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TWISTED SUMS WITH $C(K)$ SPACES

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ABSTRACT. If X is a separable Banach space, we consider the existence of non-trivial twisted sums $0 \to C(K) \to Y \to X \to 0$, where $K = [0,1]$ or ω^ω. For the case $K = [0,1]$ we show that there exists a twisted sum whose quotient map is strictly singular if and only if X contains no copy of ℓ_1. If $K = \omega^\omega$ we prove an analogue of a theorem of Johnson and Zippin (for $K = [0,1]$) by showing that all such twisted sums are trivial if X is the dual of a space with summable Szlenk index (e.g., X could be Tsirelson's space); a converse is established under the assumption that X has an unconditional finite-dimensional decomposition. We also give conditions for the existence of a twisted sum with $C(\omega^\omega)$ with strictly singular quotient map.

1. Introduction and preliminary remarks

Let X and Y be real Banach spaces. Then we say $\mathrm{Ext}(X, Y) = \{0\}$ if every short exact sequence $0 \to Y \to Z \to X \to 0$ splits; informally this means that if Z is a Banach space containing Y and so that $Z/Y \sim X$, then there is a bounded projection of Z onto Y. A space Z with a subspace isomorphic to Y so that Z/Y is isomorphic to X is often called a twisted sum of Y and X (order is important!). Thus $\mathrm{Ext}(X, Y) = \{0\}$ if and only if every twisted sum of Y and X is trivial (i.e. reduces to $Y \oplus X$).

Fundamental tools for us are the pushout and pullback constructions. These are well-known to algebraists and topologists, but less so to analysts. So we will describe them briefly in the Banach space setting. If $T : E \to X$ and $S : E \to Y$ are two operators defined on the same Banach space, then their pushout Z is defined as the quotient of $X \oplus_1 Y$ by the closure of $\{(Te, -Se) : e \in E\}$, together with the natural mappings $X \to Z$ and $Y \to Z$ (i.e., the restrictions of the quotient mapping). In case one of the mappings, say S, is the inclusion mapping from a short exact sequence, then completing the diagram gives a second short exact sequence with the same quotient space F:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \xrightarrow{S} & Y & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle T} & & \downarrow & & \| & & \\ 0 & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & F & \longrightarrow & 0. \end{array}$$

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Conversely, if we are given any commutative diagram as above, then Z must be isomorphic to the pushout of S and T; this observation will be used several times in the sequel. Note also that the operator $Y \to Z$ is an isomorphic embedding (respectively a quotient mapping) if and only if T is. Furthermore, the lower sequence splits if and only if T can be extended to Y. These well-known exercises follow from standard diagram-chasing arguments.

Dually, if $S : X \to E$ and $T : Y \to E$ are two operators into the same Banach space, then their pullback Z is defined as the subspace of all $(x, y) \in X \oplus_\infty Y$ for which $Sx = Ty$, together with the natural mappings $Z \to Y$ and $Z \to X$. In case one of the original mappings, say S, is the quotient mapping from a short exact sequence, then completing the diagram gives a second short exact sequence with the same subspace F:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & Z & \longrightarrow & Y & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle T} & & \\ 0 & \longrightarrow & F & \longrightarrow & X & \xrightarrow{S} & E & \longrightarrow & 0. \end{array}$$

Conversely, if we are given any commutative diagram as above, then Z must be isomorphic to the pullback of S and T. Note again that the operator $Z \to X$ is an isomorphic embedding (respectively a quotient mapping) if and only if T is. For further information, see [16, Chap. 1] and the references therein.

Let X be any separable Banach space and let $Q_X : \ell_1 \to X$ be any quotient map. We will keep the notation $\widetilde{X}$ for the kernel of Q_X (which is unique up to automorphism provided it is infinite dimensional, see [35], [36, p. 108] or [15, p. 382]). The following theorem is well known:

Theorem 1.1. *Suppose X and Y are separable Banach spaces. Then the following are equivalent:*

(1) $\mathrm{Ext}(X, Y) = \{0\}$.

(2) If $T : \widetilde{X} \to Y$ is a bounded operator, then there is a bounded extension $\widetilde{T} : \ell_1 \to Y$.

(3) If Z is a separable Banach space containing a subspace E so that $Z/E \sim X$ and $T : E \to Y$ is a bounded operator, then there is an extension $\widetilde{T} : Z \to Y$.

Proof. It is trivial that (3) implies (1). For (1) implies (3) we use the pushout construction:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle S} & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & W & \longrightarrow & X & \longrightarrow & 0. \end{array}$$

Now (1) implies the existence of a projection $P : W \to Y$, and then PS extends T.

That (2) is equivalent to (3) is clear from the proof of Corollary 1.1 of [26]. Alternatively, [30, Prop. 3.1] proves directly the equivalence of (1) and (2). $\square$

Remark. Of course all separability assumptions can be removed if we simply replace ℓ_1 by $\ell_1(I)$ for a suitable index set.

There is an immediate corollary, which essentially says that $\mathrm{Ext}(X, Y) = \{0\}$ is a three-space property of X:

Corollary 1.2. *Suppose Y is a Banach space and X is a Banach space with a subspace E so that* $\mathrm{Ext}(E, Y) = \{0\}$, *and* $\mathrm{Ext}(X/E, Y) = \{0\}$. *Then* $\mathrm{Ext}(X, Y) = \{0\}$.

Proof. Let $\widetilde{X}$ and Q_X be defined as above. Given $T : \widetilde{X} \to Y$, we need to find an extension to all of ℓ_1. We will apply Theorem 1.1.

If $Q : X \to X/E$ is the obvious mapping, we may choose $\widetilde{X/E}$ to be the kernel of $Q \circ Q_X$. Then $y \mapsto Q_X y$ is a quotient mapping from $\widetilde{X/E}$ onto E with kernel $\widetilde{X}$. The implication (1) $\Rightarrow$ (3) then gives us an extension $\widetilde{T} : \widetilde{X/E} \to Y$ of T, which by the implication (1) $\Rightarrow$ (2) admits a further extension $\widetilde{\widetilde{T}} : \ell_1 \to Y$. $\square$

In this paper, we consider the case when the subspace of our twisted sum is $C(K)$ for some compact metric space K. If K is uncountable, then the theorem of Milutin [40, Theorem 8.5] implies we may consider $K = [0, 1]$. The following result is due to Johnson and Zippin [26], in view of Theorem 1.1:

Theorem 1.3. *If X is isomorphic to the dual of a subspace of c_0 (so that $\widetilde{X}$ can be assumed weak*-closed), then* $\operatorname{Ext}(X, C(K)) = \{0\}$ *for every compact K.*

In [28] the following converses were found. Throughout this paper, we will use (FDD) to indicate a finite-dimensional Schauder decomposition and (UFDD) to indicate an unconditional finite-dimensional Schauder decomposition. Recall also that X is said to have the *strong Schur property* if there is a constant $c > 0$ so that for any normalized sequence (x_n) with $\|x_m - x_n\| \geq \delta > 0$ for any $m \neq n$, there exists a subsequence $(x_n)_{n \in \mathcal{M}}$ such that

$$\left\| \sum_{k \in \mathcal{M}} \alpha_k x_k \right\| \geq c\delta \sum_{k \in \mathcal{M}} |\alpha_k|$$

for any finitely supported sequence $(\alpha_k)_{k \in \mathcal{M}}$.

Theorem 1.4. *If X is separable and* $\operatorname{Ext}(X, C[0, 1]) = \{0\}$*, then X has the strong Schur property. If X also has a (UFDD), then X is isomorphic to the dual of a subspace of c_0.*

Let us remark at this point that Bourgain and Pisier [9] (cf. [16, §1.8]) showed that for any separable Banach space X that is not an $\mathcal{L}_\infty$-space there is a space Y that is an $\mathcal{L}_\infty$-space so that Y contains X as an uncomplemented subspace and Y/X has the Schur property and the Radon-Nikodým property.

Recall that an operator is called strictly singular if its restriction to an infinite-dimensional subspace of its domain is never an isomorphic embedding. In Section 2 we consider the problem of characterizing those separable spaces X for which there is a short exact sequence $0 \to C[0, 1] \to Z \to X \to 0$ so that the quotient map is strictly singular. We show in Theorem 2.3 that this is equivalent to the requirement that X contains no copy of ℓ_1.

In Section 3 we consider quantitative results for the case $K = \omega^N$. In this case $C(K)$ is isomorphic to c_0, so that $\operatorname{Ext}(X, C(K)) = \{0\}$ for every separable X by Sobczyk's theorem [43], but it is still worthwhile to consider projection constants. We need the following elementary result; we recall that Z is said to be separably injective if it is complemented in every separable superspace. As usual, I_X indicates the identity on a given Banach space X.

Proposition 1.5. *Let X be any separable Banach space, let Z be a separably injective Banach space and let k be a constant. Then the following are equivalent:*

(1) If Y is a separable Banach space and E is a closed subspace with Y/E isometric to X, then for any bounded linear operator $T : E \to Z$ and any $\varepsilon > 0$, there is an extension $\widetilde{T} : Y \to Z$ with $\|\widetilde{T}\| < k\|T\| + \varepsilon$.

(2) If $0 \longrightarrow Z \xrightarrow{j} Y \xrightarrow{q} X \longrightarrow 0$ is an (isometric) exact sequence and any $\varepsilon > 0$, then there is a linear operator $P : Y \to Z$ with $Pj = I_Z$ and $\|P\| \le k + \varepsilon$.

Proof. It is clear from the definition that if the short exact sequence is given, then we may find such a P with $\|P\| \le k + \varepsilon$. Conversely, suppose Y is a separable Banach space and E is a closed subspace with Y/E isometric to X. If $T : E \to Z$ is an operator with $\|T\| \le 1$, we form the pushout:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \xrightarrow{j} & Y & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle S} & & \| & & \\ 0 & \longrightarrow & Z & \xrightarrow{j'} & PO & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

Then, if $P : PO \to Z$ satisfies $Pj' = I_Z$, we see that $PS = \widetilde{T}$ extends T and $\|PS\| \le \|P\|$. □

Our results build on earlier work of Amir and Baker, who showed that the separable projection constant of $C(\omega^N)$ is $2N + 1$, [2], [3] and [4]. In particular, we show that, given any $\varepsilon > 0$, there is a space Z containing $C(\omega^N)$ isometrically so that $X/C(\omega^N)$ is isometric to c_0 and the norm of any projection is at least $2N + 1 - \varepsilon$. However, our main motivation in Section 3 is to provide the necessary groundwork to study the case $K = \omega^\omega$, which is done in Section 4. Here we show results parallel to Theorems 1.3 and 1.4 above. We show that if X is the dual of a space with summable Szlenk index [31], [23, §2], then $\mathrm{Ext}(X, C(\omega^\omega)) = \{0\}$, and this condition is necessary if X has a (UFDD). An example of such an X is Tsirelson's space [31].

We also consider the possibility of $\mathrm{Ext}(X, C(\omega^\omega))$ being large in the sense that there is a twisted sum $0 \to C(\omega^\omega) \to Z \to X \to 0$ for which the quotient map is strictly singular. We show that a sufficient condition for the construction of such a short exact sequence is that X has a shrinking (UFDD) and contains no subspace that is the dual of a space with summable Szlenk index. This leads to new counterexamples for several old problems.

We refer to [16] and [29] for a discussion of twisted sums in general. Let us note that in Section 3 it is important to consider twisted sums in the *isometric* category rather than the isomorphic category; hence the standard pushout and pullback constructions were defined above isometrically. Of course any isomorphic twisted sum can be equivalently renormed to an isometric twisted sum.

2. A universal twisted sum

Theorem 2.1. *Suppose X is a separable Banach space. Then there is a universal short exact sequence $0 \to C[0,1] \to Y \to X \to 0$ such that every short exact sequence $0 \to C[0,1] \to Z \to X \to 0$ can be identified with a pushout, i.e., there exist linear operators $S : C[0,1] \to C[0,1]$ and $S_1 : Y \to Z$ so that the following* *diagram commutes:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C[0,1] & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle S} & & \Big\downarrow{\scriptstyle S_1} & & \Big\| & & \\ 0 & \longrightarrow & C[0,1] & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0. \end{array}$$

Proof. Let $Q_X : \ell_1 \to X$ be a quotient mapping and let $\widetilde{X}$ be the kernel of this map. Consider the collection $\{L_j : j \in J\}$ of all linear operators $L_j : \widetilde{X} \to C[0,1]$ with $\|L_j\| \leq 1$. Then let $L : \widetilde{X} \to \ell_\infty(J : \ C[0,1])$ be defined by $L\xi = (L_j\xi)_{j\in J}$. Since L has separable range, we can find a subspace of $\ell_\infty(J : \ C[0,1])$ isomorphic to $C[0,1]$ and containing the range of L. In this way we induce a bounded linear operator $A : \widetilde{X} \to C[0,1]$ such that every bounded operator $B : \widetilde{X} \to C[0,1]$ factors through A, i.e., $B = VA$, where $V : C[0,1] \to C[0,1]$ is bounded.

Next we use the pushout construction to construct our twisted sum:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{X} & \longrightarrow & \ell_1 & \longrightarrow & X & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle A} & & \Big\downarrow{\scriptstyle A_1} & & \Big\| & & \\ 0 & \longrightarrow & C[0,1] & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & 0; \end{array}$$

it remains to verify its universality.

So let $0 \to C[0,1] \to Z \to X \to 0$ be any twisted sum of $C[0,1]$ and X. Then, using the projective property of ℓ_1, we can construct a quotient mapping $T_1 : \ell_1 \to Z$. Since it is unique up to automorphism, we may choose $\widetilde{X} = T^{-1}(C[0,1])$. If T is the restriction of T_1 to $\widetilde{X}$, then the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{X} & \longrightarrow & \ell_1 & \longrightarrow & X & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle T} & & \Big\downarrow{\scriptstyle T_1} & & \Big\| & & \\ 0 & \longrightarrow & C[0,1] & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0. \end{array}$$

This means simply that Z is obtained by the pushout of $0 \to \widetilde{X} \to \ell_1 \to X \to 0$ using T. Now we can write $T = SA$ for some $S : C[0,1] \to C[0,1]$, and it follows that Z is obtained from Y by the pushout construction using S. ☐

We need the well-known result that there is a non-trivial twisted sum of $C[0,1]$ and c_0. The first published reference we know is [22, Theorem 6]. In [1] a stronger statement about the non-existence of Lipschitz liftings is proved; a non-separable version is to be found in [18]. The example, also studied in [27], can be described as follows. Let $Q = (q_n)$ be any dense sequence in $[0,1]$. We could for example order the rationals in $(0,1)$ into a sequence (q_n), but we prefer not to be specific. Denote by D the set of all functions from $[0,1]$ into $\mathbb{R}$ that are continuous at every $t \notin Q$ and left continuous with right limits at every $t \in Q$. Routine arguments show that all such functions are bounded and that the sup-norm makes D into a Banach space. Clearly $C = C[0,1]$ is a closed subspace and D/C is isometric to c_0. More precisely, let us denote by $J : D \to c_0$ the "jump function" $Jf = \frac{1}{2}(f(q_n+) - f(q_n))$. Then J maps D onto c_0, and $d(f, C) = \|Jf\|$ for all f in D. We denote by e_n the usual basis in c_0. It is well known [6, p. 33], [27, p. 20] that D is isometric to the space of continuous functions on the Cantor set, but we do not need this representation.

Lemma 2.2. *Let (f_n) be any sequence of functions in D for which $J(f_n) = e_n$ for all n. Then the sequence (f_n) is not weakly Cauchy.*

Proof. The assumption $J(f_n) = e_n$ means that $f_n(q_n+) - f_n(q_n) = 2$ for all n. Let us assume (f_n) is weakly Cauchy and hence bounded. We first note that if I is any nonempty open interval in $(0,1)$, $\alpha \in \mathbb{R}$ and $m \in \mathbb{N}$, then there exist $n > m$ and a nonempty open interval J with $\overline{J} \subset I$ such that for some β with $|\beta - \alpha| \geq 1$ we have $|f_n(t) - \beta| \leq \frac{1}{4}$ for $t \in J$. Indeed, we just pick $n > m$ so that $q_m \in I$, and then let β be either $f_n(q_n)$ or $f_n(q_n+)$. The interval J can then be chosen using the left- or right-hand limit condition.

Now we can use this inductively to create a subsequence (f_{n_k}) of (f_n), a sequence of nonempty intervals (I_k) with $\overline{I}_{k+1} \subset I_k$, and a sequence of reals (α_k) with $|\alpha_{k+1} - \alpha_k| \geq 1$ so that $|f_{n_k}(t) - \alpha_k| \leq \frac{1}{4}$ for $t \in I_k$. If we pick $t_0 \in \bigcap_{k=1}^{\infty} I_k$ (which is nonempty by compactness), it is clear that $|f_{n_k}(t_0) - f_{n_{k+1}}(t_0)| \geq \frac{1}{2}$ for all k, and this gives us a contradiction. $\square$

Theorem 2.3. *Suppose X is a separable Banach space. Then there is a twisted sum*

$$0 \longrightarrow C[0,1] \longrightarrow Y \xrightarrow{Q} X \longrightarrow 0$$

with Q strictly singular if and only if X contains no copy of ℓ_1.

Proof. If ℓ_1 embeds into X, then, by the well-known lifting property of ℓ_1 [36, p. 107], Q cannot be strictly singular.

Conversely, suppose ℓ_1 does not embed into X. We will argue that the universal twisted sum Y given by Theorem 2.1 has a strictly singular quotient map $Q : Y \to X$. First we show that whenever E is an infinite-dimensional closed subspace of X, then there is a twisted sum $0 \to C[0,1] \to Z \to X \to 0$ so that the pullback by the inclusion $E \to X$ does not split. Since X does not contain ℓ_1, any such subspace E contains a weakly null basic sequence $(x_n)_{n=1}^{\infty}$ [36, p. 5, Remark] spanning a subspace E_0. By considering the basis expansion we thus obtain a map $T_0 : E_0 \to c_0$ so that $T_0(x_n) = e_n$, the n^{th}-basis vector in c_0. Since c_0 is separably injective, we can extend T_0 to a bounded operator $T : X \to c_0$.

We now use the twisted sum of $C[0,1]$ and c_0 constructed above and form the pullback using T:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C[0,1] & \longrightarrow & D & \xrightarrow{J} & c_0 & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle T} & & \\
0 & \longrightarrow & C[0,1] & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0.
\end{array}$$

We now need only show that the further pullback via the inclusion $E \to X$ does not split. Thus we consider

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C[0,1] & \longrightarrow & D & \xrightarrow{J} & c_0 & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle V} & & \uparrow{\scriptstyle T|_E} & & \\
0 & \longrightarrow & C[0,1] & \longrightarrow & Z_0 & \longrightarrow & E & \longrightarrow & 0.
\end{array}$$

Now if $L : E \to Z_0$ is a lifting, then VLx_n is weakly null. However, $JVLx_n = e_n$, and so we contradict Lemma 2.2.

Finally, Theorem 2.1 implies that the sequence $0 \longrightarrow C[0,1] \longrightarrow Z_0 \longrightarrow E \longrightarrow 0$ can be obtained from the sequence $0 \longrightarrow C[0,1] \longrightarrow Y \longrightarrow X \longrightarrow 0$ by first taking the pushout via $S : C[0,1] \to C[0,1]$ and then taking the pullback via $E \to X$. This procedure is equivalent to first taking the pullback via $E \to X$, and then taking the pushout via $S : C[0,1] \to C[0,1]$. Since the final sequence does not split, neither does the intermediate sequence $0 \longrightarrow C[0,1] \longrightarrow Q^{-1}(E) \longrightarrow E \longrightarrow 0$. Since E was arbitrary, we conclude that Q is strictly singular. $\square$

A simplification of this argument shows that if X is separable but fails the Schur property, then $\operatorname{Ext}(X, C[0,1]) \neq \{0\}$. Of course Theorem 1.4 is stronger.

This essentially formal construction gives an interesting corollary:

Corollary 2.4. *There is a twisted sum Y of $C[0,1]$ and c_0 that is necessarily an $\mathcal{L}_\infty$-space but is not isomorphic to a quotient of $C(K)$ for any compact K.*

Proof. Taking $X = c_0$ in Theorem 2.3 gives us an example with $Q : Y \to c_0$ strictly singular. Since c_0 is not reflexive, Q cannot be weakly compact. By a well-known result of Pełczyński [39, Theorem 1], Y cannot be isomorphic to a quotient of any $C(K)$ space. $\square$

Note here that Y^* is isomorphic to an $L_1(\mu)$-space, but Y cannot be renormed so that Y^* is isometric to an $L_1(\mu)$ by a result of Johnson and Zippin [25]. This easily gives a counterexample to the old problems 3c and 3e of Lindenstrauss and Rosenthal [35], although other much more sophisticated counterexamples have been known for some time [5], [8]. For a stronger example, see the end of §4.

3. Twisted sums with $C(\omega^N)$

If $N \in \mathbb{N}$, then the space $C(\omega^N)$ is isomorphic to c_0, and so for any separable Banach space X, we have $\operatorname{Ext}(X, C(\omega^N)) = \{0\}$. In this case it is natural to introduce the extension constant $\pi_N(X)$, which we define to be the least constant so that if Y is a separable Banach space and E is a closed subspace with Y/E isometric to X, then for any bounded linear operator $T : E \to C(\omega^N)$ and $\varepsilon > 0$, there is an extension $\widetilde{T} : Y \to C(\omega^N)$ with $\|\widetilde{T}\| < \pi_N(X)\|T\| + \varepsilon$. In view of Proposition 1.5, $\pi_N(X)$ is also the least constant such that if

$$0 \longrightarrow C(\omega^N) \xrightarrow{j} Y \xrightarrow{q} X \longrightarrow 0$$

is an (isometric) exact sequence and $\varepsilon > 0$, then there is a linear operator $P : Y \to C(\omega^N)$ with $Pj = I_{C(\omega^N)}$ and $\|P\| \leq \pi_N(X) + \varepsilon$.

The following theorem is due to Amir [2], [3] and Baker [4]:

Theorem 3.1. *For any separable Banach space X we have $\pi_N(X) \leq 2N + 1$, and there is a separable Banach space X such that $\pi_N(X) = 2N + 1$.*

In fact, it follows from the arguments in [3] that we may take $X = C(\omega^{N-1})$. The main purpose of this section is to show that X may be chosen independently of N, more precisely that $\pi_N(c_0) = 2N + 1$. This will be needed in the next section, where it will also be useful to introduce an alternative constant $\rho_N(X)$, defined as the least constant such that if $T : X \to \ell_\infty(\omega^N)$ is a bounded operator satisfying $d(Tx, C(\omega^N)) \leq \|x\|$ for $x \in X$, and $\varepsilon > 0$, there is a linear operator $L : X \to C(\omega^N)$ with $\|T - L\| \leq \rho_N(X) + \varepsilon$.

Lemma 3.2. *For any separable Banach space X we have $\rho_N(X) \leq \pi_N(X) \leq \rho_N(X) + 1$.*

Proof. First suppose Y is a Banach space containing $C(\omega^N)$ and such that $Y/C(\omega^N)$ is isometric to X. Then there is a bounded projection $P_0 : Y \to C(\omega^N)$. (We may suppose $\|P_0\| \leq 2N+1$, but this is not necessary.) We can also find a linear operator

$S : Y \to \ell_\infty(\omega^N)$ with $\|S\| = 1$ extending the identity on $C(\omega^N)$. Now $P_0 - S = Tq$ for some $T : X \to \ell_\infty(\omega^N)$, where $q : Y \to X$ is the quotient map. It is easy to check that T satisfies $d(Tx, C(\omega^N)) \le \|x\|$. Hence, for $\varepsilon > 0$, we can find a linear operator $L : X \to C(\omega^N)$ with $\|T - L\| \le \rho_N(X) + \varepsilon$. Now $P = P_0 - Lq$ is a projection onto $C(\omega^N)$. If $y \in Y$, then $Py = P_0y - Tqy + (T - L)qy = Sy + (T - L)qy$, so that $\|P\| \le 1 + \rho_N(X) + \varepsilon$. Hence $\pi_N(X) \le 1 + \rho_N(X)$.

Conversely, suppose $T : X \to \ell_\infty(\omega^N)$ is a bounded operator with

$$d(Tx, C(\omega^N)) \le \|x\|$$

for $x \in X$. Let Z be the space $X \oplus C(\omega^N)$ normed by

$$\|(x, h)\| = \max(\|x\|, \|h - Tx\|).$$

Then the map $(x, h) \to x$ defines a quotient mapping of Y onto X (since $d(Tx, C(\omega^N)) \le \|x\|$) with kernel $E = \{0\} \oplus C(\omega^N)$. Hence, if $\varepsilon > 0$, there is a projection $P : Y \to E$ with $\|P\| \le \pi_N(X) + \varepsilon$. Then P takes the form $P(x, h) = (0, h - Lx)$, where $L : X \to C(\omega^N)$ is bounded. Now if $x \in X$, we have $P(x, Tx) = (0, Tx - Lx)$, so that $\|Tx - Lx\| \le \|P\|\|x\|$. Hence $\rho_N(X) \le \pi_N(X)$. □

Lemma 3.3. *Suppose K is a compact Hausdorff space and $h \in \ell_\infty(K)$. Then*

$$d(h, C(K)) = \frac{1}{2} \sup_{s\in K} (\limsup_{t\to s} h(t) - \liminf_{t\to s} h(t)).$$

Proof. Define $f(s) = \liminf_{t\to s} h(t)$ and $g(s) = \limsup_{t\to s} h(t)$ for $s \in K$. It is routine to check that f is upper semicontinuous and that g is lower semicontinuous. If $R = \frac{1}{2} \sup_{s\in K}(g(s) - f(s))$, then a classical interpolation theorem gives us a continuous function $\widetilde{h}$ satisfying $g - R \le \widetilde{h} \le f + R$. Clearly $f \le h \le g$, and so $-R \le \widetilde{h} - h \le R$, as required. □

We now need a representation of ω^N. To this end we consider the power set of $\mathbb{N}$, i.e., $2^{\mathbb{N}}$, which is homeomorphic to the Cantor set in the standard product topology. Let $\mathcal{F}_N$ be the subset of all sets a with cardinality $|a| \le N$. Then $\mathcal{F}_N$ is homeomorphic to ω^N. Indeed, $\{\sum_{n\in a} 2^{-n} : a \in \mathcal{F}_N\}$ is order isomorphic and homeomorphic to ω^N.

Any nonempty finite subset a of $\mathbb{N}$ will be written in increasing order, i.e., $a = \{n_1, \dots, n_k\}$, where $n_1 < n_2 < \dots < n_k$. We write $\max a = n_k$. We write $a < b$ if either a is empty and b is not, or if $a = \{n_1, \dots, n_k\}$ and $b = \{m_1, \dots, m_l\}$, where $l > k$ and $m_j = n_j$ for $j \le k$. For each nonempty finite $a = \{n_1, \dots, n_k\} \in 2^{\mathbb{N}}$ we define $a- = \{n_1, \dots, n_{k-1}\} = a \setminus \{n_k\}$. We define $a+$ as the collection of all $a \vee m = \{n_1, \dots, n_k, m\}$, where $m > n_k$; $\emptyset+$ is simply $\mathbb{N}$. Although we do not need it in this section, we define here a subset $\mathcal{A}$ of $\mathcal{F}_N$ to be *full* if the following three conditions hold:

(1) $\emptyset \in \mathcal{F}_N$.
(2) If $\emptyset \ne a \in \mathcal{A}$, then $a- \in \mathcal{A}$.
(3) If $a \in \mathcal{A}$ and $|a| < N$, then $\mathcal{A} \cap a+$ is infinite.

It is then easy to see that any full subset of $\mathcal{F}_N$ is also homeomorphic to ω^N.

Next let $\mathcal{A}$ be a full subset of $\mathcal{F}_N$ and let X be a fixed separable Banach space. We consider a bounded map $a \mapsto x_a^*$ of $\mathcal{A}$ into X^*.

Lemma 3.4. *If* $T : X \to \ell_\infty(\mathcal{A})$ *is defined by* $Tx(a) = x_a^*(x)$*, then we have*

$$d(Tx, C(\mathcal{A})) \le \|x\| \qquad \forall x \in X$$

if and only if $\limsup_{b,c\to a} \|x_b^* - x_c^*\| \le 2$ *for each* $a \in A$ *with* $|a| < N$.

Proof. This follows easily from Lemma 3.3, since we require $\limsup_{b\to a} x_b^*(a) - \liminf_{b\to a} x_b^*(x) \le 2\|x\|$ for all $x \in X$. We omit the details. Note that if $|a| = N$, then any sequence converging to a will be eventually constant. □

We conclude this section with a minor variation of Amir's part of the Amir-Baker Theorem:

Theorem 3.5. *For each* N *we have* $\pi_N(c_0) = 2N + 1$.

Proof. Let us choose $\varepsilon > 0$ and $r \in \mathbb{N}$, and let $m = 2^r$. Then let G be the dyadic group $\{-1, 1\}^r$, with its usual normalized measure, and let $u_1, \ldots, u_m$ denote the characters of this group. Let $\overline{u} = \frac{1}{m}(u_1 + \cdots + u_m\}$, so that $\overline{u}$ is actually the function that is one at the identity and zero elsewhere. Let $v_k = u_k - \overline{u} \in L_\infty(G)$ and $v_k^* = u_k$, regarded as an element of $L_1(G) = L_\infty(G)^*$. Then $\|v_k\| = \|v_k^*\| = 1$ for all k, and if $j \ne k$, then $\|v_j^* - v_k^*\| = 1$.

Now consider $X = c_0(\mathcal{F}_{N-1}; L_\infty(G))$ so that X is isometric to c_0. We define a linear operator $T : X \to \ell_\infty(\mathcal{F}_N)$. Consider any element $x = (w_a)_{a\in\mathcal{F}_{N-1}} \in X$, where $w_a \in L_\infty(G)$. We define $Tx(\emptyset) = 0$, and then

$$Tx(a) = Tx(a-) + 2v_j^*(w_{a-}),$$

where $j \equiv \max a \pmod m$. Now let Z be the set of all $(x, h) \in X \oplus_\infty \ell_\infty(\mathcal{F}_N)$ such that $h - Tx \in C(\mathcal{F}_N)$, and put $E = \{(0, h) : \ h \in C(\mathcal{F}_N)\}$; it is easy to see that the quotient space Z/E is isometric to X (since $d(Tx, C(\mathcal{F}_N)) \le \|x\|$ by Lemma 3.4). Let P be a bounded projection of Z onto E, and write $P(x, Tx) = (0, Sx)$, where $S : X \to C(\mathcal{F}_N)$.

For notational purposes, if $a \in \mathcal{F}_{N-1}$ and $j \le m$, we define $H(a, j)$ to be the set of $b \ge a \vee n$, where $n > \max a$ and $n \equiv j \mod m$, and $x_{j,a} = v_j\chi_{\{a\}} \in X$. For any $a \in \mathcal{F}_N$ we put $K(a) = \{b : b \ge a\}$.

We now claim that if $a \in \mathcal{F}_{N-1}$, then there exists $j = j(a)$ so that $x = x_{j,a}$ satisfies $Sx(a) \le 0$. Indeed, $\sum_{j=1}^m x_{j,a} = 0$, and so $\sum_{j=1}^m Sx_{j,a}(a) = 0$. Considering the topology on $\mathcal{F}_N$, it follows that there exists $k = k(a) > \max a$ so that if $b \ge a \vee l$, where $l \ge k(a)$, then $Sx(b) \le \varepsilon$.

Let us take $n_1 = j(\emptyset) + mk(\emptyset)$ and then define inductively $n_2, \ldots, n_N$ so that $n_s \ge k(\{n_1, \ldots, n_{s-1}\})$ and $n_s \equiv j(\{n_1, \ldots, n_{s-1}\}) \pmod m$ for $1 < s \le N$. Let $a = \{n_1, \ldots, n_N\}$. Then we let

$$x = \sum_{\emptyset \le b < a} x_{j(b),b}.$$

It is easy to see that

$$Sx(a) \le N\varepsilon.$$

It is routine to check that if $c \ge b \vee n$, with $n \equiv j \pmod m$, then

$$T(v_{j(b)}\chi_{\{b\}})(c) = 2v_j^*(v_{j(b)}),$$

and $T(v_{j(b)}\chi_{\{b\}})(c) = 0$ for all other $c \in \mathcal{F}_N$. Since $v_j^*(v_k) = \delta_{jk} - \frac{1}{m}$, where δ_{jk} is the Kronecker delta, this implies that

$$T(x_{j(b),b}) = 2\chi_{H(b,j(b))} - \frac{2}{m}\chi_{K(b)\setminus\{b\}}.$$

Summing, we obtain

$$Tx = 2 \sum_{\emptyset \leq b < a} \left(\chi_{H(b,j(b)} - \frac{1}{m} \chi_{K(b)\setminus\{b\}} \right).$$

Let $h = \chi_{K(\emptyset)} + 2\sum_{\emptyset < b \leq a} \chi_{K(b)}$. By construction $H(b, j(b)) \subseteq K(b) \subseteq H(b-, j(b-))$ for each $b \leq a$. A short calculation then yields

$$\|Tx - h\| \leq 1 + \frac{2N}{m}.$$

Since $\|v_{j(b)}\| = 1$, we also have $\|(x, Tx - h)\| \leq 1 + \frac{2N}{m}$, and thus $\|Sx - h\| \leq \|P\|(1 + \frac{2N}{m})$. But $h(a) = 2N + 1$. Thus

$$2N + 1 - N\varepsilon \leq (h - Sx)(a) \leq \|P\|(1 + \frac{2N}{m}).$$

Since we can choose m arbitrarily large and ε arbitrarily small, this implies that $\pi_N(c_0) \geq 2N + 1$. □

4. Twisted sums with $C(\omega^\omega)$

Our motivation for studying the constants $\pi_N(X)$ comes from the following theorem:

Theorem 4.1. *Suppose X is a separable Banach space. Then* $\operatorname{Ext}(X, C(\omega^\omega)) = \{0\}$ *if and only if* $\sup_N \pi_N(X) < \infty$.

Proof. To simplify notation we will work with $C_0(\omega^\omega) = \{f \in C(\omega^\omega) : f(\omega^\omega) = 0\}$, which is clearly isomorphic to $C(\omega^\omega)$. Since $C(\omega^N)$ is isomorphic to a one-complemented subspace of $C_0(\omega^\omega)$ for each N, necessity is obvious. Conversely, suppose Y is a separable Banach space and E is a closed subspace of Y so that Y/E is isometric to X. Suppose $T : E \to C_0(\omega^\omega)$ is bounded with $\|T\| \leq 1$. Let $M = \sup_N \pi_N(X) + 1$. For $n \in \mathbb{N}$ let R_n be the restriction map $R_n : C_0(\omega^\omega) \to C(K_n)$, where $K_1 = [1, \omega]$ and $K_n = [\omega^{n-1} + 1, \omega^n]$ for $n \geq 2$.

Let F_k be an increasing sequence of finite-dimensional subspaces of Y such that $\bigcup F_k$ is dense in Y. Let G_k be finite-dimensional subspaces of E so that if $x \in F_k$, then $d(x, G_k) \leq 2d(x, E)$. Let $q : Y \to Y/E$ be the quotient map and let $q(F_k) = H_k$.

For each k let $n(k)$ be the least integer such that if $e \in (F_k + G_k) \cap E$, then $\|R_n Te\| \leq 2^{-k}\|e\|$. Then, since T maps E into $C_0(\omega^\omega)$, we see that $n(k)$ is well defined.

For fixed k, letting $n = n(k)$, we can, since $C(K_n)$ is an $\mathcal{L}_{\infty,1}$-space, find an operator $S_n : F_k + G_k \to C(K_n)$ so that $\|S_n\| \leq 2^{1-k}$ and $S_n e = R_n Te$ for $e \in E \cap (F_k + G_k)$. Also we can find an operator $V_n : Y \to C(K_n)$ such that $\|V_n\| \leq M$ and $V_n e = R_n Te$ for $e \in E$.

Now if $y \in F_k + G_k$, then there exists $g \in G_k$ so that $\|y - g\| \leq 2d(y, E)$. Then

$$\|V_n y - S_n y\| = \|V_n(y - g) - S_n(y - g)\| \leq 2(M + 2)d(y, E).$$

It follows that there is an operator $U_n : H_n \to C(K_n)$ with $\|U_n\| \leq 2M + 4$ and $U_n q = V_n - S_n$. Since $U_n(H_n)$ is finite dimensional, this may be extended to an operator $\widetilde{U}_n : X \to C(K_n)$ with $\|\widetilde{U}_n\| \leq 2M + 5$. Next set $\widetilde{T}_n = V_n - \widetilde{U}_n q$. Then $\|\widetilde{T}_n\| \leq 3M + 6$, $\widetilde{T}_n$ extends $R_n T$, and $\widetilde{T}_n y = S_n y$ for $y \in F_k + G_k$, so that $\|R_n Ty\| \leq 2^{1-k}\|y\|$ for $y \in F_k + G_k$.

We finally extend the operator T by setting

$$\widetilde{T}y(\alpha) = R_n T y(\alpha) \quad \text{if } \alpha \in K_n.$$

This provides an extension with $\|\widetilde{T}\| \leq 3M + 6$. $\square$

Next we recall some ideas from [23]. Suppose $\mathcal{A}$ is a full subset of $\mathcal{F}_N$. We say that a map $a \mapsto u_a^* : \mathcal{A} \to X^*$ is a weak*-null tree map if $u_\emptyset^* = 0$ and $\lim_{b \in a+} u_b^* = 0$ (weak*) whenever $|a| < N$. If E is a closed subspace of X^*, we will define $\alpha_N(E)$ to be the infimum of all λ such that whenever $a \mapsto u_a^*$ is a weak*-null tree map with $u_a^* \in E$ and $\|u_a^*\| \leq 1$ for all a, then there is a $b \in \mathcal{A}$ with $|b| = N$ and

$$\left\| \sum_{a \leq b} u_a^* \right\| \leq \lambda.$$

We shall say that a weak*-null tree map is strongly weak*-null if

$$\lim_{\max a \to \infty} u_a^* = 0$$

weak*. The next lemma allows us to replace weak*-null by strongly weak*-null in the above definition of $\alpha_N(E)$.

Lemma 4.2. *If $a \mapsto u_a^*$ is a bounded weak*-null tree map on a full subset $\mathcal{A}$ of $\mathcal{F}_N$, then there is a full subset $\mathcal{B}$ of $\mathcal{A}$ so that $a \mapsto u_a^*$ is strongly weak*-null on $\mathcal{A}$.*

Proof. Let (V_n) be a base of weak*-neighborhoods of 0 such that $V_{n+1} + V_{n+1} \subset V_n$ for all n. Let $\mathcal{B} = \{b \in \mathcal{A} : u_a^* \in V_{\max a}$ for each a with $\emptyset < a \leq b\}$. It is easily verified that $\mathcal{B}$ works. $\square$

Now suppose X is a separable Banach space with a finite-dimensional Schauder decomposition (F_n). We denote by $S(m,n)$, where $0 \leq m \leq n \leq \infty$ and $m < \infty$, the operator

$$S(m,n)(\sum_{k=1}^{\infty} f_k) = \sum_{k=m+1}^{n} f_k$$

if $f_k \in F_k$. Note that $S(n,n) = 0$ for all n. We say that (F_n) is *bi-monotone* if $\|S(m,n)\| \leq 1$ for all m, n.

We shall let $E(m,n)$ be the range of $S(m,n)^*$ in X^*; we refer to such subspaces as block subspaces. We let E be the closure of $\bigcup_{m<n<\infty} E(m,n)$.

Theorem 4.3. *Suppose X is a separable Banach space with a bi-monotone FDD (F_n). Then:*

(1) $\rho_{2N}(X) \leq 4\alpha_N(E)$.

(2) If (F_n) is 1-unconditional and shrinking (so that $E = X^*$), then $\alpha_N(X^*) \leq 2\rho_N(X)$.

Proof. (1) Suppose $\lambda > 0$. We define a notion of λ-*acceptable* subsets of B_E of cardinality at most N. A subset $\{x_1^*, \ldots, x_N^*\}$ of cardinality N is λ-acceptable if $\|x_1^* + \cdots + x_N^*\| \leq \lambda$. We define acceptable sets of cardinality $0 \leq k < N$ by reverse induction. For each $0 \leq k < N$, a subset $\{x_1^*, \ldots, x_k^*\}$ is λ-acceptable if there is a weak*-neighborhood V of zero so that if $x_{k+1}^* \in B_E \cap V$, then $\{x_1^*, \ldots, x_{k+1}^*\}$ is λ-acceptable. It is easily seen that if $\lambda > \alpha_N = \alpha_N(E)$, then the empty set is λ-acceptable. More precisely it is easy to show that if this fails, then one can construct a weak*-null tree map on $\mathcal{F}_N$ denoted by $a \mapsto u_a^*$ with $u_a^* \in B_E$ so that for every a with $|a| = N$ we have $\|\sum_{b\le a} u_b^*\| > \lambda$. This contradicts the definition of α_N.

Next we shall say that a collection of $k \le N$ block subspaces $\{G_1,\dots,G_k\}$ is λ-*good* if for some $\mu < \lambda$ and every $x_j^* \in B_{G_j}$ the set $\{x_1^*,\dots,x_k^*\}$ is μ-acceptable.

Claim. Suppose $\lambda > \alpha_N$. There is a function $\psi : \mathbb{N} \to \mathbb{N}$ so that if $\{G_1,\dots,G_k\}$ is a λ-good family of block subspaces of $E(0,n)$ with $k < N$, then for any block subspace G_{k+1} of $E(\psi(n),\infty)$ the collection $\{G_1,\dots,G_{k+1}\}$ is λ-good.

Let us prove the claim. Since the family of block subspaces of $E(0,n)$ is finite, it is clear there exists $\mu < \lambda$ so that every λ-good collection $\{G_1,\dots,G_k\}$ of block subspaces is actually μ-good. Then pick $\varepsilon > 0$ so that $\mu + N\varepsilon < \lambda$. Choose in each block subspace G an ε-net for the unit ball B_G. In this way we produce a finite collection $\mathcal{G}$ of μ-acceptable sets $\{x_1^*,\dots,x_k^*\}$ so that whenever $\{G_1,\dots,G_k\}$ is any λ-good collection of block subspaces of $E(0,n)$ and whenever $g_j^* \in B_{G_j}$, then there is a $\{x_1^*,\dots,x_k^*\} \in \mathcal{G}$ with $\|g_j^* - x_j^*\| \le \varepsilon$ for $1 \le j \le k$. Now it is clear from the definition of acceptability that we can find $\psi(n)$ so that if $x^* \in B_E \cap E(\psi(n),\infty)$ and $\{x_1^*,\dots,x_k^*\} \in \mathcal{G}$ with $k < N$, then $\{x_1^*,\dots,x_k^*,x^*\}$ is μ-acceptable. Now it is easy to see by a perturbation argument that if $\{G_1,\dots,G_k\}$ is λ-good with $k < N$ and each G_j is contained in $E(0,n)$, then for any block subspace G of $E(n,\infty)$ the collection $\{G_1,\dots,G_k,G\}$ is $(\mu + N\varepsilon)$-good and hence also λ-good. This proves the claim.

We now fix $\lambda > \alpha_N$ and suppose $\theta > 1$. Now suppose $Tx = (x_a^*(x))_{a\in\mathcal{F}_{2N}}$ is a linear operator $T : X \to \ell_\infty(\mathcal{F}_{2N})$ with $d(Tx, C(\mathcal{F}_{2N})) \le \|x\|$ for all $x \in X$. We use Lemma 3.4. For each $a \in A$ with $a > \emptyset$ we define $\nu = \nu(a)$ to be the greatest natural number so that if $b \in \mathcal{F}_{2N}$ and $b \ge a$, then $\|S(0,\nu)x_b^* - S(0,\nu)x_{a-}^*\| \le 2\theta$. It follows from Lemma 3.4 that $\lim_{b\in a+} \mu(b) = \infty$ for all a with $|a| < N$.

Next we inductively construct a map $\varphi : \mathcal{F}_{2N} \to \mathbb{N}$. Let $\varphi(\emptyset) = \psi(\emptyset)$. Then we define $\varphi(a)$ by induction on $|a|$. If $\nu(a) < \psi(\varphi(a-))$, we let $\varphi(a) = \varphi(a-)$. If $\nu(a) \ge \psi(\varphi(a-))$, we let $\varphi(a) = \nu(a)$.

Now we define z_a^* for $a \in \mathcal{F}_{2N}$ by putting $z_\emptyset^* = x_\emptyset^*$, and then if $|a| > 0$ we define

$$z_a^* = \sum_{\emptyset < b \le a} S(\varphi(b-),\varphi(b))^* x_{b-}^* + S(\varphi(a),\infty)^* x_a^*.$$

We claim that $a \mapsto z_a^*$ is weak*-continuous. In fact, if $b > a$, let c be the unique element in $a+$ with $a < c \le b$. Then

$$z_b^* - z_a^* = \sum_{c<d\le b} S(\varphi(d-),\varphi(d))^* x_{d-}^* - S(\varphi(c),\infty)^* x_a^*.$$

Now $\lim_{c\in a+} \mu(c) = \infty$, and so $\lim_{c\in a+} \varphi(c) = \infty$ and $\varphi(d) \ge \varphi(c)$ whenever $c \le d \le b$. Hence as $b \to a$ we have $z_b^* - z_a^* \to 0$ weak*.

Suppose now $a = \{n_1,\dots,n_k\} \in \mathcal{F}_{2N}$. Let $m_0 = \varphi(\emptyset)$, and then put $m_j = \varphi\{n_1,\dots,n_j\}$ for $1 \le j \le k$. Consider the subspaces

$$\{E(m_0,m_1), E(m_1,m_2), \dots, E(m_{k-1},m_k)\}.$$

If we delete those subspaces where $m_j = m_{j-1}$ (i.e., where the subspace reduces to $\{0\}$), then it is clear by induction that the remaining subspaces can be split into two λ-good collections by taking them alternately. Hence, if $u_j^* \in E(m_{j-1},m_j)$ with $\|u_j^*\| \le 1$ for $1 \le j \le k$, then $\|\sum_{j=1}^k u_j^*\| \le 2\lambda$.

Next we estimate $\|x_a^* - z_a^*\|$. We have

$$x_a^* - z_a^* = \sum_{\emptyset < b \leq a} S(\varphi(b-), \varphi(b))^*(x_a^* - x_{b-}^*).$$

If $\varphi(b) > \varphi(b-)$, then $\varphi(b) = \mu(b)$, and so $\|S(\varphi(b-), \varphi(b))^*(x_a^* - x_{b-}^*)\| \leq 2\theta$. By the above remarks we have

$$\|x_a^* - z_a^*\| \leq 4\lambda\theta.$$

Our conclusion is that there is a bounded operator $Lx = (z_a^*(x))_{a \in \mathcal{F}_{2N}}$ into $C(\mathcal{F}_{2N})$ with $\|L - T\| \leq 2\lambda\theta$. Thus $\rho_{2N}(X) \leq 2\alpha_N(E)$. This concludes the proof of (1).

(2) Let us suppose $a \mapsto u_a^*$ is a strongly weak*-null tree map on $\mathcal{F}_N$ with $\|u_a^*\| \leq 1$ for $a \in \mathcal{F}_N$. Let $\gamma : \mathbb{N} \to \mathbb{N}$ be any surjective map so that for each $k \in \mathbb{N}$ the set $\gamma^{-1}\{k\}$ is infinite. Let $\mathcal{A}$ be the subset of $\mathcal{F}_N$ consisting of the empty set and all $\{n_1, \ldots, n_k\}$ such that $\gamma(n_j) \geq n_{j-1}$ for $2 \leq j \leq k$. It is clear that $\mathcal{A}$ is full. Let $\sigma\{n_1, \ldots, n_k\} = \{\gamma(n_1), \ldots, \gamma(n_k)\}$ for $\{n_1, \ldots, n_k\} \in \mathcal{A}$. We then define $a \mapsto x_a^*$ for $a \in \mathcal{A}$ by

$$x_a^* = \sum_{\emptyset < b \leq a} u_{\sigma(b)}^*.$$

Note that if $d > a$ with $d \in \mathcal{A}$, then

$$x_d^* - x_a^* = u_{\sigma(c)}^* + \sum_{c < b \leq d} u_{\sigma(b)}^*,$$

where $a < c = c(d) \leq d$ and $|c| = |a| + 1$. Then it follows from the strong weak*-nullity of $a \mapsto u_a^*$ that

$$\lim_{d \to a} \sum_{c < b \leq d} u_{\sigma(b)}^* = 0$$

weak*, since $\max(\sigma(b)) \geq \max c$. Hence we have

$$\limsup_{d \to a} \|x_d^* - x_a^*\| \leq 1.$$

By Lemma 3.4 and the definition of $\rho_N(X)$, for any $\lambda > \rho_N(X)$ we can find a weak*-continuous map $a \mapsto z_a^*$ on $\mathcal{A}$ such that $\|x_a^* - z_a^*\| \leq \lambda$ for all a.

Now fix $\varepsilon > 0$. We determine an increasing sequence $n_1, \ldots, n_N$ so that $\{n_1, \ldots, n_N\} \in \mathcal{A}$ and an increasing sequence $m_1, \ldots, m_{2N} \in \mathbb{N}$ by induction. Suppose $a = \{n_1, \ldots, n_{k-1}\}$ has been chosen in $\mathcal{A}$ (where if $k = 1$, we take $a = \emptyset$) and that $m_1, \ldots, m_{2k-2}$ have been chosen. Then pick $m_{2k-1} > m_{2k-2}$ (if $k \geq 2$) so that $\|S(m_{2k-1}, \infty)^*(x_a^* - z_a^*)\| < \varepsilon/(6N)$. This is possible since the (FDD) is shrinking. Now pick $c \in \sigma(a)+$ with $\|S(0, m_{2k-1})^* u_c^*\| < \varepsilon/(6N)$; this is possible since $\lim_{c \in \sigma(a)+} u_c^* = 0$ weak*. Pick $m_{2k} > m_{2k-1}$ so that $\|S(m_{2k}, \infty)^* u_c^*\| < \varepsilon/(6N)$. Now there are infinitely many $b \in a+$ with $\sigma(b) = c$; amongst these we may choose b so that $\|S(0, m_{2k})^*(z_b^* - z_a^*)\| < \varepsilon/(6N)$, since $\lim_{b \to a} z_b^* = z_a^*$ weak*. We then let $b = \{n_1, \ldots, n_k\}$. This completes the inductive construction.

Let $a_k = \{n_1, \ldots, n_k\}$ for $0 \le k \le N$. Then

$$\begin{aligned}
&\left\|\sum_{k=1}^{N} u^*_{\sigma(a_k)}\right\| \le \frac{\varepsilon}{3} + \left\|\sum_{k=1}^{N} S(m_{2k-1}, m_{2k})^* u^*_{\sigma(a_k)}\right\| \\
&\le \frac{\varepsilon}{3} + \left\|\sum_{k=1}^{N} \Big(S(m_{2k-1}, m_{2k})^* u^*_{\sigma(a_k)} + S(m_{2k-2}, m_{2k-1})^*(z^*_{\sigma(a_k)} - z^*_{\sigma(a_{k-1})})\Big)\right\| \\
&\le \varepsilon + \left\|\sum_{k=1}^{N} (u^*_{\sigma(a_k)} + z^*_{\sigma(a_k)} - z^*_{\sigma(a_{k-1})})\right\| \\
&\le \varepsilon + \|x^*_{a_N} - z^*_{a_N} + z^*_{\emptyset} - x^*_{\emptyset}\| \\
&\le \varepsilon + 2\lambda.
\end{aligned}$$

Hence by the definition of $\alpha_N(X^*)$ we have $\alpha_N(X^*) \le 2\lambda + \varepsilon$. The theorem follows. □

We are now in a position to prove our main result:

Theorem 4.4. *(1) Suppose X is a separable Banach space with summable Szlenk index. Then* $\mathrm{Ext}(X^*, C(\omega^\omega)) = \{0\}$.

(2) If Y is a separable Banach space with $\mathrm{Ext}(Y, C(\omega^\omega)) = \{0\}$ *and Y has a (UFDD), then Y is the dual of a space X with summable Szlenk index.*

Remark. For the definition and general properties of the Szlenk index, see for example [23, §2]. The original space constructed by Tsirelson [44] is a reflexive space with summable Szlenk index [31]. Its dual is the space usually referred to nowadays as Tsirelson's space [14].

Proof. If X has a shrinking (FDD), then (1) follows directly from Theorem 4.3. We can assume via renorming that the (FDD) is bi-monotone. We consider the dual (FDD) of X^*. In this case the subspace E of X^{**} is identified with X and the condition $\sup_n \alpha_n(E) < \infty$ is equivalent (using [23, Theorem 4.10]) to the fact that X has summable Szlenk index, and this implies that $\sup_N \pi_N(X^*)$ is finite.

For the general case we use a theorem of Johnson and Rosenthal [24], [36, Theorem 1.g.2 p.48], that X has a subspace Y so that X/Y and Y both have shrinking (FDD)s. It is easy to check that having summable Szlenk index is a property that passes to quotients, and it follows from renorming results in [23] (Theorem 4.10 (ii)) that it passes also to subspaces. Thus Y and X/Y must both have summable Szlenk index. Hence we have $\mathrm{Ext}(Y^\perp, C(\omega^\omega)) = \{0\}$ and $\mathrm{Ext}(X^*/Y^\perp, C(\omega^\omega)) = \{0\}$, and so by Corollary 1.2 we have $\mathrm{Ext}(X, C(\omega^\omega)) = \{0\}$. This concludes the proof of (1).

For (2) we may assume the (UFDD) is 1-unconditional. We observe that Theorem 4.3 implies $\mathrm{Ext}(c_0, C(\omega^\omega)) \ne \{0\}$. (Direct constructions are also available.) Hence if $\mathrm{Ext}(Y, C(\omega^\omega)) = \{0\}$ and Y is separable, then Y contains no (necessarily complemented) copy of c_0. In particular, the (UFDD) of Y must be boundedly complete, and so $Y = X^*$, where $X = E$ as defined in Theorem 4.3. Then we have by Theorem 4.1, $\sup_N \pi_N(Y) < \infty$, and hence by Lemma 3.2, $\sup_N \rho_N(Y) < \infty$. Applying Theorem 4.3 (2), we obtain $\sup_n \alpha_N(X) < \infty$. It follows again from Theorem 4.10 of [23] that X has summable Szlenk index. □

If X is any separable Banach space, we define a tree map $a \mapsto v^*_a : \mathcal{F}_N \to X^*$ to be *of dense type* if the following conditions are satisfied:

(1) $v^*_\emptyset = 0$.
(2) $\|v^*_a\| \le 1$ for all $a \in \mathcal{F}_N$.
(3) For each a with $|a| < N$ there is a weak*-neighborhood V of 0 so that the weak*-closure of $\{v^*_b : b \in a+\}$ contains V.
(4) If $b_n \to a$ and $|b_n| \ge |a| + 2$ for all n, then $v^*_{b_n} \to 0$ weak*.

Next let $y^*_a = \sum_{b \le a} v^*_b$. We can define $Tx = (y^*_a(x))_{a \in \mathcal{F}_N}$, so that $T : X \to \ell_\infty(\mathcal{F}_N)$.

Lemma 4.5. *Suppose X has a (UFDD). Suppose $L : X \to C(\omega^\omega)$, and $T : X \to \ell_\infty(\mathcal{F}_N)$ is an operator induced by a tree map of dense type. Then $\rho_N(X) \le 2\|L - T\|$.*

Proof. This essentially follows from the argument in Theorem 4.3. Let $a \mapsto u^*_a$ be any strongly weak*-null tree map with $\|u^*_a\| \le 1$ for all a. Let $\gamma : \mathbb{N} \to \mathbb{N}$ be any surjective map so that for each $k \in \mathbb{N}$ the set $\gamma^{-1}\{k\}$ is infinite. Let $\mathcal{A}$ be the subset of $\mathcal{F}_N$ consisting of the empty set and all $\{n_1, \dots, n_k\}$ such that $\gamma(n_j) \ge n_{j-1}$ for $2 \le j \le k$. It is clear that $\mathcal{A}$ is full. Let $\sigma\{n_1, \dots, n_k\} = \{\gamma(n_1), \dots, \gamma(n_k)\}$ for $\{n_1, \dots, n_k\} \in \mathcal{A}$.

We now build a map $\psi : \mathcal{A} \to \mathcal{F}_N$. Define $\psi(\emptyset) = \emptyset$. If $\psi(a)$ has been defined and $|a| < N$, we define $\psi(b)$ for each $b \in a+$ so that $\psi(b) \in \psi(a)+$, ψ is one-one and $\lim_{b \in a+} u^*_{\sigma(b)} - v^*_{\psi(b)} = 0$ weak*.

Let $x^*_a = \sum_{b \le a} u^*_{\sigma(b)}$. Then we claim that $x^*_a - y^*_{\psi(a)}$ is weak*-continuous. Indeed, if $b \ge a$,

$$x^*_b - x^*_a - y^*_{\psi(b)} + y^*_{\psi(a)} = \sum_{a < c \le b} u^*_{\sigma(c)} - v^*_{\psi(c)}.$$

Now if $b_n \to a$ and we let d_n be chosen so that $b_n \le d_n \le a$ and $|d_n| = |a| + 1$, we have

$$\sum_{d_n < c < b} (u^*_{\sigma(c)} - v^*_{\psi(c)}) \to 0 \qquad \text{weak}^*$$

by the assumptions on both tree maps. On the other hand,

$$u^*_{\sigma(d_n)} - v^*_{\psi(d_n)} \to 0 \qquad \text{weak}^*$$

by construction.

Now if $Lx = (z^*_a(x))_{a \in \mathcal{F}_N}$, then $\|z^*_a - y^*_a\| \le \|L - T\|$. Now $a \mapsto z^*_{\psi(a)} + x^*_a - y^*_{\psi(a)}$ is weak*-continuous, and we can repeat the argument of Theorem 4.3 to deduce the conclusion. $\square$

It is clear that we can always construct a tree map of dense type. Simply let (V_n) be a base of weak*-neighborhoods of $\{0\}$ in X^* with $V_{n+1} + V_{n+1} \subset V_n$. Then for a with $|a| < N$, simply choose $\{u^*_{a \vee m}\}$ for $m > \max a$ to be any sequence that is weak*-dense in $V_{\max a} \cap B_{X^*}$. It is also clear that if Y is a subspace of X and $j : Y \to X$ is the inclusion, then $a \mapsto j^* u^*_a$ is a tree map of dense type in Y^*. This leads us to the following:

Proposition 4.6. *Let X be a separable Banach space with a shrinking 1-unconditional (UFDD). Then there is a bounded operator $T : X \to \ell_\infty(\omega^N)$ so that*

$$d(Tx, C(\omega^N)) \le \|x\|$$

for all $x \in X$ and so that if E is a subspace of X with a (UFDD), then $\rho_N(E) \le 2\|L - T\|$ for any bounded operator $L : E \to C(\omega^N)$.

It is obvious from Theorem 4.4 that the existence of a twisted sum $0 \to C(\omega^\omega) \to Y \to X \to 0$ with the quotient map strictly singular implies that X contains no subspace that is isomorphic to the dual of a space with summable Szlenk index. We now establish a partial converse.

Theorem 4.7. *Suppose X has a shrinking (UFDD) and contains no subspace that is isomorphic to the dual of a space with summable Szlenk index. Then there is a short exact sequence*

$$0 \longrightarrow C(\omega^\omega) \longrightarrow V \xrightarrow{q} X \longrightarrow 0$$

with q strictly singular.

Proof. We may assume X has a 1-unconditional (UFDD). For each N we construct $T_N : X \to \ell_\infty(\omega^N)$ as given in Proposition 4.6. Let Z_N be the space $X \oplus C(\omega^N)$ normed by $\|(x,h)\| = \|x\| + \|h - Tx\|$; then there is a quotient map $q_N : Z_N \to X$ defined by $q_N(x,h) = x$. We now construct an operator $S_N : \widetilde{X} \to C(\omega^N)$ in the usual way. Precisely, we fix a quotient map $Q : \ell_1 \to X$ and define $\hat{S}_N : \ell_1 \to Z_N$ so that $\|\hat{S}_N\| \le 2$ and $q_N \hat{S}_N = Q$. Now let S_N be the restriction of $\hat{S}_N$ to $\widetilde{X}$.

Let (F_n) be an increasing sequence of finite-dimensional subspaces so that $\bigcup F_n$ is dense in $\widetilde{X}$. Then, since $C(\omega^N)$ is an $\mathcal{L}_\infty$-space, we can find a finite-rank projection P_N on $C(\omega^N)$ whose range includes $S_N(F_N)$ and with $\|P_N\| \le 2$. Now let $R_N = S_N - P_N S_N$. Thus $\|R_N\| \le 6$, and $\lim_{N\to\infty} \|R_N \xi\| = 0$ for $\xi \in \widetilde{X}$.

We now define a map $R : \widetilde{X} \to W = c_0(C(\omega^N)_{N=1}^\infty)$ by $R\xi = (R_N\xi)_{N=1}^\infty$. Note that the latter space is isomorphic to $C(\omega^\omega)$. We can now construct a pushout

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{X} & \longrightarrow & \ell_1 & \xrightarrow{Q} & X & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle R} & & \big\downarrow{\scriptstyle Q_V} & & \big\| & & \\ 0 & \longrightarrow & W & \longrightarrow & V & \xrightarrow{q_X} & X & \longrightarrow & 0. \end{array}$$

We claim that q_X is strictly singular. If not, we can find a subspace E of X with a 1-unconditional shrinking (UFDD) so that there is a bounded operator $\Lambda : E \to V$ so that $q_X \Lambda = I_E$. Then on $Q^{-1}E$ we have $q_X(Q_V - \Lambda Q) = 0$, so that $Q_V - \Lambda Q : Q^{-1}(E) \to W$ is an extension of R to $Q^{-1}(E)$. It follows that there exists a uniformly bounded sequence of operators $\widetilde{R}_N : Q^{-1}(E) \to C(\omega^N)$ which extend R_N. Put $M = \sup \|\widetilde{R}_N\| < \infty$.

Note that $P_N S_N$ has an extension to $Q^{-1}(E)$ with $\|P_N S_N\| \le 5$, since it is a finite-rank operator taking values in $C(\omega^N)$. Hence S_N has an extension $\widetilde{S}_N : Q^{-1}(E) \to C(\omega^N)$ with $\|\widetilde{S}_N\| \le M + 5$. Then $\hat{S}_N - \widetilde{S}_N$ factors through an operator $e \mapsto (e, L_N e)$ from E into Z_N with norm at most $M + 7$. This implies that $\|L_N - T\| \le M + 7$, and so $\rho_N(E) \le 2M + 14$. Theorem 4.3 and [23, Theorem 4.10] then show that E must have summable Szlenk index. $\square$

It now follows that there is a twisted sum of $C(\omega^\omega)$ and c_0 so that the quotient map is strictly singular. This space is not a quotient of a $C(K)$-space, and yet its dual must be isomorphic to ℓ_1. This shows that the main result of [25] does not admit an isomorphic version. The space Y constructed in [8] also serves as a counterexample.

5. Final remarks

In [21] (cf. [29]) it is shown that $\mathrm{Ext}(\ell_2, \ell_2) \neq \{0\}$. It follows without difficulty that $\mathrm{Ext}(\ell_p, \ell_q) \neq \{0\}$ when $1 < p, q < \infty$, since each space contains uniformly complemented copies of ℓ_2^n. The following result is implicitly proved in [10], but it is heavily disguised; so we give a simple and direct diagram-chasing argument. For a nonlinear argument, see [12].

Theorem 5.1. $\mathrm{Ext}(c_0, \ell_1) \neq \{0\}$.

Proof. In fact we will argue that $\mathrm{Ext}(C[0,1], L_1) \neq \{0\}$. It then follows from local arguments that $\mathrm{Ext}(X, Y) \neq \{0\}$ whenever X is an $\mathcal{L}_\infty$-space and $Y = L_1(\mu)$ for some measure μ (see, e.g., [12, Theorem 2]). Alternatively, one may carry out the ensuing argument locally.

We begin by considering some non-trivial twisted sum of ℓ_2 and ℓ_2. By using the pushout and pullback constructions we build the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ell_2 & \xrightarrow{j_1} & Z & \xrightarrow{q_1} & \ell_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle j_4} & & \downarrow{\scriptstyle j_5} & & \| & & \\
0 & \longrightarrow & L_1 & \xrightarrow{j_2} & V & \xrightarrow{q_2} & \ell_2 & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle q_5} & & \uparrow{\scriptstyle q_4} & & \\
0 & \longrightarrow & L_1 & \xrightarrow{j_3} & W & \xrightarrow{q_3} & C[0,1] & \longrightarrow & 0.
\end{array}$$

Here linear embeddings are denoted by j and quotient maps by q. First we recall that Z is of cotype p and type q whenever $q < 2 < p$ [21, §3]. From the construction of the pushout, V is of cotype p for every $p > 2$.

We claim that the third row of this diagram cannot split. Suppose it does split. Then we can find an operator $T : C[0,1] \to W$ so that $q_3 T = I_{C[0,1]}$. Then $q_5 T : C[0,1] \to V$ must factor through some L_r-space, where $r > 2$ since V has finite cotype. (This result can be traced to Maurey [37]; cf. also [42] or [20, Theorem 11.14(b)].) Since L_r has type 2 and L_1 has cotype 2, every map from a subspace of L_r to L_1 factors through a Hilbert space (this is Maurey's generalization of Kwapien's theorem [32] and [33]) and hence extends to a bounded operator from L_r into L_1 by Maurey's Extension theorem [38] (cf. [20, Theorem 12.22]). Applying all this to $(q_5 T)^{-1}(j_2 L_1)$, we can find an operator $R : C[0,1] \to j_2(L_1)$ so that $Rf = q_5 T f$ if $q_2 q_5 T f = 0$. But $q_2 q_5 = q_4 q_3$. Then $q_5 T - R = T_1 q_4$ for some bounded operator $T_1 : \ell_2 \to V$. Thus the second row splits.

The conclusion of the argument was given in the proof of [30, Theorem 4.1]. If the second row splits, then V has cotype 2. Hence Z also has cotype 2, and also has type $p > 1$. But then Z^* is type 2 [41], and the Maurey Extension theorem guarantees that the dual exact sequence $0 \to \ell_2 \to Z^* \to \ell_2 \to 0$ splits. By reflexivity the first row splits, contrary to our choice of Z. □

Finally, let us mention a non-separable problem related to the results of this paper. If X is a separable Banach space, then $\mathrm{Ext}(X, c_0) = \{0\}$ by Sobczyk's theorem: we do not know, however, if there is a non-metrizable compact Hausdorff space K such that $\mathrm{Ext}(C(K), c_0) = \{0\}$. It is known that if Γ is uncountable, then $\mathrm{Ext}(c_0(\Gamma), c_0) \neq \{0\}$; this is essentially contained in one proof of the fact that c_0 is uncomplemented in ℓ_∞; see also [1], [19, p. 260] and [13, §3]. It was noted in [17, Theorem 3.4] that if X is any non-separable WCG-space, then $\mathrm{Ext}(X, c_0) \neq \{0\}$, and this settles the case when K is an Eberlein compact; similar arguments can be used for Corson compact spaces. At the other extreme, if K is extremally disconnected, then $C(K)$ contains a complemented ℓ_∞ and $\mathrm{Ext}(\ell_\infty, c_0) \neq \{0\}$ was shown in [12]. Finally, the case of uncountable ordinal spaces can be reduced to $K = [0, \omega_1]$, and in this case Parovičenko's theorem [7] shows that $\mathrm{Ext}(C(K), c_0) \neq \{0\}$.

References

[1] I. Aharoni and J. Lindenstrauss, Uniform equivalence between Banach spaces, Bull. Amer. Math. Soc. 84 (1978) 281-283. MR **58:**2162

[2] D. Amir, Continuous function spaces with the separable projection property. Bull. Res. Council Israel Sect. F 10F (1962) 163–164. MR **27:**566

[3] D. Amir, Projections onto continuous function spaces, Proc. Amer. Math. Soc. 15 (1964) 396-402. MR **29:**2634

[4] J. W. Baker, Projection constants for $C(S)$ spaces with the separable projection property. Proc. Amer. Math. Soc. 41 (1973), 201–204. MR **47:**9242

[5] Y. Benyamini and J. Lindenstrauss, A predual of ℓ_1 which is not isomorphic to a $C(K)$ space, Israel J. Math. 13 (1972) 246-254. MR **48:**9348

[6] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis,* Vol. 1, Amer. Math. Soc., Providence, 1999. MR **2001b:**46001

[7] A. Błaszczyk and A. Szymański, Concerning Parovičenko's theorem, Bull. Acad. Pol. Sci. 28 (1980) 311-314. MR **82j:**54042

[8] J. Bourgain and F. Delbaen, A class of special $\mathcal{L}_\infty$-spaces, Acta Math. 145 (1980) 155-176. MR **82h:**46023

[9] J. Bourgain and G. Pisier, A construction of $\mathcal{L}_\infty$-spaces and related Banach spaces, Bol. Soc. Bras. Mat. 14 (1983) 109-123. MR **86b:**46021

[10] Y. Brudnyi and N. J. Kalton, Polynomial approximation on convex subsets of $\mathbb{R}^n$, Constr. Approx. 16 (2000) 161-199. MR **2000k:**41036

[11] F. Cabello Sánchez and J. M. F. Castillo, Duality and twisted sums of Banach spaces, J. Funct. Anal. 175 (2000), 1–16. MR **2001i:**46011

[12] F. Cabello Sánchez and J. M. F. Castillo, Uniform boundedness and twisted sums of Banach spaces, Houston J. Math., to appear.

[13] F. Cabello Sánchez, J. M. F. Castillo and D. Yost, Sobczyk's Theorem from A to B, Extracta Math. 15 (2000), 391–420. MR **2002a:**46003

[14] P. G. Casazza and T. J. Shura, *Tsirelson's space,* Springer Lecture Notes in Mathematics No. 1363, 1989. MR **90b:**46030

[15] J. M. F. Castillo, Banach Spaces, à la Recherche du Temps Perdu, Extracta Math. 15 (2000) 373–390. MR **2002c:**46137

[16] J. M. F. Castillo and M. González, *Three-space problems in Banach space theory*, Springer Lecture Notes in Math. No. 1667, 1997. MR **99a:**46034

[17] J. M. F. Castillo, M. González, A. M. Plichko and D. Yost, Twisted properties of Banach spaces, Math. Scand. 89 (2001) 217–244. MR **2002m:**46014

[18] H. H. Corson, The weak topology of a Banach space, Trans. Amer. Math. Soc. 101 (1961) 1-15. MR **24:**A2220

[19] R. Deville, G. Godefroy and V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys no. 64, Longman, 1993. MR **94d:**46012

[20] J. Diestel, H. Jarchow and A. Tonge, *Absolutely summing operators.* Cambridge Studies in Advanced Mathematics, 43. Cambridge University Press, Cambridge, 1995. MR **96i:**46001

[21] P. Enflo, J. Lindenstrauss and G. Pisier, On the "three space problem", Math. Scand. 36 (1975) 199-210. MR **52:**3928

[22] C. Foiaş and I. Singer, On bases in $C([0,1])$ and $L^1([0,1])$, Rev. Roum. Math. Pures Appl. 10 (1965) 931–960. MR **34:**6509

[23] G. Godefroy, N. J. Kalton and G. Lancien, Szlenk indices and uniform homeomorphisms, Trans. Amer. Math. Soc. 353 (2001) 3895–3918.

[24] W. B. Johnson and H. P. Rosenthal, On ω^*-basic sequences and their applications in the study of Banach spaces, Studia Math. 43 (1972) 77-92. MR **46:**9696

[25] W. B. Johnson and M. Zippin, Separable L_1 preduals are quotients of $C(\Delta)$, Israel J. Math. 16 (1973) 198-202. MR **49:**3509

[26] W. B. Johnson and M. Zippin, Extension of operators from weak*-closed subspaces of ℓ_1 into $C(K)$ spaces. Studia Math. 117 (1995), 43–55. MR **96k:**46025

[27] M. I. Kadets and B. Ya. Levin, On a solution of a problem of Banach concerning topological equivalence of spaces of continuous functions (Russian), Trudy Sem. Funct. Anal. Voronezh. 3–4 (1960) 20–25.

[28] N. J. Kalton, On subspaces of c_0 and extension of operators into $C(K)$-spaces, Quart. J. Math (Oxford) 52 (2001), 313–328.

[29] N. J. Kalton and N. T. Peck, Twisted sums of sequence spaces and the three space problem, Trans. Amer. Math. Soc. 255 (1979) 1-30. MR **82g:**46021

[30] N. J. Kalton and A. Pełczyński, Kernels of surjections from $\mathcal{L}_1$-spaces with an application to Sidon sets, Math. Ann. 309 (1997) 135-158. MR **98h:**46013

[31] H. Knaust, E. Odell and T. Schlumprecht, On asymptotic structure, the Szlenk index and UKK properties in Banach spaces, Positivity 3 (1999) 173-199. MR **2001f:**46011

[32] S. Kwapien, Isomorphic characterization of inner product spaces by orthogonal series with vector coefficients, Studia Math. 44 (1972) 583-595. MR **49:**5789

[33] S. Kwapien, Isomorphic characterization of Hilbert spaces by orthogonal series with vector coefficients, Exposé No. 8, Séminaire Maurey-Schwartz, Ecole Polytéchnique, 1972-3. MR **53:**14095

[34] J. Lindenstrauss and H. P. Rosenthal, Automorphisms in c_0, ℓ_1 and m, Israel J. Math. 7 (1969) 227-239. MR **40:**3273

[35] J. Lindenstrauss and H. P. Rosenthal, The $\mathcal{L}_p$ spaces, Israel J. Math. 7 (1969) 325-349. MR **42:**5012

[36] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces, Vol. 1, Sequence spaces,* Springer-Verlag, 1977. MR **58:**17766

[37] B. Maurey, Théorèmes de factorisation pur les opérateurs linéaires à valeurs dans un espace L_p, Asterisque 11, 1974. MR **49:**9670

[38] B. Maurey, Une théorème de prolongement, C.R. Acad. Sci Paris, A279 (1974) 329-332. MR **50:**8013

[39] A. Pełczyński, On strictly singular and strictly cosingular operators I. Strictly singular and strictly cosingular operators in $C(S)$ spaces, Bull. Acad. Polon. Sci. Math. 13 (1965) 31-41. MR **31:**1563

[40] A. Pełczyński, Linear extensions, linear averagings, and their applications to linear topological classification of spaces of continuous functions. Dissertationes Math. Rozprawy Mat. 58, 1968. MR **37:**3335

[41] G. Pisier, Holomorphic semigroups and geometry of Banach spaces, Ann. Math. 115 (1982) 375-392. MR **83h:**46027

[42] H. P. Rosenthal, Some applications of p-summing operators to Banach space theory. Studia Math. 58 (1976), 21–43. MR **55:**3754

[43] A. Sobczyk, Projection of the space (m) on its subspace (c_0), Bull. Amer. Math. Soc. 47 (1941) 938–947. MR **3:**205f

[44] B. S. Tsirelson, It is impossible to embed ℓ_p or c_0 into an arbitrary Banach space (Russian) Funkts. Anal. i Prilozhen. 8, no. 2 (1974) 57-60. English translation: Funct. Anal. Appl. 8 (1974) 138–141. MR **50:**2871

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