathbb{N} \), which is homeomorphic to the Cantor set in the standard product topology. Let \( \mathcal{F}_N \) be the subset of all sets \( a \) with cardinality \( |a| \leq N \). Then \( \mathcal{F}_N \) is homeomorphic to \( \omega^N \). Indeed, \( \{\sum_{n \in a} 2^{-n} : a \in \mathcal{F}_N\} \) is order isomorphic and homeomorphic to \( \omega^N \).

Any nonempty finite subset \( a \) of \( \mathbb{N} \) will be written in increasing order, i.e., \( a = \{n_1, \ldots, n_k\} \), where \( n_1 < n_2 < \ldots < n_k \). We write \( \max a = n_k \). We write \( a < b \) if either \( a \) is empty and \( b \) is not, or if \( a = \{n_1, \ldots, n_k\} \) and \( b = \{m_1, \ldots, m_l\} \), where \( l > k \) and \( m_j = n_j \) for \( j \leq k \). For each nonempty finite \( a = \{n_1, \ldots, n_k\} \in 2^\mathbb{N} \) we define \( a^- = \{n_1, \ldots, n_{k-1}\} = a \setminus \{n_k\} \). We define \( a + \) as the collection of all \( a \) of \( m = \{n_1, \ldots, n_k, m\} \), where \( m > n_k \); \( \emptyset \) is simply \( \emptyset \). Although we do not need it in this section, we define here a subset \( \mathcal{A} \) of \( \mathcal{F}_N \) to be full if the following three conditions hold:

1. \( \emptyset \in \mathcal{F}_N \).
2. If \( \emptyset \neq a \in \mathcal{A} \), then \( a^- \in \mathcal{A} \).
3. If \( a \in \mathcal{A} \) and \( |a| < N \), then \( \mathcal{A} \cap a^+ \) is infinite.

It is then easy to see that any full subset of \( \mathcal{F}_N \) is also homeomorphic to \( \omega^N \).

Next let \( \mathcal{A} \) be a full subset of \( \mathcal{F}_N \) and let \( X \) be a fixed separable Banach space. We consider a bounded map \( a \mapsto x^*_a \) of \( \mathcal{A} \) into \( X^* \).
Lemma 3.4. If \( T : X \to \ell_\infty(A) \) is defined by \( Tx(a) = x^*_a(x) \), then we have
\[
d(Tx, C(A)) \leq \|x\| \quad \forall x \in X
\]
if and only if \( \limsup_{b,c \to a} \|x^*_b - x^*_c\| \leq 2 \) for each \( a \in A \) with \( |a| < N \).

Proof. This follows easily from Lemma 3.3 since we require \( \limsup_{b \to a} x^*_b(a) - \liminf_{b \to a} x^*_b(x) \leq 2 \|x\| \) for all \( x \in X \). We omit the details. Note that if \( |a| = N \), then any sequence converging to \( a \) will be eventually constant. \( \qed \)

We conclude this section with a minor variation of Amir’s part of the Amir-Baker Theorem:

Theorem 3.5. For each \( N \) we have \( \pi_N(c_0) = 2N + 1. \)

Proof. Let us choose \( \varepsilon > 0 \) and \( r \in \mathbb{N} \), and let \( m = 2^r \). Then let \( G \) be the dyadic group \( \{-1,1\}^r \), with its usual normalized measure, and let \( u_1, \ldots, u_m \) denote the characters of this group. Let \( \pi = \frac{1}{m} \{ u_1 + \cdots + u_m \} \), so that \( \pi \) is actually the function that is one at the identity and zero elsewhere. Let \( v_k = u_k - \pi \in L_\infty(G) \) and \( v^*_k = u_k \), regarded as an element of \( L_1(G) = L_\infty(G)^* \). Then \( \|v_k\| = \|v^*_k\| = 1 \) for all \( k \), and if \( j \neq k \), then \( \|v^*_j - v^*_k\| = 1 \).

Now consider \( X = c_0(F_{N-1}; L_\infty(G)) \) so that \( X \) is isometric to \( c_0 \). We define a linear operator \( T : X \to \ell_\infty(F_N) \). Consider any element \( x = (w_a)_{a \in F_{N-1}} \in X \), where \( w_a \in L_\infty(G) \). We define \( Tx(\emptyset) = 0 \), and then
\[
Tx(a) = Tx(a-) + 2v^*_j(w_a-),
\]
where \( j \equiv \max a \pmod{m} \). Now let \( Z \) be the set of all \( (x,h) \in X \oplus \ell_\infty(F_N) \) such that \( h - Tx \in C(F_N) \), and put \( E = \{(0,h) : h \in C(F_N)\} \); it is easy to see that the quotient space \( Z/E \) is isometric to \( X \) (since \( d(Tx, C(F_N)) \leq \|x\| \) by Lemma 3.3). Let \( P \) be a bounded projection of \( Z \) onto \( E \), and write \( P(x,Tx) = (0,Sx) \), where \( S : X \to C(F_N) \).

For notational purposes, if \( a \in F_{N-1} \) and \( j \leq m \), we define \( H(a,j) \) to be the set of \( b \geq a \wedge n \), where \( n > \max a \) and \( n \equiv j \pmod{m} \), and \( x_{j,a} = v_jx_{\{a\}} \in X \). For any \( a \in F_N \) we put \( K(a) = \{b : b \geq a\} \).

We now claim that if \( a \in F_{N-1} \), and there exists \( j = j(a) \) so that \( x = x_{j,a} \) satisfies \( Sx(a) \leq 0 \). Indeed, \( \sum_{j=1}^m x_{j,a} = 0 \), and so \( \sum_{j=1}^m Sx_{j,a}(a) = 0 \). Considering the topology on \( F_N \), it follows that there exists \( k = k(a) > \max a \) so that if \( b \geq a \lor l \), where \( l \geq k(a) \), then \( Sx(b) \leq \varepsilon \).

Let us take \( n_1 = j(\emptyset) + mk(\emptyset) \) and then define inductively \( n_2, \ldots, n_N \) so that \( n_s \geq k(n_1, \ldots, n_{s-1}) \) and \( n_s \equiv j(n_1, \ldots, n_{s-1}) \pmod{m} \) for \( 1 < s \leq N \). Let \( a = \{n_1, \ldots, n_N\} \). Then we let
\[
x = \sum\limits_{\emptyset \leq b < a} x_{j(b),b}.
\]
It is easy to see that
\[
Sx(a) \leq N\varepsilon.
\]

It is routine to check that if \( c \geq b \lor n \), with \( n \equiv j \pmod{m} \), then
\[
T(v_j(b)\chi_{\{b\}})(c) = 2v^*_j(v_{j(b)}),
\]
and \( T(v_j(b)\chi_{\{b\}})(c) = 0 \) for all other \( c \in F_N \). Since \( v^*_j(v_k) = \delta_{jk} - \frac{1}{m} \), where \( \delta_{jk} \) is the Kronecker delta, this implies that
\[
T(x_{j(b),b}) = 2\chi_{H(b,j(b))} - \frac{2}{m}\chi_{K(b)\setminus\{b\}}.
\]
Summing, we obtain
\[ Tx = 2 \sum_{\emptyset \leq b < a} \left( \chi_{H(b,j(b))} - \frac{1}{m} \chi_{K(b) \setminus \{b\}} \right). \]

Let \( h = \chi_{K(b)} + 2 \sum_{\emptyset \leq b \leq a} \chi_{K(b)}. \) By construction \( H(b,j(b)) \subseteq K(b) \subseteq H(b-,j(b-)) \)
for each \( b \leq a. \) A short calculation then yields
\[ \|Tx - h\| \leq 1 + \frac{2N}{m}. \]

Since \( \|v_{j(b)}\| = 1, \) we also have \( \|(x,Tx-h)\| \leq 1 + \frac{2N}{m}, \) and thus \( \|Sx - h\| \leq \|P\|(1 + \frac{2N}{m}). \) But \( h(a) = 2N + 1. \) Thus
\[ 2N + 1 - \nu \leq (h-Sx)(a) \leq \|P\|(1 + \frac{2N}{m}). \]

Since we can choose \( m \) arbitrarily large and \( \nu \) arbitrarily small, this implies that \( \pi_N(c_0) \geq 2N + 1. \)

\[ \square \]

4. Twisted sums with \( C(\omega^\omega) \)

Our motivation for studying the constants \( \pi_N(X) \) comes from the following theorem:

**Theorem 4.1.** Suppose \( X \) is a separable Banach space. Then \( \Ext(X,C(\omega^\omega)) = \{0\} \) if and only if \( \sup_N \pi_N(X) < \infty. \)

**Proof.** To simplify notation we will work with \( C_0(\omega^\omega) = \{ f \in C(\omega^\omega) : f(\omega^\omega) = 0 \}, \) which is clearly isomorphic to \( C(\omega^\omega) \). Since \( C(\omega^N) \) is isomorphic to a one-complemented subspace of \( C_0(\omega^\omega) \) for each \( N, \) necessity is obvious. Conversely, suppose \( Y \) is a separable Banach space and \( E \) is a closed subspace of \( Y \) so that \( Y/E \) is isometric to \( X. \) Suppose \( T : E \to C_0(\omega^\omega) \) is bounded with \( \|T\| \leq 1. \) Let \( M = \sup_N \pi_N(X) + 1. \) For \( n \in \mathbb{N} \) let \( R_n : C_0(\omega^\omega) \to C(K_n), \) where \( K_1 = [1,\omega] \) and \( K_n = [\omega^{n-1} + 1, \omega^n] \) for \( n \geq 2. \)

Let \( F_k \) be an increasing sequence of finite-dimensional subspaces of \( Y \) such that \( \bigcup F_k \) is dense in \( Y. \) Let \( G_k \) be finite-dimensional subspaces of \( E \) so that if \( x \in F_k, \) then \( d(x,G_k) \leq 2d(x,E). \) Let \( q : Y \to Y/E \) be the quotient map and let \( q(F_k) = H_k. \)

For each \( k \) let \( n(k) \) be the least integer such that if \( e \in (F_k + G_k) \cap E, \) then \( \|R_nTe\| \leq 2^{-k}\|e\|. \) Then, since \( T \) maps \( E \) into \( C_0(\omega^\omega), \) we see that \( n(k) \) is well defined.

For fixed \( k, \) letting \( n = n(k), \) we can, since \( C(K_n) \) is an \( L_{\infty,1} \)-space, find an operator \( S_n : F_k + G_k \to C(K_n) \) so that \( \|S_n\| \leq 2^{1-k} \) and \( S_n e = R_n Te \) for \( e \in E \cap (F_k + G_k). \) Also we can find an operator \( V_n : Y \to C(K_n) \) such that \( \|V_n\| \leq M \) and \( V_n e = R_n Te \) for \( e \in E. \)

Now if \( y \in F_k + G_k, \) then there exists \( g \in G_k \) so that \( \|y-g\| \leq 2d(y,E). \) Then
\[ \|V_n y - S_n y\| = \|V_n(y - g) - S_n(y - g)\| \leq 2(M + 2)d(y,E). \]

It follows that there is an operator \( U_n : H_n \to C(K_n) \) with \( \|U_n\| \leq 2M + 4 \) and \( U_n q = V_n - S_n. \) Since \( U_n(H_n) \) is finite dimensional, this may be extended to an operator \( \bar{U}_n : X \to C(K_n) \) with \( \|\bar{U}_n\| \leq 2M + 5. \) Next set \( T_n = V_n - \bar{U}_n q. \) Then \( \|T_n\| \leq 3M + 6, \) \( T_n \) extends \( R_n T, \) and \( T_n y = S_n y \) for \( y \in F_k + G_k, \) so that \( \|R_n y\| \leq 2^{1-k}\|y\| \) for \( y \in F_k + G_k. \)
We finally extend the operator $T$ by setting
\[ \tilde{T} y(\alpha) = R_n T y(\alpha) \quad \text{if } \alpha \in K_n. \]
This provides an extension with $\| \tilde{T} \| \leq 3M + 6$. \hfill \qed

Next we recall some ideas from [23]. Suppose $\mathcal{A}$ is a full subset of $\mathcal{F}_N$. We say that a map $a \mapsto \kappa_{a}^* : \mathcal{A} \to X^*$ is a weak*-null tree map if $\kappa_{a}^* \neq 0$ and $\lim_{a \in a^+} \kappa_{b}^* = 0$ (weak*) whenever $|a| < N$. If $E$ is a closed subspace of $X^*$, we will define $\alpha_N(E)$ to be the infimum of all $\lambda$ such that whenever $a \mapsto \kappa_{a}^*$ is a weak*-null tree map with $\kappa_{a}^* \in E$ and $\|\kappa_{a}^*\| \leq 1$ for all $a$, then there is a $b \in \mathcal{B}$ with $|b| = N$ and
\[
\left\| \sum_{a \leq b} \kappa_{a}^* \right\| \leq \lambda.
\]
We shall say that a weak*-null tree map is strongly weak*-null if
\[
\lim_{\max_{a \to \infty} \kappa_{a}^* = 0 \text{ weak*}. \] The next lemma allows us to replace weak*-null by strongly weak*-null in the above definition of $\alpha_N(E)$.

**Lemma 4.2.** If $a \mapsto \kappa_{a}^*$ is a bounded weak*-null tree map on a full subset $\mathcal{A}$ of $\mathcal{F}_N$, then there is a full subset $\mathcal{B}$ of $\mathcal{A}$ so that $a \mapsto \kappa_{a}^*$ is strongly weak*-null on $\mathcal{A}$.

**Proof.** Let $(V_n)$ be a base of weak*-neighborhoods of $0$ such that $V_{n+1} + V_{n+1} \subset V_n$ for all $n$. Let $\mathcal{B} = \{ b \in \mathcal{A} : \kappa_{a}^* \in V_{\max a} \text{ for each } a \text{ with } 0 < a \leq b \}$. It is easily verified that $\mathcal{B}$ works. \hfill \qed

Now suppose $X$ is a separable Banach space with a finite-dimensional Schauder decomposition $(F_n)$. We denote by $S(m,n)$, where $0 \leq m \leq n \leq \infty$ and $m < \infty$, the operator
\[
S(m,n)(\sum_{k=1}^{\infty} f_k) = \sum_{k=m+1}^{n} f_k
\]
if $f_k \in F_k$. Note that $S(n,n) = 0$ for all $n$. We say that $(F_n)$ is bi-monotone if $\|S(m,n)\| \leq 1$ for all $m,n$.

We shall let $E(m,n)$ be the range of $S(m,n)^*$ in $X^*$; we refer to such subspaces as block subspaces. We let $E$ be the closure of $\bigcup_{m<n<\infty} E(m,n)$.

**Theorem 4.3.** Suppose $X$ is a separable Banach space with a bi-monotone FDD $(F_n)$. Then:

(1) $\rho_{2N}(X) \leq 4 \alpha_N(E)$.

(2) If $(F_n)$ is 1-unconditional and shrinking (so that $E = X^*$), then $\alpha_N(X^*) \leq 2 \rho_N(X)$.

**Proof.** (1) Suppose $\lambda > 0$. We define a notion of $\lambda$-acceptable subsets of $B_E$ of cardinality at most $N$. A subset $\{x_1^*, \ldots, x_N^*\}$ of cardinality $N$ is $\lambda$-acceptable if $\|x_1^* + \cdots + x_N^*\| \leq \lambda$. We define acceptable sets of cardinality $0 \leq k < N$ by reverse induction. For each $0 \leq k < N$, a subset $\{x_1^*, \ldots, x_k^*\}$ is $\lambda$-acceptable if there is a weak*-neighborhood $V$ of zero so that if $x_{k+1}^* \in B_E \cap V$, then $\{x_1^*, \ldots, x_k^*\}$ is $\lambda$-acceptable. It is easily seen that if $\lambda > \alpha_N = \alpha_N(E)$, then the empty set is $\lambda$-acceptable. More precisely it is easy to show that if this fails, then one can construct a weak*-null tree map on $\mathcal{F}_N$ denoted by $a \mapsto \kappa_{a}^*$ with $\kappa_{a}^* \in B_E$ so that
for every $a$ with $|a| = N$ we have $\|\sum_{b \leq a} u_b^*\| > \lambda$. This contradicts the definition of $a_N$.

Next we shall say that a collection of $k \leq N$ block subspaces $\{G_1, \ldots, G_k\}$ is $\lambda$-good if for some $\mu < \lambda$ and every $x_j^* \in B_{G_j}$, the set $\{x_1^*, \ldots, x_k^*\}$ is $\mu$-acceptable.

Claim. Suppose $\lambda > a_N$. There is a function $\psi : \mathbb{N} \to \mathbb{N}$ so that if $\{G_1, \ldots, G_k\}$ is a $\lambda$-good family of block subspaces of $E(0,n)$ with $k < N$, then for any block subspace $G_{k+1}$ of $E(\psi(n), \infty)$ the collection $\{G_1, \ldots, G_{k+1}\}$ is $\lambda$-good.

Let us prove the claim. Since the family of block subspaces of $E(0,n)$ is finite, it is clear there exists $\mu < \lambda$ so that every $\lambda$-good collection $\{G_1, \ldots, G_k\}$ of block subspaces is actually $\mu$-good. Then pick $\varepsilon > 0$ so that $\mu + N\varepsilon < \lambda$. Choose in each block subspace $G$ an $\varepsilon$-net for the unit ball $B_G$. In this way we produce a finite collection $\mathcal{G}$ of $\mu$-acceptable sets $\{x_1^*, \ldots, x_k^*\}$ so that whenever $\{G_1, \ldots, G_k\}$ is any $\lambda$-good collection of block subspaces of $E(0,n)$ and whenever $y_j^* \in B_{G_j}$, there is a $\{x_1^*, \ldots, x_k^*\} \in \mathcal{G}$ with $\|y_j^* - x_j^*\| \leq \varepsilon$ for $1 \leq j \leq k$. Now it is clear from the definition of acceptability that we can find $\psi(n)$ so that $x^* \in B_{E \cap E(\psi(n), \infty)}$ and $\{x_1^*, \ldots, x_k^*\} \in \mathcal{G}$ with $k < N$, then $\{x_1^*, \ldots, x_k^*\}$ is $\mu$-acceptable. Now it is easy to see by a perturbation argument that if $\{G_1, \ldots, G_k\}$ is $\lambda$-good with $k < N$ and each $G_j$ is contained in $E(0,n)$, then for any block subspace $G$ of $E(n, \infty)$ the collection $\{G_1, \ldots, G_k, G\}$ is $(\mu + N\varepsilon)$-good and hence also $\lambda$-good. This proves the claim.

We now fix $\lambda > a_N$ and suppose $\theta > 1$. Now suppose $Tx = (x_a^*(x))_{a \in F_{2N}}$ is a linear operator $T : X \to \ell_\infty(F_{2N})$ with $d(Tx, C(F_{2N})) \leq \|x\|$ for all $x \in X$. We use Lemma 3.4. For each $a \in A$ with $a > 0$ we define $\nu = \nu(a)$ to be the greatest natural number so that if $b \in F_{2N}$ and $b \geq a$, then $\|S(0, \nu)x_b^* - S(0, \nu)x_a^*\| \leq 2\theta$. It follows from Lemma 3.4 that $\lim_{a \to +} \mu(a) = \infty$ for all $a$ with $|a| < N$.

Next we inductively construct a map $\varphi : F_{2N} \to \mathbb{N}$. Let $\varphi(\emptyset) = \psi(\emptyset)$. Then we define $\varphi(a)$ by induction on $|a|$. If $\nu(a) < \psi(\varphi(a-))$, we let $\varphi(a) = \varphi(a-)$. If $\nu(a) \geq \psi(\varphi(a-))$, we let $\varphi(a) = \nu(a)$.

Now we define $z_a^*$ for $a \in F_{2N}$ by putting $z_0^* = x_0^*$; and then if $|a| > 0$ we define

$$z_a^* = \sum_{0 < b \leq a} S(\varphi(b-), \varphi(b))^* x_b^* + S(\varphi(a), \infty)^* x_a^*.$$  

We claim that $a \mapsto z_a^*$ is weak*-continuous. In fact, if $b > a$, let $c$ be the unique element in $a+$ with $a < c \leq b$. Then

$$z_b^* - z_a^* = \sum_{c < d \leq b} S(\varphi(d-), \varphi(d))^* x_d^* - S(\varphi(c), \infty)^* x_a^*.$$  

Now $\lim_{c \to a+} \mu(c) = \infty$, and so $\lim_{c \to a+} \varphi(c) = \infty$ and $\varphi(d) \geq \varphi(c)$ whenever $c \leq d \leq b$. Hence as $b \to a$ we have $z_b^* - z_a^* \to 0$ weak*.

Suppose now $a = \{n_1, \ldots, n_k\} \in F_{2N}$. Let $m_0 = \varphi(\emptyset)$, and then put $m_j = \varphi\{n_1, \ldots, n_j\}$ for $1 \leq j \leq k$. Consider the subspaces

$$\{E(m_0, m_1), E(m_1, m_2), \ldots, E(m_{k-1}, m_k)\}.$$  

If we delete those subspaces where $m_j = m_{j-1}$ (i.e., where the subspace reduces to $\{0\}$), then it is clear by induction that the remaining subspaces can be split into two $\lambda$-good collections by taking them alternately. Hence, if $u_j^* \in E(m_{j-1}, m_j)$ with $\|u_j^*\| \leq 1$ for $1 \leq j \leq k$, then $\|\sum_{j=1}^k u_j^*\| \leq 2\lambda$.  

Next we estimate \( \|x_a^* - z_a^*\| \). We have
\[
x_a^* - z_a^* = \sum_{\emptyset \subset b \subseteq a} S(\varphi(b^-), \varphi(b))^* (x_a^* - x_{a_b}^-).
\]

If \( \varphi(b) > \varphi(b^-) \), then \( \varphi(b) = \mu(b) \), and so \( \|S(\varphi(b^-), \varphi(b))^* (x_a^* - x_{a_b}^-)\| \leq 2\theta \). By the above remarks we have

\[
\|x_a^* - z_a^*\| \leq 4\lambda\theta.
\]

Our conclusion is that there is a bounded operator \( Lx = (z_a^*(x))_{a \in \mathcal{F}_N} \) into \( C(\mathcal{F}_N) \) with \( \|L - T\| \leq 2\lambda\theta \). Thus \( \rho_{2\theta}(X) \leq 2\alpha_N(E) \). This concludes the proof of (1).

(2) Let us suppose \( a \mapsto u_a^* \) is a strongly weak* null tree map on \( \mathcal{F}_N \) with \( \|u_a^*\| \leq 1 \) for \( a \in \mathcal{F}_N \). Let \( \gamma : \mathbb{N} \to \mathbb{N} \) be any surjective map so that for each \( k \in \mathbb{N} \) the set \( \gamma^{-1}(k) \) is infinite. Let \( \mathcal{A} \) be the subset of \( \mathcal{F}_N \) consisting of the empty set and all \( \{n_1, \ldots, n_k\} \) such that \( \gamma(n_j) \geq n_{j-1} \) for \( 2 \leq j \leq k \). It is clear that \( \mathcal{A} \) is full. Let \( \sigma(n_1, \ldots, n_k) = \{\gamma(n_1), \ldots, \gamma(n_k)\} \) for \( \{n_1, \ldots, n_k\} \in \mathcal{A} \). We then define \( a \mapsto x_a^* \) for \( a \in \mathcal{A} \) by
\[
x_a^* = \sum_{\emptyset \subset b \subseteq a} u_{\sigma(b)}^*.
\]

Note that if \( d > a \) with \( d \in \mathcal{A} \), then
\[
x_d^* - x_a^* = u_{\sigma(c)}^* + \sum_{c < b \leq d} u_{\sigma(b)}^*,
\]
where \( a < c = \epsilon(d) \leq d \) and \( |c| = |a| + 1 \). Then it follows from the strongly weak*-nullity of \( a \mapsto u_a^* \) that
\[
\lim_{d \to a} \sum_{c < b \leq d} u_{\sigma(b)}^* = 0
\]
weak*, since \( \max(\sigma(b)) \geq \max c \). Hence we have
\[
\lim_{d \to a} \|x_d^* - x_a^*\| \leq 1.
\]

By Lemma 3.4 and the definition of \( \rho_N(X) \), for any \( \lambda > \rho_N(X) \) we can find a weak*-continuous map \( a \mapsto z_a^* \) on \( \mathcal{A} \) such that \( \|x_a^* - z_a^*\| \leq \lambda \) for all \( a \).

Now fix \( \epsilon > 0 \). We determine an increasing sequence \( n_1, \ldots, n_N \) so that \( \{n_1, \ldots, n_N\} \in \mathcal{A} \) and an increasing sequence \( m_1, \ldots, m_{2N} \in \mathbb{N} \) by induction. Suppose \( a = \{n_1, \ldots, n_{k-1}\} \) has been chosen in \( \mathcal{A} \) (where if \( k = 1 \), we take \( a = \emptyset \)) and that \( m_1, \ldots, m_{2k-2} \) have been chosen. Then pick \( m_{2k-1} > m_{2k-2} \) (if \( k \geq 2 \)) so that \( \|S(m_{2k-1}, \infty)^* (x_{m_1}^* - z_{m_1}^*)\| < \epsilon/(6N) \). This is possible since the (FDD) is shrinking. Now pick \( c \in \sigma(a) + \) with \( \|S(0, m_{2k-1})^* u_c^*\| < \epsilon/(6N) \); this is possible since \( \lim_{c \in \sigma(a) +} u_c^* = 0 \) weak*. Pick \( m_{2k} > m_{2k-1} \) so that \( \|S(m_{2k}, \infty)^* u_c^*\| < \epsilon/(6N) \). Now there are infinitely many \( b \in a + \) with \( \sigma(b) = c \), amongst these we may choose \( b \) so that \( \|S(0, m_{2k})^* (z_b^* - z_a^*)\| < \epsilon/(6N) \), since \( \lim_{b \to a} z_b^* = z_a^* \) weak*. We then let \( b = \{n_1, \ldots, n_k\} \). This completes the inductive construction.
Let $a_k = \{n_1, \ldots, n_k\}$ for $0 \leq k \leq N$. Then
\[
\left\| \sum_{k=1}^{N} u^*_{\sigma(a_k)} \right\| \leq \frac{\varepsilon}{3} + \left\| \sum_{k=1}^{N} S(m_{2k-1}, m_{2k})^* u^*_{\sigma(a_k)} \right\|
\]
\[
\leq \frac{\varepsilon}{3} + \left\| \sum_{k=1}^{N} \left( S(m_{2k-1}, m_{2k})^* u^*_{\sigma(a_k)} + S(m_{2k-2}, m_{2k-1})^* (z^*_{\sigma(a_k)} - z^*_{\sigma(a_k-1)}) \right) \right\|
\]
\[
\leq \frac{\varepsilon}{3} + \left\| \sum_{k=1}^{N} (u^*_{\sigma(a_k)} + z^*_{\sigma(a_k)} - z^*_{\sigma(a_k-1)}) \right\|
\]
\[
\leq \varepsilon + \| x_{a_N}^* - z_{a_N}^* + z_0^* - x_0^* \|
\]
\[
\leq \varepsilon + 2\lambda.
\]
Hence by the definition of $\alpha_N(X^*)$ we have $\alpha_N(X^*) \leq 2\lambda + \varepsilon$. The theorem follows. \qed

We are now in a position to prove our main result:

**Theorem 4.4.** (1) Suppose $X$ is a separable Banach space with summable Szlenk index. Then $\text{Ext}(X^*, C(\omega^n)) = \{0\}$.

(2) If $Y$ is a separable Banach space with $\text{Ext}(Y, C(\omega^n)) = \{0\}$ and $Y$ has a (UFDD), then $Y$ is the dual of a space $X$ with summable Szlenk index.

*Remark.* For the definition and general properties of the Szlenk index, see for example [23, 2]. The original space constructed by Tsirelson [44] is a reflexive space with summable Szlenk index [31]. Its dual is the space usually referred to nowadays as Tsirelson’s space [14].

**Proof.** If $X$ has a shrinking (FDD), then (1) follows directly from Theorem 4.3. We can assume via renorming that the (FDD) is bi-monotone. We consider the dual (FDD) of $X^*$. In this case the subspace $E$ of $X^{**}$ is identified with $X$ and the condition $\sup_n \alpha_n(E) < \infty$ is equivalent (using [23, Theorem 4.10]) to the fact that $X$ has summable Szlenk index, and this implies that $\sup_n \pi_n(X^*)$ is finite.

For the general case we use a theorem of Johnson and Rosenthal [21, 36, Theorem 1, p.48], that $X$ has a subspace $Y$ so that $X/Y$ and $Y$ both have shrinking (FDD)s. It is easy to check that having summable Szlenk index is a property that passes to quotients, and it follows from renorming results in [23, Theorem 4.10 (ii)] that it passes also to subspaces. Thus $X$ and $Y$ must both have summable Szlenk index. Hence we have $\text{Ext}(Y^\perp, C(\omega^n)) = \{0\}$ and $\text{Ext}(X^*/Y^\perp, C(\omega^n)) = \{0\}$, and so by Corollary 1.2 we have $\text{Ext}(X, C(\omega^n)) = \{0\}$. This concludes the proof of (1).

For (2) we may assume the (UFDD) is 1-unconditional. We observe that Theorem 4.3 implies $\text{Ext}(c_0, C(\omega^n)) \neq \{0\}$. (Direct constructions are also available.) Hence if $\text{Ext}(Y, C(\omega^n)) = \{0\}$ and $Y$ is separable, then $Y$ contains no (necessarily complemented) copy of $c_0$. In particular, the (UFDD) of $Y$ must be boundedly complete, and so $Y = X^*$, where $X = E$ as defined in Theorem 4.3. Then we have by Theorem 4.3 $\sup_N \pi_N(Y) < \infty$, and hence by Lemma 4.2, $\sup_N \rho_N(Y) < \infty$. Applying Theorem 4.3 (2), we obtain $\sup_n \alpha_n(X) < \infty$. It follows again from Theorem 4.10 of [23] that $X$ has summable Szlenk index. \qed

If $X$ is any separable Banach space, we define a tree map $a \mapsto v^*_a : \mathcal{F}_N \to X^*$ to be of dense type if the following conditions are satisfied:
(1) $v_0^* = 0$.
(2) $\|v_n^*\| \leq 1$ for all $a \in \mathcal{F}_N$.
(3) For each $a$ with $|a| < N$ there is a weak*-neighborhood $V$ of $0$ so that the weak*-closure of $\{v_b^* : b \in a+\}$ contains $V$.
(4) If $b_n \to a$ and $|b_n| \geq |a| + 2$ for all $n$, then $v_{b_n}^* \to 0$ weak*.

Next let $y_n^* = \sum_{b \leq a} v_b^*$. We can define $T x = (y_n^*(x))_{a \in \mathcal{F}_N}$, so that $T : X \to \ell_\infty(\mathcal{F}_N)$.

**Lemma 4.5.** Suppose $X$ has a (UFDD). Suppose $L : X \to C(\omega^n)$, and $T : X \to \ell_\infty(\mathcal{F}_N)$ is an operator induced by a tree map of dense type. Then $\rho_N(X) \leq 2\|L - T\|$.

**Proof.** This essentially follows from the argument in Theorem 4.3. Let $a \mapsto u_a^*$ be any strongly weak*-null tree map with $\|u_a^*\| \leq 1$ for all $a$. Let $\gamma : \mathbb{N} \to \mathbb{N}$ be any surjective map so that for each $k \in \mathbb{N}$ the set $\gamma^{-1}(\{k\})$ is infinite. Let $A$ be the subset of $\mathcal{F}_N$ consisting of the empty set and all $\{n_1, \ldots, n_k\}$ such that $\gamma(n_j) \geq n_j - 1$ for $2 \leq j \leq k$. It is clear that $A$ is full. Let $\sigma\{n_1, \ldots, n_k\} = \{\gamma(n_1), \ldots, \gamma(n_k)\}$ for $\{n_1, \ldots, n_k\} \in A$.

We now build a map $\psi : A \to \mathcal{F}_N$. Define $\psi(\emptyset) = \emptyset$. If $\psi(a)$ has been defined and $|a| < N$, we define $\psi(b)$ for each $b \in a+$ so that $\psi(b) \in \psi(a)+$, $\psi$ is one-one and $\lim_{a \to a+} u_{\sigma(b)}^* - v_{\psi(b)}^* = 0$ weak*.

Let $x_a^* = \sum_{b \leq a} u_{\sigma(b)}^*$. Then we claim that $x_a^* - y_{\psi(a)}^*$ is weak*-continuous. Indeed, if $b \geq a$,

$$x_b^* - x_a^* - y_{\psi(b)}^* + y_{\psi(a)}^* = \sum_{a < c \leq b} u_{\sigma(c)}^* - v_{\psi(c)}^*.$$  

Now if $b_n \to a$ and we let $d_n$ be chosen so that $b_n \leq d_n \leq a$ and $|d_n| = |a| + 1$, we have

$$\sum_{d_n < c < b} (u_{\sigma(c)}^* - v_{\psi(c)}^*) \to 0 \quad \text{weak}^*$$

by the assumptions on both tree maps. On the other hand,

$$u_{\sigma(d_n)}^* - v_{\psi(d_n)}^* \to 0 \quad \text{weak}^*$$

by construction.

Now if $Lx = (z_a^*(x))_{a \in \mathcal{F}_N}$, then $\|z_a^* - y_{\psi(a)}^*\| \leq \|L - T\|$. Now $a \mapsto z_{\psi(a)}^* + x_a^* - y_{\psi(a)}^*$ is weak*-continuous, and we can repeat the argument of Theorem 4.3 to deduce the conclusion. 

It is clear that we can always construct a tree map of dense type. Simply let $(V_n)$ be a base of weak*-neighborhoods of $\{0\}$ in $X^*$ with $V_{n+1} + V_{n+1} \subset V_n$. Then for $a$ with $|a| < N$, simply choose $u_{\sigma(a)}^* + m$ for $m > \max a$ to be any sequence that is weak*-dense in $V_{\max a} \cap B_{X^*}$. It is also clear that if $Y$ is a subspace of $X$ and $j : Y \to X$ is the inclusion, then $a \mapsto j^* u_a^*$ is a tree map of dense type in $Y^*$. This leads us to the following:

**Proposition 4.6.** Let $X$ be a separable Banach space with a shrinking 1-unconditional (UFDD). Then there is a bounded operator $T : X \to \ell_\infty(\omega^N)$ so that $d(Tx, C(\omega^N)) \leq \|x\|$ for all $x \in X$ and so that if $E$ is a subspace of $X$ with $a$ (UFDD), then $\rho_N(E) \leq 2\|L - T\|$ for any bounded operator $L : E \to C(\omega^N)$. 

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It is obvious from Theorem 4.4 that the existence of a twisted sum \(0 \to C(\omega^\omega) \to Y \to X \to 0\) with the quotient map strictly singular implies that \(X\) contains no subspace that is isomorphic to the dual of a space with summable Szlenk index. We now establish a partial converse.

**Theorem 4.7.** Suppose \(X\) has a shrinking (UFDD) and contains no subspace that is isomorphic to the dual of a space with summable Szlenk index. Then there is a short exact sequence

\[0 \to C(\omega^\omega) \to V \xrightarrow{q} X \to 0\]

with \(q\) strictly singular.

**Proof.** We may assume \(X\) has a 1-unconditional (UFDD). For each \(N\) we construct \(T_N : X \to \ell_\infty(\omega^N)\) as given in Proposition 4.6. Let \(Z_N\) be the space \(X \oplus C(\omega^N)\) normed by \(\|(x, h)\| = \|x\| + \|h - Tx\|\); then there is a quotient map \(q_N : Z_N \to X\) defined by \(q_N(x, h) = x\). We now construct an operator \(S_N : \tilde{X} \to C(\omega^N)\) in the usual way. Precisely, we fix a quotient map \(Q : \ell_1 \to X\) and define \(S_N : \ell_1 \to Z_N\) so that \(\|S_N\| \leq 2\) and \(q_NS_N = Q\). Now let \(S_N\) be the restriction of \(S_N\) to \(\tilde{X}\).

Let \((F_n)\) be an increasing sequence of finite-dimensional subspaces so that \(\bigcup F_n\) is dense in \(X\). Then, since \(C(\omega^N)\) is an \(\ell_\infty\)-space, we can find a finite-rank projection \(P_N\) on \(C(\omega^N)\) whose range includes \(S_N(F_N)\) and with \(\|P_N\| \leq 2\). Now let \(R_N = S_N - P_N S_N\). Thus \(\|R_N\| \leq 6\), and \(\lim_{N \to \infty} \|R_N \xi\| = 0\) for \(\xi \in \tilde{X}\).

We now define a map \(R : \tilde{X} \to W = c_0(C(\omega^N)_{N-1})\) by \(R \xi = (R_N \xi)_{N-1}\). Note that the latter space is isomorphic to \(C(\omega^\omega)\). We can now construct a pushout

\[
\begin{array}{ccc}
0 & \to & \tilde{X} \to \ell_1 \xrightarrow{Q} X \to 0 \\
\downarrow R & & \downarrow q_V \quad q_X \\
0 & \to & W \to V \xrightarrow{q_X} X \to 0.
\end{array}
\]

We claim that \(q_X\) is strictly singular. If not, we can find a subspace \(E\) of \(X\) with a 1-unconditional shrinking (UFDD) so that there is a bounded operator \(\Lambda : E \to V\) such that \(q_X \Lambda = I_E\). Then on \(Q^{-1}(E)\) we have \(q_X(q_V - \Lambda Q) = 0\), so that \(Q - \Lambda Q : Q^{-1}(E) \to W\) is an extension of \(R\) to \(Q^{-1}(E)\). It follows that there exists a uniformly bounded sequence of operators \(R_N : Q^{-1}(E) \to C(\omega^N)\) which extend \(R_N\). Put \(M = \sup \|R_N\| < \infty\).

Note that \(P_N S_N\) has an extension to \(Q^{-1}(E)\) with \(\|P_N S_N\| \leq 5\), since it is a finite-rank operator taking values in \(C(\omega^N)\). Hence \(S_N\) has an extension \(\tilde{S}_N : Q^{-1}(E) \to C(\omega^N)\) with \(\|S_N\| \leq M + 5\). Then \(\tilde{S}_N - \tilde{S}_N\) factors through an operator \(e \mapsto (e, L_N e)\) from \(E\) into \(Z_N\) with norm at most \(M + 7\). This implies that \(\|L_N - T\| \leq M + 7\), and so \(\rho_N(E) \leq 2M + 14\). Theorem 4.3 and [25] Theorem 4.10 then show that \(E\) must have summable Szlenk index. \(\square\)

It now follows that there is a twisted sum of \(C(\omega^\omega)\) and \(c_0\) so that the quotient map is strictly singular. This space is not a quotient of a \(C(K)\)-space, and yet its dual must be isomorphic to \(\ell_1\). This shows that the main result of [25] does not admit an isomorphic version. The space \(Y\) constructed in [8] also serves as a counterexample.
5. Final remarks

In [21] (cf. [29]) it is shown that \( \text{Ext}(\ell_2, \ell_2) \neq \{0\} \). It follows without difficulty that \( \text{Ext}(\ell_p, \ell_q) \neq \{0\} \) when \( 1 < p, q < \infty \), since each space contains uniformly complemented copies of \( \ell_2^p \). The following result is implicitly proved in [10], but it is heavily disguised; so we give a simple and direct diagram-chasing argument. For a nonlinear argument, see [12].

**Theorem 5.1.** \( \text{Ext}(c_0, \ell_1) \neq \{0\} \).

**Proof.** In fact we will argue that \( \text{Ext}(C[0,1], L_1) \neq \{0\} \). It then follows from local arguments that \( \text{Ext}(X, Y) \neq \{0\} \) whenever \( X \) is an \( \ell_\infty \)-space and \( Y = L_1(\mu) \) for some measure \( \mu \) (see, e.g., [12, Theorem 2]). Alternatively, one may carry out the ensuing argument locally.

We begin by considering some non-trivial twisted sum of \( \ell_2 \) and \( \ell_2 \). By using the pushout and pullback constructions we build the following diagram:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \ell_2 & \xrightarrow{j_1} & Z & \xrightarrow{q_1} & \ell_2 & \rightarrow & 0 \\
\downarrow{j_4} & & \downarrow{j_5} & & \downarrow{q} & & \downarrow{q} & & \\
0 & \rightarrow & L_1 & \xrightarrow{j_2} & V & \xrightarrow{q_2} & \ell_2 & \rightarrow & 0 \\
\downarrow{q} & & \downarrow{q} & & \downarrow{q} & & \downarrow{q} & & \\
0 & \rightarrow & L_1 & \xrightarrow{j_3} & W & \xrightarrow{q_3} & C[0,1] & \rightarrow & 0 \\
\end{array}
\]

Here linear embeddings are denoted by \( j \) and quotient maps by \( q \). First we recall that \( Z \) is of cotype \( p \) and type \( q \) whenever \( q < 2 < p \) [21 §3]. From the construction of the pushout, \( V \) is of cotype \( p \) for every \( p > 2 \).

We claim that the third row of this diagram cannot split. Suppose it does split. Then we can find an operator \( T : C[0,1] \rightarrow W \) so that \( q_5 T = I_{C[0,1]} \). Then \( q_5 T : C[0,1] \rightarrow V \) must factor through some \( L_r \)-space, where \( r > 2 \) since \( V \) has finite cotype. (This result can be traced to Maurey [37]; cf. also [32] or [20, Theorem 11.14(b)].) Since \( L_r \) has type 2 and \( L_1 \) has cotype 2, every map from a subspace of \( L_r \) to \( L_1 \) factors through a Hilbert space (this is Maurey’s generalization of Kwapień’s theorem [32] and [33]) and hence extends to a bounded operator from \( L_r \) into \( L_1 \) by Maurey’s Extension theorem [38] (cf. [20, Theorem 12.22]). Applying all this to \( (q_5 T)^{-1}(j_2 L_1) \), we can find an operator \( R : C[0,1] \rightarrow j_2(L_1) \) so that \( R f = q_5 T f \) if \( q_2 q_5 T f = 0 \). But \( q_2 q_5 = q_4 q_3 \). Then \( q_5 T = R = T_1 q_4 \) for some bounded operator \( T_1 : \ell_2 \rightarrow V \). Thus the second row splits.

The conclusion of the argument was given in the proof of [30, Theorem 4.1]. If the second row splits, then \( V \) has cotype 2. Hence \( Z \) also has cotype 2, and also has type \( p > 1 \). But then \( Z^* \) is type 2 [11], and the Maurey Extension theorem guarantees that the dual exact sequence \( 0 \rightarrow \ell_2 \rightarrow Z^* \rightarrow \ell_2 \rightarrow 0 \) splits. By reflexivity the first row splits, contrary to our choice of \( Z \). \( \square \)

Finally, let us mention a non-separable problem related to the results of this paper. If \( X \) is a separable Banach space, then \( \text{Ext}(X, c_0) = \{0\} \) by Sobczyk’s theorem: we do not know, however, if there is a non-metrizable compact Hausdorff space \( K \) such that \( \text{Ext}(C(K), c_0) = \{0\} \). It is known that if \( \Gamma \) is uncountable, then \( \text{Ext}(c_0(\Gamma), c_0) \neq \{0\} \), this is essentially contained in one proof of the fact that \( c_0 \) is uncomplemented in \( \ell_\infty \); see also [11, 19, p. 260] and [13, §3]. It was noted in [17, Theorem 3.4] that if \( X \) is any non-separable WCG-space, then \( \text{Ext}(X, c_0) \neq \{0\} \), and this settles the case when \( K \) is an Eberlein compact; similar arguments can...
be used for Corson compact spaces. At the other extreme, if \( K \) is extremally disconnected, then \( C(K) \) contains a complemented \( \ell_\infty \) and \( \text{Ext}(\ell_\infty, c_0) \neq \{0\} \) was shown in [12]. Finally, the case of uncountable ordinal spaces can be reduced to \( K = [0, \omega_1] \), and in this case Parovičenko’s theorem [7] shows that \( \text{Ext}(C(K), c_0) \neq \{0\} \).

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Departamento de Matemáticas, Universidad de Extremadura, Avenida de Elvas, 06071 Badajoz, Spain
E-mail address: fcabello@unex.es

Departamento de Matemáticas, Universidad de Extremadura, Avenida de Elvas, 06071 Badajoz, Spain
E-mail address: castillo@unex.es

Department of Mathematics, University of Missouri-Columbia, Columbia, Missouri 65211
E-mail address: nigel@math.missouri.edu

Department of Mathematics, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
E-mail address: dthoyost@ksu.edu.sa