TWISTED SUMS WITH $C(K)$ SPACES

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Abstract. If $X$ is a separable Banach space, we consider the existence of non-trivial twisted sums $0 \to C(K) \to Y \to X \to 0$, where $K = [0,1]$ or $\omega^\omega$. For the case $K = [0,1]$ we show that there exists a twisted sum whose quotient map is strictly singular if and only if $X$ contains no copy of $\ell_1$. If $K = \omega^\omega$ we prove an analogue of a theorem of Johnson and Zippin (for $K = [0,1]$) by showing that all such twisted sums are trivial if $X$ is the dual of a space with summable Szlenk index (e.g., $X$ could be Tsirelson’s space); a converse is established under the assumption that $X$ has an unconditional finite-dimensional decomposition. We also give conditions for the existence of a twisted sum with $C(\omega^\omega)$ with strictly singular quotient map.

1. Introduction and preliminary remarks

Let $X$ and $Y$ be real Banach spaces. Then we say $\text{Ext}(X,Y) = \{0\}$ if every short exact sequence $0 \to Y \to Z \to X \to 0$ splits; informally this means that if $Z$ is a Banach space containing $Y$ and so that $Z/Y \sim X$, then there is a bounded projection of $Z$ onto $Y$. A space $Z$ with a subspace isomorphic to $Y$ so that $Z/Y$ is isomorphic to $X$ is often called a twisted sum of $Y$ and $X$ (order is important!). Thus $\text{Ext}(X,Y) = \{0\}$ if and only if every twisted sum of $Y$ and $X$ is trivial (i.e. reduces to $Y \oplus X$).

Fundamental tools for us are the pushout and pullback constructions. These are well-known to algebraists and topologists, but less so to analysts. So we will describe them briefly in the Banach space setting. If $T : E \to X$ and $S : E \to Y$ are two operators defined on the same Banach space, then their pushout $Z$ is defined as the quotient of $X \oplus_Y Y$ by the closure of $\{(Te, -Se) : e \in E\}$, together with the natural mappings $X \to Z$ and $Y \to Z$ (i.e., the restrictions of the quotient mapping). In case one of the mappings, say $S$, is the inclusion mapping from a short exact sequence, then completing the diagram gives a second short exact sequence with the same quotient space $F$:

$$
\begin{array}{ccccccccc}
0 & \to & E & \xrightarrow{S} & Y & \xrightarrow{T} & F & \to & 0 \\
 & \ | & \downarrow & & \ | & & \ | & & \\
0 & \to & X & \to & Z & \to & F & \to & 0.
\end{array}
$$

Received by the editors June 21, 2001 and, in revised form, June 5, 2002.
2000 Mathematics Subject Classification. Primary 46B03, 46B20.
The research of the first two authors was supported in part by the DGICYT project BFM 2001-0387.
The third author was supported by NSF grant DMS-9870027.
The fourth author was supported substantially by the Junta de Extremadura, and for a few days by Research Centre Project Number Math/1420/25 from his present institution.

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Conversely, if we are given any commutative diagram as above, then $Z$ must be isomorphic to the pushout of $S$ and $T$; this observation will be used several times in the sequel. Note also that the operator $Y \to Z$ is an isomorphic embedding (respectively a quotient mapping) if and only if $T$ is. Furthermore, the lower sequence splits if and only if $T$ can be extended to $Y$. These well-known exercises follow from standard diagram-chasing arguments.

Dually, if $S : X \to E$ and $T : Y \to E$ are two operators into the same Banach space, then their pullback $Z$ is defined as the subspace of all $(x, y) \in X \oplus_\infty Y$ for which $Sx = Ty$, together with the natural mappings $Z \to Y$ and $Z \to X$. In case one of the original mappings, say $S$, is the quotient mapping from a short exact sequence, then completing the diagram gives a second short exact sequence with the same subspace $F$:

$$
\begin{array}{ccc}
0 & \to & F & \to & Z & \to & Y & \to & 0 \\
0 & \to & F & \to & X & \overset{S}{\to} & E & \to & 0.
\end{array}
$$

Conversely, if we are given any commutative diagram as above, then $Z$ must be isomorphic to the pullback of $S$ and $T$. Note again that the operator $Z \to X$ is an isomorphic embedding (respectively a quotient mapping) if and only if $T$ is. For further information, see [16, Chap. 1] and the references therein.

Let $X$ be any separable Banach space and let $Q_X : \ell_1 \to X$ be any quotient map. We will keep the notation $\tilde{X}$ for the kernel of $Q_X$ (which is unique up to automorphism provided it is infinite dimensional, see [35], [36, p. 108] or [15, p. 382]). The following theorem is well known:

**Theorem 1.1.** Suppose $X$ and $Y$ are separable Banach spaces. Then the following are equivalent:

1. $\text{Ext}(X, Y) = \{0\}$.
2. If $T : \tilde{X} \to Y$ is a bounded operator, then there is a bounded extension $\tilde{T} : \ell_1 \to Y$.
3. If $Z$ is a separable Banach space containing a subspace $E$ so that $Z/E \sim X$ and $T : E \to Y$ is a bounded operator, then there is an extension $\tilde{T} : Z \to Y$.

**Proof.** It is trivial that (3) implies (1). For (1) implies (3) we use the pushout construction:

$$
\begin{array}{ccc}
0 & \to & E & \to & Z & \to & X & \to & 0 \\
0 & \to & \tilde{Y} & \to & W & \to & X & \to & 0.
\end{array}
$$

Now (1) implies the existence of a projection $P : W \to Y$, and then $PS$ extends $T$.

That (2) is equivalent to (3) is clear from the proof of Corollary 1.1 of [20]. Alternatively, [90] Prop. 3.1 proves directly the equivalence of (1) and (2).

**Remark.** Of course all separability assumptions can be removed if we simply replace $\ell_1$ by $\ell_1(I)$ for a suitable index set.

There is an immediate corollary, which essentially says that $\text{Ext}(X, Y) = \{0\}$ is a three-space property of $X$:

**Corollary 1.2.** Suppose $Y$ is a Banach space and $X$ is a Banach space with a subspace $E$ so that $\text{Ext}(E, Y) = \{0\}$, and $\text{Ext}(X/E, Y) = \{0\}$. Then $\text{Ext}(X, Y) = \{0\}$.
Proof. Let $\widetilde{X}$ and $Q_X$ be defined as above. Given $T : \widetilde{X} \to Y$, we need to find an extension to all of $\ell_1$. We will apply Theorem 1.1.

If $Q : X \to X/E$ is the obvious mapping, we may choose $\widetilde{X}/E$ to be the kernel of $Q \circ Q_X$. Then $y \mapsto Q_X y$ is a quotient mapping from $\widetilde{X}/E$ onto $E$ with kernel $\widetilde{X}$. The implication (1) $\Rightarrow$ (3) then gives us an extension $\tilde{T} : \widetilde{X}/E \to Y$ of $T$, which by the implication (1) $\Rightarrow$ (2) admits a further extension $\tilde{T} : \ell_1 \to Y$.

In this paper, we consider the case when the subspace of our twisted sum is $C(K)$ for some compact metric space $K$. If $K$ is uncountable, then the theorem of Milutin [40, Theorem 8.5] implies we may consider $K = [0, 1]$. The following result is due to Johnson and Zippin [26], in view of Theorem 1.1:

**Theorem 1.3.** If $X$ is isomorphic to the dual of a subspace of $c_0$ (so that $\widetilde{X}$ can be assumed weak* -closed), then $\text{Ext}(X, C(K)) = \{0\}$ for every compact $K$.

In [28] the following converses were found. Throughout this paper, we will use (FDD) to indicate a finite-dimensional Schauder decomposition and (UFDD) to indicate an unconditional finite-dimensional Schauder decomposition. Recall also that $X$ is said to have the strong Schur property if there is a constant $c > 0$ so that for any normalized sequence $(x_n)$ with $\|x_m - x_n\| \geq \delta > 0$ for any $m \neq n$, there exists a subsequence $(x_{n_k})_{k \in M}$ such that

$$\sum_{k \in M} |\alpha_k x_k| \geq c\delta \sum_{k \in M} |\alpha_k|$$

for any finitely supported sequence $(\alpha_k)_{k \in M}$.

**Theorem 1.4.** If $X$ is separable and $\text{Ext}(X, C[0, 1]) = \{0\}$, then $X$ has the strong Schur property. If $X$ also has a (UFDD), then $X$ is isomorphic to the dual of a subspace of $c_0$.

Let us remark at this point that Bourgain and Pisier [9] (cf. [16, §1.8]) showed that for any separable Banach space $X$ that is not an $\ell_\infty$-space there is a space $Y$ that is an $\ell_\infty$-space so that $Y$ contains $X$ as an uncomplemented subspace and $Y/X$ has the Schur property and the Radon-Nikodym property.

Recall that an operator is called strictly singular if its restriction to an infinite-dimensional subspace of its domain is never an isomorphic embedding. In Section 2 we consider the problem of characterizing those separable spaces $X$ for which there is a short exact sequence $0 \to C[0, 1] \to Z \to X \to 0$ so that the quotient map is strictly singular. We show in Theorem 2.3 that this is equivalent to the requirement that $X$ contains no copy of $\ell_1$.

In Section 3 we consider quantitative results for the case $K = \omega^N$. In this case $C(K)$ is isomorphic to $c_0$, so that $\text{Ext}(X, C(K)) = \{0\}$ for every separable $X$ by Sobczyk’s theorem [43], but it is still worthwhile to consider projection constants. We need the following elementary result: we recall that $Z$ is said to be separably injective if it is complemented in every separable superspace. As usual, $I_X$ indicates the identity on a given Banach space $X$.

**Proposition 1.5.** Let $X$ be any separable Banach space, let $Z$ be a separably injective Banach space and let $k$ be a constant. Then the following are equivalent:

1. $\text{Ext}(X, Z) = \{0\}$;
2. $\|x_k\| \leq k$ for all $x_k$ in the separating sequence $x_k$;
3. $\text{Ext}(X, Z) = \{0\}$.

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(1) If $Y$ is a separable Banach space and $E$ is a closed subspace with $Y/E$ isometric to $X$, then for any bounded linear operator $T : E \to Z$ and any $\varepsilon > 0$, there is an extension $\tilde{T} : Y \to Z$ with $\|\tilde{T}\| < k\|T\| + \varepsilon$.

(2) If $0 \to Z \xrightarrow{j} Y \xrightarrow{o} X \to 0$ is an (isometric) exact sequence and any $\varepsilon > 0$, there is a linear operator $P : Y \to Z$ with $Pj = I_Z$ and $\|P\| \leq k + \varepsilon$.

Proof. It is clear from the definition that if the short exact sequence is given, then we may find such a $P$ with $\|P\| \leq k + \varepsilon$. Conversely, suppose $Y$ is a separable Banach space and $E$ is a closed subspace with $Y/E$ isometric to $X$. If $T : E \to Z$ is an operator with $\|T\| \leq 1$, we form the pushout:

$$
\begin{array}{cccccc}
0 & \to & E & \xrightarrow{j} & Y & \xrightarrow{T} & X & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Z & \xrightarrow{j'} & PO & \xrightarrow{s} & X & \to & 0
\end{array}
$$

Then, if $P : PO \to Z$ satisfies $Pj' = I_Z$, we see that $PS = \tilde{T}$ extends $T$ and $\|PS\| \leq \|P\|$. \qed

Our results build on earlier work of Amir and Baker, who showed that the separable projection constant of $C(\omega^N)$ is $2N + 1$, [2], [3] and [4]. In particular, we show that, given any $\varepsilon > 0$, there is a space $Z$ containing $C(\omega^N)$ isometrically so that $X/C(\omega^N)$ is isometric to $c_0$ and the norm of any projection is at least $2N + 1 - \varepsilon$. However, our main motivation in Section 3 is to provide the necessary groundwork to study the case $K = \omega^\omega$, which is done in Section 4. Here we show results parallel to Theorems 1.3 and 1.4 above. We show that if $X$ is the dual of a space with summable Szlenk index [31], [23], then $\text{Ext}(X, C(\omega^\omega)) = \{0\}$, and this condition is necessary if $X$ has a (UFDD). An example of such an $X$ is Tsirelson’s space [31].

We also consider the possibility of $\text{Ext}(X, C(\omega^\omega))$ being large in the sense that there is a twisted sum $0 \to C(\omega^\omega) \to Z \to X \to 0$ for which the quotient map is strictly singular. We show that a sufficient condition for the construction of such a short exact sequence is that $X$ has a shrinking (UFDD) and contains no subspace that is the dual of a space with summable Szlenk index. This leads to new counterexamples for several old problems.

We refer to [16] and [29] for a discussion of twisted sums in general. Let us note that in Section 4 it is important to consider twisted sums in the isometric category rather than the isomorphic category; hence the standard pushout and pullback constructions were defined above isometrically. Of course any isomorphic twisted sum can be equivalently renormed to an isometric twisted sum.

2. A Universal Twisted Sum

Theorem 2.1. Suppose $X$ is a separable Banach space. Then there is a universal short exact sequence $0 \to C[0, 1] \to Y \to X \to 0$ such that every short exact sequence $0 \to C[0, 1] \to Z \to X \to 0$ can be identified with a pushout, i.e., there exist linear operators $S : C[0, 1] \to C[0, 1]$ and $S_1 : Y \to Z$ so that the following

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diagram commutes:

\[
\begin{array}{cccc}
0 & \rightarrow & C[0,1] & \rightarrow \\
& & s & \rightarrow \\
0 & \rightarrow & C[0,1] & \rightarrow
\end{array}
\begin{array}{cccc}
Y & \rightarrow & X & \rightarrow & 0 \\
& & s_1 & \rightarrow \\
& & X & \rightarrow & 0.
\end{array}
\]

Proof. Let \(Q_X : \ell_1 \rightarrow X\) be a quotient mapping and let \(\widetilde{X}\) be the kernel of this map. Consider the collection \(\{L_j : j \in J\}\) of all linear operators \(L_j : \widetilde{X} \rightarrow C[0,1]\) with \(\|L_j\| \leq 1\). Then let \(L : \widetilde{X} \rightarrow \ell_\infty(J : C[0,1])\) be defined by \(L_x = (L_jx)_{j \in J}\). Since \(L\) has separable range, we can find a subspace of \(\ell_\infty(J : C[0,1])\) isomorphic to \(C[0,1]\) and containing the range of \(L\). In this way we induce a bounded linear operator \(A : \widetilde{X} \rightarrow C[0,1]\) such that every bounded operator \(B : \widetilde{X} \rightarrow C[0,1]\) factors through \(A\), i.e., \(B = VA\), where \(V : C[0,1] \rightarrow C[0,1]\) is bounded.

Next we use the pushout construction to construct our twisted sum:

\[
\begin{array}{cccc}
0 & \rightarrow & \widetilde{X} & \rightarrow & \ell_1 & \rightarrow & X & \rightarrow & 0 \\
& & A & \rightarrow & A_1 & \rightarrow \\
0 & \rightarrow & C[0,1] & \rightarrow & \ell_1 & \rightarrow & X & \rightarrow & 0;
\end{array}
\]

it remains to verify its universality.

So let \(0 \rightarrow C[0,1] \rightarrow Z \rightarrow X \rightarrow 0\) be any twisted sum of \(C[0,1]\) and \(X\). Then, using the projective property of \(\ell_1\), we can construct a quotient mapping \(T_1 : \ell_1 \rightarrow Z\). Since it is unique up to automorphism, we may choose \(\widetilde{X} = T_1^{-1}(C[0,1])\). If \(T\) is the restriction of \(T_1\) to \(\widetilde{X}\), then the following diagram commutes:

\[
\begin{array}{cccc}
0 & \rightarrow & \widetilde{X} & \rightarrow & \ell_1 & \rightarrow & X & \rightarrow & 0 \\
& & T & \rightarrow & T_1 & \rightarrow \\
0 & \rightarrow & C[0,1] & \rightarrow & Z & \rightarrow & X & \rightarrow & 0.
\end{array}
\]

This means simply that \(Z\) is obtained by the pushout of \(0 \rightarrow \widetilde{X} \rightarrow \ell_1 \rightarrow X \rightarrow 0\) using \(T\). Now we can write \(T = SA\) for some \(S : C[0,1] \rightarrow C[0,1]\), and it follows that \(Z\) is obtained from \(Y\) by the pushout construction using \(S\). \(\square\)

We need the well-known result that there is a non-trivial twisted sum of \(C[0,1]\) and \(c_0\). The first published reference we know is [22, Theorem 6]. In [11] a stronger statement about the non-existence of Lipschitz liftings is proved; a non-separable version is to be found in [18]. The example, also studied in [27], can be described as follows. Let \(Q = (q_n)\) be any dense sequence in \([0,1]\). We could for example order the rationals in \((0,1)\) into a sequence \((q_n)\), but we prefer not to be specific. Denote by \(D\) the set of all functions from \([0,1]\) into \(\mathbb{R}\) that are continuous at every \(t \notin Q\) and left continuous with right limits at every \(t \in Q\). Routine arguments show that all such functions are bounded and that the sup-norm makes \(D\) into a Banach space. Clearly \(C = C[0,1]\) is a closed subspace and \(D/C\) is isometric to \(c_0\). More precisely, let us denote by \(J : D \rightarrow c_0\) the “jump function” \(Jf = \frac{1}{t}(f(q_n+)-f(q_n))\). Then \(J\) maps \(D\) onto \(c_0\), and \(d(f,C) = \|Jf\|\) for all \(f\) in \(D\). We denote by \(e_n\) the usual basis in \(c_0\). It is well known [6, p. 333], [27, p. 20] that \(D\) is isometric to the space of continuous functions on the Cantor set, but we do not need this representation.

**Lemma 2.2.** Let \((f_n)\) be any sequence of functions in \(D\) for which \(J(f_n) = e_n\) for all \(n\). Then the sequence \((f_n)\) is not weakly Cauchy.
Proof. The assumption $J(f_n) = e_n$ means that $f_n(q_{n+}) - f_n(q_n) = 2$ for all $n$. Let us assume $(f_n)$ is weakly Cauchy and hence bounded. We first note that if $I$ is any nonempty open interval in $(0, 1)$, $\alpha \in \mathbb{R}$ and $m \in \mathbb{N}$, then there exist $n > m$ and a nonempty open interval $J$ with $\overline{I} \subset I$ such that for some $\beta$ with $|\beta - \alpha| \geq 1$ we have $|f_n(t) - \beta| \leq \frac{1}{4}$ for $t \in J$. Indeed, we just pick $n > m$ so that $q_m \in I$, and then let $\beta$ be either $f_n(q_n)$ or $f_n(q_{n+})$. The interval $J$ can then be chosen using the left- or right-hand limit condition.

Now we can use this inductively to create a subsequence $(f_{n_k})$ of $(f_n)$, a sequence of nonempty intervals $(I_k)$ with $\overline{I}_{k+1} \subset I_k$, and a sequence of reals $(\alpha_k)$ with $|\alpha_{k+1} - \alpha_k| \geq 1$ so that $|f_{n_k}(t) - \alpha_k| \leq \frac{1}{4}$ for $t \in I_k$. If we pick $t_0 \in \bigcap_{k=1}^{\infty} I_k$ (which is nonempty by compactness), it is clear that $|f_{n_k}(t_0) - f_{n_{k+1}}(t_0)| \geq \frac{1}{2}$ for all $k$, and this gives us a contradiction.

**Theorem 2.3.** Suppose $X$ is a separable Banach space. Then there is a twisted sum

$$0 \rightarrow C[0,1] \rightarrow Y \xrightarrow{Q} X \rightarrow 0$$

with $Q$ strictly singular if and only if $X$ contains no copy of $\ell_1$.

Proof. If $\ell_1$ embeds into $X$, then, by the well-known lifting property of $\ell_1$ ([36, p. 107]), $Q$ cannot be strictly singular.

Conversely, suppose $\ell_1$ does not embed into $X$. We will argue that the universal twisted sum $Y$ given by Theorem 2.1 has a strictly singular quotient map $Q : Y \rightarrow X$. First we show that whenever $E$ is an infinite-dimensional closed subspace of $X$, then there is a twisted sum $0 \rightarrow C[0,1] \rightarrow Z \rightarrow X \rightarrow 0$ so that the pullback by the inclusion $E \rightarrow X$ does not split. Since $X$ does not contain $\ell_1$, any such subspace $E$ contains a weakly null basic sequence $(x_n)_{n=1}^{\infty}$ ([36, p. 5, Remark] spanning a subspace $E_0$. By considering the basis expansion we thus obtain a map $T_0 : E_0 \rightarrow c_0$ so that $T_0(x_n) = e_n$, the $n$th-basis vector in $c_0$. Since $c_0$ is separably injective, we can extend $T_0$ to a bounded operator $T : X \rightarrow c_0$.

We now use the twisted sum of $C[0,1]$ and $c_0$ constructed above and form the pullback using $T$:

$$0 \rightarrow C[0,1] \xrightarrow{J} D \xrightarrow{c_0} X \rightarrow 0$$

We now need only show that the further pullback via the inclusion $E \rightarrow X$ does not split. Thus we consider

$$0 \rightarrow C[0,1] \xrightarrow{J} D \xrightarrow{c_0} E \rightarrow 0$$

Now if $L : E \rightarrow Z_0$ is a lifting, then $VLx_n$ is weakly null. However, $JVLx_n = e_n$, and so we contradict Lemma 2.2.

Finally, Theorem 2.1 implies that the sequence $0 \rightarrow C[0,1] \rightarrow Z_0 \rightarrow E \rightarrow 0$ can be obtained from the sequence $0 \rightarrow C[0,1] \rightarrow Y \rightarrow X \rightarrow 0$ by first taking the pushout via $S : C[0,1] \rightarrow C[0,1]$ and then taking the pullback via $E \rightarrow X$. This procedure is equivalent to first taking the pullback via $E \rightarrow X$, and then taking the pushout via $S : C[0,1] \rightarrow C[0,1]$. Since the final sequence does not split, neither
does the intermediate sequence $0 \rightarrow C[0, 1] \rightarrow Q^{-1}(E) \rightarrow E \rightarrow 0$. Since $E$ was arbitrary, we conclude that $Q$ is strictly singular. □

A simplification of this argument shows that if $X$ is separable but fails the Schur property, then $\text{Ext}(X, C[0, 1]) \neq \{0\}$. Of course Theorem [1.3] is stronger.

This essentially formal construction gives an interesting corollary:

**Corollary 2.4.** There is a twisted sum $Y$ of $C[0, 1]$ and $c_0$ that is necessarily an $L_\infty$-space but is not isomorphic to a quotient of $C(K)$ for any compact $K$.

*Proof.* Taking $X = c_0$ in Theorem [2.3] gives us an example with $Q : Y \rightarrow c_0$ strictly singular. Since $c_0$ is not reflexive, $Q$ cannot be weakly compact. By a well-known result of Pelczynski [39, Theorem 1], $Y$ cannot be isomorphic to a quotient of any $C(K)$ space. □

Note here that $Y^*$ is isomorphic to an $L_1(\mu)$-space, but $Y$ cannot be renormed so that $Y^*$ is isometric to an $L_1(\mu)$ by a result of Johnson and Zippin [25]. This easily gives a counterexample to the old problems 3c and 3e of Lindenstrauss and Rosenthal [35], although other much more sophisticated counterexamples have been known for some time [7], [8]. For a stronger example, see the end of §4.

3. Twisted sums with $C(\omega^N)$

If $N \in \mathbb{N}$, then the space $C(\omega^N)$ is isomorphic to $c_0$, and so for any separable Banach space $X$, we have $\text{Ext}(X, C(\omega^N)) = \{0\}$. In this case it is natural to introduce the extension constant $\pi_N(X)$, which we define to be the least constant so that if $Y$ is a separable Banach space and $E$ is a closed subspace with $Y/E$ isometric to $X$, then for any bounded linear operator $T : E \rightarrow C(\omega^N)$ and $\varepsilon > 0$, there is an extension $\bar{T} : Y \rightarrow C(\omega^N)$ with $\|T\| < \pi_N(X)\|\bar{T}\| + \varepsilon$. In view of Proposition [1.5], $\pi_N(X)$ is also the least constant such that if

$$0 \rightarrow C(\omega^N) \xrightarrow{j} Y \xrightarrow{q} X \rightarrow 0$$

is an (isometric) exact sequence and $\varepsilon > 0$, then there is a linear operator $P : Y \rightarrow C(\omega^N)$ with $Pj = \text{Id}_{C(\omega^N)}$ and $\|P\| \leq \pi_N(X) + \varepsilon$.

The following theorem is due to Amir [2], [3] and Baker [4]:

**Theorem 3.1.** For any separable Banach space $X$ we have $\pi_N(X) \leq 2N + 1$, and there is a separable Banach space $X$ such that $\pi_N(X) = 2N + 1$.

In fact, it follows from the arguments in [3] that we may take $X = C(\omega^{N-1})$. The main purpose of this section is to show that $X$ may be chosen independently of $N$, more precisely that $\pi_N(c_0) = 2N + 1$. This will be needed in the next section, where it will also be useful to introduce an alternative constant $\rho_N(X)$, defined as the least constant such that if $T : X \rightarrow \ell_\infty(\omega^N)$ is a bounded operator satisfying $d(Tx, C(\omega^N)) \leq \|x\|$ for $x \in X$, and $\varepsilon > 0$, there is a linear operator $L : X \rightarrow C(\omega^N)$ with $\|T - L\| \leq \rho_N(X) + \varepsilon$.

**Lemma 3.2.** For any separable Banach space $X$ we have $\rho_N(X) \leq \pi_N(X) \leq \rho_N(X) + 1$.

*Proof.* First suppose $Y$ is a Banach space containing $C(\omega^N)$ and such that $Y/C(\omega^N)$ is isometric to $X$. Then there is a bounded projection $P_0 : Y \rightarrow C(\omega^N)$. (We may suppose $\|P_0\| \leq 2N + 1$, but this is not necessary.) We can also find a linear operator
Lemma 3.3. Suppose \( K \) is a compact Hausdorff space and \( h \in \ell_\infty(K) \). Then

\[
d(h, C(K)) = \frac{1}{2} \sup_{s \in K} \limsup_{t \to s} h(t) - \liminf_{t \to s} h(t).
\]

Proof. Define \( f(s) = \lim_{t \to s} h(t) \) and \( g(s) = \lim_{t \to s} h(t) \) for \( s \in K \). It is routine to check that \( f \) is upper semicontinuous and that \( g \) is lower semicontinuous. If \( R = \frac{1}{2} \sup_{s \in K} (g(s) - f(s)) \), then a classical interpolation theorem gives us a continuous function \( h \) satisfying \( g - R \leq h \leq f + R \). Clearly \( f \leq h \leq g \), and so \(-R \leq h - R \leq R\), as required. 

We now need a representation of \( \omega^N \). To this end we consider the power set of \( \mathbb{N} \), i.e., \( 2^\mathbb{N} \), which is homeomorphic to the Cantor set in the standard product topology. Let \( \mathcal{F}_N \) be the subset of all sets \( a \) with cardinality \(|a| \leq N \). Then \( \mathcal{F}_N \) is homeomorphic to \( \omega^N \). Indeed, \( \{ \sum_{n=1}^N 2^{-n} : a \in \mathcal{F}_N \} \) is order isomorphic and homeomorphic to \( \omega^N \).

Any nonempty finite subset \( a \) of \( \mathbb{N} \) will be written in increasing order, i.e., \( a = \{n_1, \ldots, n_k\} \), where \( n_1 < n_2 < \ldots < n_k \). We write \( \max a = n_k \). We write \( a < b \) if either \( a \) is empty and \( b \) is not, or if \( a = \{n_1, \ldots, n_k\} \) and \( b = \{m_1, \ldots, m_l\} \), where \( l > k \) and \( m_j = n_j \) for \( j \leq k \). For each nonempty finite \( a = \{n_1, \ldots, n_k\} \in 2^\mathbb{N} \) we define \( a- = \{n_1, \ldots, n_{k-1}\} = a \setminus \{n_k\} \). We define \( a+ \) as the collection of all \( a \vee m = \{n_1, \ldots, n_k, m\} \), where \( m > n_k \); \( \emptyset \) is simply \( \mathbb{N} \). Although we do not need it in this section, we define here a subset \( \mathcal{A} \) of \( \mathcal{F}_N \) to be full if the following three conditions hold:

1. \( \emptyset \in \mathcal{F}_N \).
2. If \( \emptyset \neq a \in \mathcal{A} \), then \( a- \in \mathcal{A} \).
3. If \( a \in \mathcal{A} \) and \(|a| < N\), then \( \mathcal{A} \cap a+ \) is infinite.

It is then easy to see that any full subset of \( \mathcal{F}_N \) is also homeomorphic to \( \omega^N \).

Next let \( \mathcal{A} \) be a full subset of \( \mathcal{F}_N \) and let \( X \) be a fixed separable Banach space. We consider a bounded map \( a \mapsto x_a^* \) of \( \mathcal{A} \) into \( X^* \).
Lemma 3.4. If $T : X \to \ell_\infty(A)$ is defined by $Tx(a) = x_a^*(x)$, then we have
\[
d(Tx, C(A)) \leq \|x\| \quad \forall x \in X
\]
if and only if $\limsup_{b,c \to a} \|x_b^* - x_c^*\| \leq 2$ for each $a \in A$ with $|a| < N$.

Proof. This follows easily from Lemma 3.3 since we require $\limsup_{b,c \to a} x_b^*(a) - \liminf_{b,c \to a} x_c^*(x) \leq 2\|x\|$ for all $x \in X$. We omit the details. Note that if $|a| = N$, then any sequence converging to $a$ will be eventually constant. \qed

We conclude this section with a minor variation of Amir’s part of the Amir-Baker Theorem:

Theorem 3.5. For each $N$ we have $\pi_N(c_0) = 2N + 1$.

Proof. Let us choose $\varepsilon > 0$ and $r \in \mathbb{N}$, and let $m = 2^r$. Then let $G$ be the dyadic group $\{-1, 1\}^r$, with its usual normalized measure, and let $u_1, \ldots, u_m$ denote the characters of this group. Let $\pi = \frac{1}{m}(u_1 + \cdots + u_m)$, so that $\pi$ is actually the function that is one at the identity and zero elsewhere. Let $v_k = u_k - \pi \in \ell_\infty(G)$ and $v_k^* = u_k$, regarded as an element of $L_1(G) = \ell_\infty(G)^*$. Then $\|v_k\| = \|v_k^*\| = 1$ for all $k$, and if $k \neq k'$, then $\|v_k^* - v_k\| = 1$.

Now consider $X = c_0(F_{N-1}; \ell_\infty(G))$ so that $X$ is isometric to $c_0$. We define a linear operator $T : X \to \ell_\infty(F_N)$. Consider any element $x = (w_a)_{a \in F_{N-1}} \in X$, where $w_a \in \ell_\infty(G)$. We define $Tx(0) = 0$, and then
\[
Tx(a) = Tx(a) + 2v_a^*(w_{a-})
\]
where $j \equiv \max a \pmod{m}$. Now let $Z$ be the set of all $(x, h) \in X \oplus \ell_\infty(F_N)$ such that $h - Tx \in C(F_N)$, and put $E = \{(0, h) : h \in C(F_N)\}$; it is easy to see that the quotient space $Z/E$ is isometric to $X$ (since $d(Tx, C(F_N)) \leq \|x\|$ by Lemma 3.4). Let $P$ be a bounded projection of $Z$ onto $E$, and write $P(x, Tx) = (0, Sx)$, where $S : X \to C(F_N)$.

For notational purposes, if $a \in F_{N-1}$ and $j \leq m$, we define $H(a, j)$ to be the set of $b \geq a \vee n$, where $n > \max a$ and $n \equiv j \pmod{m}$, and $x_{j,a} = v_j \chi_{[a]}$ for all $a \in F_N$ we put $K(a) = \{b : b \geq a\}$.

We now claim that if $a \in F_{N-1}$, then there exists $j = j(a)$ so that $x = x_{j,a}$ satisfies $Sx(a) \leq 0$. Indeed, $\sum_{j=1}^m x_{j,a} = 0$, and so $\sum_{j=1}^m Sx_{j,a}(a) = 0$. Considering the topology on $F_N$, it follows that there exists $k = k(a) > \max a$ so that if $b \geq a \vee l$, where $l \geq k(a)$, then $Sx(b) \leq \varepsilon$.

Let us take $n_1 = j(0) + mk(0)$ and then define inductively $n_2, \ldots, n_N$ so that $n_s \geq k(\{n_1, \ldots, n_{s-1}\})$ and $n_s \equiv j(\{n_1, \ldots, n_{s-1}\}) \pmod{m}$ for $1 < s \leq N$. Let $a = \{n_1, \ldots, n_N\}$. Then we let
\[
x = \sum_{0 \leq b < a} x_{j(b),b}.
\]
It is easy to see that
\[
Sx(a) \leq N\varepsilon.
\]

It is routine to check that if $c \geq b \vee n$, with $n \equiv j \pmod{m}$, then
\[
T(v_{j(b)} \chi_{(b)}) = 2v_{j}^*(v_{j(b)}),
\]
and $T(v_{j(b)} \chi_{(b)}) = 0$ for all other $c \in F_N$. Since $v_{j}^*(v_{b}) = \delta_{jk} - \frac{1}{m}$, where $\delta_{jk}$ is the Kronecker delta, this implies that
\[
T(x_{j(b),b}) = 2\chi_{H(b,j(b))} - \frac{2}{m} \chi_{K(b)\setminus\{b\}}.
\]
Summing, we obtain
\[ Tx = 2 \sum_{0 \leq b < a} \left( \chi_{H(b,j(b))} - \frac{1}{m} \chi_{K(b) \setminus \{b\}} \right). \]

Let \( h = \chi_{K(b)} + 2 \sum_{0 < b < a} \chi_{K(b)}. \) By construction \( H(b,j(b)) \subseteq K(b) \subseteq H(b-, j(b-)) \) for each \( b \leq a. \) A short calculation then yields
\[ \|Tx - h\| \leq 1 + \frac{2N}{m}. \]

Since \( \|v_{j(b)}\| = 1, \) we also have \( \|(x, Tx - h)\| \leq 1 + \frac{2N}{m}, \) and thus \( \|Sx - h\| \leq \|P\|(1 + \frac{2N}{m}). \) But \( h(a) = 2N + 1. \) Thus
\[ 2N + 1 - N \leq (h - Sx)(a) \leq \|P\|(1 + \frac{2N}{m}). \]

Since we can choose \( m \) arbitrarily large and \( \varepsilon \) arbitrarily small, this implies that \( \pi_N(c_0) \geq 2N + 1. \)

\[ \square \]

4. Twisted sums with \( C(\omega^\omega) \)

Our motivation for studying the constants \( \pi_N(X) \) comes from the following theorem:

**Theorem 4.1.** Suppose \( X \) is a separable Banach space. Then \( \text{Ext}(X, C(\omega^\omega)) = \{0\} \) if and only if \( \sup_N \pi_N(X) < \infty. \)

**Proof.** To simplify notation we will work with \( C_0(\omega^\omega) = \{ f \in C(\omega^\omega) : f(\omega^\omega) = 0 \}, \) which is clearly isomorphic to \( C(\omega^\omega). \) Since \( C(\omega^N) \) is isomorphic to a one-complemented subspace of \( C_0(\omega^\omega) \) for each \( N, \) necessity is obvious. Conversely, suppose \( Y \) is a separable Banach space and \( E \) is a closed subspace of \( Y \) so that \( Y/E \) is isometric to \( X. \) Suppose \( T : E \to C_0(\omega^\omega) \) is bounded with \( \|T\| \leq 1. \) Let \( M = \sup_N \pi_N(X) + 1. \) For \( n \in \mathbb{N} \) let \( R_n : C_0(\omega^\omega) \to C(K_n), \) where \( K_1 = [1, \omega] \) and \( K_n = [\omega^{n-1} + 1, \omega^n] \) for \( n \geq 2. \)

Let \( F_k \) be an increasing sequence of finite-dimensional subspaces of \( Y \) such that \( \bigcup F_k \) is dense in \( Y. \) Let \( G_k \) be finite-dimensional subspaces of \( E \) so that if \( x \in F_k, \) then \( d(x, G_k) \leq 2d(x, E). \) Let \( q : Y \to Y/E \) be the quotient map and let \( q(F_k) = H_k. \)

For each \( k \) let \( n(k) \) be the least integer such that if \( e \in (F_k + G_k) \cap E, \) then \( \|R_n Te\| \leq 2^{-k}\|e\|. \) Then, since \( T \) maps \( E \) into \( C_0(\omega^\omega), \) we see that \( n(k) \) is well defined.

For fixed \( k, \) letting \( n = n(k), \) we can, since \( C(K_n) \) is an \( \mathcal{L}_{\infty,1}\)-space, find an operator \( S_n : F_k + G_k \to C(K_n) \) so that \( \|S_n\| \leq 2^{1-k} \) and \( S_n e = R_n Te \) for \( e \in E \cap (F_k + G_k). \) Also we can find an operator \( V_n : Y \to C(K_n) \) such that \( \|V_n\| \leq M \) and \( V_n e = R_n Te \) for \( e \in E. \)

Now if \( y \in F_k + G_k, \) then there exists \( g \in G_k \) so that \( \|y - g\| \leq 2d(y, E). \) Then
\[ \|V_n y - S_n y\| = \|V_n(y - g) - S_n(y - g)\| \leq 2(M + 2)d(y, E). \]

It follows that there is an operator \( U_n : H_n \to C(K_n) \) with \( \|U_n\| \leq 2M + 4 \) and \( U_n q = V_n - S_n. \) Since \( U_n(H_n) \) is finite dimensional, this may be extended to an operator \( \bar{U_n} : X \to C(K_n) \) with \( \|ar{U_n}\| \leq 2M + 5. \) Next set \( \bar{T_n} = V_n - \bar{U_n} q. \) Then \( \|ar{T_n}\| \leq 3M + 6, \) \( \bar{T_n} \) extends \( R_n T, \) and \( \bar{T_n} y = S_n y \) for \( y \in F_k + G_k, \) so that
\[ \|R_n T y\| \leq 2^{1-k}\|y\| \] for \( y \in F_k + G_k. \)
We finally extend the operator $T$ by setting

$$\tilde{T}y(\alpha) = R_nTy(\alpha) \quad \text{if } \alpha \in K_n.$$  

This provides an extension with $\|\tilde{T}\| \leq 3M + 6$. \hfill $\square$

Next we recall some ideas from [23]. Suppose $\mathcal{A}$ is a full subset of $\mathcal{F}_N$. We say that a map $a \mapsto u_a^* : \mathcal{A} \to X^*$ is a weak*-null tree map if $u_0^* = 0$ and $\lim_{a+b} u_b^* = 0$ (weak*) whenever $|a| < N$. If $E$ is a closed subspace of $X^*$, we will define $\alpha_N(E)$ to be the infimum of all $\lambda$ such that whenever $a \mapsto u_a^*$ is a weak*-null tree map with $u_a^* \in E$ and $\|u_a^*\| \leq 1$ for all $a$, then there is a $b \in A$ with $|b| = N$ and

$$\left\| \sum_{a \leq b} u_a^* \right\| \leq \lambda.$$

We shall say that a weak*-null tree map is strongly weak*-null if

$$\lim_{\max a \to \infty} u_a^* = 0$$

weak*. The next lemma allows us to replace weak*-null by strongly weak*-null in the above definition of $\alpha_N(E)$.

**Lemma 4.2.** If $a \mapsto u_a^*$ is a bounded weak*-null tree map on a full subset $\mathcal{A}$ of $\mathcal{F}_N$, then there is a full subset $\mathcal{B}$ of $\mathcal{A}$ so that $a \mapsto u_a^*$ is strongly weak*-null on $\mathcal{B}$.

**Proof.** Let $(V_n)$ be a base of weak* -neighborhoods of 0 such that $V_{n+1} + V_{n+1} \subset V_n$ for all $n$. Let $\mathcal{B} = \{ b \in \mathcal{A} : u_b^* \in V_{\max a} \text{ for each } a \text{ with } 0 < a \leq b \}$. It is easily verified that $\mathcal{B}$ works. \hfill $\square$

Now suppose $X$ is a separable Banach space with a finite-dimensional Schauder decomposition $(F_n)$. We denote by $S(m, n)$, where $0 \leq m \leq n \leq \infty$ and $m < \infty$, the operator

$$S(m, n)(\sum_{k=1}^{\infty} f_k) = \sum_{k=m+1}^{n} f_k$$

if $f_k \in F_k$. Note that $S(n, n) = 0$ for all $n$. We say that $(F_n)$ is bi-monotone if $\|S(m, n)\| \leq 1$ for all $m, n$.

We shall let $E(m, n)$ be the range of $S(m, n)^* \in X^*$; we refer to such subspaces as block subspaces. We let $E$ be the closure of $\bigcup_{m<n<\infty} E(m, n)$.

**Theorem 4.3.** Suppose $X$ is a separable Banach space with a bi-monotone FDD $(F_n)$. Then:

1. $\rho_{2N}(X) \leq 4\alpha_N(E)$.
2. If $(F_n)$ is 1-unconditional and shrinking (so that $E = X^*$), then $\alpha_N(X^*) \leq 2\rho_N(X)$.

**Proof.** (1) Suppose $\lambda > 0$. We define a notion of $\lambda$-acceptable subsets of $B_E$ of cardinality at most $N$. A subset $\{x_1^*, \ldots, x_N^*\}$ of cardinality $N$ is $\lambda$-acceptable if $\|x_1^* + \cdots + x_N^*\| \leq \lambda$. We define acceptable sets of cardinality $0 \leq k < N$ by reverse induction. For each $0 \leq k < N$, a subset $\{x_1^*, \ldots, x_k^*\}$ is $\lambda$-acceptable if there is a weak*-neighborhood $V$ of zero so that if $x_{k+1}^* \in B_E \cap V$, then $\{x_1^*, \ldots, x_k^*\}$ is $\lambda$-acceptable. It is easily seen that if $\lambda > \alpha_N = \alpha_N(E)$, then the empty set is $\lambda$-acceptable. More precisely it is easy to show that if this fails, then one can construct a weak*-null tree map on $\mathcal{F}_N$ denoted by $a \mapsto u_a^*$ with $u_a^* \in B_E$ so that
for every $a$ with $|a| = N$ we have $\|\sum_{b \leq a} u_{j_{b}}^{*}\| > \lambda$. This contradicts the definition of $\alpha_{N}$.

Next we shall say that a collection of $k \leq N$ block subspaces $\{G_{1}, \ldots, G_{k}\}$ is $\lambda$-good if for some $\mu < \lambda$ and every $x_{j_{b}}^{*} \in B_{G_{j_{b}}}$, the set $\{x_{j_{1}}^{*}, \ldots, x_{j_{k}}^{*}\}$ is $\mu$-acceptable.

Claim. Suppose $\lambda > \alpha_{N}$. There is a function $\psi : \mathbb{N} \to \mathbb{N}$ so that if $\{G_{1}, \ldots, G_{k}\}$ is a $\lambda$-good family of block subspaces of $E(0, n)$ with $k < N$, then for any block subspace $G_{k+1}$ of $E(\psi(n), \infty)$ the collection $\{G_{1}, \ldots, G_{k+1}\}$ is $\lambda$-good.

Let us prove the claim. Since the family of block subspaces of $E(0, n)$ is finite, it is clear there exists $\mu < \lambda$ so that every $\lambda$-good collection $\{G_{1}, \ldots, G_{k}\}$ of block subspaces is actually $\mu$-good. Then pick $\varepsilon > 0$ so that $\mu + N\varepsilon < \lambda$. Choose in each block subspace $G$ an $\varepsilon$-net for the unit ball $B_{G}$. In this way we produce a finite collection $\mathcal{G}$ of $\mu$-acceptable sets $\{x_{j_{1}}^{*}, \ldots, x_{j_{k}}^{*}\}$ so that whenever $\{G_{1}, \ldots, G_{k}\}$ is any $\lambda$-good collection of block subspaces of $E(0, n)$ and whenever $y_{j_{b}}^{*} \in B_{G_{j_{b}}}$, then there is a $\{x_{j_{1}}^{*}, \ldots, x_{j_{k}}^{*}\} \in \mathcal{G}$ with $\|y_{j_{b}}^{*} - x_{j_{b}}^{*}\| \leq \varepsilon$ for $1 \leq j \leq k$. Now it is clear from the definition of acceptability that we can find $\psi(n)$ so that if $x^{*} \in G \cap E(\psi(n), \infty)$ and $\{x_{j_{1}}^{*}, \ldots, x_{j_{k}}^{*}\} \in \mathcal{G}$ with $k < N$, then $\{x_{j_{1}}^{*}, \ldots, x_{j_{k}}^{*}, x^{*}\}$ is $\mu$-acceptable. Now it is easy to see by a perturbation argument that if $\{G_{1}, \ldots, G_{k}\}$ is $\lambda$-good with $k < N$ and each $G_{j}$ is contained in $E(0, n)$, then for any block subspace $G$ of $E(n, \infty)$ the collection $\{G_{1}, \ldots, G_{k}, G\}$ is $(\mu + N\varepsilon)$-good and hence also $\lambda$-good. This proves the claim.

We now fix $\lambda > \alpha_{N}$ and suppose $\theta > 1$. Now suppose $T_{x} = (x_{a}(x))_{a \in F_{2N}}$ is a linear operator $T : X \to \ell_{\infty}(F_{2N})$ with $d(Tx, C(F_{2N})) \leq \|x\|$ for all $x \in X$. We use Lemma 3.4. For each $a \in A$ with $a > 0$ we define $\nu = \nu(a)$ to be the greatest natural number so that if $b \in F_{2N}$ and $b \geq a$, then $\|S(0, \nu)x_{a}^{*} - S(0, \nu)x_{a}^{*}\| \leq 2\theta$. It follows from Lemma 3.4 that $\lim_{a \to \infty} \mu(b) = \infty$ for all $a$ with $|a| < N$.

Next we inductively construct a map $\varphi : F_{2N} \to \mathbb{N}$. Let $\varphi(0) = \psi(0)$. Then we define $\varphi(a)$ by induction on $|a|$. If $\nu(a) < \psi(\varphi(a)-1)$, we let $\varphi(a) = \varphi(a)$. If $\nu(a) \geq \psi(\varphi(a)-1)$, we let $\varphi(a) = \nu(a)$.

Now we define $z_{a}^{*}$ for $a \in F_{2N}$ by putting $z_{a}^{*} = x_{a}^{*}$, and then if $|a| > 0$ we define

$$z_{a}^{*} = \sum_{0 < b \leq a} S(\varphi(b-), \varphi(b))x_{b}^{*} + S(\varphi(a), \infty)x_{a}^{*}.$$ 

We claim that $a \mapsto z_{a}^{*}$ is weak*-continuous. In fact, if $b > a$, let $c$ be the unique element in $a+$ with $a < c \leq b$. Then

$$z_{b}^{*} - z_{a}^{*} = \sum_{c < d \leq b} S(\varphi(d-), \varphi(d))x_{c}^{*} + S(\varphi(c), \infty)x_{c}^{*}.$$ 

Now $\lim_{c \to a+} \mu(c) = \infty$, and so $\lim_{c \to a+} \varphi(c) = \infty$ and $\varphi(d) \geq \varphi(c)$ whenever $c \leq d \leq b$. Hence as $b \to a$ we have $z_{b}^{*} - z_{a}^{*} \to 0$ weak*.

Suppose now $a = \{n_{1}, \ldots, n_{k}\} \in F_{2N}$. Let $m_{0} = \varphi(0)$, and then put $m_{j} = \varphi\{n_{1}, \ldots, n_{j}\}$ for $1 \leq j \leq k$. Consider the subspaces

$$\{E(m_{0}, m_{1}), E(m_{1}, m_{2}), \ldots, E(m_{k-1}, m_{k})\}.$$ 

If we delete those subspaces where $m_{j} = m_{j-1}$ (i.e., where the subspace reduces to $\{0\}$), then it is clear by induction that the remaining subspaces can be split into two $\lambda$-good collections by taking them alternately. Hence, if $u_{j}^{*} \in E(m_{j-1}, m_{j})$ with $\|u_{j}^{*}\| \leq 1$ for $1 \leq j \leq k$, then $\|\sum_{j=1}^{k} u_{j}^{*}\| \leq 2\lambda$. 


Next we estimate $\|x^*_a - z^*_a\|$. We have

$$x^*_a - z^*_a = \sum_{\emptyset \neq b \subseteq a} S(\varphi(b -), \varphi(b))^*(x^*_a - x^*_{b-}).$$

If $\varphi(b) > \varphi(b-)$, then $\varphi(b) = \mu(b)$, and so $\|S(\varphi(b-), \varphi(b))^*(x^*_a - x^*_{b-})\| \leq 2\theta.$ By the above remarks we have

$$\|x^*_a - z^*_a\| \leq 4\lambda\theta.$$

Our conclusion is that there is a bounded operator $Lx = (z^*_a(x))_{a \in \mathcal{F}_{2N}}$ into $C(\mathcal{F}_{2N})$ with $\|L - T\| \leq 2\lambda\theta$. Thus $\rho_{2N}(X) \leq 2\alpha_N(E)$. This concludes the proof of (1).

(2) Let us suppose $a \mapsto u^*_a$ is a strongly weak*-null tree map on $\mathcal{F}_N$ with $\|u^*_a\| \leq 1$ for $a \in \mathcal{F}_N$. Let $\gamma : \mathbb{N} \to \mathbb{N}$ be any surjective map so that for each $k \in \mathbb{N}$ the set $\gamma^{-1}(k)$ is infinite. Let $A$ be the subset of $\mathcal{F}_N$ consisting of the empty set and all $\{n_1, \ldots, n_k\}$ such that $\gamma(n_j) \geq n_{j-1}$ for $2 \leq j \leq k$. It is clear that $A$ is full. Let $\sigma' \{n_1, \ldots, n_k\} = \{\gamma(n_1), \ldots, \gamma(n_k)\}$ for $\{n_1, \ldots, n_k\} \in A$. We then define $a \mapsto x^*_a$ for $a \in A$ by

$$x^*_a = \sum_{\emptyset \neq b \subseteq a} u^*_\sigma(b).$$

Note that if $d > a$ with $d \in A$, then

$$x^*_d - x^*_a = u^*_\sigma(c) + \sum_{c < b \leq d} u^*_\sigma(b),$$

where $a < c = e(d) \leq d$ and $|c| = |a| + 1$. Then it follows from the strong weak*-nullity of $a \mapsto u^*_a$ that

$$\lim_{d \to a} \sum_{c < b \leq d} u^*_\sigma(b) = 0$$

weak*, since $\max(\sigma(b)) \geq max c$. Hence we have

$$\lim_{d \to a} \|x^*_d - x^*_a\| \leq 1.$$ 

By Lemma 3.4 and the definition of $\rho_N(X)$, for any $\lambda > \rho_N(X)$ we can find a weak*-continuous map $a \mapsto z^*_a$ on $A$ such that $\|x^*_a - z^*_a\| \leq \lambda$ for all $a$.

Now fix $\varepsilon > 0$. We determine an increasing sequence $n_1, \ldots, n_N$ so that $\{n_1, \ldots, n_N\} \in A$ and an increasing sequence $m_1, \ldots, m_{2N} \in \mathbb{N}$ by induction. Suppose $a = \{n_1, \ldots, n_{k-1}\}$ has been chosen in $A$ (where if $k = 1$, we take $a = \emptyset$) and that $m_1, \ldots, m_{2k-2}$ have been chosen. Then pick $m_{2k-1} > m_{2k-2}$ (if $k \geq 2$) so that $\|S(m_{2k-2}, \infty)^*u^*_a\| < \varepsilon/(6N)$. This is possible since the (FDD) is shrinking. Now pick $c \in \sigma(a)$ with $\|S(0, m_{2k-1})^*u^*_a\| < \varepsilon/(6N)$; this is possible since $\lim_{c \in \sigma(a)} u^*_a = 0$ weak*. Pick $m_{2k} > m_{2k-1}$ so that $\|S(m_{2k}, \infty)^*u^*_a\| < \varepsilon/(6N)$. Now there are infinitely many $b \in a+$ with $\sigma(b) = c$; amongst these we may choose $b$ so that $\|S(0, m_{2k})^*(z^*_b - z^*_a)\| < \varepsilon/(6N)$, since $\lim_{b \to a} z^*_b = z^*_a$ weak*. We then let $b = \{n_1, \ldots, n_k\}$. This completes the inductive construction.
Let \( a_k = \{n_1, \ldots, n_k\} \) for \( 0 \leq k \leq N \). Then
\[
\left\| \sum_{k=1}^{N} u^*_\sigma(a_k) \right\| \leq \frac{\varepsilon}{3} + \left\| \sum_{k=1}^{N} S(m_{2k-1}, m_{2k}) u^*_\sigma(a_k) \right\|
\]
\[
\leq \frac{\varepsilon}{3} + \left\| \sum_{k=1}^{N} \left( S(m_{2k-1}, m_{2k}) u^*_\sigma(a_k) + S(m_{2k-2}, m_{2k-1}) (z^*_\sigma(a_k) - z^*_\sigma(a_{k-1})) \right) \right\|
\]
\[
\leq \frac{\varepsilon}{3} + \left\| u^*_\sigma(a_k) + z^*_\sigma(a_k) - z^*_\sigma(a_{k-1}) \right\|
\]
\[
\leq \varepsilon + \| x^*_a \| - z^*_a + z^*_a - x^*_a
\]
\[
\leq \varepsilon + 2\lambda.
\]

Hence by the definition of \( \alpha_N(X^*) \) we have \( \alpha_N(X^*) \leq 2\lambda + \varepsilon \). The theorem follows.

We are now in a position to prove our main result:

**Theorem 4.4.** (1) Suppose \( X \) is a separable Banach space with summable Szlenk index. Then \( \text{Ext}(X^*, C(\omega^*)) = \{0\} \).

(2) If \( Y \) is a separable Banach space with \( \text{Ext}(Y, C(\omega^*)) = \{0\} \) and \( Y \) has a (UFDD), then \( Y \) is the dual of a space \( X \) with summable Szlenk index.

**Remark.** For the definition and general properties of the Szlenk index, see for example [23] [2]. The original space constructed by Tsirelson [44] is a reflexive space with summable Szlenk index [31]. Its dual is the space usually referred to nowadays as Tsirelson’s space [14].

**Proof.** If \( X \) has a shrinking (FDD), then (1) follows directly from Theorem 4.3. We can assume via renorming that the (FDD) is bi-monotone. We consider the dual (FDD) of \( X^* \). In this case the subspace \( E \) of \( X^{**} \) is identified with \( X \) and the condition \( \sup_n \alpha_n(E) < \infty \) is equivalent (using [23] Theorem 4.10) to the fact that \( X \) has summable Szlenk index, and this implies that \( \sup_n \pi(E) \) is finite.

For the general case we use a theorem of Johnson and Rosenthal [21], [36] Theorem 1.6.1.2 p.48], that \( X \) has a subspace \( Y \) so that \( X/Y \) and \( Y \) both have shrinking (FDD)s. It is easy to check that having summable Szlenk index is a property that passes to quotients, and it follows from renorming results in [23] (Theorem 4.10 (ii)) that it passes also to subspaces. Thus \( Y \) and \( X/Y \) must both have summable Szlenk index. Hence we have \( \text{Ext}(Y^\perp, C(\omega^*)) = \{0\} \) and \( \text{Ext}(X^*/Y^\perp, C(\omega^*)) = \{0\} \), and so by Corollary 1.3 we have \( \text{Ext}(X, C(\omega^*)) = \{0\} \). This concludes the proof of (1).

For (2) we may assume the (UFDD) is 1-unconditional. We observe that Theorem 4.3 implies \( \text{Ext}(c_0, C(\omega^*)) \neq \{0\} \). (Direct constructions are also available.) Hence if \( \text{Ext}(Y, C(\omega^*)) = \{0\} \) and \( Y \) is separable, then \( Y \) contains no (necessarily complemented) copy of \( c_0 \). In particular, the (UFDD) of \( Y \) must be boundedly complete, and so \( Y = X^* \), where \( X = E \) as defined in Theorem 4.3. Then we have by Theorem 4.3 \( \sup \alpha_n(X) \) is finite, and hence by Lemma 5.2 \( \sup \pi(X) < \infty \). Applying Theorem 4.3 (2), we obtain \( \sup \alpha_n(X) < \infty \). It follows again from Theorem 4.10 of [23] that \( X \) has summable Szlenk index.

If \( X \) is any separable Banach space, we define a tree map \( a \mapsto v^*_a : \mathcal{F}_N \to X^* \) to be of **dense type** if the following conditions are satisfied:
Proof. This essentially follows from the argument in Theorem 4.3. Let $X \rightarrow a$ for Proposition 4.6.

Now if $b_n \rightarrow a$ and $|b_n| \geq |a| + 2$ for all $n$, then $v_{b_n}^* \rightarrow 0$ weak$^*$.

Next let $y_{b_n}^* = \sum_{b \leq n} v_b^*$. We can define $Tx = (y_{b_n}^*(x))_{a \in F_N}$, so that $T : X \rightarrow \ell_\infty(F_N)$.

Lemma 4.5. Suppose $X$ has a (UFDD). Suppose $L : X \rightarrow C(\omega^\omega)$, and $T : X \rightarrow \ell_\infty(F_N)$ is an operator induced by a tree map of dense type. Then $\rho_N(X) \leq 2\|L - T\|$.

Proof. This essentially follows from the argument in Theorem 4.3. Let $a \mapsto u_a^*$ be any strongly weak$^*$-null tree map with $\|u_a^*\| \leq 1$ for all $a$. Let $\gamma : N \rightarrow N$ be any surjective map so that for each $k \in N$ the set $\gamma^{-1}\{k\}$ is infinite. Let $A$ be the subset of $F_N$ consisting of the empty set and all $\{n_1, \ldots, n_k\}$ such that $\gamma(n_j) \geq n_{j-1}$ for $2 \leq j \leq k$. It is clear that $A$ is full. Let $\sigma(n_1, \ldots, n_k) = \{\gamma(n_1), \ldots, \gamma(n_k)\}$ for $\{n_1, \ldots, n_k\} \in A$.

We now build a map $\psi : A \rightarrow F_N$. Define $\psi(\emptyset) = \emptyset$. If $\psi(a)$ has been defined and $|a| < N$, we define $\psi(b)$ for each $b \in a+$ so that $\psi(b) \in \psi(a)+$, $\psi$ is one-one and $\lim_{b \in a+} u_{\sigma(b)}^* - v_{\psi(b)}^* = 0$ weak$^*$.

Let $x_a^* = \sum_{b \leq a} u_{\sigma(b)}^*$. Then we claim that $x_a^* - y_{\psi(a)}^*$ is weak$^*$-continuous. Indeed, if $b \geq a$,

$$x_b^* - x_a^* - y_{\psi(b)}^* + y_{\psi(a)}^* = \sum_{a < c \leq b} u_{\sigma(c)}^* - v_{\psi(c)}^*. $$

Now if $b_n \rightarrow a$ and we let $d_n$ be chosen so that $b_n \leq d_n \leq a$ and $|d_n| = |a| + 1$, we have

$$\sum_{d_n < c \leq b} (u_{\sigma(c)}^* - v_{\psi(c)}^*) \rightarrow 0$$

by the assumptions on both tree maps. On the other hand,

$$u_{\sigma(d_n)}^* - v_{\psi(d_n)}^* \rightarrow 0$$

by construction.

Now if $Lx = (z_a^*(x))_{a \in F_N}$, then $\|z_a^* - y_{\psi(a)}^*\| \leq \|L - T\|$. Now $a \mapsto z_{\psi(a)}^* + x_a^* - y_{\psi(a)}^*$ is weak$^*$-continuous, and we can repeat the argument of Theorem 4.3 to deduce the conclusion. 

It is clear that we can always construct a tree map of dense type. Simply let $(V_a)$ be a base of weak$^*$-neighborhoods of $\{0\}$ in $X^*$ with $V_{n+1} + V_{n+1} \subset V_n$. Then for $a$ with $|a| < N$, simply choose $\{u_{a\vee m}\}$ for $m > \max a$ to be any sequence that is weak$^*$-dense in $V_{\max a} \cap B_{X^*}$. It is also clear that if $Y$ is a subspace of $X$ and $j : Y \rightarrow X$ is the inclusion, then $a \mapsto j^* u_a^*$ is a tree map of dense type in $Y^*$. This leads us to the following:

Proposition 4.6. Let $X$ be a separable Banach space with a shrinking 1-unconditional (UFDD). Then there is a bounded operator $T : X \rightarrow \ell_\infty(\omega^N)$ so that

$$d(Tx, C(\omega^N)) \leq \|x\|$$

for all $x \in X$ and so that if $E$ is a subspace of $X$ with a (UFDD), then $\rho_N(E) \leq 2\|L - T\|$ for any bounded operator $L : E \rightarrow C(\omega^N)$. 


It is obvious from Theorem 4.4 that the existence of a twisted sum \(0 \to C(\omega^\omega) \to Y \to X \to 0\) with the quotient map strictly singular implies that \(X\) contains no subspace that is isomorphic to the dual of a space with summable Szlenk index. We now establish a partial converse.

**Theorem 4.7.** Suppose \(X\) has a shrinking (UFDD) and contains no subspace that is isomorphic to the dual of a space with summable Szlenk index. Then there is a short exact sequence

\[
0 \to C(\omega^\omega) \to V \xrightarrow{q} X \to 0
\]

with \(q\) strictly singular.

**Proof.** We may assume \(X\) has a 1-unconditional (UFDD). For each \(N\) we construct \(T_N : X \to \ell_\infty(\omega^N)\) as given in Proposition 4.6. Let \(Z_N\) be the space \(X \oplus C(\omega^N)\) normed by \(\|(x, h)\| = \|x\| + \|h - Tx\|\); then there is a quotient map \(q_N : Z_N \to X\) defined by \(q_N(x, h) = x\). We now construct an operator \(S_N : \tilde{X} \to C(\omega^N)\) in the usual way. Precisely, we fix a quotient map \(Q : \ell_1 \to X\) and define \(S_N : \ell_1 \to Z_N\) so that \(\|S_N\| \leq 2\) and \(q_N S_N = Q\). Now let \(S_N\) be the restriction of \(S_N\) to \(\tilde{X}\).

Let \((F_n)\) be an increasing sequence of finite-dimensional subspaces so that \(\bigcup F_n\) is dense in \(X\). Then, since \(C(\omega^N)\) is an \(L_\infty\)-space, we can find a finite-rank projection \(P_N\) on \(C(\omega^N)\) whose range includes \(S_N(F_N)\) and with \(\|P_N\| \leq 2\). Now let \(R_N = S_N - P_N S_N\). Thus \(\|R_N\| \leq 6\), and \(\lim_{N \to \infty} \|R_N \xi\| = 0\) for \(\xi \in \tilde{X}\).

We now define a map \(R : \tilde{X} \to W = c_0(C(\omega^N)_{N-1}^\infty)\) by \(R\xi = (R_N \xi)_{N-1}^\infty\). Note that the latter space is isomorphic to \(C(\omega^\omega)\). We can now construct a pushout

\[
\begin{array}{c}
0 \to \tilde{X} \xrightarrow{\ell_1} \ell_1 \xrightarrow{Q} X \to 0 \\
0 \to W \xrightarrow{V} V \xrightarrow{q_X} X \to 0.
\end{array}
\]

We claim that \(q_X\) is strictly singular. If not, we can find a subspace \(E\) of \(X\) with a 1-unconditional shrinking (UFDD) so that there is a bounded operator \(A : E \to V\) so that \(q_X A = I_E\). Then on \(Q^{-1}(E)\) we have \(q_X (Q V - A Q) = 0\), so that \(Q V - A Q : Q^{-1}(E) \to W\) is an extension of \(R\) to \(Q^{-1}(E)\). It follows that there exists a uniformly bounded sequence of operators \(\tilde{R}_N : Q^{-1}(E) \to C(\omega^N)\) which extend \(R_N\). Put \(M = \sup \|\tilde{R}_N\| < \infty\).

Note that \(P_N S_N\) has an extension to \(Q^{-1}(E)\) with \(\|P_N S_N\| \leq 5\), since it is a finite-rank operator taking values in \(C(\omega^N)\). Hence \(S_N\) has an extension \(\tilde{S}_N : Q^{-1}(E) \to C(\omega^N)\) with \(\|\tilde{S}_N\| \leq M + 5\). Then \(\tilde{S}_N - \tilde{S}_N\) factors through an operator \(e \mapsto (e, L_N e)\) from \(E\) into \(Z_N\) with norm at most \(M + 7\). This implies that \(\|L_N - T\| \leq M + 7\), and so \(\rho_N(E) \leq 2M + 14\). Theorem 4.5 and [25] Theorem 4.10 then show that \(E\) must have summable Szlenk index. \(\square\)

It now follows that there is a twisted sum of \(C(\omega^\omega)\) and \(c_0\) so that the quotient map is strictly singular. This space is not a quotient of a \(C(K)\)-space, and yet its dual must be isomorphic to \(\ell_1\). This shows that the main result of [25] does not admit an isomorphic version. The space \(Y\) constructed in [8] also serves as a counterexample.
5. Final remarks

In [21] (cf. [29]) it is shown that $\text{Ext}(\ell_2, \ell_2) \neq \{0\}$. It follows without difficulty that $\text{Ext}(\ell_p, \ell_q) \neq \{0\}$ when $1 < p, q < \infty$, since each space contains uniformly complemented copies of $\ell_2$. The following result is implicitly proved in [10], but it is heavily disguised; so we give a simple and direct diagram-chasing argument. For a nonlinear argument, see [12].

**Theorem 5.1.** $\text{Ext}(c_0, \ell_1) \neq \{0\}$.

**Proof.** In fact we will argue that $\text{Ext}(C[0,1], L_1) \neq \{0\}$. It then follows from local arguments that $\text{Ext}(X, Y) \neq \{0\}$ whenever $X$ is an $L_\infty$-space and $Y = L_1(\mu)$ for some measure $\mu$ (see, e.g., [12, Theorem 2]). Alternatively, one may carry out the ensuing argument locally.

We begin by considering some non-trivial twisted sum of $\ell_2$ and $\ell_2$. By using the pushout and pullback constructions we build the following diagram:

$$
\begin{array}{c}
0 \to \ell_2 \downarrow j_1 \to Z \xrightarrow{q_1} \ell_2 \to 0 \\
0 \to L_1 \xrightarrow{j_2} V \xrightarrow{q_2} \ell_2 \to 0 \\
0 \to L_1 \xrightarrow{j_3} W \xrightarrow{q_3} C[0,1] \to 0.
\end{array}
$$

Here linear embeddings are denoted by $j$ and quotient maps by $q$. First we recall that $Z$ is of cotype $p$ whenever $q < 2 < p$ [21, §3]. From the construction of the pushout, $V$ is of cotype $p$ for every $p > 2$.

We claim that the third row of this diagram cannot split. Suppose it does split. Then we can find an operator $T : C[0,1] \to W$ so that $q_3 T = I_{C[0,1]}$. Then $q_5 T : C[0,1] \to V$ must factor through some $L_r$-space, where $r > 2$ since $V$ has finite cotype. (This result can be traced to Maurey [37]; cf. also [32] or [20, Theorem 11.14(b)].) Since $L_r$ has type 2 and $L_1$ has cotype 2, every map from a subspace of $L_r$ to $L_1$ factors through a Hilbert space (this is Maurey’s generalization of Kwapien’s theorem [32] and [33]) and hence extends to a bounded operator from $L_r$ into $L_1$ by Maurey’s Extension theorem [38] (cf. [20, Theorem 12.22]). Applying all this to $(q_5 T)^{-1} (j_2 L_1)$, we can find an operator $R : C[0,1] \to j_2(L_1)$ so that $R f = q_3 T f$ if $q_2 q_5 T f = 0$. But $q_2 q_5 = q_4 q_3$. Then $q_5 T - R = T_1 q_4$ for some bounded operator $T_1 : U \to V$. Thus the second row splits.

The conclusion of the argument was given in the proof of [39, Theorem 4.1]. If the second row splits, then $V$ has cotype 2. Hence $Z$ also has cotype 2, and also has type $p > 1$. But then $Z^*$ is type 2 [41], and the Maurey Extension theorem guarantees that the dual exact sequence $0 \to \ell_2 \to Z^* \to \ell_2 \to 0$ splits. By reflexivity the first row splits, contrary to our choice of $Z$. \hfill $\square$

Finally, let us mention a non-separable problem related to the results of this paper. If $X$ is a separable Banach space, then $\text{Ext}(X, c_0) = \{0\}$ by Sobczyk’s theorem: we do not know, however, if there is a non-metrizable compact Hausdorff space $K$ such that $\text{Ext}(C(K), c_0) = \{0\}$. It is known that if $\Gamma$ is uncountable, then $\text{Ext}(c_0(\Gamma), c_0) \neq \{0\}$; this is essentially contained in one proof of the fact that $c_0(\Gamma)$ is uncomplemented in $\ell_\infty$; see also [11], [19, p. 260] and [13, §3]. It was noted in [17, Theorem 3.4] that if $X$ is any non-separable WCG-space, then $\text{Ext}(X, c_0) \neq \{0\}$, and this settles the case when $K$ is an Eberlein compact; similar arguments can
be used for Corson compact spaces. At the other extreme, if \( K \) is extremally disconnected, then \( C(K) \) contains a complemented \( \ell_\infty \) and \( \text{Ext}(\ell_\infty, c_0) \neq \{0\} \) was shown in [1]. Finally, the case of uncountable ordinal spaces can be reduced to \( K = [0, \omega_1] \), and in this case Parovičenko's theorem [7] shows that \( \text{Ext}(C(K), c_0) \neq \{0\} \).

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