

EXAMPLES FOR THE MOD p MOTIVIC COHOMOLOGY OF CLASSIFYING SPACES

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ABSTRACT. Let BG be the classifying space of a compact Lie group G . Some examples of computations of the motivic cohomology $H^{*,*}(BG; \mathbb{Z}/p)$ are given, by comparing with $H^*(BG; \mathbb{Z}/p)$, $CH^*(BG)$ and $BP^*(BG)$.

1. INTRODUCTION

Let p be a prime number and k a subfield of the complex number field \mathbb{C} . Let k contain a primitive p -th root of unity. Given a scheme X of finite type over k , the mod p motivic cohomology $H^{*,*}(X; \mathbb{Z}/p) = \bigoplus_{m,n} H^{m,n}(X; \mathbb{Z}/p)$ has been defined by Suslin and Voevodsky ([Vo1], [Vo2]). When X is smooth, the subring $H^{2*,*}(X; \mathbb{Z}/p) = \bigoplus_n H^{2n,n}(X; \mathbb{Z}/p)$ is identified with the classical mod p Chow ring $CH^*(X)/p$ of algebraic cycles on X .

The inclusion $t_{\mathbb{C}} : k \subset \mathbb{C}$ induces a natural transformation (realization map) $t_{\mathbb{C}}^{m,n} : H^{m,n}(X; \mathbb{Z}/p) \rightarrow H^m(X(\mathbb{C}); \mathbb{Z}/p)$, where $X(\mathbb{C})$ is the complex variety of \mathbb{C} -valued points of X . Let us write the coimage of $t_{\mathbb{C}}^{*,*}$ as

$$(1.1) \quad h^{*,*}(X; \mathbb{Z}/p) = \bigoplus_{m,n} H^{m,n}(X; \mathbb{Z}/p) / \text{Ker}(t_{\mathbb{C}}^{m,n}).$$

It is known that there is an element $\tau \in H^{0,1}(\text{Spec}(k); \mathbb{Z}/p)$ with $t_{\mathbb{C}}^{*,*}(\tau) = 1$. Then we have the bigraded $\mathbb{Z}/p[\tau]$ -algebra monomorphism

$$(1.2) \quad h^{*,*}(X; \mathbb{Z}/p) \hookrightarrow H^*(X(\mathbb{C}); \mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau, \tau^{-1}]$$

where the bidegree of $x \in H^n(X(\mathbb{C}); \mathbb{Z}/p)$ is given by (n, n) . If $k = \mathbb{C}$ and the Beilinson-Lichtenbaum condition [Vo2] is satisfied for p , then we also have the injection $H^*(X(\mathbb{C}); \mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau] \hookrightarrow h^{*,*}(X; \mathbb{Z}/p)$.

When $x \in H^{m,n}(X; \mathbb{Z}/p)$, define the weight of x by $w(x) = 2n - m$. Clearly $w(x) = 0$ if and only if $x \in CH^*(X)/p$. Voevodsky has extended the Steenrod algebra A_p^* of cohomology operations to the case of motivic cohomology. Among them, we have the Milnor primitive operation

$$Q_i : H^{*,*}(X; \mathbb{Z}/p) \rightarrow H^{*+2p^i-1, *+p^i-1}(X; \mathbb{Z}/p),$$

so that it is sent to the usual Milnor operation Q_i by the realization map $t_{\mathbb{C}}^*$. Hence $w(Q_i) = -1$, and the Q_i ($0 \leq i$) form an exterior algebra $\Lambda(Q_0, Q_1, \dots) \subset A_p^*$ also for the motivic cohomology. To simplify the notation, let us write the exterior algebra $Q(n) = \Lambda(Q_0, \dots, Q_n)$ for $n \geq 0$ and $Q(-1) = \mathbb{Z}/p$.

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In this paper we are mainly concerned with the following case. For $n \geq -1$, let G_n be a \mathbb{Z}/p -module and $Q(n)G_n$ the free $Q(n)$ -module generated by G_n . Moreover, the scheme X satisfies the assumption that there is a \mathbb{Z}/p -module injection

$$(1.3) \quad j_{\mathbb{C}} : H^*(X(\mathbb{C}); \mathbb{Z}/p) \hookrightarrow \bigoplus_{n=-1}^{\infty} Q(n)G_n \quad \text{with } j_{\mathbb{C}}^{-1}(Q_0 \dots Q_n G_n) \subset \text{Im}(t_{\mathbb{C}}^{2*,*})$$

such that $p_n j_{\mathbb{C}} : H^*(X(\mathbb{C}) : \mathbb{Z}/p) \rightarrow Q(n)G_n$ is the $Q(n)$ -module map and $p'_n p_n j_{\mathbb{C}} : H^*(X(\mathbb{C}) : \mathbb{Z}/p) \rightarrow Q_0 \dots Q_{n-1} G_n$ is a surjection for each n , where $p_n : \bigoplus Q(n)G_n \rightarrow Q(n)G_n$ and $p'_n : Q(n)G_n \rightarrow Q_0 \dots Q_{n-1} G_n$ are the projections. (We do not assume a $Q(n)$ -module structure on the right-hand side module in (1.3).)

We take the weight on the right-hand side by putting $w(x) = n + 1$ for every $x \in G_n$ (simply write $w(G_n) = n + 1$), so that $w(Q_0 \dots Q_n x) = 0$. Then we get the injection of bigraded \mathbb{Z}/p -modules

$$(1.4) \quad j : h^{*,*}(X; \mathbb{Z}/p) \hookrightarrow \bigoplus_{n=-1}^{\infty} Q(n)G_n \otimes \mathbb{Z}/p[\tau]$$

such that the composition $(p_n \otimes \mathbb{Z}/p[\tau])j : h^{*,*}(X; \mathbb{Z}/p) \rightarrow Q(n)G_n \otimes \mathbb{Z}/p[\tau]$ is the bigraded $Q(n)$ -module map.

The above argument has its counterpart in the BP -theory of $X(\mathbb{C})$. As we know, $BP^*(-)$ is the cohomology theory with the coefficient ring $BP^* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$, $|v_i| = -2(p^i - 1)$. Let us write $BP^*/(p, v_1, \dots, v_{m-1})$ as $P(m)^*$. The Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*} = H^*(X(\mathbb{C})) \otimes BP^* \implies BP^*(X(\mathbb{C}))$$

has the differential

$$(1.5) \quad d_{2p^i-1}(x) = Q_i(x) \otimes v_i \quad \text{mod}(M_i),$$

where M_i is the ideal of $E_{2p^i-1}^{*,*}$ generated by elements in $(p, v_1, \dots, v_{i-1})E_2^{*,*}$. We assume here that nonzero differentials are all of the form (1.5) and that $H^*(X(\mathbb{C}))$ has no higher p -torsion. Then we easily see that (1.3) implies

$$(1.6) \quad E_{\infty}^{*,*} \cong \bigoplus_{n=-1}^{\infty} P(n+1)^* \tilde{G}_n \oplus B \quad \text{with } \tilde{G} = Q_0 \dots Q_n G_n,$$

where $P(n+1)^* \tilde{G}_n$ is the free $P(n+1)^*$ -module generated by elements in \tilde{G}_n and B is the BP^* -submodule of $E_{\infty}^{*,*}$ of generators in $\text{Ideal}(p, v_1, \dots)E_2^{*,*}$. Conversely, by the same assumption, if $\tilde{G}_n \subset \text{Im}(t_{\mathbb{C}}^{2*,*})$, then the isomorphism (1.6) implies the existence of the injections $j_{\mathbb{C}}$ in (1.3) and so j in (1.4).

Let $\rho : BP(X(\mathbb{C})) \otimes_{BP^*} \mathbb{Z}/p \rightarrow H^*(X(\mathbb{C}); \mathbb{Z}/p)$ be the Thom map. Then (1.6) and $\tilde{G}_n \subset \text{Im}(t_{\mathbb{C}}^{2*,*})$ imply that

$$\text{Im}(t_{\mathbb{C}}^{2*,*}) = \text{Im}(\rho) \cong \bigoplus_{n=-1}^{\infty} \tilde{G}_n \subset BP^*(X(\mathbb{C})) \otimes_{BP^*} \mathbb{Z}/p.$$

More generally, B. Totaro [To1], [To2] constructed the modified cycle map

$$(1.7) \quad \tilde{cl}^* : CH^*(X)/p \rightarrow BP^*(X(\mathbb{C})) \otimes_{BP^*} \mathbb{Z}/p$$

in such a way that the composition $\rho \tilde{cl}^*$ is the realization map $t_{\mathbb{C}}^{2*,*}$. If a BP^* -module generator of B in (1.6) is represented by transfer of a Chern class, then

this element gives a nonzero element in $\text{Ker}(t_{\mathbb{C}}^{2*,*})$ by the modified cycle map \tilde{cl}^* . Using this argument, Totaro found nonzero elements in $\text{Ker}(t_{\mathbb{C}}^{2*,*})$ when X is the classifying space $BSO(4)$.

The motivic cohomology of the classifying space is defined as follows. Let G be a linear algebraic group over k . Let V be a representation of G such that G acts freely on $V - S$ for some closed subset S . Then $(V - S)/G$ exists as a quasi-projective variety over k . Following Totaro [To1] and Voevodsky, define

$$(1.8) \quad H^{*,*}(BG; \mathbb{Z}/p) = \lim_{\dim(V), \text{codim}(S) \rightarrow \infty} H^{*,*}((V - S)/G; \mathbb{Z}/p).$$

The topological space $BG(\mathbb{C}) = \lim((V - S)/G)(\mathbb{C})$ is the usual classifying space BG . Hence we write the \mathbb{C} -value points $BG(\mathbb{C})$ simply as BG .

We will show that the isomorphism (1.6) is satisfied when $X = BG$ for the following cases: $O(n), SO(4), D_8, G_2, Spin(7)$ for $p = 2$, PGL_3, F_4 for $p = 3$ and the extraspecial p -group p_+^{1+2} of order p^3 and of exponent p for odd primes. (However note that $H^*(Bp_+^{1+2})$ has p^2 -torsion.)

Hence we will prove (1.4) for these BG . Moreover, when $k = \mathbb{C}$ and $G = O(3)$ for $p = 2$, PGL_3 for $p = 3$, p_+^{1+2} and $(\mathbb{Z}/p)^n$ for all primes, we will show that

$$(1.9) \quad h^{*,*}(BG; \mathbb{Z}/p) \cong \bigoplus Q(n)G_n \otimes \mathbb{Z}/p[\tau].$$

S. Wilson [RWY] first constructed the decomposition (1.3) so that $j_{\mathbb{C}}$ is an isomorphism for $X = BO(n)$, and next computed $BP^*(BO(n))$. However, it is unknown whether j in (1.4) is an isomorphism or not for $X = BO(n)$, $n \geq 4$.

The contents of this paper are as follows. The aim of §§2 and 3 is a short introduction to motivic cohomology for algebraic topologists unfamiliar with it. In these sections, we concentrate on the computation of $H^*(B(\mathbb{Z}/p)^n; \mathbb{Z}/p)$. In §4, we deal with the study of $h^{*,*}(X; \mathbb{Z}/p)$, making no use of $BP^*(BG)$ but Milnor’s operation Q_i instead. In §5, we give an account of $h^{*,*}(BG; \mathbb{Z}/p)$ expressed in term of $BP^*(BG)$. Also in this section we give some results on $\text{Ker}(t_{\mathbb{C}}^{*,*})$. The motivic cohomology of the Eilenberg-MacLane space $K(\mathbb{Z}/p(n), n)$ is studied in §6. In §7, we give some comments on algebraic cobordism theory and algebraic BP -theory.

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2. CHOW RING, MILNOR K -THEORY, ÉTALE COHOMOLOGY

We use the category Spc of (algebraic) spaces, along with schemes A , their quotients A_1/A_2 and $\text{colim}(A_\alpha)$, all defined by Voevodsky [Vo2], [MoVo]. Here schemes are defined over a field k with $ch(k) = 0$. The motivic cohomology is the double indexed cohomology defined by Suslin and Voevodsky, directly related with the Chow ring and Milnor K -theory.

(CH) For a smooth scheme X we have $H^{2n,n}(X) \cong CH^n(X)$, the classical Chow group of codim n cycles on X .

(MK) $H^{n,n}(Spec(k)) \cong K_n^M(k)$, the Milnor K -group for the field k .

For a smooth variety X with $\dim(X) = n$, the Chow ring is the sum $CH^*(X) = \bigoplus_i CH^i(X)$, where

$$CH^i(X) = \{(n - i) \text{ cycles in } X\} / (\text{rational equivalence}).$$

Here the rational equivalence $a \equiv b$ is defined if there is a codimension i subvariety W in $X \times \mathbb{P}^1$ such that $a = p_*f^*(0)$ and $b = p_*f^*(1)$, where \mathbb{P}^1 is the projective line and p (resp. f) is the projection on the first (resp. second) factor.

For $k = \mathbb{C}$, if X has a cellular decomposition, i.e., $X = X_n \supset X_{n-1} \supset \dots \supset X_0$ with $X_i - X_{i-1} = \bigcup \mathbb{A}^{n_{ij}}$, where $\mathbb{A}^{n_{ij}}$ is the affine space of dimension n_{ij} , then $CH^*(X) \cong H^*(X(\mathbb{C}))$, the singular cohomology theory of \mathbb{C} -rational points of X . For example, $CH^*(\mathbb{P}^n) \cong H^*(\mathbb{C}\mathbb{P}^n)$ for projective spaces \mathbb{P}^n . Since Spc contains *colimit*, we can consider the infinite projective space $\mathbb{P}^\infty = B\mathbb{G}_m$ and the infinite lens space $\lim_n (\mathbb{A}^n - \{0\}/\mathbb{Z}/p) = L_p^\infty = B\mathbb{Z}/p$. The Chow rings of classifying spaces of abelian groups are given in [To1]:

$$(2.1) \quad CH^*(\mathbb{P}^\infty) \cong H^{2*,*}(\mathbb{P}^\infty) \cong \mathbb{Z}[y], \quad CH^*(B\mathbb{Z}/p) \cong H^{2*,*}(B\mathbb{Z}/p) \cong \mathbb{Z}[y]/(py),$$

with $deg(y) = (2, 1)$. For products of these spaces we have

$$(2.2) \quad CH^*((\mathbb{P}^\infty)^n) \cong \mathbb{Z}[y_1, \dots, y_n],$$

$$(2.3) \quad CH^*((B\mathbb{Z}/p)^n) \cong \mathbb{Z}[y_1, \dots, y_n]/(py_1, \dots, py_n).$$

Here note that $CH^*(X) \not\cong H^{even}(X(\mathbb{C}))$ for the last case. Even if $H^*(X(\mathbb{C}))$ is generated by even dimensional elements, there are cases that $CH^*(X) \not\cong H^*(X(\mathbb{C}))$, e.g., K3-surfaces have the cohomology $H^2(X(\mathbb{C})) \cong \mathbb{Z}^{22}$, but there is a K3-surface such that $CH^1(X) \cong \mathbb{Z}^i$ for each $1 \leq i \leq 20$.

Milnor K -theory is the graded ring $\bigoplus_n K_n^M(k)$ defined by $K_n^M(k) = (k^*)^{\otimes n}/J$, where the ideal J is generated by elements $a \otimes (1 - a)$ for $a \in k^* - \{1\}$. Here the addition of k^* is given by the multiplication in the field k . Hence $K_0^M(k) = \mathbb{Z}$ and $K_1^M(k) = k^*$. Hilbert's Theorem 90, which essentially says that the Galois cohomology $H^1(G(k_s/k); k_s^*) = 0$, implies the isomorphism $K_1^M(k)/p \cong k^*/(k^*)^p \cong H^1(G(k_s/k); \mathbb{Z}/p)$ for $1/p \in k$. Similarly we can define a map (the norm residue map) for any extension F of k of finite type,

$$(BK) \quad K_n^M(F)/p \rightarrow H^n(G(F_s/F); \mu_p^{\otimes n}),$$

where $\mu_p^{\otimes n}$ is the discrete $G(F_s/F)$ -module of n -th tensor power of the group of p -roots of 1. The Bloch-Kato conjecture is that this map is an isomorphism for all field k , and the Milnor conjecture is its $p = 2$ case. This conjecture is solved when $n = 2$ by Merkurjev and Suslin [MeSu], and for $p = 2$ by Voevodsky [Vo1].

Notice that $H^n(G(k_s/k); \mu_p^{\otimes n}) \cong H_{et}^n(Spec(k), \mu_p^{\otimes n})$, the étale cohomology of the point. The étale cohomology $H_{et}^*(X; \mathbb{Z}/p)$ has the following properties:

(E.1) If k contains a primitive p -th root of 1, then there is the additive isomorphism

$$H_{et}^m(X, \mu_p^{\otimes n}) \cong H_{et}^m(X; \mathbb{Z}/p).$$

(E.2) For smooth X over $k = \mathbb{C}$,

$$H_{et}^m(X; \mathbb{Z}/p^N) \cong H^m(X(\mathbb{C}); \mathbb{Z}/p^N) \quad \text{for all } N \geq 1.$$

The last cohomology is the usual mod p ordinary cohomology of \mathbb{C} -rational points of X . Of course $H_{et}^*(Spec(\mathbb{C}); \mathbb{Z}/p) \cong \mathbb{Z}/p$. It is known that

$$K_*^M(\mathbb{R})/2 \cong H_{et}^*(Spec(\mathbb{R}); \mathbb{Z}/2) \cong \mathbb{Z}/2[\rho]$$

with $deg(\rho) = 1$ for the real number field \mathbb{R} . Let F_v be a local field with residue field k_v of $ch(k_v) \neq 2$. Then $K_*^M(F_v)/2 \cong H_{et}^*(Spec(F_v); \mathbb{Z}/2) \cong \Lambda(\alpha, \beta)$ with $deg(\alpha) = deg(\beta) = 1$. Thus we know that $\bigoplus_m H^{m,m}(pt; \mathbb{Z}/2)$ for these cases.

3. THE REALIZATION MAP

In this section we consider the relation to the usual ordinary cohomology. Let R be \mathbb{Z} or \mathbb{Z}/p . The motivic cohomology has the following properties [Vo2].

- (C1) $H^{*,*}(X; R)$ is a bigraded ring natural in X .
- (C2) When $k \subset \mathbb{C}$, there are maps (realization maps)

$$t_{\mathbb{C}}^{m,n} : H^{m,n}(X; R) \rightarrow H^m(X(\mathbb{C}); R)$$

which sum up to $t_{\mathbb{C}}^{*,*} = \bigoplus_{m,n} t_{\mathbb{C}}^{m,n}$, the natural ring homomorphism.

- (C3) There are the (Bockstein, reduced powers) operations

$$\begin{aligned} \beta : H^{*,*}(X; \mathbb{Z}/p) &\rightarrow H^{*+1,*}(X; \mathbb{Z}/p), \\ P^i : H^{*,*}(X; \mathbb{Z}/p) &\rightarrow H^{*+2(p-1)i, *+(p-1)i}(X; \mathbb{Z}/p), \end{aligned}$$

which commutes with the realization map $t_{\mathbb{C}}$ when $k \subset \mathbb{C}$.

- (C4) For the projective space \mathbb{P}^n , there is an isomorphism

$$H^{*,*}(X \times \mathbb{P}^n / \mathbb{P}^{n-1}; R) \cong H^{*,*}(X; R)\{1, y'\}$$

with $\deg(y') = (2n, n)$ and $t_{\mathbb{C}}(y') \neq 0$ for $k \subset \mathbb{C}$.

We recall the Lichtenbaum motivic cohomology [Vo2]. Lichtenbaum defined the similar cohomology $H_L^{*,*}(X; R)$ by using the étale topology, while $H^{*,*}(X; R)$ is defined using the Nisnevich topology. Since Nisnevich covers are restricted étale covers, there is the natural map $H^{*,*}(X; R) \rightarrow H_L^{*,*}(X; R)$. We say that the $B(n, p)$ condition holds if

$$H^{m,n}(X; \mathbb{Z}_{(p)}) \cong H_L^{m,n}(X; \mathbb{Z}_{(p)}) \quad \text{for all } m \leq n + 1$$

and all smooth X . The Beilinson-Lichtenbaum conjecture is that $B(n, p)$ holds for all n and p . It is known that the condition $B(n, p)$ is equivalent to the Bloch-Kato conjecture (BK) for degree n and prime p . Hence $B(n, p)$ holds for $n \leq 2$ or $p = 2$. Moreover, Suslin and Voevodsky have proved

(L-E)
$$H_L^{m,n}(X; \mathbb{Z}/p) \cong H_{\text{ét}}^m(X; \mu_p^{\otimes n}).$$

Now we compute $H^{*,*}(pt; \mathbb{Z}/p) = H^{*,*}(\text{Spec}(k); \mathbb{Z}/p)$. For a smooth X , the following dimensional condition is known:

- (C5) For a smooth X , if $H^{m,n}(X; R) \neq 0$, then

$$m \leq n + \dim(X), \quad m \leq 2n \text{ and } m \geq 0.$$

For the rest of this paper, we assume that k contains a primitive p -th root of 1 and $B(n, p)$ holds for all n , but $X = \text{Spec}(k)$. Then

$$H^{m,n}(pt; \mathbb{Z}/p) \cong H_{\text{ét}}^m(pt; \mu_p^{\otimes n}) \cong H_{\text{ét}}^m(pt; \mathbb{Z}/p) \quad \text{if } m \leq n,$$

and $H^{m,n}(pt; \mathbb{Z}/p) \cong 0$ for $m > n$. Let $\tau \in H^{0,1}(pt; \mathbb{Z}/p)$ be the element corresponding to a generator of $H_{\text{ét}}^0(\text{Spec}(k); \mu_p) \cong H_{\text{ét}}^0(\text{Spec}(k); \mathbb{Z}/p)$. Then we get the isomorphism

$$H^{*,*}(\text{Spec}(k); \mathbb{Z}/p) \cong H_{\text{ét}}^*(\text{Spec}(k); \mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau]$$

since $\tau : H_{\text{ét}}^m(pt; \mu_p^{\otimes n}) \cong H_{\text{ét}}^m(pt; \mu_p^{\otimes(n+1)})$. In particular, for the real number field \mathbb{R} and a local field F_v with the residue field k_v of $\text{ch}(k_v) \neq 2$ we have

(3.1)
$$H^{*,*}(\text{Spec}(\mathbb{R}); \mathbb{Z}/2) \cong \mathbb{Z}/2[\rho, \tau] \quad \text{with } \deg(\rho) = (1, 1),$$

(3.2)
$$H^{*,*}(\text{Spec}(F_v); \mathbb{Z}/2) \cong \mathbb{Z}/2[\tau] \otimes \Lambda(\alpha, \beta) \quad \text{with } \deg(\alpha) = \deg(\beta) = (1, 1).$$

For $k = \mathbb{C}$, we know that $K_n^M(\mathbb{C})/p \cong 0$ for $n > 0$, and hence

$$(3.3) \quad H^{*,*}(Spec(\mathbb{C}); \mathbb{Z}/p) \cong \mathbb{Z}/p[\tau] \quad \text{with } deg(\tau) = (0, 1).$$

When $k = \mathbb{C}$, if the $B(n, p)$ condition holds for X , then it is immediate that

$$(3.4) \quad [\tau^{-1}]H^{*,*}(X; \mathbb{Z}/p) \cong H^*(X(\mathbb{C}); \mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau, \tau^{-1}],$$

where the degree is defined by $deg(x) = (m, m)$ if $x \in H^m(X(\mathbb{C}); \mathbb{Z}/p)$.

Next we compute the cohomology of \mathbb{P}^∞ and $B\mathbb{Z}/p$. For any (algebraic) map $f : X \rightarrow Y$ in the category Spc , we can construct the cofiber sequence

$$X \rightarrow Y \rightarrow cone(f) = Y/X,$$

which induces the long exact sequence (Voevodsky [Ve2])

$$(3.5) \quad H^{*,*}(X; R) \leftarrow H^{*,*}(Y; R) \leftarrow H^{*,*}(Y/X; R) \leftarrow H^{*-1,*}(X; R).$$

In particular, we get the Mayer-Vietoris, Gysin and blow-up long exact sequences.

By the cofiber sequence $\mathbb{P}^{n-1} \rightarrow \mathbb{P}^n \rightarrow \mathbb{P}^n/\mathbb{P}^{n-1}$ and (C4), we can inductively see that

$$(3.6) \quad H^{*,*}(\mathbb{P}^n; \mathbb{Z}/p) \cong H^{*,*}(pt; \mathbb{Z}/p) \otimes \mathbb{Z}/p[y]/(y^{n+1}) \quad \text{with } deg(y) = (2, 1).$$

When $k = \mathbb{C}$, since $B(1, p)$ always holds, $H^{1,1}(L_p^n; \mathbb{Z}/p) \cong H^1(L_p^n; \mathbb{Z}/p)$. Hence there is an element $x' \in H^{1,1}(L_p^n; \mathbb{Z}/p)$ with $t_{\mathbb{C}}(x') = x \in H^1(L_p^n; \mathbb{Z}/p)$. This also holds for general k [Vo3]. The lens space is identified with the sphere bundle associated with the line bundle

$$(\mathbb{A}^n - \{0\}) \times_{(\mathbb{A} - \{0\})} \mathbb{A} \rightarrow (\mathbb{A}^n - \{0\})/(\mathbb{A} - \{0\}) = \mathbb{P}^n.$$

Here $(\mathbb{A}^n - \{0\}) \times_{(\mathbb{A} - \{0\})} \mathbb{A}$ is the identification such that $(z_i, z) \sim (a^{-1}z_i, a^p z) \in (\mathbb{A}^n - \{0\}) \times \mathbb{A}$ for $(z_i) \in \mathbb{A}^n, z \in \mathbb{A}, a \in \mathbb{A} - \{0\}$. Hence we get the cofiber-
ing $L_p^n \rightarrow \mathbb{P}^n \xrightarrow{\times p} \mathbb{P}^n$. Thus we get the additive isomorphism $H^{*,*}(L_p^n; \mathbb{Z}/p) \cong H^{*,*}(\mathbb{P}^n; \mathbb{Z}/p)\{1, x\}$. This induces the ring isomorphism for $p = odd$

$$(3.7) \quad H^{*,*}(L_p^n; \mathbb{Z}/p) \cong \mathbb{Z}/p[y]/(y^{n+1}) \otimes \Lambda(x) \otimes H^{*,*}(pt; \mathbb{Z}/p) \quad \text{with } deg(x) = (1, 1).$$

However, note that when $p = 2$ we get $x^2 = y\tau + x\rho$ [Vo3], where $\rho \in H^{1,1}(pt; \mathbb{Z}/p) \cong k^*/k^{2*}$ represents -1 . (Hence $\rho = 0$ when $\sqrt{-1} \in k^*$.) This is proved by the well-known fact that $\{a, a\} = \{a, -1\}$ in the Milnor K -theory $K_2^M(k)$.

We say that a space X satisfies the Künneth formula for a space Y if

$$H^{*,*}(X \times Y; \mathbb{Z}/p) \cong H^{*,*}(X; \mathbb{Z}/p) \otimes_{H^{*,*}(pt; \mathbb{Z}/p)} H^{*,*}(Y; \mathbb{Z}/p).$$

By the above cofiber sequences, we can easily see that \mathbb{P}^∞ and $B\mathbb{Z}/p$ satisfy the Künneth formula for all spaces. In particular, we have the ring isomorphisms

$$(3.8) \quad H^{*,*}((\mathbb{P}^\infty)^n; \mathbb{Z}/p) \cong \mathbb{Z}/p[y_1, \dots, y_n] \otimes H^{*,*}(pt; \mathbb{Z}/p),$$

$$(3.9) \quad H^{*,*}((B\mathbb{Z}/p)^n; \mathbb{Z}/p) \cong \mathbb{Z}/p[y_1, \dots, y_n] \otimes \Lambda(x_1, \dots, x_n) \otimes H^{*,*}(pt; \mathbb{Z}/p)$$

(when $p = 2, x_i^2 = y_i\tau + x_i\rho$).

This fact is used to define the reduced power operation P^i in (C3). Since a Sylow p -subgroup of the symmetric group S_p of p letters is isomorphic to \mathbb{Z}/p , we have the isomorphism

$$H^{*,*}(BS_p; \mathbb{Z}/p) \cong H^{*,*}(B\mathbb{Z}/p; \mathbb{Z}/p)^{F_p^*} \cong \mathbb{Z}/p[Y] \otimes \Lambda(W) \otimes H^{*,*}(pt; \mathbb{Z}/p),$$

identifying $Y = y^{p-1}$ and $W = xy^{p-2}$. If X is smooth (and suppose p is odd, to simplify arguments), we can define the reduced powers (of Chow rings) as follows. Consider maps

$$\begin{aligned} H^{2*,*}(X; \mathbb{Z}/p) &\xrightarrow{i_1} H^{2p*,p*}(X^p \times_{S_p} ES_p) \\ &\xrightarrow{\Delta^*} H^{*,*}(X \times BS_p; \mathbb{Z}/p) \cong H^{*,*}(X; \mathbb{Z}/p) \otimes_{H^{*,*}(pt; \mathbb{Z}/p)} H^{*,*}(BS_p; \mathbb{Z}/p), \end{aligned}$$

where i_1 is the Gysin map for the p -th external power, and Δ is the diagonal map. For $\text{deg}(x) = (2n, n)$, the reduced powers are defined as

$$(3.10) \quad \Delta^* i_1(x) = \sum P^i(x) \otimes Y^{n-i} + \beta P^i(x) \otimes WY^{n-i-1}.$$

Hence $\text{deg}(P^i) = \text{deg}(Y^i) = \text{deg}(y^{i(p-1)}) = (2i(p-1), i(p-1))$. Voevodsky defined i_1 for nonsmooth X also, and by using suspensions maps he defined reduced powers for all degree elements in $H^{*,*}(X; \mathbb{Z}/p)$ for all X [Vo3].

Moreover, we can see (Hu-Kříž [HK]) that

$$(3.11) \quad H^{*,*}(BGL_n; \mathbb{Z}/p) \cong \mathbb{Z}/p[c_1, \dots, c_n] \otimes H^{*,*}(pt; \mathbb{Z}/p),$$

where the Chern class c_i with $\text{deg}(c_i) = (2i, i)$ is identified with the elementary symmetric polynomial in $H^{*,*}((\mathbb{P}^\infty)^n; \mathbb{Z}/p)$. So we can define the Chern class $\rho^*(c_i) \in H^{2*,*}(BG; \mathbb{Z}/p)$ for each representation $\rho : G \rightarrow GL_n$.

4. $H^{*,*}(X; \mathbb{Z}/p) / \text{Ker}(t_{\mathbb{C}})$ AND THE OPERATION Q_i

In this section we assume that X is smooth and $k = \mathbb{C}$. Even in this case the motivic cohomology $H^{*,*}(X; \mathbb{Z}/p)$ seems difficult, in general. Hence we consider a bigraded ring which is computable only by using the algebraic topology of $H^*(X(\mathbb{C}); \mathbb{Z}/p)$. Define a bidegree algebra by

$$(4.1) \quad h^{*,*}(X; \mathbb{Z}/p) = \bigoplus_{m,n} H^{m,n}(X; \mathbb{Z}/p) / \text{Ker}(t_{\mathbb{C}}^{m,n}).$$

Since $t_{\mathbb{C}}^{*,*}(\tau) = 1$, it is almost immediate that there is the injection of bidegree $\mathbb{Z}/p[\tau]$ -algebras

$$h^{*,*}(X; \mathbb{Z}/p) \hookrightarrow H^*(X(\mathbb{C}); \mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau, \tau^{-1}],$$

where the bidegree of $x \in H^n(X(\mathbb{C}); \mathbb{Z}/p)$ is (n, n) . (This also holds when $k \subset \mathbb{C}$ and k has a primitive p -th root of 1.)

Suppose the $B(n, p)$ condition holds. By the isomorphisms (B, p) , (L-E), (E1) and (E2), we have

$$H^{n,n}(X; \mathbb{Z}/p) \cong H_L^{n,n}(X; \mathbb{Z}/p) \cong H_{et}^n(X; \mu_p^{\otimes n}) \cong H_{et}^n(X; \mathbb{Z}/p) \cong H^n(X(\mathbb{C}); \mathbb{Z}/p).$$

Hence we get the injection of bidegree $\mathbb{Z}/p[\tau]$ -algebras

$$H^*(X(\mathbb{C}); \mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau] \hookrightarrow h^{*,*}(X; \mathbb{Z}/p).$$

Thus there exist a \mathbb{Z}/p -basis $\{a_I\}$ of $H^*(X(\mathbb{C}); \mathbb{Z}/p)$ and a $|\frac{1}{2}a_I| \geq t_I \geq 0$ such that

$$h^{*,*}(X; \mathbb{Z}/p) \cong \bigoplus_I \mathbb{Z}/p[\tau] \{ \tau^{-t_I} a_I \}.$$

Remark. Let $F_i = \text{Im}(\bigoplus_m t_{\mathbb{C}}^{m,i})$. When the $B(n, p)$ condition is satisfied, we have $\bigcup_i F_i = H^*(X(\mathbb{C}); \mathbb{Z}/p)$. We also have the interesting bigraded ring

$$grH^*(X(\mathbb{C}); \mathbb{Z}/p) = \bigoplus F_{i+1}/F_i \cong h^{*,*}(X; \mathbb{Z}/p)/(\text{Im } \tau),$$

so that $\mathbb{Z}/p[\tau] \otimes grH^*(X(\mathbb{C}); \mathbb{Z}/p)$ is additively isomorphic to $h^{*,*}(X; \mathbb{Z}/p)$, while the ring structures are different.

Here we recall the Milnor primitive operations $Q_0 = \beta$ and $Q_i = [Q_{i-1}, P^{p^{i-1}}]$:

$$Q_i : H^{*,*}(X; \mathbb{Z}/p) \rightarrow H^{*+2p^i-1, *+p^i-1}(X; \mathbb{Z}/p),$$

which is derivative, $Q_i(xy) = Q_i(x)y + xQ_i(y)$. Note also that $Q_i(\tau) = 0$, because of the dimension of $H^{*,*}(pt; \mathbb{Z}/p) \cong \mathbb{Z}/p[\tau]$.

Lemma 4.1. *If $0 \neq Q_{i_1} \dots Q_{i_s} x \in H^{2*,*}(X; \mathbb{Z}/p)$, then x is a $\mathbb{Z}/p[\tau]$ -module generator.*

Proof. If $x = x'\tau$, then $\tau Q_{i_1} \dots Q_{i_s}(x') \neq 0$. But

$$Q_{i_1} \dots Q_{i_s}(x') = 0 \in H^{2*,* - 1}(X; \mathbb{Z}/p),$$

since $H^{m,n}(X; \mathbb{Z}/p) = 0$ for $m > 2n$. □

Define the weight by $w(x) = 2n - m$ for an element $x \in H^{m,n}(X; \mathbb{Z}/p)$, so that $w(x') = 0$ for $x' \in CH^*(X)/p$. Of course we get $w(xy) = w(x) + w(y)$, $w(P^i x) = w(x)$ and $w(Q_i(x)) = w(x) - 1$.

Corollary 4.2. *Suppose that $B(n, p)$ holds. If $x \in H^n(X(\mathbb{C}); \mathbb{Z}/p)$ and $Q_{i_1} \dots Q_{i_n}(x) \neq 0$, then there is a $\mathbb{Z}/p[\tau]$ -module generator $x' \in H^{n,n}(X; \mathbb{Z}/p)$ so that $t_{\mathbb{C}}(x') = x$ and, for each $0 \leq k \leq n$, $Q_{i_1} \dots Q_{i_k}(x')$ is also a $\mathbb{Z}/p[\tau]$ -module generator of $H^{*,*}(X; \mathbb{Z}/p)$.*

Proof. By the $B(n, p)$ condition, $t_{\mathbb{C}}^{n,n} : H^{n,n}(X; \mathbb{Z}/p) \cong H^n(X(\mathbb{C}); \mathbb{Z}/p)$. Hence there is an element $x' \in H^{n,n}(X; \mathbb{Z}/p)$ with $t_{\mathbb{C}}(x') = x$. This means $w(x') = n$ and $w(Q_{i_1} \dots Q_{i_n}(x)) = 0$. From the above lemma, we get the corollary. □

Lemma 4.3. *Suppose that $B(n, p)$ holds. If there is an $s > 0$ with $p^s H^{n+1}(X(\mathbb{C}))_{(p)} \subset t_{\mathbb{C}}(H^{n+1,n}(X)_{(p)})$, then*

$$\text{Im}(H^{n+1}(X(\mathbb{C})) \rightarrow H^{n+1}(X(\mathbb{C}); \mathbb{Z}/p)) = \text{Im}((H^{n+1,n}(X) \rightarrow H^{n+1}(X(\mathbb{C}); \mathbb{Z}/p)).$$

Proof. Consider the following diagram:

$$\begin{array}{ccccccc} H_L^{n+1,n}(X) & \xrightarrow{(1)} & H_L^{n+1,n}(X; \mathbb{Z}/p^N) & \longrightarrow & H_L^{n+2,n}(X) & \xrightarrow{p^N} & H_L^{n+2,n}(X) \\ (2) \downarrow & & \cong \downarrow & & \downarrow & & \downarrow \\ H^{n+1}(X(\mathbb{C})) & \xrightarrow{(3)} & H^{n+1}(X(\mathbb{C}); \mathbb{Z}/p^N) & \longrightarrow & H^{n+2}(X(\mathbb{C})) & \xrightarrow{p^N} & H^{n+2}(X(\mathbb{C})) \end{array}$$

where $H^*(-)$ means $H^*(-; \mathbb{Z})_{(p)}$ and the rows are exact.

Let $H^{n+i}(X(\mathbb{C})) \cong F_i \oplus T_i$ and $H_L^{n+i,n}(X) \cong F'_i \oplus T'_i \oplus D_i$, where F_i, F'_i are free, T_i, T'_i are non- p -divisible torsion and D_i are p -divisible submodules. Take N and s so that $p^N > p^s > |T_i|, |T'_i|$ for $i = 1, 2$. Hence $H_L^{n+1,n}(X; \mathbb{Z}/p^N) \cong H^{n+1}(X(\mathbb{C}); \mathbb{Z}/p^N) \cong F_1/p^N \oplus T_1 \oplus T_2$.

By the $B(n, p)$ condition, $H^{n+1,n}(X) \cong H_L^{n+1,n}(X)$, and the map (2) is identified with the realization map. So $p^s(F_1 \oplus T_1) = p^s F_1 \subset \text{Image}(2)$. Therefore there is the quotient map $F_1/p^s \oplus T_1 \oplus T_2 \rightarrow \text{Coker}(1)$. On the other hand,

$\text{Ker}(p^N)|H_L^{n+2,n}(X) \cong (\text{Ker}(p^N)|D_2) \oplus T'_2 \cong (\mathbb{Z}/p^N)^k \oplus T'_2$. Hence if $k \neq 0$, then it is a contradiction to $\text{Ker}(p^N) = \text{Coker}(1)$. Hence we get $\text{Coker}(1) \cong T'_2$ and hence $\text{Im}(3)(2) = F_1/p^N \oplus T_1$. \square

Corollary 4.4. *Suppose that $B(n, p)$ holds and $t_{\mathbb{C}}^{n+1,n} \otimes \mathbb{Q} : H^{n+1,n}(X) \otimes \mathbb{Q} \rightarrow H^{n+1}(X(\mathbb{C})) \otimes \mathbb{Q}$ is epic. If $x \in \text{Im}(H^{n+1}(X(\mathbb{C})) \rightarrow H^{n+1}(X(\mathbb{C}); \mathbb{Z}/p))$ and $Q_{i_1} \dots Q_{i_{n-1}}(x) \neq 0$, then there is an element $x' \in H^{n+1,n}(X)_{(p)}$ so that $t_{\mathbb{C}}(x') = x$ and, for each $0 \leq k \leq n - 1$, $Q_{i_1} \dots Q_{i_k}(x)$ is also a $\mathbb{Z}/p[\tau]$ -module generator of $H^{*,*}(X; \mathbb{Z}/p)$.*

Here we mention the case $n = 1$. Totaro showed [To2] that $CH^*(BG) \otimes \mathbb{Q} \cong H^*(BG) \otimes \mathbb{Q}$ for any complex algebraic group G . Hence $CH^1(BG) \rightarrow H^2(BG)$ is epic; indeed, he also showed that this map is an isomorphism. As for $K3$ -surfaces, $CH^*(X) \otimes \mathbb{Q} \rightarrow H^*(X(\mathbb{C})) \otimes \mathbb{Q}$ is not epic and $H_L^{3,1}(X)$ contains p -divisible elements.

Now we consider some examples. The mod 2 cohomology of $BO(n)$ is $H^*(BO(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, \dots, w_n]$, where the Stiefel-Whitney class w_i restricts the elementary symmetric polynomial in $H^*(B(\mathbb{Z}/2)^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, \dots, x_n]$. Each element w_i^2 is represented by the Chern class c_i of the induced representation $O(n) \subset U(n)$. Hence $c_i \in CH^*(BO(n); \mathbb{Z}/2) = H^{2*,*}(BO(n); \mathbb{Z}/2)$.

Proposition 4.5. $h^{*,*}(BO(n); \mathbb{Z}/2) \supset \mathbb{Z}/2[c_1, \dots, c_n] \otimes \Delta(w_1, \dots, w_n) \otimes \mathbb{Z}/2[\tau]$, where $\text{deg}(c_i) = (2i, i)$, $\text{deg}(w_i) = (i, i)$ and $w_i^2 = \tau^i c_i$.

Since $Q_{i-1} \dots Q_0(w_i) \neq 0$, each w_i is a $\mathbb{Z}/2[\tau]$ -module generator. However, even $h^{*,*}(BO(n); \mathbb{Z}/2)$ seems very complicated. Consider the case $X = BO(3)$. The cohomology operations act by

$$\begin{aligned} w_2 &\xrightarrow{Sq^1} w_1w_2 + w_3 \xrightarrow{Sq^2} w_2w_1^3 + w_1^2w_3 + w_1w_2^2 + w_2w_3 \xrightarrow{Sq^1} w_1^2w_2^2 + w_3^2, \\ w_3 &\xrightarrow{Sq^1} w_3w_1 \xrightarrow{Sq^2} w_1w_2w_3. \end{aligned}$$

Theorem 4.6. *There is the isomorphism*

$$h^{*,*}(BO(3); \mathbb{Z}/2) \cong \mathbb{Z}/2[c_1, c_2, c_3] \{1, w_1, w_2, Q_0w_2, Q_1w_2, w_3, Q_0w_3, Q_1w_3\} \otimes \mathbb{Z}/2[\tau].$$

where $Q_0w_2 = \tau^{-1}(w_1w_2 + w_3), \dots, Q_1w_3 = \tau^{-2}w_1w_2w_3$.

W. S. Wilson ([RWY], [KY]) found a good $Q(i) = \Lambda(Q_0, \dots, Q_i)$ -module decomposition for $X = BO(n)$, namely,

$$(4.2) \quad H^*(X; \mathbb{Z}/2) = \bigoplus_{i=-1}^{\infty} Q(i)G_i \quad \text{with} \quad Q_0 \dots Q_i G_i \in t_{\mathbb{C}}(CH^*(X)).$$

Here G_{k-1} is quite complicated; namely, it is generated by symmetric functions

$$\Sigma x_1^{2i_1+1} \dots x_k^{2i_k+1} x_{k+1}^{2j_1} \dots x_{k+q}^{2j_q}, \quad k + q \leq n,$$

with $0 \leq i_1 \leq \dots \leq i_k$ and $0 \leq j_1 \leq \dots \leq j_q$; and if the number of j equal to j_u is odd, then there is some $s \leq k$ such that $2i_s + 2^s < 2j_u < 2i_s + 2^{s+1}$.

Then $w(G_i) \geq i + 1$ in $h^{*,*}(X; \mathbb{Z}/p)$, and so we have

Proposition 4.7. *Letting $w(G_i) = i + 1$, we have the monomorphism*

$$h^{*,*}(BO(n); \mathbb{Z}/2) \subset \left(\bigoplus_i Q(i)G_i \right) \otimes \mathbb{Z}/2[\tau].$$

One interesting problem is whether the above injection is really an isomorphism. The similar decomposition holds for $X = (B\mathbb{Z}/p)^n$, and the above injection is an isomorphism. (See Lemma 5.6 below.) The case $X = BO(3)$ is also an isomorphism. Since the direct decomposition of $BO(3) \cong BSO(3) \times B\mathbb{Z}/2$ is complicated, we only write here that of $SO(3)$:

$$(4.3) \quad \begin{aligned} H^*(BSO(3); \mathbb{Z}/2) &\cong \mathbb{Z}/2[w_2, w_3] \cong \mathbb{Z}/2[c_2, c_3]\{1, w_2, w_3 = Q_0w_2, w_2w_3 = Q_1w_2\} \\ &\cong \mathbb{Z}/2[c_2, c_3]\{w_2, Q_0w_2, Q_1w_2, c_3 = Q_0Q_1w_2\} \oplus \mathbb{Z}/2[c_2] \\ &\cong \mathbb{Z}/2[c_2, c_3]Q(1)\{w_2\} \oplus \mathbb{Z}/2[c_2]. \end{aligned}$$

Since there is the isomorphism $O(2n + 1) \cong SO(2n + 1) \times \mathbb{Z}/2$, the cohomology of $BSO(2n + 1)$ is reduced from that of $BO(2n + 1)$. However, the situation for $BO(2n)$ is different. In the next section, we will study $BSO(4)$ for details.

The extraspecial 2-group 2_+^{1+2n} is the n -th central product of the dihedral group D_8 of order 8. It has a central extension

$$(4.4) \quad 0 \rightarrow \mathbb{Z}/2 \rightarrow G \rightarrow V = \bigoplus_{i=1}^{2n} \mathbb{Z}/2 \rightarrow 0.$$

Let $H^*(BV; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, \dots, x_{2n}]$. Then Quillen proved [Q]

$$(4.5) \quad H^*(BG; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, \dots, x_{2n}]/(f, Q_0f, \dots, Q_{n-2}f) \otimes \mathbb{Z}/2[w_{2n}].$$

Here w_{2n} is the Stiefel-Whitney class of the real 2^n -dimensional irreducible representation which restricts nonzero on the center, and $f = \sum_i x_{2i-1}x_{2i} \in H^2(BV; \mathbb{Z}/2)$ represents the central extension (4.4).

Letting $y_i = x_i^2$ in $H^*(BG; \mathbb{Z}/2)$, we can write $f^2 = \sum y_{2i-1}y_{2i}$ and

$$\begin{aligned} (Q_{k-1}f)^2 &= Q_0Q_kf = \sum y_{2i-1}^{2^k}y_{2i} - y_{2i-1}y_{2i}^{2^k}, \\ Q_{k-1}f &= \sum y_{2i-1}^{2^{k-1}}x_{2i} - x_{2i-1}y_{2i}^{2^{k-1}}. \end{aligned}$$

Now we consider the motivic cohomology $H^{*,*}(BG; \mathbb{Z}/2)$ and change $y_i = \tau^{-1}x_i^2$. Since $f = 0 \in H^{2,2}(BG; \mathbb{Z}/2)$, we can see that $Q_{k-1}f = 0$ and $Q_kQ_0(f) = 0$ also in $H^{*,*}(BG; \mathbb{Z}/2)$. However, for general n , $\sum y_{2i}y_{2i-1} \neq 0$ in $H^{*,*}(BG; \mathbb{Z}/2)$. Let

$$(4.6) \quad \begin{aligned} A &= (\mathbb{Z}/2[y_1, \dots, y_{2n}, c_{2^n}]/(Q_0Q_kf, \dots, Q_0Q_nf) \\ &\quad \otimes \Delta(x_1, \dots, x_{2n}, w_{2^n})/(f, Q_0f, \dots, Q_{n-2}f)) \otimes \mathbb{Z}/2[\tau]. \end{aligned}$$

Lemma 4.8. For $G = 2_+^{1+2n}$, there is a map $A \rightarrow H^{*,*}(BG; \mathbb{Z}/2)$ which induces the injection $A/(f^2) \subset h^{*,*}(BG; \mathbb{Z}/2)$.

When $m = 0, 1, -1 \pmod 8$ and $m > 0$, we say that $Spin(m)$ is real type [Q]. When $Spin(m)$ is real type, from Quillen, we know that $H^*(BSpin(m); \mathbb{Z}/2) \subset H^*(BG; \mathbb{Z}/2)$, where $G = 2_+^{2h+1}$ and h is the Hurwitz number (for details see [Q]).

Corollary 4.9. Let $G = Spin(m)$ be real type with Hurwitz number h , and let

$$\begin{aligned} A &= (\mathbb{Z}/2[c_2, c_3, \dots, c_m, c_{2^h}]/((Q_1Q_0w_2), \dots, (Q_hQ_0w_2)) \\ &\quad \otimes \Delta(w_2, \dots, w_m, w_{2^h})/(w_2, Q_0w_2, \dots, Q_{h-2}w_2)) \otimes \mathbb{Z}/2[\tau], \end{aligned}$$

where $w_i, i \leq m$ (resp. w_{2^h}) is the Stiefel-Whitney class of the usual $SO(m)$ representation (resp. of the irreducible 2^h -dimensional spin representation). Then we have a map $A \rightarrow H^{*,*}(BG; \mathbb{Z}/2)$ which induces the injection

$$A/(c_2) \subset h^{*,*}(BG; \mathbb{Z}/2).$$

We study $Spin(7)$ and the exceptional Lie group G_2 . The cohomology of G_2 is given by $H^*(BG_2; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_4, w_6, w_7]$, where w_i is the Stiefel-Whitney class of the inclusion $G_2 \subset SO(7)$. The cohomology $H^*(BSpin(7); \mathbb{Z}/2) \cong H^*(BG_2; \mathbb{Z}/2) \otimes \mathbb{Z}/2[w_8]$.

Corollary 4.10. *Let $A = \mathbb{Z}/2[c_2, c_4, c_6, c_7] \otimes \Delta(w_4, w_6, w_7) \otimes \mathbb{Z}/2[\tau]$. Then there is the map $A \rightarrow H^{*,*}(BG_2; \mathbb{Z}/2)$ which induces the injection $A/(c_2) \subset h^{*,*}(BG_2; \mathbb{Z}/2)$.*

Remark. Similar facts hold for $BSpin(7)$ tensoring $\mathbb{Z}/2[c_8]$.

The cohomology operations are given by

$$w_4 \xrightarrow{Sq^2} w_6 \xrightarrow{Sq^1} w_7 \xrightarrow{Sq^4} w_4w_7 \xrightarrow{Sq^2} w_7w_6 \xrightarrow{Sq^1} w_7^2, \\ Q_1Q_0(w_4w_6) = w_7^2, \quad Q_2Q_1Q_0(w_4w_6w_7) = w_7^4.$$

Proposition 4.11. *Let $w(w_4) = 2, w(w_{(4,6)}) = 2$ and $w(w_{(4,6,7)}) = 3$ with $t_{\mathbb{C}}(w_{(i_1, \dots, i_n)}) = w_{i_1} \dots w_{i_n}$. Then $h^{*,*}(BG_2; \mathbb{Z}/2)$ is a subalgebra of*

$$\mathbb{Z}/p[\tau] \otimes \mathbb{Z}/2[c_4, c_6, c_7] \otimes \mathbb{Z}/2\{1, w_4, Sq^2w_4, Q_1w_4, Q_2w_4, Sq^2Q_2w_4, w_{(4,6)}, w_{(4,6,7)}\}.$$

Remark. If $t_{\mathbb{C}}^{4,3} \otimes \mathbb{Q}$ is epic, then we can take $w_4 \in h^{4,3}(BG; \mathbb{Z}/2)$, i.e., $w(w_4) = 2$.

The kernel $\text{Ker}(t_{\mathbb{C}})^{2*,*}$ is not so big for $X = BG_2$. Indeed, it is known [Y3] that

$$CH^*(BG_2)/2 \cong \mathbb{Z}/2[c_2, c_4, c_6, c_7]/(rc_2^2, c_2c_7), \quad \text{where } r = 0 \text{ or } 1.$$

The cohomology operations are given in $H^*(BSO(7); \mathbb{Z}/2)$ by

$$Q_1Q_0w_2 = w_3^2, \quad Q_2Q_0w_2 = w_5^2, \quad Q_3Q_0w_2 = w_7^2w_2^2 + w_6^2w_3^2 + w_5^2w_4^2.$$

Hence we have $c_3 = 0, c_5 = 0$ and $c_2c_7 = 0$ in $CH^*(BG_2)/2$, but $c_2 \neq 0$.

From here we consider the case $p = \text{odd}$. One of the easiest examples is the case $G = PGL_3$ and $p = 3$. The mod 3 cohomology is given by ([KY], [Ve1])

$$(\mathbb{Z}/3[y_2]\{y^2\} \oplus \mathbb{Z}/3\{1, y_2, y_3, y_7\}[y_8]) \otimes \mathbb{Z}/3[y_{12}]$$

It is known that y_2^2, y_2^3, y_8^2 and y_{12} are represented by Chern classes. Moreover, $Q_1Q_0(y_2) = y_8$. Hence these elements are in the Chow ring; namely,

$$h^{2*,*}(BPGL_3; \mathbb{Z}/3) \cong (\mathbb{Z}/3[y_2]\{y_2^2\} \oplus \mathbb{Z}/3[y_8]) \otimes \mathbb{Z}/3[y_{12}].$$

The cohomology operations are given by

$$(4.7) \quad y_2 \xrightarrow{\beta} y_3 \xrightarrow{P^1} y_7 \xrightarrow{\beta} y_8.$$

Thus we get $h^{*,*}(PGL_3; \mathbb{Z}/3)$ completely.

Theorem 4.12. *Letting $w(y_2) = 2$, we have the isomorphism*

$$h^{*,*}(BPGL_3; \mathbb{Z}/3) \cong (\mathbb{Z}/3[y_2]\{y^2\} \oplus \mathbb{Z}/3\{1\} \oplus \mathbb{Z}/3[y_8] \otimes Q(1)\{y_2\}) \otimes \mathbb{Z}/3[y_{12}, \tau].$$

Next consider the extraspecial p -group $G = p_+^{1+2n}$. When $n > 2$, even the cohomology rings $H^*(BG; \mathbb{Z}/p)$ are unknown, while it contains the subring [TeY1]

$$(4.8) \quad R = \mathbb{Z}/p[y_1, \dots, y_{2n}, c_{p^n}]/(Q_1Q_0f, \dots, Q_nQ_0f),$$

where $f = \sum^n x_{2i-1}x_{2i}$ for $\beta x_i = y_i$ and $Q_kQ_0f = \sum y_{2i-1}y_{2i}^{p^k} - y_{2i-1}^p y_{2i}$. Since $f = 0 \in H^{2,2}(BG; \mathbb{Z}/p)$, we have

Proposition 4.13. *There is the injection*

$$R \otimes \mathbb{Z}/p[\tau] \hookrightarrow H^{*,*}(Bp_+^{1+2n}; \mathbb{Z}/p).$$

We consider here other arguments for a different but similar group. Let \tilde{p}_+^{1+2n} be the central product of p_+^{1+2n} and the circle, i.e. $\tilde{p}_+^{1+2n} = p_+^{1+2n} \times_C S^1$, identifying $C \cong \mathbb{Z}/p \subset S^1$, where C is the center. Let us write

$$(4.9) \quad e_A = \prod_{0 \neq (\lambda_1, \lambda_3, \dots, \lambda_{2n-1})} (\lambda_1 y_1 + \dots + \lambda_{2n-1} y_{2n-1}).$$

If we localize by inverting e_A , then the cohomology of \tilde{p}_+^{1+2n} is expressed easily [Y2] as

$$(4.10) \quad [e_A^{-1}]H^*(B\tilde{p}_+^{1+2n}; \mathbb{Z}/p) \cong [e_A^{-1}]R \otimes \Lambda(x_1, x_3, \dots, x_{2n-1}), \quad \beta(x_i) = y_i.$$

Theorem 4.14. *Letting $w(x_i) = 1$, we have the ring isomorphism*

$$[e_A^{-1}]h^{*,*}(B\tilde{p}_+^{1+2n}; \mathbb{Z}/p) \cong [e_A^{-1}]R \otimes \mathbb{Z}/p[\tau] \otimes \Lambda(x_1, x_3, \dots, x_{2n-1}).$$

Proof. There is the splitting abelian subgroup $(\mathbb{Z}/p)^n \cong A \subset \tilde{p}_+^{1+2n}$ such that

$$h^{*,*}(BA; \mathbb{Z}/p) \cong \mathbb{Z}/p[\tau, y_1, y_3, \dots, y_{2n-1}] \otimes \Lambda(x_1, x_3, \dots, x_{2n-1}).$$

Each monomial $x_{i_1} \dots x_{i_s}$, $1 \leq i_1, \dots, i_s \leq 2n - 1$, is a $\mathbb{Z}/p[\tau]$ -module generator in the above cohomology, hence also in the cohomology of $B\tilde{p}_+^{1+2n}$. \square

We consider the case $n = 1$ here. Let us write $E = p_+^{1+2}$ for each odd prime p . The ordinary cohomology is known by Lewis [Ly], [TeY2]; namely,

$$\begin{aligned} H^{even}(BE)/p &\cong (\mathbb{Z}/p[y_1, y_2]/(y_1^p y_2 - y_1 y_2^p) \oplus \mathbb{Z}/p\{c_2, \dots, c_{p-1}\}) \otimes \mathbb{Z}/p[c_p], \\ H^{odd}(BE) &\cong \mathbb{Z}/p[y_1, y_2, c_p]\{a_1, a_2\}/(y_1 a_2 - y_2 a_1, y_1^p a_2 - y_2^p a_1), \quad |a_i| = 3. \end{aligned}$$

It is also known that $Q_1(a_i) = y_i c_p$ and $order(c_p) = p^2$.

The group 2_+^{1+2} is the dihedral group D_8 of order 8. The integral cohomologies are

$$H^{even}(BD_8)/2 \cong \mathbb{Z}/2[y_1, y_2, c_2]/(y_1 y_2), \quad H^{odd}(BD_8) \cong H^{even}(BD_8)/2\{e\}$$

where $c_2 = w_2^2$, $e = (x_1 + x_2)w_2$ in $H^*(BD_8; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, x_2, w_2]/(x_1 x_2)$ and $Q_1 e = (y_1 + y_2)c_2$, $order(c_2) = 4$.

Theorem 4.15. *For all primes p , we have the isomorphisms*

$$h^{*,*}(Bp_+^{1+2}; \mathbb{Z}/p) \cong (\{1, \partial_p^{-1}\}(H^*(Bp_+^{1+2})/p) - \{\partial_p^{-1}1\}) \otimes \mathbb{Z}/p[\tau],$$

where $w(H^{even}(Bp_+^{1+2})/p) = 0, w(H^{odd}(Bp_+^{1+2})) = 1$ and $w(\partial_p^{-1}(x)) = w(x) + 1$.

Proof. We will prove this only for odd primes, since the proof for $p = 2$ is similar. Since all elements in $H^{even}(BE)$ are generated by Chern classes, we have the isomorphism $h^{2*,*}(BG; \mathbb{Z}/p) \cong H^{even}(BE)/p$. We know $H^{odd}(BE; \mathbb{Z}/p)$ is generated as an $H^{even}(BE)/p$ -module by two elements a_1, a_2 such that $Q_1 a_i = y_i c_p$ [TeY2].

The mod p cohomology is written additively, $H^*(BE; \mathbb{Z}/p) \cong \{1, \partial_p^{-1}\}H^*(BE)/p$. Here ∂_p is the (higher) Bockstein operator. All elements in $H^{odd}(BE)$ are just p -torsion, and we can take $a'_i \in H^2(BE; \mathbb{Z}/p)$ such that $\beta(a'_i) = a_i$. Thus we take $a'_i \in H^{2,2}(BE; \mathbb{Z}/p)$ so that $a_i \in H^{3,2}(BE; \mathbb{Z}/p)$.

Next consider elements $x = \partial_p^{-1}(y)$, $y \in H^{even}(BE)/p$. If $y \in (\text{Ideal}(y_1, y_2))$, then $\partial_p^{-1}(y) = \sum x_i b_i$ for $b_i \in H^{even}(BE)/p$, and hence we can take $w(\partial_p^{-1}(y)) = 1$. For other elements $y = c_i c_p^n$, $2 \leq i \leq p - 1$, it is known [Ly] that $c_i = \text{Cor}_M^E(u^i)$

with $0 \neq u \in H^2(B\mathbb{Z}/p; \mathbb{Z}/p)$ for a maximal abelian subgroup $M \cong \mathbb{Z}/p \times \mathbb{Z}/p$. Hence $y \in H^{2*,*}(BE; \mathbb{Z}/p)$ is also p -torsion. Considering the exact sequence

$$\rightarrow H^{2^{*-1},*}(BE; \mathbb{Z}/p^N) \rightarrow H^{2*,*}(BE) \xrightarrow{p^N} H^{2*,*}(BE) \rightarrow,$$

we get $w(\partial_p^{-1}(y)) = 1$. The element $y = c_p^n$ is p^2 -torsion in $H^*(BE; \mathbb{Z}/p)$. Note that $\text{Cor}_M^E(u^{pn}) = pc_p^n + k$ with $k \in \text{Ideal}(y_1, y_2)$. Thus $y \in H^{2*,*}(BE; \mathbb{Z}/p)$ is also p^2 -torsion. Then we can take $w(\partial_p^{-1}(y)) = 1$. This completes the proof. \square

We easily compute the following results.

Corollary 4.16. *For each prime p , there is an isomorphism*

$$h^{*,*}(Bp_+^{1+2}; \mathbb{Z}/p) \cong \mathbb{Z}/p[\tau] \otimes (\mathbb{Z}/p\{1\} \oplus Q'(0)G'_0 \oplus Q(0)G_0 \oplus Q(1)G_1),$$

where $Q'(0) = \Lambda(\beta_{p^2})$, β_{p^2} is the p^2 -torsion Bockstein operator, and if $p = 2$, then

$$\begin{cases} G'_0 \cong \mathbb{Z}/2[c_2]\{x_1w_2\}, & \beta_4(x_1w_2) = c_2, \\ G_0 \cong \mathbb{Z}/2[y_1]\{x_1\} \oplus \mathbb{Z}/2[y_2]\{x_2\} \oplus \mathbb{Z}/2[c_2]\{x_1c_2\}, \\ G_1 \cong \mathbb{Z}/2[y_1, y_2, c_2]/(y_1y_2)\{w_2\}, \end{cases}$$

and if p is an odd prime, then

$$\begin{cases} G'_0 \cong \mathbb{Z}/p[c_p]\{c'_p\}, & \beta_{p^2}(c'_p) = c_p, \\ G_0 \cong \mathbb{Z}/p[y_1, y_2]\{x_1, x_2\}/(y_2x_1 - y_1x_2, y_2^p x_1 - y_1^p x_2) \oplus \mathbb{Z}/p[c_p]\{c'_2, \dots, c'_{p-1}\}, \\ G_1 \cong \mathbb{Z}/p[y_1, y_2, c_p]\{a'_1, a'_2\}/(y_2a'_1 - y_1a'_2, y_2^p a'_1 - y_1^p a'_2), & \beta(a'_i) = a_i, \beta(c'_i) = c_i. \end{cases}$$

5. BP -THEORY AND $\text{Ker } t_{\mathbb{C}}^{2*,*}$

In this section, we always assume $k = \mathbb{C}$. Even this case it seems difficult to know $\text{Ker } t_{\mathbb{C}}$. For Chow rings $CH^*(X)$, Totaro found a good way to get nonzero elements in $\text{Ker } t_{\mathbb{C}}$. Let $MU^*(-)$ (resp. $BP^*(-)$) be the complex cobordism theory (resp. Brown-Peterson theory) with the coefficient ring $MU^* = MU^*(pt) = \mathbb{Z}[x_1, \dots]$, $|x_i| = -2i$ (resp. $BP^* = \mathbb{Z}_{(p)}[v_1, \dots]$, $|v_i| = -2(p^i - 1)$). The Thom map induces $\rho : MU^*(X(\mathbb{C})) \otimes_{MU^*} \mathbb{Z} \rightarrow H^*(X(\mathbb{C}); \mathbb{Z})$. Totaro constructed [To1] the map

$$(5.1) \quad \tilde{cl} : CH^*(X) \rightarrow MU^*(X(\mathbb{C})) \otimes_{MU^*} \mathbb{Z}$$

such that the composition $\rho\tilde{cl}$ is the usual cycle map $cl = t_{\mathbb{C}}^{2*,*}$, which is also the realization map.

In this section, hereafter, X is just a topological space, e.g., $X(\mathbb{C})$, to simplify the notation. Since $BP^*(X) \cong BP^* \otimes_{MU^*(p)} MU^*(X)_{(p)}$, the similar fact holds for BP -theory. Let $P(n)^* = BP^*/(p, v_1, \dots, v_{n-1})$, e.g., $P(0)^* = BP^*$, $P(1)^* = BP^*/p$ and $P(\infty)^* = \mathbb{Z}/p$. Then there are cohomology theories $P(n)^*(-)$ with the coefficient $P(n)^*(pt) \cong P(n)^*$, e.g., $P(0)^*(X) = BP^*(X)$, $P(1)^*(X) = BP^*(X; \mathbb{Z}/p)$ and $P(\infty)^*(X) = H^*(X; \mathbb{Z}/p)$. Hence there are maps of cohomology theories

$$\begin{aligned} cl_p : CH^*(-)/p &\rightarrow BP^*(-) \otimes_{BP^*} \mathbb{Z}/p \rightarrow \dots \rightarrow P(n)^*(-) \otimes_{P(n)^*} \mathbb{Z}/p \\ &\rightarrow P(n+1)^*(-) \otimes_{P(n+1)^*} \mathbb{Z}/p \rightarrow \dots \rightarrow H^*(-; \mathbb{Z}/p) \end{aligned}$$

such that the composition is the cycle map $cl_p = t_{\mathbb{C}}$. The Morava K -theory is defined by $K(n)^*(X) = P(n)^*(X) \otimes_{P(n)^*} K(n)^*$, where $K(n)^* = \mathbb{Z}/p[v_n, v_n^{-1}]$. In

general, $K(n)^*(X) \not\cong K(n)^* \otimes_{BP^*} BP^*(X)$. However, when $K(n)^{odd}(X) = 0$, it is known [RWY] that

$$P(n)^*(X) \cong BP^*(X) \otimes_{BP^*} P(n)^*, \quad K(n)^*(X) \cong BP^*(X) \otimes_{BP^*} K(n)^*.$$

We know that $K(n)^{odd}(BG) = 0$ for many cases, while Kríž showed $K(n)^*(BG') \neq 0$ for some fine group G' .

One useful tool for computing $BP^*(X)$ is the Atiyah-Hirzebruch spectral sequence [TeY2], [KY]

$$E_2^{*,*} = H^*(X) \otimes BP^* \implies BP^*(X).$$

It is known that $d_{2p^i-1}(x) = v_i \otimes Q_i(x) \text{ mod } (M_i)$, where M_i is the ideal of $E_{2p^i-1}^{*,*}$ generated by elements in $(p, v_1, \dots, v_{i-1})E_2^{*,*}$. Here we assume that $H^*(X)$ has no higher p -torsion and that

(5.2) All nonzero differentials are of the form

$$d_{2p^i-1}(x) = v_i \otimes Q_i(x) \text{ mod } (M_i).$$

Let us write

$$(5.3) \quad grBP^*(X) \cong E_\infty^{*,*} \cong A \oplus B$$

where A (resp. B) is a BP^* -module generated by nonzero elements in $H^*(X)/p$ (resp. $pH^*(X) \oplus E_\infty^{*,*minus}$), so that $B \subset \text{Ker}(\rho_p)$. We can write

$$A \cong \bigoplus_{n=-1}^\infty P(n+1)^* \tilde{G}_n$$

by the prime invariant ideal theorem of Landweber; if $P(n)^*/(a)$ is a $BP^*(BP)$ -module, then $a = v_n^s$ for some $s \geq 1$.

Take a nonzero element $\tilde{g}_n \in \tilde{G}_n$ for $n \geq 2$. Since \tilde{g}_n is (p, \dots, v_n) -torsion, there is $g_{(n,s)} \in E_{2p^s-1}^{*,*0}$ such that $d_{2p^s-1}(g_{(n,s)}) = v_s \otimes \tilde{g}_n$ for each $1 \leq s \leq n$. Let the BP^* -module in $E_{2p^s-1}^{*,*}$ generated by $g_{(n,s)}$ be isomorphic to a $P(s'+1)^*$ -free module for $s' < s$. Here note that if $s' \neq s-1$, then $\text{Ideal}(v_{s'+1}, \dots, v_{s-1})\{g_{(n,s)}\} \subset \text{Ker}(d_{2p^s-1})$. In any case, we can take $g_{(n,s,t)} \in H^*(X)/p$ for $t < s'$ such that $d_{2p^t-1}(g_{(n,s,t)}) = v_t \otimes g_{(n,s)}$. Continuing this argument we can take

$$\tilde{g}_n \xleftarrow{Q_{s_1}} g_{(n,s_1)} \xleftarrow{Q_{s_2}} g_{(n,s_1,s_2)} \xleftarrow{\dots} \dots \xleftarrow{Q_{s_m}} g_{(n,s_1,\dots,s_m)}$$

for some $(n > s_1 > \dots > s_m)$.

Lemma 5.1. *Let $H^*(X)_{(p)}$ have no higher p -torsion. Suppose (5.2) holds, and $A = \bigoplus_{n=-1} P(n+1)^* \tilde{G}_n$ in (5.3). Then there is the injection*

$$H^*(X; \mathbb{Z}/p) \hookrightarrow \bigoplus_n Q(n)G_n \quad \text{with } Q_0 \dots Q_n G_n = \tilde{G}_n.$$

Proof. Let H be a \mathbb{Z}/p -module generated by elements $g_{(n,s_1,\dots,s_m)}$ discussed above. Define the map $j_C : H \hookrightarrow \bigoplus Q(n)G_n$ by

$$j_C(g_{(n,s_1,\dots,s_m)}) = Q_{s_m}^{-1} \dots Q_{s_1}^{-1}(\tilde{g}_n) = Q_0 \dots \hat{Q}_{s_m} \dots \hat{Q}_{s_1} \dots Q_n(g_n), \quad Q_0 \dots Q_n g_n = \tilde{g}_n.$$

Suppose $x \in H^*(X)_{(p)} - H$. Then by the assumption (5.3), x is not a permanent cycle. Hence $d_{2p^i-1}(x) \neq 0$ for some i , and so $Q_i(x) \neq 0$. Let t be a largest number such that $Q_{i_t} \dots Q_{i_1} Q_{i_t} x = \tilde{g} \neq 0$. Since $Q_j(\tilde{g}) = 0$ for all j , we know \tilde{g} is a permanent cycle. This element $\tilde{g} \in E_\infty^{*,*0}$ generates a $P(N+1)^*$ -module for $N = \max(i_s, \dots, i_1, i)$. This means $x = (Q_i^{-1} Q_{i_1}^{-1} \dots Q_{i_s}^{-1} \tilde{g}) \in H$. \square

Let us write $Q(i, n) = \Lambda(Q_i, \dots, Q_n)$, so that $Q(0, n) = Q(n)$.

Lemma 5.2. *Let $H^*(X)_{(p)}$ have no higher p -torsion.*

(1) *If (5.2) is satisfied and, in (5.3),*

$$A = \bigoplus_{n=-1} P(n+1)^* \tilde{G}_n \quad \text{and} \quad B \cong \bigoplus_{s=0} BP^* \{p, v_1, \dots, v_s\} \tilde{K}_s,$$

then we have the isomorphisms

$$H^*(X)/p \cong (\tilde{G}_{-1} \oplus \tilde{G}_0 \oplus \bigoplus_{n=1} Q(1, n)G'_n - \bigoplus_{s=0} (Q(1, s)K'_s - \tilde{K}'_s)),$$

$$H^*(X; \mathbb{Z}/p) \cong (\bigoplus_{n=-1} Q(n)G_n - \bigoplus_{s=0} (Q(s)K_s - \tilde{K}_s))$$

with $Q_0 \dots Q_n G_n = \tilde{G}_n$, $Q_0 G_n = G'_n$ and $Q_0 \dots Q_s K_s = \tilde{K}_s$, $Q_0 K_s = K'_s$.

(2) *If $Q_0 \dots Q_n G_n \in \text{Im}(\rho)$ and the degrees of \tilde{K}_s and \tilde{G}_n are even, then the converse also holds.*

Proof. (1) Let $0 \neq x \in \tilde{K}_s$. Since x is not a permanent cycle, $d_{2p^i-1}(x) \neq 0$ and $Q_i(x) \neq 0$. Since $\{p, \dots, v_s\} \tilde{K}_s$ are permanent cycles, we know $Q_i(x) \in E_{2p^i-1}^{*,*}$ is a $P(s+1)^*$ -module, that is, $i = s+1$ by the Landweber invariant prime ideal theorem, and

$$\bigoplus Q(n)G_n \supset Q(s)K_s.$$

Since $v_i x$ generates a free BP^* -module, $x \notin \text{Im}(Q_j)$ for all j . Hence we get the injection

$$H^*(X; \mathbb{Z}/p) \hookrightarrow \bigoplus Q(n)G_n - (Q(s)K_s - \tilde{K}_s).$$

Let $x = Q_{i_1} \dots Q_{i_k} g_n$ be in the right-hand side of the above injection, and such that $0 \neq Q_i(x) \in H^*(X; \mathbb{Z}/p)$ but $x \notin H^*(X; \mathbb{Z}/p)$. If $Q_i(x)$ is not a permanent cycle, then $v_i Q_i(x)$ is permanent, so $Q_i(x)$ must be in \tilde{K}_s and hence $x \in Q(s)K_s$; this is a contradiction. Otherwise $Q_i(x) = \tilde{g}_n$ generates a $P(n)^*$ -module and $Q_i(x)$ must be $\text{Im}(Q_j)$ for all $j \leq n$. Hence $x \in H^*(X; \mathbb{Z}/p)$.

(2) By induction on i , we assume $E_{2p^i-1}^{*,*} \cong C(i) \oplus D(i)$, where

$$C(i) = P(i)^* (\bigoplus_{i \leq n} Q(i, n) Q_{i-1} \dots Q_0 G_n - \bigoplus_{i \leq s} Q(i, n) Q_{i-1} \dots Q_0 K_s) \oplus \bigoplus_{i-1 \leq s} BP^* \tilde{K}_s,$$

$$D(i) = \bigoplus_{n \leq i-1} P(n+1)^* \tilde{G}_n \oplus \bigoplus_{s \leq i-2} BP^* \{p, \dots, v_s\} \tilde{K}_s.$$

Here elements of \tilde{K}_s and $D(i)$ are even dimensional. Hence all odd dimensional elements generate free $P(i)^*$ -modules. Note that if $i > j$, then there are no non-trivial maps from $P(i)^*$ -modules to free $P(j)^*$ -modules. We also note that there is no possibility that $d_t(v_k x) = v_i y$ for $x \in \tilde{K}_s$ and $y \in E_t^{odd,*}$, $t < 2p^j - 1$. Indeed there is the map i^* of spectral sequences from that for $BP^*(X)$ to that for $P(i)^*(X)$; in the last spectral sequence $E_{2p^i-1}^{*,*} \cong P(i)^* \otimes H^*(X; \mathbb{Z}/p)$ and $i^*(v_i y) \neq 0$. Hence the next nonzero differential must be of the form $d_{2p^i-1}(x) = v_i \otimes Q_i(x)$. Therefore we have

$$E_{2p^i}^{*,*} \cong C(i+1) \oplus D(i) \oplus P(i+1) Q_i \dots Q_0 G_i \oplus BP^* \{p, \dots, v_{i-1}\} \tilde{K}_{i-1}.$$

The last term is computed from $Q_i \tilde{K}_{i-1} \neq 0$ and $\text{Ker } d_{2p^i-1} | BP^* \{ \tilde{K}_{i-1} \} = BP^* \{p, \dots, v_{i-1}\} \tilde{K}_{i-1}$, since $Q_i \tilde{K}_{i-1}$ is $P(i)^*$ -free in $E_{2p^i-1}^{*,*}$. \square

The classifying spaces of groups $BO(n), SO(4), G_2, Spin(m), m \leq 9$ for $p = 2$ and PGL_3, F_4 for $p = 3$, and $(\mathbb{Z}/p)^n$ satisfy the assumption of the above lemma. However $SO(6)$ does not satisfy the above lemma [I].

We will show that the isomorphism (1) in Lemma 5.2 approximates $h^{*,*}(X; \mathbb{Z}/p)$. Let $Ih^{*,*}(X)$ be a $\mathbb{Z}/p[\tau]$ -submodule of $h^{*,*}(X; \mathbb{Z}/p)$ generated by image from $h^{*,*}(X)/p$. The following theorem is almost immediate.

Theorem 5.3. *Suppose that (1) in Lemma 5.2 holds. Then we have the injection*

$$Ih^{*,*} \hookrightarrow ((G_{-1}/p \bigoplus_{n=1} Q(1, n)G'_n) - (\bigoplus_{s=1} Q(1, s)K'_s - \tilde{K}'_s)) \otimes \mathbb{Z}/p[\tau],$$

$$h^{*,*}(X; \mathbb{Z}/p) \hookrightarrow (\bigoplus_{n=1} Q(n)G_n - (\bigoplus_{s=1} Q(s)K_s - \tilde{K}_s)) \otimes \mathbb{Z}/p[\tau],$$

with $w(G_n) = n + 1, w(G'_n) = n$. Moreover, if some BP^* -module generator in $\text{Ideal}(p, \dots, v_1)\tilde{K}_s \subset E_{\infty}^{*,*}$ is represented by transfer of a Chern class, then $\text{Ker}(t_{\mathbb{C}}^{2*,*})$ contains a nonzero element.

The $P(m)^*(-)$ version of above facts also holds, if we consider the spectral sequence

$$E_2^{*,*} = H^*(X; \mathbb{Z}/p) \otimes P(m)^* \implies P(m)^*(X).$$

(5.3)' Let $E_{\infty}^{*,*} = A \oplus B$, where A (resp. B) is the $P(m)^*$ -module generated by generators in $E_{\infty}^{*,0}$ (resp. in $E_{\infty}^{*,\text{minus}}$).

Lemma 5.4. (1) *If (5.2) is satisfied and, in (5.3)',*

$$A \cong \bigoplus_{n=-1} P(m+n+1)^*\tilde{G}_n(m), \quad B \cong \bigoplus_{s=0} P(m)^*\{v_m, \dots, v_s\}K_s(m),$$

then we have the isomorphism

$$H^*(X; \mathbb{Z}/p) \cong (\bigoplus_{n=-1} Q(m, n+m)G_n(m)) - (\bigoplus_{s=0} Q(m, m+s)K_s(m) - \tilde{K}_s(m))$$

with $Q_m \dots Q_{m+n}G_n(m) = \tilde{G}_n(m)$ and $Q_m \dots Q_{m+s}K_s(m) = \tilde{K}_s(m)$.

(2) *If $Q_m \dots Q_{m+n}G_n(m) \in \text{Im}(\rho)$ and $|\tilde{K}_s(m)| = \text{even}$, then the converse also holds.*

The $P(m)^*$ -versions also hold for $G = (\mathbb{Z}/p)^n, BO(n), BSO(4), p_+^{1+2}$. One application for the above lemma is the following.

Corollary 5.5. *Let $H^*(X; \mathbb{Z}/p)$ (resp. $H^*(Y; \mathbb{Z}/p)$) have the decomposition of Lemma 5.2 (1) (resp. Lemma 5.4 (1) for all $m \geq 0$). Then $H^*(X \times Y; \mathbb{Z}/p)$ also has decomposition similar to that of Lemma 5.2 (1).*

Proof. We get the following isomorphism:

$$Q(n-1)G_{n-1} \otimes H^*(Y; \mathbb{Z}/p)$$

$$\cong Q(n-1)G_{n-1} \otimes (Q(n, n+k)G_k(n) - \bigoplus Q(n, n+t)K_t(n) - \tilde{K}_t(n))$$

$$\cong (Q(n+k)G_{n-1} \otimes G_k(n)) - (Q(n+t)G_{n-1} \otimes K_t(n) - Q(n-1)G_{n-1} \otimes \tilde{K}_s(n)),$$

since each Q_i is derivative. □

Lemma 5.6. *If $H^*(X; \mathbb{Z}/p) \cong \bigoplus Q(n)G_n$, then $H^*(X \times B\mathbb{Z}/p) \cong \bigoplus Q(n)G'_n$, where*

$$G'_n \cong \mathbb{Z}/p[y]/(y^{p^n})G_n \oplus \mathbb{Z}/p[y]G_{n-1}\{x\}.$$

Proof. Since we have the decomposition

$$H^*(B\mathbb{Z}/p; \mathbb{Z}/p) \cong \mathbb{Z}/p[y]/(y^{p^n}) \oplus \mathbb{Z}/p[y]Q(n, n)\{x\},$$

we get the lemma. □

When $X = (B\mathbb{Z}/p)^n$, inductively we get the decomposition $H^*((B\mathbb{Z}/p)^n; \mathbb{Z}/p) \cong \bigoplus Q(n)G_n$. Hence $B = 0$ and

$$grBP^*(X) \cong \bigoplus P(n+1)^*\tilde{G}_n, \quad H^{*,*}(X; \mathbb{Z}/p) \cong \bigoplus Q(n)G_n \otimes \mathbb{Z}/p[\tau].$$

Of course these are given in (3.9). The similar facts also hold for $X = BO(n)$. Moreover, W. S. Wilson proved [RWY] that

$$BP^*(BO(n)) \cong BP^*[c_1, \dots, c_n]/(c_1 - c_1^*, \dots, c_n - c_n^*),$$

where c_i^* is the complex conjugate of the Chern class of the usual complex representation. The cohomology $h^{*,*}(BO(n))$ is studied in (4.2).

Next consider the case $X = BSO(4)$. The mod 2 cohomology is $H^*(X; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3, w_4]$. The cohomology operation acts as

$$Q_0w_2 = w_3, \quad Q_1w_3 = w_3^2, \quad Q_1w_4 = w_4w_3, \quad Q_1Q_2w_4 = w_3^2w_4^2.$$

The integral cohomology is written as

$$H^*(X)_{(2)} \cong Z_{(2)}[w_2^2, w_4] \otimes (Z_{(2)}\{1\} \oplus \mathbb{Z}/2[w_3]\{w_3\}).$$

In the Atiyah-Hirzebruch spectral sequence, nonzero differentials are $d_{2i+1-1}(x) = v_i \otimes Q_i(x)$ for $i = 1, 2$. We can compute

$$\begin{aligned} E_\infty^{*,*} &\cong E_8^{*,*} \cong Z_{(2)}[c_2] \otimes (BP^*[c_4]\{1, 2w_4\} \oplus P(2)^*[c_3]\{c_3\} \oplus P(3)^*[c_3, c_4]\{c_3c_4\}), \\ BP^*(X) \otimes_{BP^*} Z_{(2)} &\cong Z_{(2)}[c_2, c_4] \otimes (Z_{(2)}\{1, 2w_4\} \oplus \mathbb{Z}/2[c_3]\{c_3\}). \end{aligned}$$

Hence the assumption of (1) in Lemma 5.2 is satisfied by

$$\begin{aligned} \tilde{G}'_{-1} &\cong \mathbb{Z}/2[c_2, c_4], \quad \tilde{G}'_1 = \mathbb{Z}/2[c_2, c_3]\{c_3\} \quad \tilde{G}'_2 = \mathbb{Z}/2[c_2, c_3, c_4]\{c_3c_4\}, \\ \tilde{K}'_0 &= \mathbb{Z}/2[c_2, c_4]\{2w_4\}. \end{aligned}$$

Therefore we get

Proposition 5.7. *Let $w(w_4) = 2$. Then the bidegree $\mathbb{Z}/2[\tau]$ -module $Ih^{*,*}(BSO(4))$ (resp. $h^{*,*}(BSO(4); \mathbb{Z}/2)$) is isomorphic to a bidegree $\mathbb{Z}/2[\tau]$ -submodule of*

$$\mathbb{Z}/2[\tau, c_2] \otimes (\mathbb{Z}/2[c_4]\{1\} \oplus \mathbb{Z}/2[c_3] \otimes Q(1, 1)\{w_3\} \oplus \mathbb{Z}/2[c_3, c_4] \otimes Q(1, 2)\{w_4\})$$

(resp. $\mathbb{Z}/2[\tau, c_2] \otimes (\mathbb{Z}/2[c_4]\{1\} \oplus \mathbb{Z}/2[c_3] \otimes Q(1)\{w_2\} \oplus \mathbb{Z}/2[c_3, c_4] \otimes (Q(2) - \mathbb{Z}/p)\{a\})$,

where $Q_0a = w_4$).

Remark. If $w_4 \in H^{4,3}(BSO(4))$, then $Ih^{*,*}(BSO(4))$ is isomorphic to the $\mathbb{Z}/2[\tau]$ -module in the above proposition.

Remark. For this case, we have $K_0 = \mathbb{Z}/2[c_2]\{a\}$ and $Q_0K_0 = K'_0$ in Lemma 5.2. Indeed, $Q_0a = w_4$. However, $w_4 \notin \text{Im}(Q_0)$ in $h^{*,*}(BSO(4); \mathbb{Z}/2)$, because a itself does not exist in $h^{*,*}(BSO(4); \mathbb{Z}/2)$.

We know that the element corresponding to $2w_4$ is represented by a Chern class c'_2 of some representation, and this means the Totaro's cycle map \tilde{cl} is epic. Indeed, Totaro and Pandharipande showed that this map is isomorphic, namely,

$$CH^*(BSO(4))_{(2)} \cong Z_{(2)}[c_2, c_3, c_4, c'_2]/(2c_3, c_3c'_2, c'^2_2 - 4c_4).$$

Next consider the $P(1)^*$ -version for $BSO(4)$. By using the computations of Q_iw_j [I] and the Atiyah-Hirzebruch spectral sequence, we can prove that

$$\begin{aligned} grP(1)^*(BSO(4)) &\cong P(1)^*[c_4]\{1, v_1w_2w_4\} \oplus P(2)^*\{c_3\} \\ &\oplus P(3)^*[c_3]\{c^2_3, c_3c_4\} \oplus P(3)^*[c_4]\{c_3c^2_4\} \oplus P(4)^*[c_3, c_4]\{c^2_3c^2_4\} \end{aligned}$$

We have another decomposition of $H^*(BSO(4); \mathbb{Z}/2)$.

Proposition 5.8.

$$\begin{aligned} H^*(BSO(4); \mathbb{Z}/2) &\cong \mathbb{Z}/2[c_4] \oplus Q(1, 1)\{w_3\} \oplus \mathbb{Z}/2[c_3] \otimes (Q(1, 2)\{w_2, w_4\}) \\ &\oplus \mathbb{Z}/2[c_4] \otimes (Q(1, 2)\{c_4w_4\}) \oplus \mathbb{Z}/2[c_3, c_4] \otimes (Q(1, 3)\{Q^{-1}_1w_2w_4\} - \{Q^{-1}_1w_2w_4\}). \end{aligned}$$

We consider the relation between $grBP^*(X)$ and $grP(1)^*(X)$. When $X = BSO(4)$, it is known [KY] that $K(n)^{odd}(X) = 0$, and hence

$$P(m)^*(X) \cong P(m)^* \otimes_{BP^*} BP^*(X).$$

Therefore no $P(m)^*(X)$ is v_m -torsion. Of course we have already seen that for the $grBP^*(-)$ -versions the above facts do not hold. If there is a relation $pa_0 + v_1a_1 + v_2a_2 + \dots = 0 \in BP^*(X)$, then it is known [Y1] that there is $y \in H^*(X; \mathbb{Z}/p)$ such that $Q_i(y) = \rho(a_i)$, where $\rho : BP^*(X) \rightarrow H^*(X; \mathbb{Z}/p)$ is the Thom map. In $H^*(BSO(4); \mathbb{Z}/2)$, we see that

$$Q_0(w_2w_3) = c_3, \quad Q_1(w_2w_3) = 0, \quad Q_2(w_2w_3) = c^2_3.$$

Hence we have the relation $2c_3 + v_2c^2_3 + \dots = 0 \in BP^*(BSO(2))$. This shows that c^2_3 is $P(2)^*$ -free in $grBP^*(BSO(4))$, but it is a $P(3)^*$ -free module in

$$grP(1)^*(BSO(4)) = gr(BP^*(BSO(4))/2).$$

We also see that for $x = c_3w_3w_4 + c_4w_2w_3$

$$Q_0(x) = c_3c_4, \quad Q_1(x) = Q_2(x) = 0, \quad Q_3(x) = c^2_3c^2_4.$$

This means that $2c_3c_4 + v_3c^2_3c^2_4 + \dots = 0 \in BP^*(BSO(4))$. Hence $c^2_3c^2_4$ is a $P(3)^*$ -free module in $grBP^*(BSO(4))$ but is a $P(4)^*$ -free module in $gr(BP^*(BSO(4))/2)$.

Next consider the case $X = BSO(6)$. In this case the assumption (5.3) is not satisfied. In fact, Inoue computed [I]

$$grBP^*(BSO(6)) \cong \bigoplus_{n=-1}^4 P(n+1)^*\tilde{G}_n \oplus P(2)^*/(v^2_2)\tilde{G}'_1 \oplus BP^*\{2\}\tilde{K}_0.$$

(For details, see [I].) In particular, he showed that

$$d_5(2w_6) = v^2_1w_6w_5, \quad d_{11}(v_1 \otimes w_6w_5) = v^2_2w^2_6w^2_5.$$

However, even this case we can show that

$$H^*(BSO(6); \mathbb{Z}/2) \subset \bigoplus Q(n)G_n \oplus Q(1)G'_1.$$

Moreover, R. Field [F] announced that

$$CH^*(BSO(2n)) \cong Z_{(2)}[c_2, \dots, c_{2n}, y_n]/(2c_{odd}, c_{odd}y_n, y^2_n - (-1)^n 2^{2n-2}c_{2n})$$

with $\deg(y_n) = 2n$. Hence $\text{Ideal}(y_n) \subset \text{Ker}(t_{\mathbb{C}})$. However, y_n is not represented by a Chern class of any representation for $n > 2$. We also note that $BP^*(BSO(2n))$ are not known for $n > 3$.

The cases $X = BG_2, BSpin(7)$ are quite similar to the case $X = BSO(4)$. Indeed, $CH^*(BG_2)/2$ and $h^{*,*}(BG_2; \mathbb{Z}/2)$ have been discussed in §4, and

$$grBP^*(BG_2) \cong Z_{(2)}[c_4, c_6] \otimes (BP^*\{1, 2w_4\} \oplus P(3)^*[c_7]\{c_7\}).$$

The infinite term of the spectral sequence for $BP^*(BSpin(7))$ is computed by

$$Z_{(2)}[c_4, c_6] \otimes (BP^*[c_8]\{1, 2w_4, 2w_8, 2w_4w_8, v_1w_8\} \oplus P(3)^*[c_7]\{c_7\} \oplus P(4)^*[c_7, c_8]\{c_7c_8\}).$$

Therefore we obtain

Corollary 5.9. *Let $w(w_8) = 2$. Then the cohomology $Ih^{*,*}(BSpin(7))$ (resp. $h^{*,*}(BSpin(7); \mathbb{Z}/2)$) is isomorphic to a $\mathbb{Z}/2[\tau]$ -submodule of $\mathbb{Z}/2[\tau, c_4, c_6] \otimes A$ (resp. $\mathbb{Z}/2[\tau, c_4, c_6] \otimes B$), where*

$$A = \mathbb{Z}/2[c_8] \oplus \mathbb{Z}/2[c_7] \otimes Q(1, 2)\{w_4\} \oplus \mathbb{Z}/2[c_7, c_8] \otimes (Q(1, 3) - \mathbb{Z}/p)\{b\},$$

$$B = (\mathbb{Z}/2[c_8] \oplus \mathbb{Z}/2[c_7](Q(2) - \mathbb{Z}/p)\{a\} \oplus \mathbb{Z}/2[c_7, c_8](Q(3) - Q(1) + Q_0Q_1 - Q_2)\{c\})$$

with $Q_1b = w_8, Q_0a = w_4, Q_1Q_0c = w_8, Q_2Q_0c = w_4w_8$.

The algebra $BP^*(BSpin(7)) \otimes_{BP^*} Z_{(2)}$ is isomorphic to

$$Z_{(2)}[c_4, c_6, c_8] \otimes (Z_{(2)}\{1, 2w_4, 2w_8, 2w_4w_8\} \oplus \mathbb{Z}/2\{v_1w_8\} \oplus \mathbb{Z}/2[c_7]\{c_7\}).$$

It is known that $2w_2, 2w_8, 2w_4w_8$ are represented by Chern classes but v_1w_8 is not. However, Totaro has shown that the cycle map cl is epic for this case also (see [ScY], [Y3]).

Corollary 5.10. *There is an epimorphism*

$$CH^*(BSpin(7)) \rightarrow Z_{(2)}[c_4, c_6, c'_8] \otimes (Z_{(2)}\{1, c'_2, c'_4, c'_6\} \oplus \mathbb{Z}/2\{\xi_3\} \oplus \mathbb{Z}/2[c_7]\{c_7\}),$$

where c'_i is the i -th Chern class of complexification of the spin representation Δ and ξ_3 is a 6-dimensional element which is not represented by Chern classes. Thus c'_2, c'_4, c'_6 are in $\text{Ker}(\rho_2)$ and $\xi_3 \in \text{Ker}(\rho)$.

Next we consider the case $p = \text{odd}$. The cases PGL_3 and p_+^{1+2} are easy, and $Ih^{*,*}(BG)$ are given. For example, for $E = p_+^{1+2}$

$$grBP^*(BE) \cong BP^* \otimes H^{\text{even}}(BE)/(v_1Q_1H^{\text{odd}}(BE)).$$

Finally we consider the case $G = F_4, p = 3$, whose Chow ring is still unknown. The mod 3 cohomology of F_4 is isomorphic to $H^*(BF_4; \mathbb{Z}/3) \cong C \otimes D$ [Tod] with $D = Z_{(3)}[x_{36}, x_{48}]$ and

$$C = \mathbb{Z}/3[x_4, x_8] \otimes \{1, x_{20}, x_{20}^2\} \oplus \mathbb{Z}/3[x_{26}] \otimes \Lambda(x_9) \otimes \{1, x_{20}, x_{21}, x_{25}\},$$

where two terms of C have the intersection $\{1, x_{20}\}$. Then we can prove [KY]

$$grBP^*(BF_4) \cong D \otimes (BP^*\{1, 3x_4\} \oplus BP^* \otimes E \oplus P(3)^*[x_{26}]\{x_{26}\})$$

with $E = Z_{(3)}[x_4, x_8]\{ab|a, b \in \{x_4, x_8, x_{20}\}\}$. Therefore we obtain

Corollary 5.11. *Let $w(E) = 0$ and $w(x_4) = 2$. Then $Ih^{*,*}(BF_4)$ is a $\mathbb{Z}/3[\tau]$ -submodule of*

$$D \otimes (\mathbb{Z}/3\{1\} \oplus E \oplus \mathbb{Z}/3[x_{26}] \otimes Q(1, 2)\{x_4\}) \otimes \mathbb{Z}/3[\tau].$$

The element $3x_4$ can be proved to be represented by a Chern class, and $x_{26} = Q_2Q_1x_4$. The element x_{36} is also represented by a Chern class, and $P^3x_{36} = x_{48}$. If we can prove that $E/3 \subset \text{Im}(cl_p)$ and $x \in H^{4,3}(BF_4, \mathbb{Z}/3)$, then the above module is just $Ih^{*,*}(BF_4)$ for $p = 3$.

Let G be a simply connected Lie group. Then $H^3(G; \mathbb{Z}) \cong \mathbb{Z}$ and $H^4(G; \mathbb{Z}) \cong 0$. Suppose that $H^*(G; \mathbb{Z})$ has p -torsion. Then it is known that there is an element $x' \in H^3(G; \mathbb{Z})$ such that $0 \neq Q_1x' \in H^{2p+2}(G; \mathbb{Z}/p)$. Taking the classifying space, we get an element $x \in H^4(BG; \mathbb{Z})$ such that $Q_1x \neq 0$ in $H^{2p+3}(BG; \mathbb{Z}/p)$. By Totaro [To2] it is known that $CH^*(BG) \otimes \mathbb{Q} \cong H^*(BG) \otimes \mathbb{Q}$. Hence there is an $s \geq 1$ such that $p^s x_4 \in H^4(BG)$ is in $\text{Im}(cl)$. Thus there is a nonzero element $c \in CH^2(BG)/p$ with $t_{\mathbb{C}}^{2*,*}(c) = 0$. For the groups G_2 or $Spin(7)$ for $p = 2$ and $G = F_4$ for $p = 3$, we can take $s = 1$, since px_4 is represented by the second Chern class c_2 .

Proposition 5.12. *Let $p = 2, 3$ or 5 . There is a classifying space $B\tilde{G}$ such that for all m, n with $3 \leq n + 1 < m \leq 2n$, the kernel $\text{Ker}(t_{\mathbb{C}}^{m,n})$ is nonzero.*

Proof. Let $\tilde{G} = G \times (\mathbb{Z}/p)^\infty$, where $G = G_2, p = 2, G = F_4, p = 3$ or $G = E_8, p = 5$. Recall that $(B\mathbb{Z}/p)^n$ satisfies the Künneth formula for all spaces. For $\mathbb{Z}/p[\tau]$ -module generators $x \in H^{*,*}((B\mathbb{Z}/p)^\infty; \mathbb{Z}/p)$, the elements xc are all nonzero and all in $\text{Ker } t_{\mathbb{C}}$. □

6. HOMOTOPY CATEGORY

From the category Spc , Voevodsky constructed [Vo1], [Vo2], [MoVo] the $(\mathbb{A}^1, \text{algebraic})$ homotopy category Hot and the stable homotopy category $SHot$. There are two different types of spheres in Spc :

$$(6.1) \quad S_s^1 = \mathbb{A}^1/\{0, 1\} \quad \text{and} \quad S_t^1 = \mathbb{A}^1 - \{0\}.$$

The Tate object is $T = \mathbb{A}^1/(\mathbb{A}^1 - 0) \cong \mathbb{P}^1 \cong S_t^1 \wedge S_s^1$ in Hot . The category $SHot$ is defined by T as the suspension, e.g., $E = \{E_i\}, E_i \in Spt$ is a spectrum if there is a map $T \wedge E_i \rightarrow E_{i+1}$.

Let Σ_T^∞ be the functor from Spc to T -spectra that takes X to $\{T^i \wedge X\}$. If E is a T -spectrum, then the motivic (generalized) cohomology $E^{*,*}(-)$ is defined by

$$(6.2) \quad E^{m,n}(X) = \text{Hom}_{SHot}(\Sigma_T^\infty(X), S_s^{m-n} \wedge S_t^n \wedge E),$$

$$(6.3) \quad E_{m,n}(X) = \text{Hom}_{SHot}(S_s^{m-n} \wedge S_t^n, \Sigma_T^\infty(X) \wedge E),$$

where $\text{Hom}_{SHot}(-, -)$ is the homomorphism defined on $SHot$.

The realization map $t_{\mathbb{C}}$ is originally defined as the functor $t_{\mathbb{C}} : X \rightarrow X(\mathbb{C})$ from Hot to the category of homotopy spaces. Note that this induces

$$(6.4) \quad t_{\mathbb{C}} : E^{m,n}(X) \rightarrow (t_{\mathbb{C}}E)^m(X(\mathbb{C})).$$

The spectrum for the ordinary motivic cohomology is defined as follows. Let $L(X; R)$ for $R = \mathbb{Z}$ or \mathbb{Z}/p be the presheaf sending a connected U to the free R -module generated by the set of all closed irreducible $W \subset U \times X$ such that the projection $W \rightarrow U$ is finite and surjective. The Eilenberg-MacLane spectrum is defined as

$$K(R(n), 2n) = L(\mathbb{A}^n; R)/L(\mathbb{A}^n - \{0\}; R).$$

Voevodsky proved that $K(R(n), 2n)$ is the Ω -spectrum for the suspension T , namely, $K(R(n), 2n) \cong \Omega_T K(R(n+1), 2n+2)$ in *Hot*. Define also, for $m < 2n$,

$$(6.5) \quad K(R(n), m) = \Omega_{S^1}^{2n-m}(R(n), 2n).$$

Thus the ordinary motivic cohomology is defined by

$$(6.6) \quad H^{m,n}(X; R) = \text{Hom}_{\text{Hot}}(X, K(R(n), m)).$$

Question 6.1. Let $k \subset \mathbb{C}$, and let $0 \neq \tau_n \in H^{n,n}(K(\mathbb{Z}/p(n), n); \mathbb{Z}/p)$ (resp. $\tau'_{n+1} \in H^{n+1,n}(K(\mathbb{Z}_{(p)}(n), n+1); \mathbb{Z}/p)$) be the fundamental class (representing the identity map). Then are there isomorphisms

$$h^{2*,*}(K(\mathbb{Z}/p(n), n); \mathbb{Z}/p) \cong \mathbb{Z}/p[Q_{i_{n-1}} \dots Q_{i_1} Q_0 \tau_n | 0 < i_1 < \dots < i_{n-1}],$$

$$h^{2*,*}(K(\mathbb{Z}_{(p)}(n), n+1); \mathbb{Z}/p) \cong \mathbb{Z}/p[Q_{i_{n-1}} \dots Q_{i_1} \tau'_{n+1} | 0 < i_1 < \dots < i_{n-1}]?$$

It is well known that the dual A_{p*} of the (topological) Steenrod algebra A_p^* is isomorphic to $\mathbb{Z}/p[\xi_1, \dots] \otimes \Lambda(\tau_0, \dots)$, $|\xi_i| = 2(p^i - 1)$, $|\tau_i| = 2p^i - 1$. Let $P^J \in A_p^*$ (resp. $Q^I \in A_p^*$) be the dual of $\xi_1^{j_1} \dots$ (resp. $\tau_0^{i_0} \dots$, $i_k = 0$ or 1), so that $A_p^* \cong \mathbb{Z}/p\{P^J Q^I\}$. Note that $Q^I = \pm Q_0^{i_0} \dots$. Define $m(J) = \sum_{k=1} j_k$ and $m(I) = \sum_{k=0} i_k$. Then it is also known [Ta] that

$$(6.7) \quad H^*(K(\mathbb{Z}/p, n); \mathbb{Z}/p) \cong \mathbb{Z}/p[Q^I P^J \tau_n | m(I) + 2m(J) < n + i_0].$$

On the other hand, suppose that $Q^I P^J \tau_n \in H^{m,n}(K(\mathbb{Z}/p(n), n); \mathbb{Z}/p)$ for $m \geq 2n$, i.e., $w(Q^I P^J \tau) \leq 0$. Since $w(P^J) = 0$ and $w(Q_i) = -1$, we see that

$$0 \geq w(Q^I P^J \tau_n) = n - m(I).$$

This implies $m(J) = 0, m(I) = n$ and $i_0 \neq 0$. Hence we know that $Q^I P^J \tau$ is the form of the ring generator of the polynomial in the above question.

Remark. Let us write the above as $A = \mathbb{Z}/p[Q_{i_{n-1}} \dots Q_{i_1} Q_0 \tau | 0 < i_1 < \dots < i_{n-1}]$. By Tamanoi [Ta], the image $\rho_p(K(\mathbb{Z}/p, n)) = A \subset H^*(K(\mathbb{Z}/p, n); \mathbb{Z}/p)$. Moreover, there is [RWY] the isomorphism $BP^*(K(\mathbb{Z}/p, n)) \otimes_{BP^*} \mathbb{Z}/p \cong A$.

7. ALGEBRAIC COBORDISM

Let BGL denote the infinite Grassmannian, the union of $GL_N(\infty)$ over N . The corresponding generalized cohomology theory is the algebraic K -theory. The motivic cobordism theory $MGL^{*,*}(-)$ is the generalized cohomology theory defined by the Thom spectrum $MGL = \{Th(E_n \rightarrow GL_n)\}_n$ identifying $Th(E \oplus O) \cong T \wedge Th(E)$ and $E_n \oplus O \rightarrow E_n$ for the trivial line bundle O . It is known (Hu-Kříž [HK], Vezzosi [Ve2]) that

$$(7.1) \quad MGL^{*,*}((\mathbb{P}^\infty)^n) \cong MGL^{*,*}(pt)[y_1, \dots, y_n],$$

$$(7.2) \quad MGL^{*,*}(BGL_n) \cong MGL^{*,*}(pt)[c_1, \dots, c_n],$$

where the c_i are identified with the elementary symmetric polynomials in the y_i 's. Hence the Chern classes are also defined in $MGL^{2*,*}(BG)$. The realization maps

$$t_{\mathbb{C}}^{2*,*} : MGL^{2*,*}(BG)_{(p)} \rightarrow MU^*(BG)_{(p)}$$

are epic for $G = O(n), SO(4), G_2$ for $p = 2$ and p_+^{1+2} for all primes, because the $MU^*(BG)_{(p)}$ are generated by Chern classes.

For a smooth scheme X over $k \subset \mathbb{C}$, Levine and Morel [LM1], [LM2] constructed the algebraic cobordism theory $\Omega^*(X)$ such that there are natural maps

$$(7.3) \quad \rho_H : \Omega^*(X) \rightarrow H^{2*,*}(X), \quad \rho_{MGL} : \Omega^*(X) \rightarrow MGL^{2*,*}(X)$$

with $\rho_H = \rho_{(MGL,H)}\rho_{MGL}$ for the algebraic Thom map $\rho_{(MGL,H)} : MGL^{*,*}(X) \rightarrow H^{*,*}(X)$. Moreover, they proved that

$$(7.4) \quad \rho_H \otimes_{\Omega^*} \mathbb{Z} : \Omega^*(X) \otimes_{\Omega^*} \mathbb{Z} \cong H^{2*,*}(X), \quad t_{\mathbb{C}}^{2*,*}\rho_{MGL} : \Omega^*(pt) \cong MU^{2*}(pt).$$

This implies the motivic version of the Totaro cycle map \tilde{cl} :

$$(7.5) \quad \rho_{MGL}(\rho_H \otimes_{\Omega^*} \mathbb{Z})^{-1} : CH^*(X) \rightarrow MGL^{2*,*}(X) \otimes_{MGL^{2*,*}} \mathbb{Z},$$

and moreover $t_{\mathbb{C}}^{2*,*}\rho_{MGL}(\rho_H \otimes_{\Omega^*} \mathbb{Z})^{-1}$ is the Totaro cycle map \tilde{cl} . Thus the Thom map $\rho_{(MGL,H)}^{2*,*} : MGL^{2*,*}(X) \rightarrow H^{2*,*}(X)$ is always epic.

For groups $G = (\mathbb{Z}/p)^n, O(n)$, we can easily prove that

$$(7.6) \quad \Omega^*(BG) \cong MU^*(BG).$$

Hence in these cases $MGL^{2*,*}(BG)$ contains $MG^*(BG)$ as a splitting subring.

Corollary 7.1. *Let $\tilde{cl}_p : CH^*(BG)/p \rightarrow MU^*(BG) \otimes_{MU^*} \mathbb{Z}/p$ be epic. Then $t_{\mathbb{C}}^{2*,*} : MGL^{2*,*}(X)/p \rightarrow MU^*(BG)/p$ is epic, and $\text{Im } \rho_{(MGL,h)} \subset \mathbb{Z}/p[\tau] \otimes h^{2*,*}(X; \mathbb{Z}/p)$, where $\rho_{(MGL,h)} : MGL^{*,*}(X) \rightarrow h^{*,*}(X; \mathbb{Z}/p)$ is the induced Thom map.*

The modified cycle maps \tilde{cl} are epic also for the groups $Spin(7)$ for $p = 2$ and PGL_3 for $p = 3$.

By the Thom isomorphism, we get $MGL^{*,*}(BGL) \cong MGL^{*,*}(MGL)$. This means that the Steenrod algebra of $MGL^{*,*}(-)$ is generated as an $MGL^{*,*}(pt)$ -module by the Landweber-Novikov operation S_{α} :

$$(7.7) \quad MGL^{*,*}(MGL) \cong MGL^{*,*}(pt)\{S_{\alpha} | \alpha = (i_1, \dots, i_n), i_j \geq 0\}.$$

Here $S_{\alpha} : MGL^{*,*}(X) \rightarrow MGL^{*+2|\alpha|, *+|\alpha|}(X)$ and $|\alpha| = \sum_k i_k k$. These operations satisfy the Cartan formula

$$(7.8) \quad S_{\alpha}(xy) = \sum_{\alpha=\beta+\gamma} S_{\beta}(x)S_{\gamma}(y),$$

and $S_{\alpha}|MU^*(pt)$ is the usual Landweber-Novikov operation.

Kříž, Hu and Vezzosi construct algebraic Brown-Peterson theory $ABP^{*,*}(-)$ by using a modified Quillen argument. Here we note that we can also construct algebraic BP-theory by using the technique of Novikov(5.4 in [N]). Recall that $MU_{(p)}^* \cong \mathbb{Z}_{(p)}[x_1, \dots], |x_i| = -2i$. Define

$$(7.9) \quad \Delta_{x_i} = \sum_{q \geq 1} (x_i/S_{\Delta_i}(x_i))^{q-1} S_{q\Delta_i},$$

where $\Delta_i = (0, \dots, 0, 1, 0, \dots, 0)$ (1 in i -th place). Note that $\Delta_{x_i}(x_i) = S_{\Delta_i}(x_i) = 1$ if $i \neq p^j - 1$. Then we can easily prove that $\pi_i = 1 - x_i \Delta_{x_i}$ is a multiplicative projection such that $\pi_i(x_j) = (1 - \delta_{ij})x_j$. Essentially composing (for details, see p. 587 in [N]) the π_i for all $i \neq p^j - 1$, we get the multiplicative projection $\Phi : MGL_{(p)} \rightarrow MGL_{(p)}$ such that

$$(7.10) \quad \Phi(x_i) = \begin{cases} x_i & (\text{if } i = p^j - 1 \text{ for some } j), \\ 0 & (\text{otherwise}). \end{cases}$$

Define the algebraic Brown-Peterson spectrum by $\Phi MGL = ABP$. Of course $t_{\mathbb{C}}(ABP) = BP$

Theorem 7.2. *Identify $BP^* = MU_{(p)}^*/(x_i | i \neq p^j - 1)$. Then*

$$ABP^{*,*}(X) \cong BP^* \otimes_{MU_{(p)}^*} MGL^{*,*}(X)_{(p)}.$$

Proof. Since $\pi_{x_i}(a) = (1 - x_i \Delta_{x_i})a = a \bmod(x_i)$, we get $\Phi(a) = a \bmod(x_i | i \neq p^j - 1)$ for all $a \in MGL^{*,*}(X)$. The isomorphism is proved, since $ABP^{*,*}(X) \subset MGL^{*,*}(X)_{(p)}$ by the property $\Phi^2 = \Phi$. \square

Since $ABP^{*,*}(pt) \cong BP^* \otimes_{MU_{(p)}^*} MGL^{*,*}(pt)$, we can write the above isomorphism as

$$ABP^{*,*}(X) \cong ABP^{*,*} \otimes_{MGL_{(p)}^{*,*}} MGL^{*,*}(X)_{(p)}.$$

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