TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 355, Number 11, Pages 4427–4450
S 0002-9947(03)03177-5
Article electronically published on July 2, 2003

EXAMPLES FOR THE MOD p MOTIVIC COHOMOLOGY OF CLASSIFYING SPACES

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ABSTRACT. Let BG be the classifying space of a compact Lie group G. Some examples of computations of the motivic cohomology $H^{*,*}(BG;\mathbb{Z}/p)$ are given, by comparing with $H^*(BG;\mathbb{Z}/p)$, $CH^*(BG)$ and $BP^*(BG)$.

1. INTRODUCTION

Let p be a prime number and k a subfield of the complex number field $\mathbb{C}$. Let k contain a primitive p-th root of unity. Given a scheme X of finite type over k, the mod p motivic cohomology $H^{*,*}(X;\mathbb{Z}/p)=\bigoplus_{m,n}H^{m,n}(X;\mathbb{Z}/p)$ has been defined by Suslin and Voevodsky ([Vo1], [Vo2]). When X is smooth, the subring $H^{2*,*}(X;\mathbb{Z}/p)=\bigoplus_n H^{2n,n}(X;\mathbb{Z}/p)$ is identified with the classical mod p Chow ring $CH^*(X)/p$ of algebraic cyles on X.

The inclusion $t_{\mathbb{C}}: k\subset\mathbb{C}$ induces a natural transformation (realization map) $t_{\mathbb{C}}^{m,n}: H^{m,n}(X;\mathbb{Z}/p)\to H^m(X(\mathbb{C});\mathbb{Z}/p)$, where $X(\mathbb{C})$ is the complex variety of $\mathbb{C}$-valued points of X. Let us write the coimage of $t_{\mathbb{C}}^{*,*}$ as

$$h^{*,*}(X;\mathbb{Z}/p)=\bigoplus_{m,n}H^{m,n}(X;\mathbb{Z}/p)/\operatorname{Ker}(t_{\mathbb{C}}^{m,n}). \tag{1.1}$$

It is known that there is an element $\tau\in H^{0,1}(Spec(k);\mathbb{Z}/p)$ with $t_{\mathbb{C}}^{*,*}(\tau)=1$. Then we have the bigraded $\mathbb{Z}/p[\tau]$-algebra monomorphism

$$h^{*,*}(X;\mathbb{Z}/p)\hookrightarrow H^*(X(\mathbb{C});\mathbb{Z}/p)\otimes\mathbb{Z}/p[\tau,\tau^{-1}] \tag{1.2}$$

where the bidegree of $x\in H^n(X(\mathbb{C});\mathbb{Z}/p)$ is given by (n,n). If $k=\mathbb{C}$ and the Beilinson-Lichtenbaum condition [Vo2] is satisfied for p, then we also have the injection $H^*(X(\mathbb{C});\mathbb{Z}/p)\otimes\mathbb{Z}/p[\tau]\hookrightarrow h^{*,*}(X;\mathbb{Z}/p)$.

When $x\in H^{m,n}(X;\mathbb{Z}/p)$, define the weight of x by $w(x)=2n-m$. Clearly $w(x)=0$ if and only if $x\in CH^*(X)/p$. Voevodsky has extended the Steenrod algebra A_p^* of cohomology operations to the case of motivic cohomology. Among them, we have the Milnor primitive operation

$$Q_i: H^{*,*}(X;\mathbb{Z}/p)\to H^{*+2p^i-1,*+p^i-1}(X;\mathbb{Z}/p),$$

so that it is sent to the usual Milnor operation Q_i by the realization map $t_{\mathbb{C}}^*$. Hence $w(Q_i)=-1$, and the Q_i $(0\le i)$ form an exterior algebra $\Lambda(Q_0,Q_1,...)\subset A_p^*$ also for the motivic cohomology. To simplify the notation, let us write the exterior algebra $Q(n)=\Lambda(Q_0,...,Q_n)$ for $n\ge 0$ and $Q(-1)=\mathbb{Z}/p$.

Received by the editors January 10, 2002.
2000 *Mathematics Subject Classification.* Primary 55P35, 57T25; Secondary 55R35, 57T05.
Key words and phrases. Motivic cohomology, Chow ring, BP-theory.

In this paper we are mainly concerned with the following case. For $n \geq -1$, let G_n be a $\mathbb{Z}/p$-module and $Q(n)G_n$ the free $Q(n)$-module generated by G_n. Moreover, the scheme X satisfies the assumption that there is a $\mathbb{Z}/p$-module injection

$$(1.3) \quad j_{\mathbb{C}} : H^*(X(\mathbb{C});\mathbb{Z}/p) \hookrightarrow \bigoplus_{n=-1}^{\infty} Q(n)G_n \quad \text{with } j_{\mathbb{C}}^{-1}(Q_0...Q_nG_n) \subset Im(t_{\mathbb{C}}^{2*,*})$$

such that $p_nj_{\mathbb{C}} : H^*(X(\mathbb{C}) : \mathbb{Z}/p) \to Q(n)G_n$ is the $Q(n)$-module map and $p'_np_nj_{\mathbb{C}} : H^*(X(\mathbb{C}) : \mathbb{Z}/p) \to Q_0...Q_{n-1}G_n$ is a surjection for each n, where $p_n : \bigoplus Q(n)G_n \to Q(n)G_n$ and $p'_n : Q(n)G_n \to Q_0...Q_{n-1}G_n$ are the projections. (We do not assume a $Q(n)$-module structure on the right-hand side module in (1.3).)

We take the weight on the right-hand side by putting $w(x) = n+1$ for every $x \in G_n$ (simply write $w(G_n) = n+1$), so that $w(Q_0...Q_nx) = 0$. Then we get the injection of bigraded $\mathbb{Z}/p$-modules

$$(1.4) \qquad j : h^{*,*}(X;\mathbb{Z}/p) \hookrightarrow \bigoplus_{n=-1}^{\infty} Q(n)G_n \otimes \mathbb{Z}/p[\tau]$$

such that the composition $(p_n \otimes \mathbb{Z}/p[\tau])j : h^{*,*}(X;\mathbb{Z}/p) \to Q(n)G_n \otimes \mathbb{Z}/p[\tau]$ is the bigraded $Q(n)$-module map.

The above argument has its counterpart in the BP-theory of $X(\mathbb{C})$. As we know, $BP^*(-)$ is the cohomology theory with the coefficient ring $BP^* = \mathbb{Z}_{(p)}[v_1, v_2, ...]$, $|v_i| = -2(p^i - 1)$. Let us write $BP^*/(p, v_1, ..., v_{m-1})$ as $P(m)^*$. The Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*} = H^*(X(\mathbb{C})) \otimes BP^* \Longrightarrow BP^*(X(\mathbb{C}))$$

has the differential

$$(1.5) \qquad d_{2p^i-1}(x) = Q_i(x) \otimes v_i \qquad \mathrm{mod}(M_i),$$

where M_i is the ideal of $E_{2p^i-1}^{*,*}$ generated by elements in $(p, v_1, ..., v_{i-1})E_2^{*,*}$ We assume here that nonzero differentials are all of the form (1.5) and that $H^*(X(\mathbb{C}))$ has no higher p-torsion. Then we easily see that (1.3) implies

$$(1.6) \qquad E_\infty^{*,*} \cong \bigoplus_{n=-1}^{\infty} P(n+1)^*\tilde{G}_n \ \oplus \ B \qquad \text{with} \quad \tilde{G} = Q_0...Q_nG_n,$$

where $P(n+1)^*\tilde{G}_n$ is the free $P(n+1)^*$-module generated by elements in $\tilde{G}_n$ and B is the BP^*-submodule of $E_\infty^{*,*}$ of generators in Ideal$(p, v_1, ...)E_2^{*,*}$. Conversely, by the same assumption, if $\tilde{G}_n \subset \mathrm{Im}(t_{\mathbb{C}}^{2*,*})$, then the isomorphism (1.6) implies the existence of the injections $j_{\mathbb{C}}$ in (1.3) and so j in (1.4).

Let $\rho : BP(X(\mathbb{C})) \otimes_{BP^*} \mathbb{Z}/p \to H^*(X(\mathbb{C});\mathbb{Z}/p)$ be the Thom map. Then (1.6) and $\tilde{G}_n \subset \mathrm{Im}(t_{\mathbb{C}}^{2*,*})$ imply that

$$\mathrm{Im}(t_{\mathbb{C}}^{2*,*}) = \mathrm{Im}(\rho) \cong \bigoplus_{n=-1}^{\infty} \tilde{G}_n \subset BP^*(X(\mathbb{C})) \otimes_{BP^*} \mathbb{Z}/p.$$

More generally, B. Totaro [To1], [To2] constructed the modified cycle map

$$(1.7) \qquad \tilde{cl}^* : CH^*(X)/p \to BP^*(X(\mathbb{C})) \otimes_{BP^*} \mathbb{Z}/p$$

in such a way that the composition $\rho\tilde{cl}^*$ is the realization map $t_{\mathbb{C}}^{2*,*}$. If a BP^*-module generator of B in (1.6) is represented by transfer of a Chern class, then this element gives a nonzero element in $\mathrm{Ker}(t_{\mathbb{C}}^{2*,*})$ by the modified cycle map $\tilde{cl}^*$. Using this argument, Totaro found nonzero elements in $\mathrm{Ker}(t_{\mathbb{C}}^{2*,*})$ when X is the classifying space $BSO(4)$.

The motivic cohomology of the classifying space is defined as follows. Let G be a linear algebraic group over k. Let V be a representation of G such that G acts freely on $V-S$ for some closed subset S. Then $(V-S)/G$ exists as a quasi-projective variety over k. Following Totaro [To1] and Voevodsky, define

$$(1.8)\qquad H^{*,*}(BG;\mathbb{Z}/p)=\lim_{dim(V),codim(S)\to\infty} H^{*,*}((V-S)/G;\mathbb{Z}/p).$$

The topological space $BG(\mathbb{C})=\lim((V-S)/G)(\mathbb{C})$ is the usual classifying space BG. Hence we write the $\mathbb{C}$-value points $BG(\mathbb{C})$ simply as BG.

We will show that the isomorphism (1.6) is satisfied when $X=BG$ for the following cases: $O(n), SO(4), D_8, G_2, Spin(7)$ for $p=2$, PGL_3, F_4 for $p=3$ and the extraspecial p-group p_+^{1+2} of order p^3 and of exponent p for odd primes. (However note that $H^*(Bp_+^{1+2})$ has p^2-torsion.)

Hence we will prove (1.4) for these BG. Moreover, when $k=\mathbb{C}$ and $G=O(3)$ for $p=2$, PGL_3 for $p=3$, p_+^{1+2} and $(\mathbb{Z}/p)^n$ for all primes, we will show that

$$(1.9)\qquad h^{*,*}(BG;\mathbb{Z}/p)\cong \bigoplus Q(n)G_n\otimes\mathbb{Z}/p[\tau].$$

S. Wilson [RWY] first constructed the decomposition (1.3) so that $j_{\mathbb{C}}$ is an isomorphism for $X=BO(n)$, and next computed $BP^*(BO(n))$. However, it is unknown whether j in (1.4) is an isomorphism or not for $X=BO(n)$, $n\geq 4$.

The contents of this paper are as follows. The aim of §§2 and 3 is a short introduction to motivic cohomology for algebraic topologists unfamiliar with it. In these sections, we concentrate on the computation of $H^*(B(\mathbb{Z}/p)^n;\mathbb{Z}/p)$. In §4, we deal with the study of $h^{*,*}(X;\mathbb{Z}/p)$, making no use of $BP^*(BG)$ but Milnor's operation Q_i instead. In §5, we give an account of $h^{*,*}(BG;\mathbb{Z}/p)$ expressed in term of $BP^*(BG)$. Also in this section we give some results on $\mathrm{Ker}(t_{\mathbb{C}}^{*,*})$. The motivic cohomology of the Eilenberg-MacLane space $K(\mathbb{Z}/p(n),n)$ is studied in §6. In §7, we give some comments on algebraic cobordism theory and algebraic BP-theory.

Finally, we want to express our deep gratitude to Professors Burt Totaro, Tokushi Nakamura, and Toshio Kawashima for their kind help in correcting many errors in the first draft of this paper.

2. Chow ring, Milnor K-theory, étale cohomology

We use the category Spc of (algebraic) spaces, along with schemes A, their quotients A_1/A_2 and $\mathrm{colim}(A_\alpha)$, all defined by Voevodsky [Vo2], [MoVo]. Here schemes are defined over a field k with $ch(k)=0$. The motivic cohomology is the double indexed cohomology defined by Suslin and Voevodsky, directly related with the Chow ring and Milnor K-theory.

(CH) For a smooth scheme X we have $H^{2n,n}(X)\cong CH^n(X)$, the classical Chow group of codim n cycles on X.

(MK) $H^{n,n}(Spec(k))\cong K_n^M(k)$, the Milnor K-group for the field k.

For a smooth variety X with $dim(X)=n$, the Chow ring is the sum $CH^*(X)=\bigoplus_i CH^i(X)$, where

$$CH^i(X)=\{(n-i)\text{ cycles in }X\}/(\text{ rational equivalence}).$$

Here the rational equivalence $a \equiv b$ is defined if there is a codimension i subvariety W in $X \times \mathbb{P}^1$ such that $a = p_* f^*(0)$ and $b = p_* f^*(1)$, where $\mathbb{P}^1$ is the projective line and p (resp. f) is the projection on the first (resp. second) factor.

For $k = \mathbb{C}$, if X has a cellular decomposition, i.e., $X = X_n \supset X_{n-1} \supset ... \supset X_0$ with $X_i - X_{i-1} = \bigcup \mathbb{A}^{n_{ij}}$, where $\mathbb{A}^{n_{ij}}$ is the affine space of dimension n_{ij}, then $CH^*(X) \cong H^*(X(\mathbb{C}))$, the singular cohomology theory of $\mathbb{C}$-rational points of X. For example, $CH^*(\mathbb{P}^n) \cong H^*(\mathbb{CP}^n)$ for projective spaces $\mathbb{P}^n$. Since Spc contains *colimit*, we can consider the infinite projective space $\mathbb{P}^\infty = B\mathbb{G}_m$ and the infinite lens space $\lim_n(\mathbb{A}^n - \{0\}/\mathbb{Z}/p) = L_p^\infty = B\mathbb{Z}/p$. The Chow rings of classifying spaces of abelian groups are given in [To1]:

(2.1)

$$CH^*(\mathbb{P}^\infty) \cong H^{2*,*}(\mathbb{P}^\infty) \cong \mathbb{Z}[y], \quad CH^*(B\mathbb{Z}/p) \cong H^{2*,*}(B\mathbb{Z}/p) \cong \mathbb{Z}[y]/(py),$$

with $deg(y) = (2,1)$. For products of these spaces we have

$$CH^*((\mathbb{P}^\infty)^n) \cong \mathbb{Z}[y_1, ..., y_n], \tag{2.2}$$

$$CH^*((B\mathbb{Z}/p)^n) \cong \mathbb{Z}[y_1, ..., y_n]/(py_1, ..., py_n). \tag{2.3}$$

Here note that $CH^*(X) \not\cong H^{even}(X(\mathbb{C}))$ for the last case. Even if $H^*(X(\mathbb{C}))$ is generated by even dimensional elements, there are cases that $CH^*(X) \not\cong H^*(X(\mathbb{C}))$, e.g., K3-surfaces have the cohomology $H^2(X(\mathbb{C})) \cong \mathbb{Z}^{22}$, but there is a K3-surface such that $CH^1(X) \cong \mathbb{Z}^i$ for each $1 \le i \le 20$.

Milnor K-theory is the graded ring $\bigoplus_n K_n^M(k)$ defined by $K_n^M(k) = (k^*)^{\otimes n}/J$, where the ideal J is generated by elements $a \otimes (1-a)$ for $a \in k^* - \{1\}$. Here the addition of k^* is given by the multiplication in the field k. Hence $K_0^M(k) = \mathbb{Z}$ and $K_1^M(k) = k^*$. Hilbert's Theorem 90, which essentially says that the Galois cohomology $H^1(G(k_s/k); k_s^*) = 0$, implies the isomorphism $K_1^M(k)/p \cong k^*/(k^*)^p \cong H^1(G(k_s/k); \mathbb{Z}/p)$ for $1/p \in k$. Similarly we can define a map (the norm residue map) for any extension F of k of finite type,

$$K_n^M(F)/p \to H^n(G(F_s/F); \mu_p^{\otimes n}), \tag{BK}$$

where $\mu_p^{\otimes n}$ is the discrete $G(F_s/F)$-module of n-th tensor power of the group of p-roots of 1. The Bloch-Kato conjecture is that this map is an isomorphism for all field k, and the Milnor conjecture is its $p=2$ case. This conjecture is solved when $n = 2$ by Merkurjev and Suslin [MeSu], and for $p = 2$ by Voevodsky [Vo1].

Notice that $H^n(G(k_s/k); \mu_p^{\otimes n}) \cong H_{et}^n(Spec(k), \mu_p^{\otimes n})$, the étale cohomology of the point. The étale cohomology $H_{et}^*(X; \mathbb{Z}/p)$ has the following properties:

(E.1) If k contains a primitive p-th root of 1, then there is the additive isomorphism

$$H_{et}^m(X, \mu_p^{\otimes n}) \cong H_{et}^m(X; \mathbb{Z}/p).$$

(E.2) For smooth X over $k = \mathbb{C}$,

$$H_{et}^m(X; \mathbb{Z}/p^N) \cong H^m(X(\mathbb{C}); \mathbb{Z}/p^N) \quad \text{for all } N \ge 1.$$

The last cohomology is the usual mod p ordinary cohomology of $\mathbb{C}$-rational points of X. Of course $H_{et}^*(Spec(\mathbb{C}); \mathbb{Z}/p) \cong \mathbb{Z}/p$. It is known that

$$K_*^M(\mathbb{R})/2 \cong H_{et}^*(Spec(\mathbb{R}); \mathbb{Z}/2) \cong \mathbb{Z}/2[\rho]$$

with $deg(\rho) = 1$ for the real number field $\mathbb{R}$. Let F_v be a local field with residue field k_v of $ch(k_v) \neq 2$. Then $K_*^M(F_v)/2 \cong H_{et}^*(Spec(F_v); \mathbb{Z}/2) \cong \Lambda(\alpha, \beta)$ with $deg(\alpha) = deg(\beta) = 1$. Thus we know that $\bigoplus_m H^{m,m}(pt; \mathbb{Z}/2)$ for these cases.

3. The realization map

In this section we consider the relation to the usual ordinary cohomology. Let R be $\mathbb{Z}$ or $\mathbb{Z}/p$. The motivic cohomology has the following properties [Vo2].

(C1) $H^{*,*}(X;R)$ is a bigraded ring natural in X.

(C2) When $k \subset \mathbb{C}$, there are maps (realization maps)

$$t_{\mathbb{C}}^{m,n} : H^{m,n}(X;R) \to H^m(X(\mathbb{C});R)$$

which sum up to $t_{\mathbb{C}}^{*,*} = \bigoplus_{m,n} t_{\mathbb{C}}^{m,n}$, the natural ring homomorphism.

(C3) There are the (Bockstein, reduced powers) operations

$$\beta : H^{*,*}(X;\mathbb{Z}/p) \to H^{*+1,*}(X;\mathbb{Z}/p),$$
$$P^i : H^{*,*}(X;\mathbb{Z}/p) \to H^{*+2(p-1)i,*+(p-1)i}(X;\mathbb{Z}/p),$$

which commutes with the realization map $t_{\mathbb{C}}$ when $k \subset \mathbb{C}$.

(C4) For the projective space $\mathbb{P}^n$, there is an isomorphism

$$H^{*,*}(X \times \mathbb{P}^n/\mathbb{P}^{n-1};R) \cong H^{*,*}(X;R)\{1,y'\}$$

with $deg(y') = (2n,n)$ and $t_{\mathbb{C}}(y') \neq 0$ for $k \subset \mathbb{C}$.

We recall the Lichtenbaum motivic cohomology [Vo2]. Lichtenbaum defined the similar cohomology $H_L^{*,*}(X;R)$ by using the étale topology, while $H^{*,*}(X;R)$ is defined using the Nisnevich topology. Since Nisnevich covers are restricted étale covers, there is the natural map $H^{*,*}(X;R) \to H_L^{*,*}(X;R)$. We say that the $B(n,p)$ condition holds if

$$H^{m,n}(X;Z_{(p)}) \cong H_L^{m,n}(X;Z_{(p)}) \quad \text{for all } m \leq n+1$$

and all smooth X. The Beilinson-Lichtenbaum conjecture is that $B(n,p)$ holds for all n and p. It is known that the condition $B(n,p)$ is equivalent to the Bloch-Kato conjecture (BK) for degree n and prime p. Hence $B(n,p)$ holds for $n \leq 2$ or $p = 2$. Moreover, Suslin and Voevodsky have proved

(L-E) $$H_L^{m,n}(X;\mathbb{Z}/p) \cong H_{et}^m(X;\mu_p^{\otimes n}).$$

Now we compute $H^{*,*}(pt;\mathbb{Z}/p) = H^{*,*}(Spec(k);\mathbb{Z}/p)$. For a smooth X, the following dimensional condition is known:

(C5) For a smooth X, if $H^{m,n}(X;R) \ncong 0$, then

$$m \leq n + dim(X), \qquad m \leq 2n \text{ and } m \geq 0.$$

For the rest of this paper, we assume that k contains a primitive p-th root of 1 and $B(n,p)$ holds for all n, but $X = Spec(k)$. Then

$$H^{m,n}(pt;\mathbb{Z}/p) \cong H_{et}^m(pt;\mu_p^{\otimes n}) \cong H_{et}^m(pt;\mathbb{Z}/p) \qquad \text{if } m \leq n,$$

and $H^{m,n}(pt;\mathbb{Z}/p) \cong 0$ for $m > n$. Let $\tau \in H^{0,1}(pt;\mathbb{Z}/p)$ be the element corresponding to a generator of $H_{et}^0(Spec(k);\mu_p) \cong H_{et}^0(Spec(k);\mathbb{Z}/p)$. Then we get the isomorphism

$$H^{*,*}(Spec(k);\mathbb{Z}/p) \cong H_{et}^*(Spec(k);\mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau]$$

since $\tau : H_{et}^m(pt;\mu_p^{\otimes n}) \cong H_{et}^m(pt;\mu_p^{\otimes(n+1)})$. In particular, for the real number field $\mathbb{R}$ and a local field F_v with the residue field k_v of $ch(k_v) \neq 2$ we have

(3.1) $$H^{*,*}(Spec(\mathbb{R});\mathbb{Z}/2) \cong \mathbb{Z}/2[\rho,\tau] \quad \text{with } deg(\rho) = (1,1),$$

(3.2) $$H^{*,*}(Spec(F_v);\mathbb{Z}/2) \cong \mathbb{Z}/2[\tau] \otimes \Lambda(\alpha,\beta) \quad \text{with } deg(\alpha) = deg(\beta) = (1,1).$$

For $k=\mathbb{C}$, we know that $K_n^M(\mathbb{C})/p \cong 0$ for $n>0$, and hence

$$(3.3)\qquad H^{*,*}(Spec(\mathbb{C});\mathbb{Z}/p)\cong \mathbb{Z}/p[\tau] \quad \text{with } deg(\tau)=(0,1).$$

When $k=\mathbb{C}$, if the $B(n,p)$ condition holds for X, then it is immediate that

$$(3.4)\qquad [\tau^{-1}]H^{*,*}(X;\mathbb{Z}/p)\cong H^*(X(\mathbb{C});\mathbb{Z}/p)\otimes \mathbb{Z}/p[\tau,\tau^{-1}],$$

where the degree is defined by $deg(x)=(m,m)$ if $x\in H^m(X(\mathbb{C});\mathbb{Z}/p)$.

Next we compute the cohomology of $\mathbb{P}^\infty$ and $B\mathbb{Z}/p$. For any (algebraic) map $f: X\to Y$ in the category Spc, we can construct the cofiber sequence

$$X\to Y\to cone(f)=Y/X,$$

which induces the long exact sequence (Voevodsky [Ve2])

$$(3.5)\qquad H^{*,*}(X;R)\leftarrow H^{*,*}(Y;R)\leftarrow H^{*,*}(Y/X:R)\leftarrow H^{*-1,*}(X;R).$$

In particular, we get the Mayer-Vietoris, Gysin and blow-up long exact sequences.

By the cofiber sequence $\mathbb{P}^{n-1}\to\mathbb{P}^n\to\mathbb{P}^n/\mathbb{P}^{n-1}$ and (C4), we can inductively see that

$$(3.6)\qquad H^{*,*}(\mathbb{P}^n;\mathbb{Z}/p)\cong H^{*,*}(pt;\mathbb{Z}/p)\otimes\mathbb{Z}/p[y]/(y^{n+1}) \quad \text{with } deg(y)=(2,1).$$

When $k=\mathbb{C}$, since $B(1,p)$ always holds, $H^{1,1}(L_p^n;\mathbb{Z}/p)\cong H^1(L_p^n;\mathbb{Z}/p)$. Hence there is an element $x'\in H^{1,1}(L_p^n;\mathbb{Z}/p)$ with $t_{\mathbb{C}}(x')=x\in H^1(L_p^n;\mathbb{Z}/p)$. This also holds for general k [Vo3]. The lens space is identified with the sphere bundle associated with the line bundle

$$(\mathbb{A}^n-\{0\})\times_{(\mathbb{A}-\{0\})}\mathbb{A}\to(\mathbb{A}^n-\{0\})/(\mathbb{A}-\{0\})=\mathbb{P}^n.$$

Here $(\mathbb{A}^n-\{0\})\times_{(\mathbb{A}-\{0\})}\mathbb{A}$ is the identification such that $(z_i,z)\sim(a^{-1}z_i,a^pz)\in(\mathbb{A}^n-\{0\})\times\mathbb{A}$ for $(z_i)\in\mathbb{A}^n,\ z\in\mathbb{A},\ a\in\mathbb{A}-\{0\}$. Hence we get the cofibering $L_p^n\to\mathbb{P}^n\xrightarrow{\times p}\mathbb{P}^n$. Thus we get the additive isomorphism $H^{*,*}(L_p^n;\mathbb{Z}/p)\cong H^{*,*}(\mathbb{P}^n;\mathbb{Z}/p)\{1,x\}$. This induces the ring isomorphism for $p=odd$

$$(3.7)\quad H^{*,*}(L_p^n;\mathbb{Z}/p)\cong\mathbb{Z}/p[y]/(y^{n+1})\otimes\Lambda(x)\otimes H^{*,*}(pt;\mathbb{Z}/p) \quad \text{with } deg(x)=(1,1).$$

However, note that when $p=2$ we get $x^2=y\tau+x\rho$ [Vo3], where $\rho\in H^{1,1}(pt;\mathbb{Z}/p)\cong k^*/k^{2*}$ represents -1. (Hence $\rho=0$ when $\sqrt{-1}\in k^*$.) This is proved by the well-known fact that $\{a,a\}=\{a,-1\}$ in the Milnor K-theory $K_2^M(k)$.

We say that a space X satisfies the Künneth formula for a space Y if

$$H^{*,*}(X\times Y;\mathbb{Z}/p)\cong H^{*,*}(X;\mathbb{Z}/p)\otimes_{H^{*,*}(pt;\mathbb{Z}/p)}H^{*,*}(Y;\mathbb{Z}/p).$$

By the above cofiber sequences, we can easily see that $\mathbb{P}^\infty$ and $B\mathbb{Z}/p$ satisfy the Künneth formula for all spaces. In particular, we have the ring isomorphisms

$$(3.8)\qquad H^{*,*}((\mathbb{P}^\infty)^n;\mathbb{Z}/p)\cong\mathbb{Z}/p[y_1,...,y_n]\otimes H^{*,*}(pt;\mathbb{Z}/p),$$

$$(3.9)\qquad H^{*,*}((B\mathbb{Z}/p)^n;\mathbb{Z}/p)\cong\mathbb{Z}/p[y_1,...,y_n]\otimes\Lambda(x_1,...,x_n)\otimes H^{*,*}(pt;\mathbb{Z}/p)$$

(when $p=2$, $x_i^2=y_i\tau+x_i\rho$).

This fact is used to define the reduced power operation P^i in (C3). Since a Sylow p-subgroup of the symmetric group S_p of p letters is isomorphic to $\mathbb{Z}/p$, we have the isomorphism

$$H^{*,*}(BS_p;\mathbb{Z}/p)\cong H^{*,*}(B\mathbb{Z}/p;\mathbb{Z}/p)^{F_p^*}\cong\mathbb{Z}/p[Y]\otimes\Lambda(W)\otimes H^{*,*}(pt;\mathbb{Z}/p),$$

identifying $Y = y^{p-1}$ and $W = xy^{p-2}$. If X is smooth (and suppose p is odd, to simplify arguments), we can define the reduced powers (of Chow rings) as follows. Consider maps

$$\begin{aligned} &H^{2*,*}(X;\mathbb{Z}/p) \xrightarrow{i_!} H^{2p*,p*}(X^p \times_{S_p} ES_p) \\ &\xrightarrow{\Delta^*} H^{*,*}(X \times BS_p;\mathbb{Z}/p) \cong H^{*,*}(X;\mathbb{Z}/p) \otimes_{H^{*,*}(pt;\mathbb{Z}/p)} H^{*,*}(BS_p;\mathbb{Z}/p), \end{aligned}$$

where $i_!$ is the Gysin map for the p-th external power, and Δ is the diagonal map. For $deg(x) = (2n, n)$, the reduced powers are defined as

$$\Delta^* i_!(x) = \sum P^i(x) \otimes Y^{n-i} + \beta P^i(x) \otimes WY^{n-i-1}. \tag{3.10}$$

Hence $deg(P^i) = deg(Y^i) = deg(y^{i(p-1)}) = (2i(p-1), i(p-1))$. Voevodsky defined $i_!$ for nonsmooth X also, and by using suspensions maps he defined reduced powers for all degree elements in $H^{*,*}(X;\mathbb{Z}/p)$ for all X [Vo3].

Moreover, we can see (Hu-Kříž [HK]) that

$$H^{*,*}(BGL_n;\mathbb{Z}/p) \cong \mathbb{Z}/p[c_1,...,c_n] \otimes H^{*,*}(pt;\mathbb{Z}/p), \tag{3.11}$$

where the Chern class c_i with $deg(c_i) = (2i, i)$ is identified with the elementary symmetric polynomial in $H^{*,*}((\mathbb{P}^\infty)^n;\mathbb{Z}/p)$. So we can define the Chern class $\rho^*(c_i) \in H^{2*,*}(BG;\mathbb{Z}/p)$ for each representation $\rho : G \to GL_n$.

4. $H^{*,*}(X;\mathbb{Z}/p)/\operatorname{Ker}(t_{\mathbb{C}})$ and the operation Q_i

In this section we assume that X is smooth and $k = \mathbb{C}$. Even in this case the motivic cohomology $H^{*,*}(X;\mathbb{Z}/p)$ seems difficult, in general. Hence we consider a bigraded ring which is computable only by using the algebraic topology of $H^*(X(\mathbb{C});\mathbb{Z}/p)$. Define a bidegree algebra by

$$h^{*,*}(X;\mathbb{Z}/p) \ = \ \bigoplus_{m,n} H^{m,n}(X;\mathbb{Z}/p)/\operatorname{Ker}(t_{\mathbb{C}}^{m,n}). \tag{4.1}$$

Since $t_{\mathbb{C}}^{*,*}(\tau) = 1$, it is almost immediate that there is the injection of bidegree $\mathbb{Z}/p[\tau]$-algebras

$$h^{*,*}(X;\mathbb{Z}/p) \hookrightarrow H^*(X(\mathbb{C});\mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau,\tau^{-1}],$$

where the bidegree of $x \in H^n(X(\mathbb{C});\mathbb{Z}/p)$ is (n,n). (This also holds when $k \subset \mathbb{C}$ and k has a primitive p-th root of 1.)

Suppose the $B(n,p)$ condition holds. By the isomorphisms (B,p), (L-E), (E1) and (E2), we have

$$H^{n,n}(X;\mathbb{Z}/p) \cong H_L^{n,n}(X;\mathbb{Z}/p) \cong H_{et}^n(X;\mu_p^{\otimes n}) \cong H_{et}^n(X;\mathbb{Z}/p) \cong H^n(X(\mathbb{C});\mathbb{Z}/p).$$

Hence we get the injection of bidegree $\mathbb{Z}/p[\tau]$-algebras

$$H^*(X(\mathbb{C});\mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau] \hookrightarrow h^{*,*}(X;\mathbb{Z}/p).$$

Thus there exist a $\mathbb{Z}/p$-basis $\{a_I\}$ of $H^*(X(\mathbb{C});\mathbb{Z}/p)$ and a $|\frac{1}{2}a_I| \geq t_I \geq 0$ such that

$$h^{*,*}(X;\mathbb{Z}/p) \cong \bigoplus_I \mathbb{Z}/p[\tau]\{\tau^{-t_I}a_I\}.$$

Remark. Let $F_i = \mathrm{Im}(\bigoplus_m t_{\mathbb{C}}^{m,i})$. When the $B(n,p)$ condition is satisfied, we have $\bigcup_i F_i = H^*(X(\mathbb{C});\mathbb{Z}/p)$. We also have the interesting bigraded ring

$$grH^*(X(\mathbb{C});\mathbb{Z}/p) = \bigoplus F_{i+1}/F_i \cong h^{*,*}(X;\mathbb{Z}/p)/(\mathrm{Im}\,\tau),$$

so that $\mathbb{Z}/p[\tau]\otimes grH^*(X(\mathbb{C});\mathbb{Z}/p)$ is additively isomorphic to $h^{*,*}(X;\mathbb{Z}/p)$, while the ring structures are different.

Here we recall the Milnor primitive operations $Q_0 = \beta$ and $Q_i = [Q_{i-1}, P^{p^{i-1}}]$:

$$Q_i : H^{*,*}(X;\mathbb{Z}/p) \to H^{*+2p^i-1,*+p^i-1}(X;\mathbb{Z}/p),$$

which is derivative, $Q_i(xy) = Q_i(x)y + xQ_i(y)$. Note also that $Q_i(\tau) = 0$, because of the dimension of $H^{*,*}(pt;\mathbb{Z}/p) \cong \mathbb{Z}/p[\tau]$.

Lemma 4.1. *If $0 \neq Q_{i_1}...Q_{i_s}x \in H^{2*,*}(X;\mathbb{Z}/p)$, then x is a $\mathbb{Z}/p[\tau]$-module generator.*

Proof. If $x = x'\tau$, then $\tau Q_{i_1}...Q_{i_s}(x') \neq 0$. But

$$Q_{i_1}...Q_{i_s}(x') = 0 \in H^{2*,*-1}(X;\mathbb{Z}/p),$$

since $H^{m,n}(X;\mathbb{Z}/p) = 0$ for $m > 2n$. $\square$

Define the weight by $w(x) = 2n - m$ for an element $x \in H^{m,n}(X;\mathbb{Z}/p)$, so that $w(x') = 0$ for $x' \in CH^*(X)/p$. Of course we get $w(xy) = w(x) + w(y)$, $w(P^ix) = w(x)$ and $w(Q_i(x)) = w(x) - 1$.

Corollary 4.2. *Suppose that $B(n,p)$ holds. If $x \in H^n(X(\mathbb{C});\mathbb{Z}/p)$ and $Q_{i_1}...Q_{i_n}(x) \neq 0$, then there is a $\mathbb{Z}/p[\tau]$-module generator $x' \in H^{n,n}(X;\mathbb{Z}/p)$ so that $t_{\mathbb{C}}(x') = x$ and, for each $0 \leq k \leq n$, $Q_{i_1}...Q_{i_k}(x')$ is also a $\mathbb{Z}/p[\tau]$-module generator of $H^{*,*}(X;\mathbb{Z}/p)$.*

Proof. By the $B(n,p)$ condition, $t_{\mathbb{C}}^{n,n} : H^{n,n}(X;\mathbb{Z}/p) \cong H^n(X(\mathbb{C});\mathbb{Z}/p)$. Hence there is an element $x' \in H^{n,n}(X;\mathbb{Z}/p)$ with $t_{\mathbb{C}}(x') = x$. This means $w(x') = n$ and $w(Q_{i_1}...Q_{i_n}(x)) = 0$. From the above lemma, we get the corollary. $\square$

Lemma 4.3. *Suppose that $B(n,p)$ holds. If there is an $s > 0$ with $p^sH^{n+1}(X(\mathbb{C}))_{(p)} \subset t_{\mathbb{C}}(H^{n+1,n}(X)_{(p)})$, then*

$$\mathrm{Im}(H^{n+1}(X(\mathbb{C})) \to H^{n+1}(X(\mathbb{C});\mathbb{Z}/p)) = \mathrm{Im}((H^{n+1,n}(X) \to H^{n+1}(X(\mathbb{C});\mathbb{Z}/p)).$$

Proof. Consider the following diagram:

$$\begin{array}{ccccccc}
H_L^{n+1,n}(X) & \xrightarrow{(1)} & H_L^{n+1,n}(X;\mathbb{Z}/p^N) & \longrightarrow & H_L^{n+2,n}(X) & \xrightarrow{p^N} & H_L^{n+2,n}(X) \\
{\scriptstyle (2)}\downarrow & & {\scriptstyle \cong}\downarrow & & \downarrow & & \downarrow \\
H^{n+1}(X(\mathbb{C})) & \xrightarrow{(3)} & H^{n+1}(X(\mathbb{C});\mathbb{Z}/p^N) & \longrightarrow & H^{n+2}(X(\mathbb{C})) & \xrightarrow{p^N} & H^{n+2}(X(\mathbb{C}))
\end{array}$$

where $H^*(-)$ means $H^*(-;\mathbb{Z})_{(p)}$ and the rows are exact.

Let $H^{n+i}(X(\mathbb{C})) \cong F_i \oplus T_i$ and $H_L^{n+i,n}(X) \cong F_i' \oplus T_i' \oplus D_i$, where F_i, F_i' are free, T_i, T_i' are non-p-divisible torsion and D_i are p-divisible submodules. Take N and s so that $p^N > p^s > |T_i|, |T_i'|$ for $i = 1, 2$. Hence $H_L^{n+1,n}(X;\mathbb{Z}/p^N) \cong H^{n+1}(X(\mathbb{C});\mathbb{Z}/p^N) \cong F_1/p^N \oplus T_1 \oplus T_2$.

By the $B(n,p)$ condition, $H^{n+1,n}(X) \cong H_L^{n+1,n}(X)$, and the map (2) is identified with the realization map. So $p^s(F_1 \oplus T_1) = p^sF_1 \subset \mathrm{Image}(2)$. Therefore there is the quotient map $F_1/p^s \oplus T_1 \oplus T_2 \to \mathrm{Coker}(1)$. On the other hand,

$\mathrm{Ker}(p^N)|H_L^{n+2,n}(X) \cong (\mathrm{Ker}(p^N)|D_2) \oplus T_2' \cong (\mathbb{Z}/p^N)^k \oplus T_2'$. Hence if $k \neq 0$, then it is a contradiction to $\mathrm{Ker}(p^N) = \mathrm{Coker}(1)$. Hence we get $\mathrm{Coker}(1) \cong T_2'$ and hence $\mathrm{Im}(3)(2) = F_1/p^N \oplus T_1$. □

Corollary 4.4. *Suppose that $B(n,p)$ holds and $t_{\mathbb{C}}^{n+1,n} \otimes \mathbb{Q} : H^{n+1,n}(X) \otimes \mathbb{Q} \to H^{n+1}(X(\mathbb{C})) \otimes \mathbb{Q}$ is epic. If $x \in \mathrm{Im}(H^{n+1}(X(\mathbb{C})) \to H^{n+1}(X(\mathbb{C});\mathbb{Z}/p))$ and $Q_{i_1}...Q_{i_{n-1}}(x) \neq 0$, then there is an element $x' \in H^{n+1,n}(X)_{(p)}$ so that $t_{\mathbb{C}}(x') = x$ and, for each $0 \leq k \leq n-1$, $Q_{i_1}...Q_{i_k}(x)$ is also a $\mathbb{Z}/p[\tau]$-module generator of $H^{*,*}(X;\mathbb{Z}/p)$.*

Here we mention the case $n = 1$. Totaro showed [To2] that $CH^*(BG) \otimes \mathbb{Q} \cong H^*(BG) \otimes \mathbb{Q}$ for any complex algebraic group G. Hence $CH^1(BG) \to H^2(BG)$ is epic; indeed, he also showed that this map is an isomorphism. As for $K3$-surfaces, $CH^*(X) \otimes \mathbb{Q} \to H^*(X(\mathbb{C})) \otimes \mathbb{Q}$ is not epic and $H_L^{3,1}(X)$ contains p-divisible elements.

Now we consider some examples. The mod 2 cohomology of $BO(n)$ is $H^*(BO(n);\mathbb{Z}/2) \cong \mathbb{Z}/2[w_1,...,w_n]$, where the Stiefel-Whitney class w_i restricts the elementary symmetric polynomial in $H^*(B(\mathbb{Z}/2)^n;\mathbb{Z}/2) \cong \mathbb{Z}/2[x_1,...,x_n]$. Each element w_i^2 is represented by the Chern class c_i of the induced representation $O(n) \subset U(n)$. Hence $c_i \in CH^*(BO(n);\mathbb{Z}/2) = H^{2*,*}(BO(n);\mathbb{Z}/2)$.

Proposition 4.5. $h^{*,*}(BO(n);\mathbb{Z}/2) \supset \mathbb{Z}/2[c_1,...,c_n] \otimes \Delta(w_1,...,w_n) \otimes \mathbb{Z}/2[\tau]$, *where $deg(c_i) = (2i,i)$, $deg(w_i) = (i,i)$ and $w_i^2 = \tau^i c_i$.*

Since $Q_{i-1}...Q_0(w_i) \neq 0$, each w_i is a $\mathbb{Z}/2[\tau]$-module generator. However, even $h^{*,*}(BO(n);\mathbb{Z}/2)$ seems very complicated. Consider the case $X = BO(3)$. The cohomology operations act by

$$\begin{array}{ccccccc} w_2 & \xrightarrow{Sq^1} & w_1w_2 + w_3 & \xrightarrow{Sq^2} & w_2w_1^3 + w_1^2w_3 + w_1w_2^2 + w_2w_3 & \xrightarrow{Sq^1} & w_1^2w_2^2 + w_3^2, \\ w_3 & \xrightarrow{Sq^1} & w_3w_1 & \xrightarrow{Sq^2} & w_1w_2w_3. & & \end{array}$$

Theorem 4.6. *There is the isomorphism*

$$h^{*,*}(BO(3);\mathbb{Z}/2) \cong \mathbb{Z}/2[c_1,c_2,c_3]\{1, w_1, w_2, Q_0w_2, Q_1w_2, w_3, Q_0w_3, Q_1w_3\} \otimes \mathbb{Z}/2[\tau].$$

where $Q_0w_2 = \tau^{-1}(w_1w_2 + w_3), ..., Q_1w_3 = \tau^{-2}w_1w_2w_3$.

W. S. Wilson ([RWY], [KY]) found a good $Q(i) = \Lambda(Q_0,...,Q_i)$-module decomposition for $X = BO(n)$, namely,

$$H^*(X;\mathbb{Z}/2) = \bigoplus_{i=-1}^{\infty} Q(i)G_i \quad \text{with} \quad Q_0...Q_iG_i \in t_{\mathbb{C}}(CH^*(X)). \tag{4.2}$$

Here G_{k-1} is quite complicated; namely, it is generated by symmetric functions

$$\Sigma x_1^{2i_1+1}...x_k^{2i_k+1}x_{k+1}^{2j_1}...x_{k+q}^{2j_q}, \quad k + q \leq n,$$

with $0 \leq i_1 \leq ... \leq i_k$ and $0 \leq j_1 \leq ... \leq j_q$; and if the number of j equal to j_u is odd, then there is some $s \leq k$ such that $2i_s + 2^s < 2j_u < 2i_s + 2^{s+1}$.

Then $w(G_i) \geq i + 1$ in $h^{*,*}(X;\mathbb{Z}/p)$, and so we have

Proposition 4.7. *Letting $w(G_i) = i + 1$, we have the monomorphism*

$$h^{*,*}(BO(n);\mathbb{Z}/2) \subset (\bigoplus_i Q(i)G_i) \otimes \mathbb{Z}/2[\tau].$$

One interesting problem is whether the above injection is really an isomorphism. The similar decomposition holds for $X = (B\mathbb{Z}/p)^n$, and the above injection is an isomorphism. (See Lemma 5.6 below.) The case $X = BO(3)$ is also an isomorphism. Since the direct decomposition of $BO(3) \cong BSO(3) \times B\mathbb{Z}/2$ is complicated, we only write here that of $SO(3)$:

(4.3)

$$\begin{aligned} H^*(BSO(3);\mathbb{Z}/2) &\cong \mathbb{Z}/2[w_2, w_3] \cong \mathbb{Z}/2[c_2, c_3]\{1, w_2, w_3 = Q_0w_2, w_2w_3 = Q_1w_2\} \\ &\cong \mathbb{Z}/2[c_2, c_3]\{w_2, Q_0w_2, Q_1w_2, c_3 = Q_0Q_1w_2\} \oplus \mathbb{Z}/2[c_2] \\ &\cong \mathbb{Z}/2[c_2, c_3]Q(1)\{w_2\} \oplus \mathbb{Z}/2[c_2]. \end{aligned}$$

Since there is the isomorphism $O(2n+1) \cong SO(2n+1) \times \mathbb{Z}/2$, the cohomology of $BSO(2n+1)$ is reduced from that of $BO(2n+1)$. However, the situation for $BO(2n)$ is different. In the next section, we will study $BSO(4)$ for details.

The extraspecial 2-group 2_+^{1+2n} is the n-th central product of the dihedral group D_8 of order 8. It has a central extension

$$0 \to \mathbb{Z}/2 \to G \to V = \bigoplus^{2n} \mathbb{Z}/2 \to 0. \tag{4.4}$$

Let $H^*(BV;\mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, ..., x_{2n}]$. Then Quillen proved [Q]

$$H^*(BG;\mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, ..., x_{2n}]/(f, Q_0f, ..., Q_{n-2}f) \otimes \mathbb{Z}/2[w_{2^n}]. \tag{4.5}$$

Here w_{2^n} is the Stiefel-Whitney class of the real 2^n-dimensional irreducible representation which restricts nonzero on the center, and $f = \sum_i x_{2i-1}x_{2i} \in H^2(BV;\mathbb{Z}/2)$ represents the central extension (4.4).

Letting $y_i = x_i^2$ in $H^*(BG;\mathbb{Z}/2)$, we can write $f^2 = \sum y_{2i-1}y_{2i}$ and

$$\begin{aligned} (Q_{k-1}f)^2 = Q_0Q_kf &= \sum y_{2i-1}^{2^k}y_{2i} - y_{2i-1}y_{2i}^{2^k}, \\ Q_{k-1}f &= \sum y_{2i-1}^{2^{k-1}}x_{2i} - x_{2i-1}y_{2i}^{2^{k-1}}. \end{aligned}$$

Now we consider the motivic cohomology $H^{*,*}(BG;\mathbb{Z}/2)$ and change $y_i = \tau^{-1}x_i^2$. Since $f = 0 \in H^{2,2}(BG;\mathbb{Z}/2)$, we can see that $Q_{k-1}f = 0$ and $Q_kQ_0(f) = 0$ also in $H^{*,*}(BG;\mathbb{Z}/2)$. However, for general n, $\sum y_{2i}y_{2i-1} \neq 0$ in $H^{*,*}(BG;\mathbb{Z}/2)$. Let

$$\begin{aligned} A = (&\mathbb{Z}/2[y_1, ..., y_{2n}, c_{2^n}]/(Q_0Q_kf, ..., Q_0Q_nf) \\ &\otimes \Delta(x_1, ..., x_{2n}, w_{2^n})/(f, Q_0f, .., Q_{n-2}f)) \otimes \mathbb{Z}/2[\tau]. \end{aligned} \tag{4.6}$$

Lemma 4.8. *For $G = 2_+^{1+2n}$, there is a map $A \to H^{*,*}(BG;\mathbb{Z}/2)$ which induces the injection $A/(f^2) \subset h^{*,*}(BG;\mathbb{Z}/2)$.*

When $m = 0, 1, -1 \mod 8$ and $m > 0$, we say that $Spin(m)$ is *real type* [Q]. When $Spin(m)$ is real type, from Quillen, we know that $H^*(BSpin(m);\mathbb{Z}/2) \subset H^*(BG;\mathbb{Z}/2)$, where $G = 2_+^{2h+1}$ and h is the Hurwitz number (for details see [Q]).

Corollary 4.9. *Let $G = Spin(m)$ be real type with Hurwitz number h, and let*

$$\begin{aligned} A = (&\mathbb{Z}/2[c_2, c_3, , ..., c_m, c_{2^h}]/((Q_1Q_0w_2), ..., (Q_hQ_0w_2)) \\ &\otimes \Delta(w_2, ..., w_m, w_{2^h})/(w_2, Q_0w_2, ..., Q_{h-2}w_2)) \otimes \mathbb{Z}/2[\tau], \end{aligned}$$

where $w_i, i \leq m$ (resp. w_{2^h}) is the Stiefel-Whitney class of the usual $SO(m)$ representation (resp. of the irreducible 2^h-dimensional spin representation). Then we have a map $A \to H^{*,*}(BG;\mathbb{Z}/2)$ which induces the injection

$$A/(c_2) \subset h^{*,*}(BG;\mathbb{Z}/2).$$

We study $Spin(7)$ and the exceptional Lie group G_2. The cohomology of G_2 is given by $H^*(BG_2;\mathbb{Z}/2)\cong \mathbb{Z}/2[w_4,w_6,w_7]$, where w_i is the Stiefel-Whitney class of the inclusion $G_2\subset SO(7)$. The cohomology $H^*(BSpin(7);\mathbb{Z}/2)\cong H^*(BG_2;\mathbb{Z}/2)\otimes \mathbb{Z}/2[w_8]$.

Corollary 4.10. *Let $A=\mathbb{Z}/2[c_2,c_4,c_6,c_7]\otimes\Delta(w_4,w_6,w_7)\otimes\mathbb{Z}/2[\tau]$. Then there is the map $A\to H^{*,*}(BG_2;\mathbb{Z}/2)$ which induces the injection $A/(c_2)\subset h^{*,*}(BG_2;\mathbb{Z}/2)$.*

Remark. Similar facts hold for $BSpin(7)$ tensoring $\mathbb{Z}/2[c_8]$.

The cohomology operations are given by

$$w_4 \xrightarrow{Sq^2} w_6 \xrightarrow{Sq^1} w_7 \xrightarrow{Sq^4} w_4w_7 \xrightarrow{Sq^2} w_7w_6 \xrightarrow{Sq^1} w_7^2,$$
$$Q_1Q_0(w_4w_6)=w_7^2,\quad Q_2Q_1Q_0(w_4w_6w_7)=w_7^4.$$

Proposition 4.11. *Let $w(w_4)=2, w(w_{(4,6)})=2$ and $w(w_{(4,6,7)})=3$ with $t_{\mathbb{C}}(w_{(i_1,\dots,i_n)})=w_{i_1}\dots w_{i_n}$. Then $h^{*,*}(BG_2;\mathbb{Z}/2)$ is a subalgebra of*

$$\mathbb{Z}/p[\tau]\otimes\mathbb{Z}/2[c_4,c_6,c_7]\otimes\mathbb{Z}/2\{1,w_4,Sq^2w_4,Q_1w_4,Q_2w_4,Sq^2Q_2w_4,w_{(4,6)},w_{(4,6,7)}\}.$$

Remark. If $t_{\mathbb{C}}^{4,3}\otimes\mathbb{Q}$ is epic, then we can take $w_4\in h^{4,3}(BG;\mathbb{Z}/2)$, *i.e.*, $w(w_4)=2$.

The kernel $\mathrm{Ker}(t_{\mathbb{C}})^{2*,*}$ is not so big for $X=BG_2$. Indeed, it is known [Y3] that

$$CH^*(BG_2)/2\cong\mathbb{Z}/2[c_2,c_4,c_6,c_7]/(rc_2^2,c_2c_7),\quad \text{where } r=0 \text{ or } 1.$$

The cohomology operations are given in $H^*(BSO(7);\mathbb{Z}/2)$ by

$$Q_1Q_0w_2=w_3^2,\quad Q_2Q_0w_2=w_5^2,\quad Q_3Q_0w_2=w_7^2w_2^2+w_6^2w_3^2+w_5^2w_4^2.$$

Hence we have $c_3=0, c_5=0$ and $c_2c_7=0$ in $CH^*(BG_2)/2$, but $c_2\neq 0$.

From here we consider the case $p=odd$. One of the easiest examples is the case $G=PGL_3$ and $p=3$. The mod 3 cohomology is given by ([KY], [Ve1])

$$(\mathbb{Z}/3[y_2]\{y^2\}\oplus\mathbb{Z}/3\{1,y_2,y_3,y_7\}[y_8])\otimes\mathbb{Z}/3[y_{12}]$$

It is known that y_2^2,y_2^3,y_8^2 and y_{12} are represented by Chern classes. Moreover, $Q_1Q_0(y_2)=y_8$. Hence these elements are in the Chow ring; namely,

$$h^{2*,*}(BPGL_3;\mathbb{Z}/3)\cong(\mathbb{Z}/3[y_2]\{y_2^2\}\oplus\mathbb{Z}/3[y_8])\otimes\mathbb{Z}/3[y_{12}].$$

The cohomology operations are given by

$$y_2 \xrightarrow{\beta} y_3 \xrightarrow{P^1} y_7 \xrightarrow{\beta} y_8. \tag{4.7}$$

Thus we get $h^{*,*}(PGL_3;\mathbb{Z}/3)$ completely.

Theorem 4.12. *Letting $w(y_2)=2$, we have the isomorphism*

$$h^{*,*}(BPGL_3;\mathbb{Z}/3)\cong(\mathbb{Z}/3[y_2]\{y^2\}\oplus\mathbb{Z}/3\{1\}\oplus\mathbb{Z}/3[y_8]\otimes Q(1)\{y_2\})\otimes\mathbb{Z}/3[y_{12},\tau].$$

Next consider the extraspecial p-group $G=p_+^{1+2n}$. When $n>2$, even the cohomology rings $H^*(BG;\mathbb{Z}/p)$ are unknown, while it contains the subring [TeY1]

$$R=\mathbb{Z}/p[y_1,\dots,y_{2n},c_{p^n}]/(Q_1Q_0f,\dots,Q_nQ_0f), \tag{4.8}$$

where $f=\sum^n x_{2i-1}x_{2i}$ for $\beta x_i=y_i$ and $Q_kQ_0f=\sum y_{2i-1}y_{2i}^{p^k}-y_{2i-1}^{p^k}y_{2i}$. Since $f=0\in H^{2,2}(BG;\mathbb{Z}/p)$, we have

Proposition 4.13. *There is the injection*

$$R\otimes\mathbb{Z}/p[\tau]\hookrightarrow H^{*,*}(Bp_+^{1+2n};\mathbb{Z}/p).$$

We consider here other arguments for a different but similar group. Let $\tilde{p}_+^{1+2n}$ be the central product of p_+^{1+2n} and the circle, i.e. $\tilde{p}_+^{1+2n} = p_+^{1+2n} \times_C S^1$, identifying $C \cong \mathbb{Z}/p \subset S^1$, where C is the center. Let us write

$$e_A = \prod_{0 \neq (\lambda_1, \lambda_3, \ldots, \lambda_{2n-1})} (\lambda_1 y_1 + \cdots + \lambda_{2n-1} y_{2n-1}). \tag{4.9}$$

If we localize by inverting e_A, then the cohomology of $\tilde{p}_+^{1+2n}$ is expressed easily [Y2] as

$$[e_A^{-1}]H^*(B\tilde{p}_+^{1+2n}; \mathbb{Z}/p) \cong [e_A^{-1}]R \otimes \Lambda(x_1, x_3, ..., x_{2n-1}), \qquad \beta(x_i) = y_i. \tag{4.10}$$

Theorem 4.14. *Letting $w(x_i) = 1$, we have the ring isomorphism*

$$[e_A^{-1}]h^{*,*}(B(\tilde{p}_+^{1+2n}); \mathbb{Z}/p) \cong [e_A^{-1}]R \otimes \mathbb{Z}/p[\tau] \otimes \Lambda(x_1, x_3, \ldots, x_{2n-1}).$$

Proof. There is the splitting abelian subgroup $(\mathbb{Z}/p)^n \cong A \subset \tilde{p}_+^{1+2n}$ such that

$$h^{*,*}(BA; \mathbb{Z}/p) \cong \mathbb{Z}/p[\tau, y_1, y_3, ..., y_{2n-1}] \otimes \Lambda(x_1, x_3, ..., x_{2n-1}).$$

Each monomial $x_{i_1} ... x_{i_s}$, $1 \le i_1, ..., i_s \le 2n-1$, is a $\mathbb{Z}/p[\tau]$-module generator in the above cohomology, hence also in the cohomology of $B\tilde{p}_+^{1+2n}$. $\square$

We consider the case $n = 1$ here. Let us write $E = p_+^{1+2}$ for each odd prime p. The ordinary cohomology is known by Lewis [Ly], [TeY2]; namely,

$$H^{even}(BE)/p \cong (\mathbb{Z}/p[y_1, y_2]/(y_1^p y_2 - y_1 y_2^p) \oplus \mathbb{Z}/p\{c_2, ..., c_{p-1}\}) \otimes \mathbb{Z}/p[c_p],$$
$$H^{odd}(BE) \cong \mathbb{Z}/p[y_1, y_2, c_p]\{a_1, a_2\}/(y_1 a_2 - y_2 a_1, y_1^p a_2 - y_2^p a_1), \quad |a_i| = 3.$$

It is also known that $Q_1(a_i) = y_i c_p$ and $order(c_p) = p^2$.

The group 2_+^{1+2} is the dihedral group D_8 of order 8. The integral cohomologies are

$$H^{even}(BD_8)/2 \cong \mathbb{Z}/2[y_1, y_2, c_2]/(y_1 y_2), \quad H^{odd}(BD_8) \cong H^{even}(BD_8)/2\{e\}$$

where $c_2 = w_2^2$, $e = (x_1 + x_2)w_2$ in $H^*(BD_8; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, x_2, w_2]/(x_1 x_2)$ and $Q_1 e = (y_1 + y_2)c_2$, $order(c_2) = 4$.

Theorem 4.15. *For all primes p, we have the isomorphisms*

$$h^{*,*}(Bp_+^{1+2}; \mathbb{Z}/p) \cong (\{1, \partial_p^{-1}\}(H^*(Bp_+^{1+2})/p) - \{\partial_p^{-1}1\}) \otimes \mathbb{Z}/p[\tau],$$

where $w(H^{even}(Bp_+^{1+2})/p) = 0, w(H^{odd}(Bp_+^{1+2})) = 1$ and $w(\partial_p^{-1}(x)) = w(x) + 1$.

Proof. We will prove this only for odd primes, since the proof for $p = 2$ is similar. Since all elements in $H^{even}(BE)$ are generated by Chern classes, we have the isomorphism $h^{2*,*}(BG; \mathbb{Z}/p) \cong H^{even}(BE)/p$. We know $H^{odd}(BE; \mathbb{Z}/p)$ is generated as an $H^{even}(BE)/p$-module by two elements a_1, a_2 such that $Q_1 a_i = y_i c_p$ [TeY2].

The mod p cohomology is written additively, $H^*(BE; \mathbb{Z}/p) \cong \{1, \partial_p^{-1}\}H^*(BE)/p$. Here ∂_p is the (higher) Bockstein operator. All elements in $H^{odd}(BE)$ are just p-torsion, and we can take $a_i' \in H^2(BE; \mathbb{Z}/p)$ such that $\beta(a_i') = a_i$. Thus we take $a_i' \in H^{2,2}(BE; \mathbb{Z}/p)$ so that $a_i \in H^{3,2}(BE; \mathbb{Z}/p)$.

Next consider elements $x = \partial_p^{-1}(y)$, $y \in H^{even}(BE)/p$. If $y \in (\mathrm{Ideal}(y_1, y_2))$, then $\partial_p^{-1}(y) = \sum x_i b_i$ for $b_i \in H^{even}(BE)/p$, and hence we can take $w(\partial_p^{-1}(y)) = 1$. For other elements $y = c_i c_p^n$, $2 \le i \le p-1$, it is known [Ly] that $c_i = \mathrm{Cor}_M^E(u^i)$

with $0 \neq u \in H^2(B\mathbb{Z}/p;\mathbb{Z}/p)$ for a maximal abelian subgroup $M \cong \mathbb{Z}/p \times \mathbb{Z}/p$. Hence $y \in H^{2*,*}(BE;\mathbb{Z}/p)$ is also p-torsion. Considering the exact sequence

$$\to H^{2*-1,*}(BE;\mathbb{Z}/p^N) \to H^{2*,*}(BE) \xrightarrow{p^N} H^{2*,*}(BE) \to,$$

we get $w(\partial_p^{-1}(y)) = 1$. The element $y = c_p^n$ is p^2-torsion in $H^*(BE;\mathbb{Z}/p)$. Note that $\mathrm{Cor}_M^E(u^{pn}) = pc_p^n + k$ with $k \in \mathrm{Ideal}(y_1, y_2)$. Thus $y \in H^{2*,*}(BE;\mathbb{Z}/p)$ is also p^2-torsion. Then we can take $w(\partial_p^{-1}(y)) = 1$. This completes the proof. $\square$

We easily compute the following results.

Corollary 4.16. *For each prime p, there is an isomorphism*

$$h^{*,*}(Bp_+^{1+2};\mathbb{Z}/p) \cong \mathbb{Z}/p[\tau] \otimes (\mathbb{Z}/p\{1\} \oplus Q'(0)G_0' \oplus Q(0)G_0 \oplus Q(1)G_1),$$

where $Q'(0) = \Lambda(\beta_{p^2})$, β_{p^2} is the p^2-torsion Bockstein operator, and if $p = 2$, then

$$\begin{cases} G_0' \cong \mathbb{Z}/2[c_2]\{x_1w_2\}, \quad \beta_4(x_1w_2) = c_2, \\ G_0 \cong \mathbb{Z}/2[y_1]\{x_1\} \oplus \mathbb{Z}/2[y_2]\{x_2\} \oplus \mathbb{Z}/2[c_2]\{x_1c_2\}, \\ G_1 \cong \mathbb{Z}/2[y_1, y_2, c_2]/(y_1y_2)\{w_2\}, \end{cases}$$

and if p is an odd prime, then

$$\begin{cases} G_0' \cong \mathbb{Z}/p[c_p]\{c_p'\}, \quad \beta_{p^2}(c_p') = c_p, \\ G_0 \cong \mathbb{Z}/p[y_1, y_2]\{x_1, x_2\}/(y_2x_1 - y_1x_2, y_2^px_1 - y_1^px_2) \oplus \mathbb{Z}/p[c_p]\{c_2', ..., c_{p-1}'\}, \\ G_1 \cong \mathbb{Z}/p[y_1, y_2, c_p]\{a_1', a_2'\}/(y_2a_1' - y_1a_2', y_2^pa_1' - y_1^pa_2'), \quad \beta(a_i') = a_i, \ \beta(c_i') = c_i. \end{cases}$$

5. BP-theory and $\mathrm{Ker}\, t_{\mathbb{C}}^{2*,*}$

In this section, we always assume $k = \mathbb{C}$. Even this case it seems difficult to know $\mathrm{Ker}\, t_{\mathbb{C}}$. For Chow rings $CH^*(X)$, Totaro found a good way to get nonzero elements in $\mathrm{Ker}\, t_{\mathbb{C}}$. Let $MU^*(-)$ (resp. $BP^*(-)$) be the complex cobordism theory (resp. Brown-Peterson theory) with the coefficient ring $MU^* = MU^*(pt) = \mathbb{Z}[x_1, ...],\ |x_i| = -2i$ (resp. $BP^* = \mathbb{Z}_{(p)}[v_1, ...],\ |v_i| = -2(p^i - 1)$). The Thom map induces $\rho : MU^*(X(\mathbb{C})) \otimes_{MU^*} \mathbb{Z} \to H^*(X(\mathbb{C});\mathbb{Z})$. Totaro constructed [To1] the map

$$\tilde{cl} : CH^*(X) \to MU^*(X(\mathbb{C})) \otimes_{MU^*} \mathbb{Z} \tag{5.1}$$

such that the composition $\rho\tilde{cl}$ is the usual cycle map $cl = t_{\mathbb{C}}^{2*,*}$, which is also the realization map.

In this section, hereafter, X is just a topological space, e.g., $X(\mathbb{C})$, to simplify the notation. Since $BP^*(X) \cong BP^* \otimes_{MU^*_{(p)}} MU^*(X)_{(p)}$, the similar fact holds for BP-theory. Let $P(n)^* = BP^*/(p, v_1, ..., v_{n-1})$, e.g., $P(0)^* = BP^*$, $P(1)^* = BP^*/p$ and $P(\infty)^* = \mathbb{Z}/p$. Then there are cohomology theories $P(n)^*(-)$ with the coefficient $P(n)^*(pt) \cong P(n)^*$, e.g., $P(0)^*(X) = BP^*(X)$, $P(1)^*(X) = BP^*(X;\mathbb{Z}/p)$ and $P(\infty)^*(X) = H^*(X;\mathbb{Z}/p)$. Hence there are maps of cohomology theories

$$\begin{aligned} cl_p : CH^*(-)/p \to BP^*(-) \otimes_{BP^*} \mathbb{Z}/p \to ... \to P(n)^*(-) \otimes_{P(n)^*} \mathbb{Z}/p \\ \to P(n+1)^*(-) \otimes_{P(n+1)^*} \mathbb{Z}/p \to ... \to H^*(-;\mathbb{Z}/p) \end{aligned}$$

such that the composition is the cycle map $cl_p = t_{\mathbb{C}}$. The Morava K-theory is defined by $K(n)^*(X) = P(n)^*(X) \otimes_{P(n)^*} K(n)^*$, where $K(n)^* = \mathbb{Z}/p[v_n, v_n^{-1}]$. In general, $K(n)^*(X) \not\cong K(n)^* \otimes_{BP^*} BP^*(X)$. However, when $K(n)^{odd}(X) = 0$, it is known [RWY] that

$$P(n)^*(X) \cong BP^*(X) \otimes_{BP^*} P(n)^*, \quad K(n)^*(X) \cong BP^*(X) \otimes_{BP^*} K(n)^*.$$

We know that $K(n)^{odd}(BG) = 0$ for many cases, while Kříž showed $K(n)^*(BG') \neq 0$ for some fine group G'.

One useful tool for computing $BP^*(X)$ is the Atiyah-Hirzebruch spectral sequence [TeY2], [KY]

$$E_2^{*,*} = H^*(X) \otimes BP^* \Longrightarrow BP^*(X).$$

It is known that $d_{2p^i-1}(x) = v_i \otimes Q_i(x) \bmod(M_i)$, where M_i is the ideal of $E_{2p^i-1}^{*,*}$ generated by elements in $(p, v_1, ..., v_{i-1})E_2^{*,*}$. Here we assume that $H^*(X)$ has no higher p-torsion and that

(5.2) All nonzero differentials are of the form

$$d_{2p^i-1}(x) = v_i \otimes Q_i(x) \bmod(M_i).$$

Let us write

$$grBP^*(X) \cong E_\infty^{*,*} \cong A \oplus B \tag{5.3}$$

where A (resp. B) is a BP^*-module generated by nonzero elements in $H^*(X)/p$ (resp. $pH^*(X) \oplus E_\infty^{*,minus}$), so that $B \subset \mathrm{Ker}(\rho_p)$. We can write

$$A \cong \bigoplus_{n=-1}^{\infty} P(n+1)^* \tilde{G}_n$$

by the prime invariant ideal theorem of Landweber; if $P(n)^*/(a)$ is a $BP^*(BP)$-module, then $a = v_n^s$ for some $s \geq 1$.

Take a nonzero element $\tilde{g}_n \in \tilde{G}_n$ for $n \geq 2$. Since $\tilde{g}_n$ is $(p, ..., v_n)$-torsion, there is $g_{(n,s)} \in E_{2p^s-1}^{*,0}$ such that $d_{2p^s-1}(g_{(n,s)}) = v_s \otimes \tilde{g}_n$ for each $1 \leq s \leq n$. Let the BP^*-module in $E_{2p^s-1}^{*,*}$ generated by $g_{(n,s)}$ be isomorphic to a $P(s'+1)^*$-free module for $s' < s$. Here note that if $s' \neq s-1$, then $\mathrm{Ideal}(v_{s'+1}, .., v_{s-1})\{g_{n,s}\} \subset \mathrm{Ker}(d_{2p^s-1})$. In any case, we can take $g_{(n,s,t)} \in H^*(X)/p$ for $t < s'$ such that $d_{2p^t-1}(g_{(n,s,t)}) = v_t \otimes g_{(n,s)}$. Continuing this argument we can take

$$\tilde{g}_n \xleftarrow{Q_{s_1}} g_{(n,s_1)} \xleftarrow{Q_{s_2}} g_{(n,s_1,s_2)} \longleftarrow \ \dots \ \xleftarrow{Q_{s_m}} g_{(n,s_1,...,s_m)}$$

for some $(n > s_1 > ... > s_m)$.

Lemma 5.1. *Let $H^*(X)_{(p)}$ have no higher p-torsion. Suppose (5.2) holds, and $A = \bigoplus_{n=-1} P(n+1)^*\tilde{G}_n$ in (5.3). Then there is the injection*

$$H^*(X;\mathbb{Z}/p) \hookrightarrow \bigoplus_n Q(n)G_n \quad \text{with } Q_0...Q_nG_n = \tilde{G}_n.$$

Proof. Let H be a $\mathbb{Z}/p$-module generated by elements $g_{(n,s_1,...,s_m)}$ discussed above. Define the map $j_{\mathbb{C}} : H \hookrightarrow \oplus Q(n)G_n$ by

$$j_{\mathbb{C}}(g_{(n,s_1,...,s_m)}) = Q_{s_m}^{-1}...Q_{s_1}^{-1}(\tilde{g}_n) = Q_0...\hat{Q}_{s_m}....\hat{Q}_{s_1}...Q_n(g_n), \quad Q_0...Q_ng_n = \tilde{g}_n.$$

Suppose $x \in H^*(X)_{(p)} - H$. Then by the assumption (5.3), x is not a permanent cycle. Hence $d_{2p^i-1}(x) \neq 0$ for some i, and so $Q_i(x) \neq 0$. Let t be a largest number such that $Q_{i_t}...Q_{i_1}Q_ix = \tilde{g} \neq 0$. Since $Q_j(\tilde{g}) = 0$ for all j, we know $\tilde{g}$ is a permanent cycle. This element $\tilde{g} \in E_\infty^{*,0}$ generates a $P(N+1)^*$-module for $N = \max(i_s, ..., i_1, i)$. This means $x = (Q_i^{-1}Q_{i_1}^{-1}...Q_{i_s}^{-1}\tilde{g}) \in H$. □

Let us write $Q(i,n) = \Lambda(Q_i, ..., Q_n)$, so that $Q(0,n) = Q(n)$.

Lemma 5.2. *Let $H^*(X)_{(p)}$ have no higher p-torsion.*

(1) *If (5.2) is satisfied and, in (5.3),*

$$A = \bigoplus_{n=-1} P(n+1)^*\tilde{G}_n \quad \text{and } B \cong \bigoplus_{s=0} BP^*\{p, v_1, .., v_s\}\tilde{K}_s,$$

then we have the isomorphisms

$$H^*(X)/p \cong (\tilde{G}_{-1} \oplus \tilde{G}_0 \oplus \bigoplus_{n=1} Q(1,n)G'_n - \bigoplus_{s=0}(Q(1,s)K'_s - \tilde{K}'_s)),$$

$$H^*(X;\mathbb{Z}/p) \cong (\bigoplus_{n=-1} Q(n)G_n - \bigoplus_{s=0}(Q(s)K_s - \tilde{K}_s))$$

with $Q_0...Q_nG_n = \tilde{G}_n$, $Q_0G_n = G'_n$ and $Q_0...Q_sK_s = \tilde{K}_s$, $Q_0K_s = K'_s$.

(2) *If $Q_0...Q_nG_n \in \mathrm{Im}(\rho)$ and the degrees of $\tilde{K}_s$ and $\tilde{G}_n$ are even, then the converse also holds.*

Proof. (1) Let $0 \neq x \in \tilde{K}_s$. Since x is not a permanent cycle, $d_{2p^i-1}(x) \neq 0$ and $Q_i(x) \neq 0$. Since $\{p, ..., v_s\}\tilde{K}_s$ are permanent cycles, we know $Q_i(x) \in E^{*,*}_{2p^i-1}$ is a $P(s+1)^*$-module, that is, $i = s+1$ by the Landweber invariant prime ideal theorem, and

$$\bigoplus Q(n)G_n \supset Q(s)K_s.$$

Since v_ix generates a free BP^*-module, $x \notin \mathrm{Im}(Q_j)$ for all j. Hence we get the injection

$$H^*(X;\mathbb{Z}/p) \hookrightarrow \bigoplus Q(n)G_n - (Q(s)K_s - \tilde{K}_s).$$

Let $x = Q_{i_1}...Q_{i_k}g_n$ be in the right-hand side of the above injection, and such that $0 \neq Q_i(x) \in H^*(X;\mathbb{Z}/p)$ but $x \notin H^*(X;\mathbb{Z}/p)$. If $Q_i(x)$ is not a permanent cycle, then $v_iQ_i(x)$ is permanent, so $Q_i(x)$ must be in $\tilde{K}_s$ and hence $x \in Q(s)K_s$; this is a contradiction. Otherwise $Q_i(x) = \tilde{g}_n$ generates a $P(n)^*$-module and $Q_i(x)$ must be $\mathrm{Im}(Q_j)$ for all $j \leq n$. Hence $x \in H^*(X;\mathbb{Z}/p)$.

(2) By induction on i, we assume $E^{*,*}_{2p^i-1} \cong C(i) \oplus D(i)$, where

$$C(i) = P(i)^*(\bigoplus_{i \leq n} Q(i,n)Q_{i-1}...Q_0G_n - \bigoplus_{i \leq s} Q(i,n)Q_{i-1}...Q_0K_s) \oplus \bigoplus_{i-1 \leq s} BP^*\tilde{K}_s,$$

$$D(i) = \bigoplus_{n \leq i-1} P(n+1)^*\tilde{G}_n \oplus \bigoplus_{s \leq i-2} BP^*\{p, ..., v_s\}\tilde{K}_s.$$

Here elements of $\tilde{K}_s$ and $D(i)$ are even dimensional. Hence all odd dimensional elements generate free $P(i)^*$-modules. Note that if $i > j$, then there are no nontrivial maps from $P(i)^*$-modules to free $P(j)^*$-modules. We also note that there is no possibility that $d_t(v_kx) = v_iy$ for $x \in \tilde{K}_s$ and $y \in E^{odd,*}_t$, $t < 2p^j - 1$. Indeed there is the map i^* of spectral sequences from that for $BP^*(X)$ to that for $P(i)^*(X)$; in the last spectral sequence $E^{*,*}_{2p^i-1} \cong P(i)^* \otimes H^*(X;\mathbb{Z}/p)$ and $i^*(v_iy) \neq 0$. Hence the next nonzero differential must be of the form $d_{2p^i-1}(x) = v_i \otimes Q_i(x)$. Therefore we have

$$E^{*,*}_{2p^i} \cong C(i+1) \oplus D(i) \oplus P(i+1)Q_i...Q_0G_i \oplus BP^*\{p, ..., v_{i-1}\}\tilde{K}_{i-1}.$$

The last term is computed from $Q_i\tilde{K}_{i-1} \neq 0$ and $\mathrm{Ker}\, d_{2p^i-1}|BP^*\{\tilde{K}_{i-1}\} = BP^*\{p, ..., v_{i-1}\}\tilde{K}_{i-1}$, since $Q_i\tilde{K}_{i-1}$ is $P(i)^*$-free in $E^{*,*}_{2p^i-1}$. □

The classifying spaces of groups $BO(n), SO(4), G_2, Spin(m), m \leq 9$ for $p = 2$ and PGL_3, F_4 for $p = 3$, and $(\mathbb{Z}/p)^n$ satisfy the assumption of the above lemma. However $SO(6)$ does not satisfy the above lemma [I].

We will show that the isomorphism (1) in Lemma 5.2 approximates $h^{*,*}(X; \mathbb{Z}/p)$. Let $Ih^{*,*}(X)$ be a $\mathbb{Z}/p[\tau]$-submodule of $h^{*,*}(X; \mathbb{Z}/p)$ generated by image from $h^{*,*}(X)/p$. The following theorem is almost immediate.

Theorem 5.3. *Suppose that (1) in Lemma 5.2 holds. Then we have the injection*

$$Ih^{*,*} \hookrightarrow ((G_{-1}/p \bigoplus_{n=1} Q(1,n)G'_n) - (\bigoplus_{s=1} Q(1,s)K'_s - \tilde{K}'_s)) \otimes \mathbb{Z}/p[\tau],$$

$$h^{*,*}(X; \mathbb{Z}/p) \hookrightarrow (\bigoplus_{n=1} Q(n)G_n - (\bigoplus_{s=1} Q(s)K_s - \tilde{K}_s)) \otimes \mathbb{Z}/p[\tau],$$

with $w(G_n) = n + 1$, $w(G'_n) = n$. Moreover, if some BP^*-module generator in $\mathrm{Ideal}(p, .., v_1)\tilde{K}_s \subset E_\infty^{*,*}$ *is represented by transfer of a Chern class, then* $\mathrm{Ker}(t_{\mathbb{C}}^{2*,*})$ *contains a nonzero element.*

The $P(m)^*(-)$ version of above facts also holds, if we consider the spectral sequence

$$E_2^{*,*} = H^*(X; \mathbb{Z}/p) \otimes P(m)^* \Longrightarrow P(m)^*(X).$$

$(5.3)'$ Let $E_\infty^{*,*} = A \oplus B$, where A (resp. B) is the $P(m)^*$-module generated by generators in $E_\infty^{*,0}$ (resp. in $E_\infty^{*,minus}$).

Lemma 5.4. (1) *If (5.2) is satisfied and, in $(5.3)'$,*

$$A \cong \bigoplus_{n=-1} P(m + n + 1)^* \tilde{G}_n(m), \quad B \cong \bigoplus_{s=0} P(m)^*\{v_m, ..., v_s\}K_s(m),$$

then we have the isomorphism

$$H^*(X; \mathbb{Z}/p) \cong (\bigoplus_{n=-1} Q(m, n + m)G_n(m)) - \bigoplus_{s=0} Q(m, m + s)K_s(m) - \tilde{K}_s(m))$$

with $Q_m ... Q_{m+n}G_n(m) = \tilde{G}_n(m)$ and $Q_m ... Q_{m+s}K_s(m) = \tilde{K}_s(m)$.

(2) *If $Q_m ... Q_{m+n}G_n(m) \in \mathrm{Im}(\rho)$ and $|\tilde{K}_s(m)| = even$, then the converse also holds.*

The $P(m)^*$-versions also hold for $G = (\mathbb{Z}/p)^n, BO(n), BSO(4), p_+^{1+2}$. One application for the above lemma is the following.

Corollary 5.5. *Let $H^*(X; \mathbb{Z}/p)$ (resp. $H^*(Y; \mathbb{Z}/p)$) have the decomposition of Lemma 5.2 (1) (resp. Lemma 5.4 (1) for all $m \geq 0$). Then $H^*(X \times Y; \mathbb{Z}/p)$ also has decomposition similar to that of Lemma 5.2 (1).*

Proof. We get the following isomorphism:

$$Q(n-1)G_{n-1} \otimes H^*(Y; \mathbb{Z}/p)$$

$$\cong Q(n-1)G_{n-1} \otimes (Q(n, n+k)G_k(n) - \bigoplus Q(n, n+t)K_t(n) - \tilde{K}_t(n)))$$

$$\cong (Q(n+k)G_{n-1} \otimes G_k(n)) - (Q(n+t)G_{n-1} \otimes K_t(n) - Q(n-1)G_{n-1} \otimes \tilde{K}_s(n)),$$

since each Q_i is derivative. □

Lemma 5.6. *If $H^*(X; \mathbb{Z}/p) \cong \bigoplus Q(n)G_n$, then $H^*(X \times B\mathbb{Z}/p) \cong \bigoplus Q(n)G'_n$, where*

$$G'_n \cong \mathbb{Z}/p[y]/(y^{p^n})G_n \oplus \mathbb{Z}/p[y]G_{n-1}\{x\}.$$

Proof. Since we have the decomposition

$$H^*(B\mathbb{Z}/p;\mathbb{Z}/p)\cong \mathbb{Z}/p[y]/(y^{p^n})\oplus \mathbb{Z}/p[y]Q(n,n)\{x\},$$

we get the lemma. □

When $X=(B\mathbb{Z}/p)^n$, inductively we get the decomposition $H^*((B\mathbb{Z}/p)^n;\mathbb{Z}/p)\cong \bigoplus Q(n)G_n$. Hence $B=0$ and

$$grBP^*(X)\cong \bigoplus P(n+1)^*\tilde{G}_n,\quad H^{*,*}(X;\mathbb{Z}/p)\cong \bigoplus Q(n)G_n\otimes \mathbb{Z}/p[\tau].$$

Of course these are given in (3.9). The similar facts also hold for $X=BO(n)$. Moreover, W. S. Wilson proved [RWY] that

$$BP^*(BO(n))\cong BP^*[c_1,...,c_n]/(c_1-c_1^*,...,c_n-c_n^*),$$

where c_i^* is the complex conjugate of the Chern class of the usual complex representation. The cohomology $h^{*,*}(BO(n))$ is studied in (4.2).

Next consider the case $X=BSO(4)$. The mod 2 cohomology is $H^*(X;\mathbb{Z}/2)\cong \mathbb{Z}/2[w_2,w_3,w_4]$. The cohomology operation acts as

$$Q_0w_2=w_3,\ Q_1w_3=w_3^2,\ Q_1w_4=w_4w_3,\ Q_1Q_2w_4=w_3^2w_4^2.$$

The integral cohomology is written as

$$H^*(X)_{(2)}\cong Z_{(2)}[w_2^2,w_4]\otimes (Z_{(2)}\{1\}\oplus \mathbb{Z}/2[w_3]\{w_3\}).$$

In the Atiyah-Hirzebruch spectral sequence, nonzero differentials are $d_{2^{i+1}-1}(x)=v_i\otimes Q_i(x)$ for $i=1,2$. We can compute

$$E_\infty^{*,*}\cong E_8^{*,*}\cong Z_{(2)}[c_2]\otimes (BP^*[c_4]\{1,2w_4\}\oplus P(2)^*[c_3]\{c_3\}\oplus P(3)^*[c_3,c_4]\{c_3c_4\}),$$
$$BP^*(X)\otimes_{BP^*}Z_{(2)}\cong Z_{(2)}[c_2,c_4]\otimes (Z_{(2)}\{1,2w_4\}\oplus \mathbb{Z}/2[c_3]\{c_3\}).$$

Hence the assumption of (1) in Lemma 5.2 is satisfied by

$$\tilde{G}'_{-1}\cong \mathbb{Z}/2[c_2,c_4],\quad \tilde{G}'_1=\mathbb{Z}/2[c_2,c_3]\{c_3\}\quad \tilde{G}'_2=\mathbb{Z}/2[c_2,c_3,c_4]\{c_3c_4\},$$
$$\tilde{K}'_0=\mathbb{Z}/2[c_2,c_4]\{2w_4\}.$$

Therefore we get

Proposition 5.7. *Let $w(w_4)=2$. Then the bidegree $\mathbb{Z}/2[\tau]$-module $Ih^{*,*}(BSO(4))$ (resp. $h^{*,*}(BSO(4);\mathbb{Z}/2)$) is isomorphic to a bidegree $\mathbb{Z}/2[\tau]$- submodule of*

$$\mathbb{Z}/2[\tau,c_2]\otimes (\mathbb{Z}/2[c_4]\{1\}\oplus \mathbb{Z}/2[c_3]\otimes Q(1,1)\{w_3\}\oplus \mathbb{Z}/2[c_3,c_4]\otimes Q(1,2)\{w_4\}))$$

(resp. $\mathbb{Z}/2[\tau,c_2]\otimes(\mathbb{Z}/2[c_4]\{1\}\oplus\mathbb{Z}/2[c_3]\otimes Q(1)\{w_2\}\oplus\mathbb{Z}/2[c_3,c_4]\otimes(Q(2)-\mathbb{Z}/p)\{a\})$, *where $Q_0a=w_4$.*

Remark. If $w_4\in H^{4,3}(BSO(4))$, then $Ih^{*,*}(BSO(4))$ is isomorphic to the $\mathbb{Z}/2[\tau]$-module in the above proposition.

Remark. For this case, we have $K_0=\mathbb{Z}/2[c_2]\{a\}$ and $Q_0K_0={K'}_0$ in Lemma 5.2. Indeed, $Q_0a=w_4$. However, $w_4\notin \mathrm{Im}(Q_0)$ in $h^{*,*}(BSO(4);\mathbb{Z}/2)$, because a itself does not exist in $h^{*,*}(BSO(4);\mathbb{Z}/2)$.

We know that the element corresponding to $2w_4$ is represented by a Chern class c_2' of some representation, and this means the Totaro's cycle map $\tilde{cl}$ is epic. Indeed, Totaro and Pandharipande showed that this map is isomorphic, namely,

$$CH^*(BSO(4))_{(2)} \cong Z_{(2)}[c_2, c_3, c_4, c_2']/(2c_3, c_3c_2', {c_2'}^2 - 4c_4).$$

Next consider the $P(1)^*$-version for $BSO(4)$. By using the computations of Q_iw_j [I] and the Atiyah-Hirzebruch spectral sequence, we can prove that

$$grP(1)^*(BSO(4)) \cong P(1)^*[c_4]\{1, v_1w_2w_4\} \oplus P(2)^*\{c_3\}$$
$$\oplus P(3)^*[c_3]\{c_3^2, c_3c_4\} \oplus P(3)^*[c_4]\{c_3c_4^2\} \oplus P(4)^*[c_3, c_4]\{c_3^2c_4^2\}$$

We have another decomposition of $H^*(BSO(4); \mathbb{Z}/2)$.

Proposition 5.8.

$$H^*(BSO(4); \mathbb{Z}/2) \cong \mathbb{Z}/2[c_4] \oplus Q(1,1)\{w_3\} \oplus \mathbb{Z}/2[c_3] \otimes (Q(1,2)\{w_2, w_4\})$$
$$\oplus \mathbb{Z}/2[c_4] \otimes (Q(1,2)\{c_4w_4\}) \oplus \mathbb{Z}/2[c_3, c_4] \otimes (Q(1,3)\{Q_1^{-1}w_2w_4\} - \{Q_1^{-1}w_2w_4\}).$$

We consider the relation between $grBP^*(X)$ and $grP(1)^*(X)$. When $X = BSO(4)$, it is known [KY] that $K(n)^{odd}(X) = 0$, and hence

$$P(m)^*(X) \cong P(m)^* \otimes_{BP^*} BP^*(X).$$

Therefore no $P(m)^*(X)$ is v_m-torsion. Of course we have already seen that for the $grBP^*(-)$-versions the above facts do not hold. If there is a relation $pa_0 + v_1a_1 + v_2a_2 + ... = 0 \in BP^*(X)$, then it is known [Y1] that there is $y \in H^*(X; \mathbb{Z}/p)$ such that $Q_i(y) = \rho(a_i)$, where $\rho : BP^*(X) \to H^*(X; \mathbb{Z}/p)$ is the Thom map. In $H^*(BSO(4); \mathbb{Z}/2)$, we see that

$$Q_0(w_2w_3) = c_3,\ Q_1(w_2w_3) = 0,\ Q_2(w_2w_3) = c_3^2.$$

Hence we have the relation $2c_3 + v_2c_3^2 + ... = 0 \in BP^*(BSO(2))$. This shows that c_3^2 is $P(2)^*$-free in $grBP^*(BSO(4))$, but it is a $P(3)^*$-free module in

$$grP(1)^*(BSO(4)) = gr(BP^*(BSO(4))/2).$$

We also see that for $x = c_3w_3w_4 + c_4w_2w_3$

$$Q_0(x) = c_3c_4, \quad Q_1(x) = Q_2(x) = 0, \quad Q_3(x) = c_3^2c_4^2.$$

This means that $2c_3c_4 + v_3c_3^2c_4^2 + ... = 0 \in BP^*(BSO(4))$. Hence $c_3^2c_4^2$ is a $P(3)^*$-free module in $grBP^*(BSO(4))$ but is a $P(4)^*$-free module in $gr(BP^*(BSO(4))/2)$.

Next consider the case $X = BSO(6)$. In this case the assumption (5.3) is not satisfied. In fact, Inoue computed [I]

$$grBP^*(BSO(6)) \cong \bigoplus_{n=-1}^{4} P(n+1)^*\tilde{G}_n \oplus P(2)^*/(v_2^2)\tilde{G}_1' \oplus BP^*\{2\}\tilde{K}_0.$$

(For details, see [I].) In particular, he showed that

$$d_5(2w_6) = v_1^2w_6w_5, \quad d_{11}(v_1 \otimes w_6w_5) = v_2^2w_6^2w_5^2.$$

However, even this case we can show that

$$H^*(BSO(6); \mathbb{Z}/2) \subset \bigoplus Q(n)G_n \oplus Q(1)G_1'.$$

Moreover, R. Field [F] announced that

$$CH^*(BSO(2n)) \cong Z_{(2)}[c_2, ..., c_{2n}, y_n]/(2c_{odd}, c_{odd}y_n, y_n^2 - (-1)^n 2^{2n-2}c_{2n})$$

with $deg(y_n) = 2n$. Hence $\mathrm{Ideal}(y_n) \subset \mathrm{Ker}(t_{\mathbb{C}})$. However, y_n is not represented by a Chern class of any representation for $n > 2$. We also note that $BP^*(BSO(2n))$ are not known for $n > 3$.

The cases $X = BG_2, BSpin(7)$ are quite similar to the case $X = BSO(4)$. Indeed, $CH^*(BG_2)/2$ and $h^{*,*}(BG_2;\mathbb{Z}/2)$ have been discussed in §4, and

$$grBP^*(BG_2) \cong Z_{(2)}[c_4, c_6] \otimes (BP^*\{1, 2w_4\} \oplus P(3)^*[c_7]\{c_7\}).$$

The infinite term of the spectral sequence for $BP^*(BSpin(7))$ is computed by

$$Z_{(2)}[c_4, c_6] \otimes (BP^*[c_8]\{1, 2w_4, 2w_8, 2w_4w_8, v_1w_8\} \\ \oplus P(3)^*[c_7]\{c_7\} \oplus P(4)^*[c_7, c_8]\{c_7c_8\}).$$

Therefore we obtain

Corollary 5.9. *Let $w(w_8) = 2$. Then the cohomology $Ih^{*,*}(BSpin(7))$ (resp. $h^{*,*}(BSpin(7);\mathbb{Z}/2)$) is isomorphic to a $\mathbb{Z}/2[\tau]$-submodule of $\mathbb{Z}/2[\tau, c_4, c_6] \otimes A$ (resp. $\mathbb{Z}/2[\tau, c_4, c_6] \otimes B$), where*

$$A = \mathbb{Z}/2[c_8] \oplus \mathbb{Z}/2[c_7] \otimes Q(1,2)\{w_4\} \oplus \mathbb{Z}/2[c_7, c_8] \otimes (Q(1,3) - \mathbb{Z}/p)\{b\},$$

$$B = (\mathbb{Z}/2[c_8] \oplus \mathbb{Z}/2[c_7](Q(2) - \mathbb{Z}/p)\{a\} \oplus \mathbb{Z}/2[c_7, c_8](Q(3) - Q(1) + Q_0Q_1 - Q_2)\{c\}$$

with $Q_1 b = w_8,\ Q_0 a = w_4,\ Q_1Q_0 c = w_8,\ Q_2Q_0 c = w_4w_8.$

The algebra $BP^*(BSpin(7)) \otimes_{BP^*} Z_{(2)}$ is isomorphic to

$$Z_{(2)}[c_4, c_6, c_8] \otimes (Z_{(2)}\{1, 2w_4, 2w_8, 2w_4w_8\} \oplus \mathbb{Z}/2\{v_1w_8\} \oplus \mathbb{Z}/2[c_7]\{c_7\}).$$

It is known that $2w_2, 2w_8, 2w_4w_8$ are represented by Chern classes but v_1w_8 is not. However, Totaro has shown that the cycle map $\tilde{cl}$ is epic for this case also (see [ScY], [Y3]).

Corollary 5.10. *There is an epimorphism*

$$CH^*(BSpin(7)) \to Z_{(2)}[c_4, c_6, c_8'] \otimes (Z_{(2)}\{1, c_2', c_4', c_6'\} \oplus \mathbb{Z}/2\{\xi_3\} \oplus \mathbb{Z}/2[c_7]\{c_7\}),$$

where c_i' is the i-th Chern class of complexification of the spin representation Δ and ξ_3 is a 6-dimensional element which is not represented by Chern classes. Thus c_2', c_4', c_6' are in $\mathrm{Ker}(\rho_2)$ and $\xi_3 \in \mathrm{Ker}(\rho)$.

Next we consider the case $p = odd$. The cases PGL_3 and p_+^{1+2} are easy, and $Ih^{*,*}(BG)$ are given. For example, for $E = p_+^{1+2}$

$$grBP^*(BE) \cong BP^* \otimes H^{even}(BE)/(v_1Q_1H^{odd}(BE)).$$

Finally we consider the case $G = F_4, p = 3$, whose Chow ring is still unknown. The mod 3 cohomology of F_4 is isomorphic to $H^*(BF_4;\mathbb{Z}/3) \cong C \otimes D$ [Tod] with $D = Z_{(3)}[x_{36}, x_{48}]$ and

$$C = \mathbb{Z}/3[x_4, x_8] \otimes \{1, x_{20}, x_{20}^2\} \oplus \mathbb{Z}/3[x_{26}] \otimes \Lambda(x_9) \otimes \{1, x_{20}, x_{21}, x_{25}\},$$

where two terms of C have the intersection $\{1, x_{20}\}$. Then we can prove [KY]

$$grBP^*(BF_4) \cong D \otimes (BP^*\{1, 3x_4\} \oplus BP^* \otimes E \oplus P(3)^*[x_{26}]\{x_{26}\})$$

with $E = Z_{(3)}[x_4, x_8]\{ab | a, b \in \{x_4, x_8, x_{20}\}\}$. Therefore we obtain

Corollary 5.11. *Let $w(E) = 0$ and $w(x_4) = 2$. Then $Ih^{*,*}(BF_4)$ is a $\mathbb{Z}/3[\tau]$-submodule of*

$$D \otimes (\mathbb{Z}/3\{1\} \oplus E \oplus \mathbb{Z}/3[x_{26}] \otimes Q(1,2)\{x_4\})) \otimes \mathbb{Z}/3[\tau].$$

The element $3x_4$ can be proved to be represented by a Chern class, and $x_{26} = Q_2Q_1x_4$. The element x_{36} is also represented by a Chern class, and $P^3x_{36} = x_{48}$. If we can prove that $E/3 \subset \mathrm{Im}(cl_p)$ and $x \in H^{4,3}(BF_4, \mathbb{Z}/3)$, then the above module is just $Ih^{*,*}(BF_4)$ for $p = 3$.

Let G be a simply connected Lie group. Then $H^3(G;\mathbb{Z}) \cong \mathbb{Z}$ and $H^4(G;\mathbb{Z}) \cong 0$. Suppose that $H^*(G;\mathbb{Z})$ has p-torsion. Then it is known that there is an element $x' \in H^3(G;\mathbb{Z})$ such that $0 \neq Q_1x' \in H^{2p+2}(G;\mathbb{Z}/p)$. Taking the classifying space, we get an element $x \in H^4(BG;Z)$ such that $Q_1x \neq 0$ in $H^{2p+3}(BG;\mathbb{Z}/p)$. By Totaro [To2] it is known that $CH^*(BG) \otimes \mathbb{Q} \cong H^*(BG) \otimes \mathbb{Q}$. Hence there is an $s \geq 1$ such that $p^sx_4 \in H^4(BG)$ is in $\mathrm{Im}(cl)$. Thus there is a nonzero element $c \in CH^2(BG)/p$ with $t_{\mathbb{C}}^{2*,*}(c) = 0$. For the groups G_2 or $Spin(7)$ for $p = 2$ and $G = F_4$ for $p = 3$, we can take $s = 1$, since px_4 is represented by the second Chern class c_2.

Proposition 5.12. *Let $p = 2$, 3 or 5. There is a classifying space $B\tilde{G}$ such that for all m, n with $3 \leq n+1 < m \leq 2n$, the kernel $\mathrm{Ker}(t_{\mathbb{C}}^{m,n})$ is nonzero.*

Proof. Let $\tilde{G} = G \times (\mathbb{Z}/p)^\infty$, where $G = G_2, p = 2$, $G = F_4, p = 3$ or $G = E_8, p = 5$. Recall that $(B\mathbb{Z}/p)^n$ satisfies the Künneth formula for all spaces. For $\mathbb{Z}/p[\tau]$-module generators $x \in H^{*,*}((B\mathbb{Z}/p)^\infty;\mathbb{Z}/p)$, the elements xc are all nonzero and all in $\mathrm{Ker}\, t_{\mathbb{C}}$. □

6. Homotopy category

From the category Spc, Voevodsky constructed [Vo1], [Vo2], [MoVo] the ($\mathbb{A}^1$, algebraic) homotopy category Hot and the stable homotopy category $SHot$. There are two different types of spheres in Spc:

$$S_s^1 = \mathbb{A}^1/\{0,1\} \quad \text{and} \quad S_t^1 = \mathbb{A}^1 - \{0\}. \tag{6.1}$$

The Tate object is $T = \mathbb{A}^1/(\mathbb{A}^1 - 0) \cong \mathbb{P}^1 \cong S_t^1 \wedge S_s^1$ in Hot. The category $SHot$ is defined by T as the suspension, e.g., $E = \{E_i\}$, $E_i \in Spt$ is a spectrum if there is a map $T \wedge E_i \to E_{i+1}$.

Let Σ_T^∞ be the functor from Spc to T-spectra that takes X to $\{T^i \wedge X\}$. If E is a T-spectrum, then the motivic (generalized) cohomology $E^{*,*}(-)$ is defined by

$$E^{m,n}(X) = \mathrm{Hom}_{SHot}(\Sigma_T^\infty(X), S_s^{m-n} \wedge S_t^n \wedge E), \tag{6.2}$$

$$E_{m,n}(X) = \mathrm{Hom}_{SHot}(S_s^{m-n} \wedge S_t^n, \Sigma_T^\infty(X) \wedge E), \tag{6.3}$$

where $\mathrm{Hom}_{SHot}(-,-)$ is the homomorphism defined on $SHot$.

The realization map $t_{\mathbb{C}}$ is originally defined as the functor $t_{\mathbb{C}} : X \to X(\mathbb{C})$ from Hot to the category of homotopy spaces. Note that this induces

$$t_{\mathbb{C}} : E^{m,n}(X) \to (t_{\mathbb{C}}E)^m(X(\mathbb{C})). \tag{6.4}$$

The spectrum for the ordinary motivic cohomology is defined as follows. Let $L(X;R)$ for $R = \mathbb{Z}$ or $\mathbb{Z}/p$ be the presheaf sending a connected U to the free R-module generated by the set of all closed irreducible $W \subset U \times X$ such that the projection $W \to U$ is finite and surjective. The Eilenberg-MacLane spectrum is defined as

$$K(R(n), 2n) = L(\mathbb{A}^n;R)/L(\mathbb{A}^n - \{0\};R).$$

Voevodsky proved that $K(R(n),2n)$ is the Ω-spectrum for the suspension T, namely, $K(R(n),2n) \cong \Omega_T K(R(n+1),2n+2)$ in Hot. Define also, for $m < 2n$,

$$K(R(n),m) = \Omega_{S^1}^{2n-m}(R(n),2n). \tag{6.5}$$

Thus the ordinary motivic cohomology is defined by

$$H^{m,n}(X;R) = \mathrm{Hom}_{Hot}(X, K(R(n),m)). \tag{6.6}$$

Question 6.1. Let $k \subset \mathbb{C}$, and let $0 \neq \tau_n \in H^{n,n}(K(\mathbb{Z}/p(n),n);\mathbb{Z}/p)$ (resp. $\tau'_{n+1} \in H^{n+1,n}(K(\mathbb{Z}_{(p)}(n),n+1);\mathbb{Z}/p)$) be the fundamental class (representing the identity map). Then are there isomorphisms

$$h^{2*,*}(K(\mathbb{Z}/p(n),n);\mathbb{Z}/p) \cong \mathbb{Z}/p[Q_{i_{n-1}}...Q_{i_1}Q_0\tau_n | 0 < i_1 < ... < i_{n-1}],$$
$$h^{2*,*}(K(Z_{(p)}(n),n+1);\mathbb{Z}/p) \cong \mathbb{Z}/p[Q_{i_{n-1}}...Q_{i_1}\tau'_{n+1} | 0 < i_1 < ... < i_{n-1}]?$$

It is well known that the dual A_{p*} of the (topological) Steenrod algebra A_p^* is isomorphic to $\mathbb{Z}/p[\xi_1,...] \otimes \Lambda(\tau_0,...)$, $|\xi_i| = 2(p^i-1)$, $|\tau_i| = 2p^i-1$. Let $P^J \in A_p^*$ (resp. $Q^I \in A_p^*$) be the dual of $\xi_1^{j_1}...$ (resp. $\tau_0^{i_0}...$, $i_k = 0$ or 1), so that $A_p^* \cong \mathbb{Z}/p\{P^J Q^I\}$. Note that $Q^I = \pm Q_0^{i_0}....$ Define $m(J) = \sum_{k=1} j_k$ and $m(I) = \sum_{k=0} i_k$. Then it is also known [Ta] that

$$H^*(K(\mathbb{Z}/p,n);\mathbb{Z}/p) \cong \mathbb{Z}/p[Q^I P^J \tau_n | m(I) + 2m(J) < n + i_0]. \tag{6.7}$$

On the other hand, suppose that $Q^I P^J \tau_n \in H^{m,n}(K(\mathbb{Z}/p(n),n);\mathbb{Z}/p)$ for $m \geq 2n$, i.e., $w(Q^I P^J \tau) \leq 0$. Since $w(P^j) = 0$ and $w(Q_i) = -1$, we see that

$$0 \geq w(Q^I P^J \tau_n) = n - m(I).$$

This implies $m(J) = 0, m(I) = n$ and $i_0 \neq 0$. Hence we know that $Q^I P^J \tau$ is the form of the ring generator of the polynomial in the above question.

Remark. Let us write the above as $A = \mathbb{Z}/p[Q_{i_{n-1}}...Q_{i_1}Q_0\tau | 0 < i_1 < ... < i_{n-1}]$. By Tamanoi [Ta], the image $\rho_p(K(\mathbb{Z}/p,n)) = A \subset H^*(K(\mathbb{Z}/p,n);\mathbb{Z}/p)$. Moreover, there is [RWY] the isomorphism $BP^*(K(\mathbb{Z}/p,n)) \otimes_{BP^*} \mathbb{Z}/p \cong A$.

7. Algebraic cobordism

Let BGL denote the infinite Grassmannian, the union of $GL_N(\infty)$ over N. The corresponding generalized cohomology theory is the algebraic K-theory. The motivic cobordism theory $MGL^{*,*}(-)$ is the generalized cohomology theory defined by the Thom spectrum $MGL = \{Th(E_n \to GL_n)\}_n$ identifying $Th(E \oplus O) \cong T \wedge Th(E)$ and $E_n \oplus O \to E_n$ for the trivial line bundle O. It is known (Hu-Kříž [HK], Vezzosi [Ve2]) that

$$MGL^{*,*}((\mathbb{P}^\infty)^n) \cong MGL^{*,*}(pt)[y_1,...,y_n], \tag{7.1}$$
$$MGL^{*,*}(BGL_n) \cong MGL^{*,*}(pt)[c_1,...,c_n], \tag{7.2}$$

where the c_i are identified with the elementary symmetric polynomials in the y_i's. Hence the Chern classes are also defined in $MGL^{2*,*}(BG)$. The realization maps

$$t_{\mathbb{C}}^{2*,*} : MGL^{2*,*}(BG)_{(p)} \to MU^*(BG)_{(p)}$$

are epic for $G = O(n), SO(4), G_2$ for $p = 2$ and p_+^{1+2} for all primes, because the $MU^*(BG)_{(p)}$ are generated by Chern classes.

For a smooth scheme X over $k \subset \mathbb{C}$, Levine and Morel [LM1], [LM2] constructed the algebraic cobordism theory $\Omega^*(X)$ such that there are natural maps

$$(7.3) \qquad \rho_H : \Omega^*(X) \to H^{2*,*}(X), \qquad \rho_{MGL} : \Omega^*(X) \to MGL^{2*,*}(X)$$

with $\rho_H = \rho_{(MGL,H)}\rho_{MGL}$ for the algebraic Thom map $\rho_{(MGL,H)} : MGL^{*,*}(X) \to H^{*,*}(X)$. Moreover, they proved that

$$(7.4) \quad \rho_H \otimes_{\Omega^*} \mathbb{Z} : \Omega^*(X) \otimes_{\Omega^*} \mathbb{Z} \cong H^{2*,*}(X), \qquad t_{\mathbb{C}}^{2*,*}\rho_{MGL} : \Omega^*(pt) \cong MU^{2*}(pt).$$

This implies the motivic version of the Totaro cycle map $\tilde{cl}$:

$$(7.5) \qquad \rho_{MGL}(\rho_H \otimes_{\Omega^*} \mathbb{Z})^{-1} : CH^*(X) \to MGL^{2*,*}(X) \otimes_{MGL^{2*,*}} \mathbb{Z},$$

and moreover $t_{\mathbb{C}}^{2*,*}\rho_{MGL}(\rho_H \otimes_{\Omega^*} \mathbb{Z})^{-1}$ is the Totaro cycle map $\tilde{cl}$. Thus the Thom map $\rho^{2*,*}_{(MGL,H)} : MGL^{2*,*}(X) \to H^{2*,*}(X)$ is always epic.

For groups $G = (\mathbb{Z}/p)^n$, $O(n)$, we can easily prove that

$$(7.6) \qquad \Omega^*(BG) \cong MU^*(BG).$$

Hence in these cases $MGL^{2*,*}(BG)$ contains $MG^*(BG)$ as a splitting subring.

Corollary 7.1. *Let* $\tilde{cl}_p : CH^*(BG)/p \to MU^*(BG) \otimes_{MU^*} \mathbb{Z}/p$ *be epic. Then* $t_{\mathbb{C}}^{2*,*} : MGL^{2*,*}(X)/p \to MU^*(BG)/p$ *is epic, and* $\operatorname{Im} \rho_{(MGL,h)} \subset \mathbb{Z}/p[\tau] \otimes h^{2*,*}(X;\mathbb{Z}/p)$, *where* $\rho_{(MGL,h)} : MGL^{*,*}(X) \to h^{*,*}(X;\mathbb{Z}/p)$ *is the induced Thom map.*

The modified cycle maps $\tilde{cl}$ are epic also for the groups $Spin(7)$ for $p = 2$ and PGL_3 for $p = 3$.

By the Thom isomorphism, we get $MGL^{*,*}(BGL) \cong MGL^{*,*}(MGL)$. This means that the Steenrod algebra of $MGL^{*,*}(-)$ is generated as an $MGL^{*,*}(pt)$-module by the Landweber-Novikov operation S_α:

$$(7.7) \qquad MGL^{*,*}(MGL) \cong MGL^{*,*}(pt)\{S_\alpha | \alpha = (i_1, ..., i_n),\ i_j \geq 0\}.$$

Here $S_\alpha : MGL^{*,*}(X) \to MGL^{*+2|\alpha|,*+|\alpha|}(X)$ and $|\alpha| = \sum_k i_k k$. These operations satisfy the Cartan formula

$$(7.8) \qquad S_\alpha(xy) = \sum_{\alpha=\beta+\gamma} S_\beta(x) S_\gamma(y),$$

and $S_\alpha | MU^*(pt)$ is the usual Landweber-Novikov operation.

Kříž, Hu and Vezzosi construct algebraic Brown-Peterson theory $ABP^{*,*}(-)$ by using a modified Quillen argument. Here we note that we can also construct algebraic BP-theory by using the technique of Novikov(5.4 in [N]). Recall that $MU^*_{(p)} \cong \mathbb{Z}_{(p)}[x_1, ...], |x_i| = -2i$. Define

$$(7.9) \qquad \Delta_{x_i} = \sum_{q \geq 1} (x_i / S_{\Delta_i}(x_i))^{q-1} S_{q\Delta_i},$$

where $\Delta_i = (0, ..., 0, 1, 0, ..., 0)$ (1 in i-th place). Note that $\Delta_{x_i}(x_i) = S_{\Delta_i}(x_i) = 1$ if $i \neq p^j - 1$. Then we can easily prove that $\pi_i = 1 - x_i \Delta_{x_i}$ is a multiplicative projection such that $\pi_i(x_j) = (1 - \delta_{ij})x_j$. Essentially composing (for details, see p. 587 in [N]) the π_i for all $i \neq p^j - 1$, we get the multiplicative projection $\Phi : MGL_{(p)} \to MGL_{(p)}$ such that

$$(7.10) \qquad \Phi(x_i) = \begin{cases} x_i & (\text{if } i = p^j - 1 \text{ for some } j), \\ 0 & (\text{otherwise}). \end{cases}$$

Define the algebraic Brown-Peterson spectrum by $\Phi MGL = ABP$. Of course $t_{\mathbb{C}}(ABP) = BP$

Theorem 7.2. *Identify $BP^* = MU^*_{(p)}/(x_i|i \neq p^j - 1)$. Then*

$$ABP^{*,*}(X) \cong BP^* \otimes_{MU^*_{(p)}} MGL^{*,*}(X)_{(p)}.$$

Proof. Since $\pi_{x_i}(a) = (1 - x_i\Delta_{x_i})a = a \bmod(x_i)$, we get $\Phi(a) = a \bmod(x_i|i \neq p^j - 1)$ for all $a \in MGL^{*,*}(X)$. The isomorphism is proved, since $ABP^{*,*}(X) \subset MGL^{*,*}(X)_{(p)}$ by the property $\Phi^2 = \Phi$. $\square$

Since $ABP^{*,*}(pt) \cong BP^* \otimes_{MU^*_{(p)}} MGL^{*,*}(pt)$, we can write the above isomorphism as

$$ABP^{*,*}(X) \cong ABP^{*,*} \otimes_{MGL^{*,*}_{(p)}} MGL^{*,*}(X)_{(p)}.$$

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