EXAMPLES FOR THE MOD $p$ MOTIVIC COHOMOLOGY
OF CLASSIFYING SPACES

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Abstract. Let $BG$ be the classifying space of a compact Lie group $G$. Some examples of computations of the motivic cohomology $H^{*,*}(BG;\mathbb{Z}/p)$ are given, by comparing with $H^*(BG;\mathbb{Z}/p)$, $CH^*(BG)$ and $BP^*(BG)$.

1. Introduction

Let $p$ be a prime number and $k$ a subfield of the complex number field $\mathbb{C}$. Let $k$ contain a primitive $p$-th root of unity. Given a scheme $X$ of finite type over $k$, the mod $p$ motivic cohomology $H^{*,*}(X;\mathbb{Z}/p)$ has been defined by Suslin and Voevodsky ([Vo1], [Vo2]). When $X$ is smooth, the subring $H^{2*,*}(X;\mathbb{Z}/p) = \bigoplus_n H^{2n,n}_p(X;\mathbb{Z}/p)$ is identified with the classical mod $p$ Chow ring $CH^{*,*}(X)/p$ of algebraic cycles on $X$.

The inclusion $t_\mathbb{C} : k \subset \mathbb{C}$ induces a natural transformation (realization map) $t_\mathbb{C}^{m,n} : H^{m,n}_p(X;\mathbb{Z}/p) \rightarrow H^m(X(\mathbb{C});\mathbb{Z}/p)$, where $X(\mathbb{C})$ is the complex variety of $\mathbb{C}$-valued points of $X$. Let us write the coimage of $t_\mathbb{C}^{*,*}$ as

$$h^{*,*}(X;\mathbb{Z}/p) = \bigoplus_{m,n} H^{m,n}_p(X;\mathbb{Z}/p)/\text{Ker}(t_\mathbb{C}^{m,n}).$$

It is known that there is an element $\tau \in H^{0,1}(\text{Spec}(k);\mathbb{Z}/p)$ with $t_\mathbb{C}^{*,*}(\tau) = 1$. Then we have the bigraded $\mathbb{Z}/p[\tau]$-algebra monomorphism

$$h^{*,*}(X;\mathbb{Z}/p) \hookrightarrow H^*(X(\mathbb{C});\mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau,\tau^{-1}]$$

where the bidegree of $x \in H^n(X(\mathbb{C});\mathbb{Z}/p)$ is given by $(n,n)$. If $k = \mathbb{C}$ and the Beilinson-Lichtenbaum condition $[Vo2]$ is satisfied for $p$, then we also have the injection $H^*(X(\mathbb{C});\mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau] \hookrightarrow h^{*,*}(X;\mathbb{Z}/p)$.

When $x \in H^{m,n}(X;\mathbb{Z}/p)$, define the weight of $x$ by $w(x) = 2n - m$. Clearly $w(x) = 0$ if and only if $x \in CH^*(X)/p$. Voevodsky has extended the Steenrod algebra $A^n_p$ of cohomology operations to the case of motivic cohomology. Among them, we have the Milnor primitive operation

$$Q_i : H^{*,*}(X;\mathbb{Z}/p) \rightarrow H^{*,*+2p^{i-1},*+p^i-1}(X;\mathbb{Z}/p),$$

so that it is sent to the usual Milnor operation $Q_i$ by the realization map $t_\mathbb{C}$. Hence $w(Q_i) = -1$, and the $Q_i (0 \leq i)$ form an exterior algebra $\Lambda(Q_0, Q_1, \ldots) \subset A^n_p$ also for the motivic cohomology. To simplify the notation, let us write the exterior algebra $Q(n) = Q(0, \ldots, Q_n)$ for $n \geq 0$ and $Q(-1) = \mathbb{Z}/p$. 

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In this paper we are mainly concerned with the following case. For \( n \geq 1 \), let \( G_n \) be a \( \mathbb{Z}/p \)-module and \( Q(n)G_n \) the free \( Q(n) \)-module generated by \( G_n \). Moreover, the scheme \( X \) satisfies the assumption that there is a \( \mathbb{Z}/p \)-module injection

\[
(1.3) \quad j_C : H^*(X(\mathbb{C}); \mathbb{Z}/p) \hookrightarrow \bigoplus_{n=-1}^{\infty} Q(n)G_n \quad \text{with} \quad j_C^{-1}(Q_0\ldots Q_n G_n) \subset \text{Im}(t_C^{2*,*})
\]

such that \( p_n j_C : H^*(X(\mathbb{C}); \mathbb{Z}/p) \rightarrow Q(n)G_n \) is the \( Q(n) \)-module map and \( p'_n p_n j_C : H^*(X(\mathbb{C}); \mathbb{Z}/p) \rightarrow Q_0\ldots Q_{n-1}G_n \) is a surjection for each \( n \), where \( p_n : \bigoplus Q(n)G_n \rightarrow Q(n)G_n \) and \( p'_n : Q(n)G_n \rightarrow Q_0\ldots Q_{n-1}G_n \) are the projections. (We do not assume a \( Q(n) \)-module structure on the right-hand side module in (1.3).)

We take the weight on the right-hand side by putting \( w(x) = n + 1 \) for every \( x \in G_n \) (simply write \( w(G_n) = n + 1 \)), so that \( w(Q_0\ldots Q_n x) = 0 \). Then we get the injection of bigraded \( \mathbb{Z}/p \)-modules

\[
(1.4) \quad j : h^{*,*}(X; \mathbb{Z}/p) \hookrightarrow \bigoplus_{n=-1}^{\infty} Q(n)G_n \otimes \mathbb{Z}/p[\tau]
\]

such that the composition \( (p_n \otimes \mathbb{Z}/p[\tau])j : h^{*,*}(X; \mathbb{Z}/p) \rightarrow Q(n)G_n \otimes \mathbb{Z}/p[\tau] \) is the bigraded \( Q(n) \)-module map.

The above argument has its counterpart in the \( BP \)-theory of \( X(\mathbb{C}) \). As we know, \( BP^*(-) \) is the cohomology theory with the coefficient ring \( BP^* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots] \), \( |v_1| = -2(p^i - 1) \). Let us write \( BP^*/(p, v_1, \ldots, v_{m-1}) \) as \( P(m)^* \). The Atiyah-Hirzebruch spectral sequence

\[
E_2^{*,*} = H^*(X(\mathbb{C})) \otimes BP^* \Rightarrow BP^*(X(\mathbb{C}))
\]

has the differential

\[
(1.5) \quad d_{2p^i-1}(x) = Q_i(x) \otimes v_i \mod(M_i),
\]

where \( M_i \) is the ideal of \( E_2^{2p^i-1} \) generated by elements in \( (p, v_1, \ldots, v_{i-1})E_2^{2*,*} \). We assume here that nonzero differentials are all of the form (1.5) and that \( H^*(X(\mathbb{C})) \) has no higher \( p \)-torsion. Then we easily see that (1.3) implies

\[
(1.6) \quad E_\infty^{*,*} \cong \bigoplus_{n=-1}^{\infty} P(n+1)^*\hat{G}_n \oplus B \quad \text{with} \quad \hat{G} = Q_0\ldots Q_n G_n,
\]

where \( P(n+1)^*\hat{G}_n \) is the free \( P(n+1)^* \)-module generated by elements in \( \hat{G}_n \) and \( B \) is the \( BP^* \)-submodule of \( E_\infty^{2*,*} \) of generators in \( \text{Ideal}(p, v_1, \ldots)E_2^{2*,*} \). Conversely, by the same assumption, if \( \hat{G}_n \subset \text{Im}(t_C^{2*,*}) \), then the isomorphism (1.6) implies the existence of the injections \( j_C \) in (1.3) and so \( j \) in (1.4).

Let \( \rho : BP(X(\mathbb{C})) \otimes B_{BP^*} \mathbb{Z}/p \rightarrow H^*(X(\mathbb{C}); \mathbb{Z}/p) \) be the Thom map. Then (1.6) and \( \hat{G}_n \subset \text{Im}(t_C^{2*,*}) \) imply that

\[
\text{Im}(t_C^{2*,*}) = \text{Im}(\rho) \cong \bigoplus_{n=-1}^{\infty} \hat{G}_n \subset BP^*(X(\mathbb{C})) \otimes B_{BP^*} \mathbb{Z}/p.
\]

More generally, B. Totaro [To1], [To2] constructed the modified cycle map

\[
(1.7) \quad c_B^* : CH^*(X)/p \rightarrow BP^*(X(\mathbb{C})) \otimes B_{BP^*} \mathbb{Z}/p
\]

in such a way that the composition \( \rho c_B^* \) is the realization map \( t_C^{2*,*} \). If a \( BP^* \)-module generator of \( B \) in (1.6) is represented by transfer of a Chern class, then
this element gives a nonzero element in $\text{Ker}(t^2_0)$ by the modified cycle map $t^2_0$. Using this argument, Totaro found nonzero elements in $\text{Ker}(t^2_0)$ when $X$ is the classifying space $BSO(4)$.

The motivic cohomology of the classifying space is defined as follows. Let $G$ be a linear algebraic group over $k$. Let $V$ be a representation of $G$ such that $G$ acts freely on $V - S$ for some closed subset $S$. Then $(V - S)/G$ exists as a quasi-projective variety over $k$. Following Totaro [To1] and Voevodsky, define

$$H^{*,*}(BG; \mathbb{Z}/p) = \lim_{\dim(V), \text{codim}(S) \to \infty} H^{*,*}((V - S)/G; \mathbb{Z}/p).$$

The topological space $BG(\mathbb{C}) = \lim((V - S)/G)(\mathbb{C})$ is the usual classifying space $BG$. Hence we write the $\mathbb{C}$-value points $BG(\mathbb{C})$ simply as $BG$.

We will show that the isomorphism (1.6) is satisfied when $X = BG$ for the following cases: $O(n), SO(4), Ds, G_2, Spin(7)$ for $p = 2$, $PGL_3, F_4$ for $p = 3$ and the extraspecial $p$-group $p_{1+2}$ of order $p^3$ and of exponent $p$ for odd primes. (However note that $H^*(BP_{1+2})$ has $p^2$-torsion."

Hence we will prove (1.4) for these $BG$. Moreover, when $k = \mathbb{C}$ and $G = O(3)$ for $p = 2$, $PGL_3$ for $p = 3$, $p_{1+2}$ and $(\mathbb{Z}/p)^n$ for all primes, we will show that

$$h^{*,*}(BG; \mathbb{Z}/p) \cong \bigoplus Q(n)G_n \otimes \mathbb{Z}/p[\tau].$$

S. Wilson [RWY] first constructed the decomposition (1.3) so that $j_2$ is an isomorphism for $X = BO(n)$, and next computed $BP^*(BO(n))$. However, it is unknown whether $j$ in (1.4) is an isomorphism or not for $X = BO(n)$, $n \geq 4$.

The contents of this paper are as follows. The aim of §§2 and 3 is a short introduction to motivic cohomology for algebraic topologists unfamiliar with it. In these sections, we concentrate on the computation of $H^*(B(\mathbb{Z}/p)^n; \mathbb{Z}/p)$. In §4, we deal with the study of $h^{*,*}(X; \mathbb{Z}/p)$, making no use of $BP^*(BG)$ but Milnor’s operation $Q_i$ instead. In §5, we give an account of $h^{*,*}(BG; \mathbb{Z}/p)$ expressed in term of $BP^*(BG)$. Also in this section we give some results on $\text{Ker}(t^2_0)$. The motivic cohomology of the Eilenberg-MacLane space $K(\mathbb{Z}/p(n), n)$ is studied in §6. In §7, we give some comments on algebraic cobordism theory and algebraic $BP$-theory.

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2. CHOW RING, MILNOR K-THEORY, ÉTALE COHOMOLOGY

We use the category $\text{Spa}$ of (algebraic) spaces, along with schemes $A$, their quotients $A_1/A_2$ and colim$(A_\alpha)$, all defined by Voevodsky [Vo2], [MoVo]. Here schemes are defined over a field $k$ with $ch(k) = 0$. The motivic cohomology is the double indexed cohomology defined by Suslin and Voevodsky, directly related with the Chow ring and Milnor $K$-theory.

(CH) For a smooth scheme $X$ we have $H^{2n,n}(X) \cong CH^n(X)$, the classical Chow group of codim $n$ cycles on $X$.

(MK) $H^{n,n}(\text{Spec}(k)) \cong K^M_n(k)$, the Milnor $K$-group for the field $k$.

For a smooth variety $X$ with $\dim(X) = n$, the Chow ring is the sum $CH^*(X) = \bigoplus_i CH^i(X)$, where

$$CH^i(X) = \{(n - i) \text{ cycles in } X \}/(\text{ rational equivalence}).$$
Here the rational equivalence $a \equiv b$ is defined if there is a codimension $i$ subvariety $W$ in $X \times \mathbb{P}^1$ such that $a = p_* f^*(0)$ and $b = p_* f^*(1)$, where $\mathbb{P}^1$ is the projective line and $p$ (resp. $f$) is the projection on the first (resp. second) factor.

For $k = \mathbb{C}$, if $X$ has a cellular decomposition, i.e., $X = X_n \supset X_{n-1} \supset \ldots \supset X_0$ with $X_i - X_{i-1} = \bigcup A_{n_{ij}}$, where $A_{n_{ij}}$ is the affine space of dimension $n_{ij}$, then $CH^*(X) \cong H^*(X(\mathbb{C}))$, the singular cohomology theory of $\mathbb{C}$-rational points of $X$. For example, $CH^*(\mathbb{P}^n) \cong H^*(\mathbb{C}\mathbb{P}^n)$ for projective spaces $\mathbb{P}^n$. Since $SPc$ contains colimit, we can consider the infinite projective space $\mathbb{P}^\infty = B\mathbb{G}_m$ and the infinite lens space $\lim_n (\mathbb{A}^n - \{0\}/\mathbb{Z}/p) = L^\infty = B\mathbb{Z}/p$. The Chow rings of classifying spaces of abelian groups are given in [To1]:

\[(2.1)\]
\[CH^*(\mathbb{P}^\infty) \cong H^{2*}\ast(\mathbb{P}^\infty) \cong \mathbb{Z}[y], \quad CH^*(B\mathbb{Z}/p) \cong H^{2*}\ast(B\mathbb{Z}/p) \cong \mathbb{Z}[y]/(py),\]

with $\text{deg}(y) = (2, 1)$. For products of these spaces we have

\[(2.2)\]
\[CH^*((\mathbb{P}^\infty)^n) \cong \mathbb{Z}[y_1, \ldots, y_n],\]

\[(2.3)\]
\[CH^*((B\mathbb{Z}/p)^n) \cong \mathbb{Z}[y_1, \ldots, y_n]/(py_1, \ldots, py_n).\]

Here note that $CH^*(X) \not\cong H^{\text{even}}(X(\mathbb{C}))$ for the last case. Even if $H^*(X(\mathbb{C}))$ is generated by even dimensional elements, there are cases that $CH^*(X) \not\cong H^*(X(\mathbb{C}))$, e.g., K3-surfaces have the cohomology $H^2(X(\mathbb{C})) \cong \mathbb{Z}^{22}$, but there is a K3-surface such that $CH^3(X) \cong \mathbb{Z}^2$ for each $1 \leq i \leq 20$.

Milnor $K$-theory is the graded ring $\bigoplus_n K^M_n(k)$ defined by $K^M_n(k) = (k^\ast)^{\otimes n}/J$, where the ideal $J$ is generated by elements $a \otimes (1 - a)$ for $a \in k^\ast - \{1\}$. Here the addition of $k^\ast$ is given by the multiplication in the field $k$. Hence $K^M_0(k) = \mathbb{Z}$ and $K^M_1(k) = k^\ast$. Hilbert’s Theorem 90, which essentially says that the Galois cohomology $H^1(G(k_s/k); k^\ast) = 0$, implies the isomorphism $K^M_1(k)/p \cong k^\ast/(k^\ast)^p \cong H^1(G(k_s/k); \mathbb{Z}/p)$ for $1/p \in k$. Similarly we can define a map (the norm residue map) for any extension $F$ of $k$ of finite type,

\[(BK)\]
\[K^M_n(F)/p \rightarrow H^n(G(F_s/F); \mu_p^\otimes),\]

where $\mu_p^\otimes$ is the discrete $G(F_s/F)$-module of $n$-th tensor power of the group of $p$-roots of 1. The Bloch-Kato conjecture is that this map is an isomorphism for all field $k$, and the Milnor conjecture is its $p = 2$ case. This conjecture is solved when $n = 2$ by Merkurjev and Suslin [MSu], and for $p = 2$ by Voevodsky [Yo1].

Notice that $H^n(G(k_s/k); \mu_p^\otimes) \cong H^n_{et}(\text{Spec}(k), \mu_p^\otimes)$, the étale cohomology of the point. The étale cohomology $H_{et}^\ast(X; \mathbb{Z}/p)$ has the following properties:

(E.1) If $k$ contains a primitive $p$-th root of 1, then there is the additive isomorphism

\[H^n_{et}(X, \mu_p^\otimes) \cong H^n_{et}(X; \mathbb{Z}/p).\]

(E.2) For smooth $X$ over $k = \mathbb{C}$,

\[H^n_{et}(X; \mathbb{Z}/p^N) \cong H^n(X(\mathbb{C}); \mathbb{Z}/p^N) \quad \text{for all} \ N \geq 1.\]

The last cohomology is the usual mod $p$ ordinary cohomology of $\mathbb{C}$-rational points of $X$. Of course $H^n_{et}(\text{Spec}(\mathbb{C}); \mathbb{Z}/p) \cong \mathbb{Z}/p$. It is known that

\[K^M_n(\mathbb{R})/2 \cong H^\ast_n(\text{Spec}(\mathbb{R}); \mathbb{Z}/2) \cong \mathbb{Z}/2[p]\]

with $\text{deg}(p) = 1$ for the real number field $\mathbb{R}$. Let $F_v$ be a local field with residue field $k_v$ of $ch(k_v) \neq 2$. Then $K^M_n(F_v)/2 \cong H^\ast_n(\text{Spec}(F_v); \mathbb{Z}/2) \cong \Lambda(\alpha, \beta)$ with $\text{deg}(\alpha) = \text{deg}(\beta) = 1$. Thus we know that $\bigoplus_m H^{m,m}(pt; \mathbb{Z}/2)$ for these cases.
3. The realization map

In this section we consider the relation to the usual ordinary cohomology. Let \( R \) be \( \mathbb{Z} \) or \( \mathbb{Z}/p \). The motivic cohomology has the following properties \[ Vo2. \]

(C1) \( H^{*,*}(X; R) \) is a bigraded ring natural in \( X \).

(C2) When \( k \subset \mathbb{C} \), there are maps (realization maps) \( t^m_n : H^{m,n}(X; R) \rightarrow H^m(X(\mathbb{C}); R) \) which sum up to \( t^*_m = \bigoplus n t^m_n \), the natural ring homomorphism.

(C3) There are the (Bockstein, reduced powers) operations

\[
\begin{align*}
\beta & : H^{*,*}(X; \mathbb{Z}/p) \rightarrow H^{*,*+1}(X; \mathbb{Z}/p), \\
\rho^i & : H^{*,*}(X; \mathbb{Z}/p) \rightarrow H^{*,*+2(p-1)i+1}(X; \mathbb{Z}/p),
\end{align*}
\]

which commutes with the realization map \( t^{\mathbb{C}} \) when \( k \subset \mathbb{C} \).

(C4) For the projective space \( \mathbb{P}^n \), there is an isomorphism

\[
H^{*,*}(X \times \mathbb{P}^n/\mathbb{P}^{n-1}; R) \cong H^{*,*}(X; R)\{1, y'\}
\]

with \( \deg(y') = (2n, n) \) and \( t^{\mathbb{C}}(y') \neq 0 \) for \( k \subset \mathbb{C} \).

We recall the Lichtenbaum motivic cohomology \[ Vo2. \]. Lichtenbaum defined the similar cohomology \( H^{*,*}_L(X; R) \) by using the étale topology, while \( H^{*,*}(X; R) \) is defined using the Nisnevich topology. Since Nisnevich covers are restricted étale covers, there is the natural map \( H^{*,*}(X; R) \rightarrow H^{*,*}_L(X; R) \). We say that the \( B(n, p) \) condition holds if

\[
H^{m,n}(X; Z_{(p)}) \cong H^{m,n}_L(X; Z_{(p)}) \quad \text{for all } m \leq n + 1
\]

and all smooth \( X \). The Beilinson-Lichtenbaum conjecture is that \( B(n, p) \) holds for all \( n \) and \( p \). It is known that the condition \( B(n, p) \) is equivalent to the Bloch-Kato conjecture (BK) for degree \( n \) and prime \( p \). Hence \( B(n, p) \) holds for \( n \leq 2 \) or \( p = 2 \). Moreover, Suslin and Voevodsky have proved

\[
H^{m,n}_L(X; \mathbb{Z}/p) \cong H^{m,n}_L(X; \mu_p^{\otimes n}). 
\]

Now we compute \( H^{*,*}(pt; \mathbb{Z}/p) = H^{*,*}(\Spec(k); \mathbb{Z}/p) \). For a smooth \( X \), the following dimensional condition is known:

(C5) For a smooth \( X \), if \( H^{m,n}(X; R) \not\cong 0 \), then

\[
m \leq n + \dim(X), \quad m \leq 2n \text{ and } m \geq 0.
\]

For the rest of this paper, we assume that \( k \) contains a primitive \( p \)-th root of 1 and \( B(n, p) \) holds for all \( n \), but \( X = \Spec(k) \). Then

\[
H^{m,n}(pt; \mathbb{Z}/p) \cong H^{m,n}_L(pt; \mu_p^{\otimes n}) \cong H^{m,n}_L(pt; \mathbb{Z}/p) \quad \text{if } m \leq n,
\]

and \( H^{m,n}(pt; \mathbb{Z}/p) \cong 0 \) for \( m > n \). Let \( \tau \in H^{0,1}(pt; \mathbb{Z}/p) \) be the element corresponding to a generator of \( H^0_\et(\Spec(k); \mu_p) \cong H^0_\et(\Spec(k); \mathbb{Z}/p) \). Then we get the isomorphism

\[
H^{*,*}(\Spec(k); \mathbb{Z}/p) \cong H^*_\et(\Spec(k); \mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau]
\]

since \( \tau : H^{m}_\et(pt; \mu_p^{\otimes n}) \cong H^{m}_\et(pt; \mu_p^{\otimes (n+1)}) \). In particular, for the real number field \( \mathbb{R} \) and a local field \( \mathbb{F}_q \) with the residue field \( k_\tau \), of \( ch(k_\tau) \neq 2 \) we have

\[
(3.1) \quad H^{*,*}(\Spec(\mathbb{R}); \mathbb{Z}/2) \cong \mathbb{Z}/2[\rho, \tau] \quad \text{with } \deg(\rho) = (1, 1),
\]

(3.2) \quad \begin{align*}
H^{*,*}(\Spec(\mathbb{F}_q); \mathbb{Z}/2) & \cong \mathbb{Z}/2[\tau] \otimes \Lambda(\alpha, \beta) \quad \text{with } \deg(\alpha) = \deg(\beta) = (1, 1).
\end{align*}
For \( k = \mathbb{C} \), we know that \( K_n^M(\mathbb{C})/p \cong 0 \) for \( n > 0 \), and hence
\[
H^{\ast,\ast}(Spec(\mathbb{C}); \mathbb{Z}/p) \cong \mathbb{Z}/p[\tau] \quad \text{with} \quad deg(\tau) = (0, 1).
\]

When \( k = \mathbb{C} \), if the \( B(n, p) \) condition holds for \( X \), then it is immediate that
\[
[r^{-1}]H^{\ast,\ast}(X; \mathbb{Z}/p) \cong H^{\ast}(X(\mathbb{C}); \mathbb{Z}/p) \otimes \mathbb{Z}/p[r, \tau^{-1}],
\]
where the degree is defined by \( deg(x) = (m, m) \) if \( x \in H^m(X(\mathbb{C}); \mathbb{Z}/p) \).

Next we compute the cohomology of \( \mathbb{P}^n \) and \( B\mathbb{Z}/p \). For any (algebraic) map \( f : X \to Y \) in the category \( Spec \), we can construct the cofiber sequence
\[
X \to Y \to cone(f) = Y/X,
\]
which induces the long exact sequence (Voevodsky [Ve2])
\[
H^{\ast,\ast}(X; R) \leftarrow H^{\ast,\ast}(Y; R) \leftarrow H^{\ast,\ast}(Y/X : R) \leftarrow H^{-1,\ast}(X; R).
\]
In particular, we get the Mayer-Vietoris, Gysin and blow-up long exact sequences.

By the cofiber sequence \( \mathbb{P}^{n-1} \to \mathbb{P}^n \to \mathbb{P}^n/\mathbb{P}^{n-1} \) and (C4), we can inductively see that
\[
H^{\ast,\ast}(\mathbb{P}^n; \mathbb{Z}/p) \cong H^{\ast,\ast}(pt; \mathbb{Z}/p) \otimes \mathbb{Z}/p[y]/(y^{n+1}) \quad \text{with} \quad deg(y) = (2, 1).
\]

When \( k = \mathbb{C} \), since \( B(1, p) \) always holds, \( H^{1,1}(L_p^n; \mathbb{Z}/p) \cong H^1(L_p^n; \mathbb{Z}/p) \). Hence there is an element \( x' \in H^{1,1}(L_p^n; \mathbb{Z}/p) \) with \( t_c(x') = x \in H^1(L_p^n; \mathbb{Z}/p) \). This also holds for general \( k \) [Ve63]. The lens space is identified with the sphere bundle associated with the line bundle
\[
(A^n - \{0\}) \times_{(\mathbb{A} - \{0\})} \mathbb{A} \to (A^n - \{0\})/(\mathbb{A} - \{0\}) = \mathbb{P}^n.
\]
Here \( (A^n - \{0\}) \times_{(\mathbb{A} - \{0\})} \mathbb{A} \) is the identification such that \( (z_i, z) \sim (a^{-1}z_i, a^pz) \in (A^n - \{0\}) \times \mathbb{A} \) for \( (z_i) \in A^n, \ z \in \mathbb{A}, \ a \in \mathbb{A} - \{0\} \). Hence we get the cofibering \( L_p^n \to \mathbb{P}^n \xrightarrow{p^p} \mathbb{P}^n \). Thus we get the additive isomorphism \( H^{\ast,\ast}(L_p^n; \mathbb{Z}/p) \cong H^{\ast,\ast}(\mathbb{P}^n; \mathbb{Z}/p) \{1, x\} \). This induces the ring isomorphism for \( p = \text{odd} \)
\[
H^{\ast,\ast}(L_p^n; \mathbb{Z}/p) \cong H^\ast(pt; \mathbb{Z}/p) \otimes H^{\ast,\ast}(\mathbb{P}^n; \mathbb{Z}/p) \quad \text{with} \quad deg(x) = (1, 1).
\]

However, note that when \( p = 2 \) we get \( x^2 = y^2 + xp \) [Ve64], where \( p \in H^1(pt; \mathbb{Z}/p) \cong k^*/k^2 \) represents \(-1\). (Hence \( p = 0 \) when \( \sqrt{-1} \in k^* \) ) This is proved by the well-known fact that \( \{a, a\} = \{a, -1\} \) in the Milnor \( K \)-theory \( K_2^M(k) \).

We say that a space \( X \) satisfies the K"unneth formula for a space \( Y \) if
\[
H^{\ast,\ast}(X \times Y; \mathbb{Z}/p) \cong H^{\ast,\ast}(X; \mathbb{Z}/p) \otimes_{H^{\ast,\ast}(pt; \mathbb{Z}/p)} H^{\ast,\ast}(Y; \mathbb{Z}/p).
\]

By the above cofiber sequences, we can easily see that \( \mathbb{P}^n \) and \( B\mathbb{Z}/p \) satisfy the K"unneth formula for all spaces. In particular, we have the ring isomorphisms
\[
H^{\ast,\ast}((\mathbb{P}^n)^\ast; \mathbb{Z}/p) \cong \mathbb{Z}/p[y_1, \ldots, y_n] \otimes H^{\ast,\ast}(pt; \mathbb{Z}/p),
\]
\[
H^{\ast,\ast}((B\mathbb{Z}/p)^n; \mathbb{Z}/p) \cong \mathbb{Z}/p[y_1, \ldots, y_n] \otimes \Lambda(x_1, \ldots, x_n) \otimes H^{\ast,\ast}(pt; \mathbb{Z}/p)
\]
when \( p = 2, x_i^2 = y_i + x_i \).

This fact is used to define the reduced power operation \( P^i \) in (C3). Since a Sylow \( p \)-subgroup of the symmetric group \( S_p \) of \( p \) letters is isomorphic to \( \mathbb{Z}/p \), we have the isomorphism
\[
H^{\ast,\ast}(BS_p; \mathbb{Z}/p) \cong H^{\ast,\ast}(B\mathbb{Z}/p; \mathbb{Z}/p)^{[p]} \cong \mathbb{Z}/p[Y] \otimes \Lambda(W) \otimes H^{\ast,\ast}(pt; \mathbb{Z}/p),
\]
identifying $Y = y^{p-1}$ and $W = xy^{p-2}$. If $X$ is smooth (and suppose $p$ is odd, to simplify arguments), we can define the reduced powers (of Chow rings) as follows. Consider maps
\[
H^{2s,s}_c(X; \mathbb{Z}/p) \xrightarrow{\iota} H^{2p,s,p}(X^p \times S_p, ES_p)
\]
where $\iota$ is the Gysin map for the $p$-th external power, and $\Delta$ is the diagonal map. For $deg(x) = (2n, n)$, the reduced powers are defined as
\[
\Delta^i \iota(x) = \sum P^i(x) \otimes Y^{n-i} + \beta P^i(x) \otimes W^{n-i-1}.
\]
Hence $deg(P^i) = deg(Y^i) = deg(y^{i(p-1)} = (2i(p-1), i(p-1))$. Voevodsky defined $\iota$ for nonsmooth $X$ also, and by using suspensions maps he defined reduced powers for all degree elements in $H^{*,*}(X; \mathbb{Z}/p)$ for all $X$.

Moreover, we can see (Hu-Kňž [HK]) that
\[
H^{*,*}(BG_{n}; \mathbb{Z}/p) \cong \mathbb{Z}/p[c_1, ..., c_n] \otimes H^{*,*}(pt; \mathbb{Z}/p),
\]
where the Chern class $c_i$ with $deg(c_i) = (2i, i)$ is identified with the elementary symmetric polynomial in $H^{*,*}(\mathbb{P}^\infty, Z/p)$. So we can define the Chern class $c_i \in H^{2s,s}(BG; \mathbb{Z}/p)$ for each representation $\rho : G \rightarrow GL_n$.

4. $H^{*,*}(X; \mathbb{Z}/p)/ \text{Ker}(t_C)$ AND THE OPERATION $Q_i$

In this section we assume that $X$ is smooth and $k = \mathbb{C}$. Even in this case the motivic cohomology $H^{*,*}(X; \mathbb{Z}/p)$ seems difficult, in general. Hence we consider a bigraded ring which is computable only by using the algebraic topology of $H^{*,*}(X(\mathbb{C}); \mathbb{Z}/p)$. Define a bidegree algebra by
\[
h^{m,n}(X; \mathbb{Z}/p) = \bigoplus_{m,n} H^{m,n}(X; \mathbb{Z}/p)/ \text{Ker}(t_C^{m,n}).
\]
Since $t_C^{(*)(\tau)} = 1$, it is almost immediate that there is the injection of bidegree $\mathbb{Z}/p[\tau]$-algebras
\[
h^{*,*}(X; \mathbb{Z}/p) \hookrightarrow H^{*,*}(X(\mathbb{C}); \mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau, \tau^{-1}],
\]
where the bidegree of $x \in H^n(X(\mathbb{C}); \mathbb{Z}/p)$ is $(n, n)$. (This also holds when $k \subset \mathbb{C}$ and $k$ has a primitive $p$-th root of 1.)

Suppose the $B(n, p)$ condition holds. By the isomorphisms $(B, p)$, (L-E), (E1) and (E2), we have
\[
H^{n,n}(X; \mathbb{Z}/p) \cong H^{n,n}_c(X; \mathbb{Z}/p) \cong H^{n,n}(X, \mu_p^{\infty}) \cong H^{n,n}(X; \mathbb{Z}/p) \cong H^n(X(\mathbb{C}); \mathbb{Z}/p).
\]
Hence we get the injection of bidegree $\mathbb{Z}/p[\tau]$-algebras
\[
H^{*,*}(X(\mathbb{C}); \mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau] \hookrightarrow h^{*,*}(X; \mathbb{Z}/p).
\]
Thus there exist a $\mathbb{Z}/p$-basis $\{a_I\}$ of $H^*(X(\mathbb{C}); \mathbb{Z}/p)$ and a $\frac{1}{t_I}a_I \geq t_I \geq 0$ such that
\[
h^{*,*}(X; \mathbb{Z}/p) \cong \bigoplus_I \mathbb{Z}/p[\tau^{-t_I}a_I].
\]
Remark. Let \( F_i = \text{Im}(\bigoplus_{k} t_{i+k}^0) \). When the \( B(n, p) \) condition is satisfied, we have
\[
\bigcup_i F_i = H^*(X; \mathbb{C})/\mathbb{Z}/p\biggm{\wedge}.
\]
We also have the interesting bigraded ring
\[
gr H^*(X; \mathbb{C}/\mathbb{Z}/p) = \bigoplus_{i=1}^{\infty} F_{i+1}/F_i \cong h^{*-\ast}(X; \mathbb{C}/\mathbb{Z}/p)/(\text{Im} \tau),
\]
so that \( \mathbb{Z}/p(1) \otimes gr H^*(X; \mathbb{C}/\mathbb{Z}/p) \) is additively isomorphic to \( h^{*-\ast}(X; \mathbb{C}/\mathbb{Z}/p) \), while the ring structures are different.

Here we recall the Milnor primitive operations \( Q_0 = \beta \) and \( Q_i = [Q_{i-1}, P^{p^{i-1}}] : Q_i : H^{*,\ast}(X; \mathbb{C}/\mathbb{Z}/p) \to H^{*-2p^{i-1}+\ast}(X; \mathbb{C}/\mathbb{Z}/p) \),
which is derivative, \( Q_i(xy) = Q_i(x)y + xQ_i(y) \). Note also that \( Q_1(\tau) = 0 \), because of the dimension of \( H^{*,\ast}(pt; \mathbb{C}/\mathbb{Z}/p) \cong \mathbb{Z}/p[1] \).

**Lemma 4.1.** If \( n \neq Q_{i_1} \ldots Q_{i_u} x \in H^{2*,s}(X; \mathbb{Z}/p) \), then \( x \) is a \( \mathbb{Z}/p[\tau] \)-module generator.

**Proof.** If \( x = x' \tau \), then \( \tau Q_{i_1} \ldots Q_{i_u}(x') = 0 \). But
\[ Q_{i_1} \ldots Q_{i_u}(x') = 0 \in H^{2*,s-1}(X; \mathbb{Z}/p), \]
so that \( H^{m,n}(X; \mathbb{Z}/p) = 0 \) for \( m > 2n \).

Define the weight by \( w(x) = 2n - m \) for an element \( x \in H^{m,n}(X; \mathbb{Z}/p) \), so that \( w(x') = 0 \) for \( x' \in CH^*(X)/p \). Of course we get \( w(xy) = w(x) + w(y) \), \( w(P^i x) = w(x) \) and \( w(Q_i(x)) = w(x) - 1 \).

**Corollary 4.2.** Suppose that \( B(n, p) \) holds. If \( x \in H^n(X; \mathbb{C}/\mathbb{Z}/p) \) and \( Q_{i_1} \ldots Q_{i_u}(x) \neq 0 \), then there is a \( \mathbb{Z}/p[\tau] \)-module generator \( x' \in H^{n,n}(X; \mathbb{Z}/p) \) so that \( tC(x') = x \) and, for each \( 0 \leq k \leq n \), \( Q_{i_1} \ldots Q_{i_u}(x') \) is also a \( \mathbb{Z}/p[\tau] \)-module generator of \( H^{n*,s}(X; \mathbb{Z}/p) \).

**Proof.** By the \( B(n, p) \) condition, \( t_{i+n}^n : H^{n,n}(X; \mathbb{Z}/p) \cong H^n(X; \mathbb{Z}/p) \). Hence there is an element \( x' \in H^{n,n}(X; \mathbb{Z}/p) \) with \( tC(x') = x \). This means \( w(x') = n \) and \( w(Q_{i_1} \ldots Q_{i_u}(x)) = 0 \). From the above lemma, we get the corollary.

**Lemma 4.3.** Suppose that \( B(n, p) \) holds. If there is an \( s > 0 \) with \( p^s H^{n+1}(X; \mathbb{C}) / (p) \subset tC(H^{n+1,n}(X; \mathbb{C})) \), then
\[ \text{Im}(H^{n+1}(X; \mathbb{C}) \to H^{n+1}(X; \mathbb{C}) / (p)) = \text{Im}((H^{n+1,n}(X) \to H^{n+1}(X; \mathbb{C}) / (p)). \]

**Proof.** Consider the following diagram:
\[
\begin{array}{cccccc}
H^{n+1,n}(X) & \xrightarrow{(1)} & H^{n+1,n}(X; \mathbb{Z}/p^N) & \longrightarrow & H^{n+2,n}(X) & \xrightarrow{p^N} & H^{n+2,n}(X) \\
\downarrow & & \cong & & \downarrow & & \downarrow \\
H^{n+1}(X; \mathbb{C}) & \xrightarrow{(3)} & H^{n+1}(X; \mathbb{Z}/p^N) & \longrightarrow & H^{n+2}(X; \mathbb{C}) & \xrightarrow{p^N} & H^{n+2}(X; \mathbb{C})
\end{array}
\]
where \( H^*(-) \) means \( H^*(-; \mathbb{Z}/p) \) and the rows are exact.

Let \( H^{n+i}(X; \mathbb{C}) \cong F_i \oplus T_i \) and \( H^{n+i,n}(X) \cong F_i' \oplus T_i' \oplus D_i \), where \( F_i, F_i' \) are free, \( T_i, T_i' \) are non-\( p \)-divisible torsion and \( D_i \) are \( p \)-divisible submodules. Take \( N \) and \( s \) so that \( p^N > p^s > |T_i|, |T_i'| \) for \( i = 1, 2 \). Hence \( H^{n+1,n}(X; \mathbb{Z}/p^N) \cong H^{n+1}(X; \mathbb{Z}/p^N) \cong F_i/p^N \oplus T_i \oplus T_2. \)

By the \( B(n, p) \) condition, \( H^{n+1,n}(X) \cong H^{n+1,n}(X) \), and the map (2) is identified with the realization map. So \( p^s(F_i \oplus T_1) = p^s F_i \subset \text{Image}(2) \). Therefore there is the quotient map \( F_i/p^s \oplus T_1 \oplus T_2 \to \text{Coker}(1) \). On the other hand,
Ker\( (p^N) | H^{n+2,n}_L(X) \cong (\text{Ker}(p^N) | D_2) \oplus T'_L \cong (\mathbb{Z}/p^N)^k \oplus T'_2. \) Hence if \( k \neq 0 \), then it is a contradiction to \( \text{Ker}(p^N) = \text{Coker}(1) \). Hence we get \( \text{Coker}(1) \cong T'_2 \) and hence \( \text{Im}(3) = F_1/p^N \oplus T_1. \)

**Corollary 4.4.** Suppose that \( B(n,p) \) holds and \( t'^{n+1,n}_C : H^{n+1,n}(X) \otimes \mathbb{Q} \to H^{n+1}(X(\mathbb{C})) \otimes \mathbb{Q} \) is epic. If \( x \in \text{Im}(H^{n+1}(X(\mathbb{C})) \to H^{n+1}(X(\mathbb{C}); \mathbb{Z}/p)) \) and \( Q_{i_1}...Q_{i_{n-1}}(x) \neq 0 \), then there is an element \( x' \in H^{n+1,n}(X)_{(p)} \) so that \( t_C(x') = x \) and, for each \( 0 \leq k \leq n-1 \), \( Q_{i_1}...Q_{i_k}(x) \) is also a \( \mathbb{Z}/p[\tau] \)-module generator of \( H^{*,*}(X; \mathbb{Z}/p). \)

Here we mention the case \( n = 1 \). Totaro showed \([102]\) that \( CH^*(BG) \otimes \mathbb{Q} \cong H^*(BG) \otimes \mathbb{Q} \) for any complex algebraic group \( G \). Hence \( CH^1(BG) \to H^2(BG) \) is epic; indeed, he also showed that this map is an isomorphism. As for \( K \)-surfaces, \( CH^*(X) \otimes \mathbb{Q} \to H^*(X(\mathbb{C})) \otimes \mathbb{Q} \) is not epic and \( H^3_L(X) \) contains \( p \)-divisible elements.

Now we consider some examples. The mod 2 cohomology of \( BO(n) \) is \( H^*(BO(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1,...,w_n] \), where the Stiefel-Whitney class \( w_i \) restricts the elementary symmetric polynomial in \( H^*(B[2]^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1,...,x_n] \). Each element \( w_i^2 \) is represented by the Chern class \( c_i \) of the induced representation \( O(n) \subset U(n) \). Hence \( c_i \in CH^*(BO(n); \mathbb{Z}/2) \cong H^{2*,*}(BO(n); \mathbb{Z}/2) \).

**Proposition 4.5.** \( h^{*,*}(BO(n); \mathbb{Z}/2) \supset \mathbb{Z}/2[c_1,...,c_n] \otimes \Delta(w_1,...,w_n) \otimes \mathbb{Z}/2[\tau] \), where \( \deg(c_i) = (2i, i), \deg(w_i) = (i, i) \) and \( w_i^2 = \tau^i c_i. \)

Since \( Q_{i-1}...Q_0(w_i) \neq 0 \), each \( w_i \) is a \( \mathbb{Z}/2[\tau] \)-module generator. However, even \( h^{*,*}(BO(n); \mathbb{Z}/2) \) seems very complicated. Consider the case \( X = BO(3) \). The cohomology operations act by

\[
\begin{align*}
0 \to w_2 & \xrightarrow{s_0} w_1 w_2 + w_3 \xrightarrow{s_0^2} w_2 w_2^2 + w_3^2, \\
& \quad \xrightarrow{s_0^3} w_3 w_3 + w_3^2 \\
& \quad \xrightarrow{s_0^4} w_1 w_2 w_3.
\end{align*}
\]

**Theorem 4.6.** There is the isomorphism

\[
h^{*,*}(BO(3); \mathbb{Z}/2) \cong \mathbb{Z}/2[c_1,c_2,c_3][1, w_1, w_2, Q_0 w_2, Q_1 w_2, w_3, Q_0 w_3, Q_1 w_3] \otimes \mathbb{Z}/2[\tau],
\]

where \( Q_0 w_2 = \tau^{-1}(w_1 w_2 + w_3), ..., Q_1 w_3 = \tau^{-2} w_1 w_2 w_3 \).

W. S. Wilson \((RWY], [KY]\) found a good \( Q(i) = \Lambda(Q_0,...,Q_i) \)-module decomposition for \( X = BO(n) \), namely,

\[
h^{*,*}(X; \mathbb{Z}/2) = \bigoplus_{i=-1}^\infty Q(i)G_i \quad \text{with} \quad Q_0...Q_i G_i \in t_C(CH^*(X)).
\]

Here \( G_{-1} \) is quite complicated; namely, it is generated by symmetric functions \( \sum x_1^{2i_1}...x_n^{2i_n} \) for \( 0 \leq i_1 \leq ... \leq i_n \) and the number of \( i_n \) equal to \( j_u \); if the number of \( j_u \) is odd, then there is some \( s \leq k \) such that \( 2i_s + 2^s < 2j_u < 2i_s + 2^{s+1}. \)

Then \( w(G_i) \geq i + 1 \) in \( h^{*,*}(X; \mathbb{Z}/p) \), and so we have

**Proposition 4.7.** Letting \( w(G_i) = i + 1 \), we have the monomorphism

\[
h^{*,*}(BO(n); \mathbb{Z}/2) \subset \bigoplus_i Q(i)G_i \otimes \mathbb{Z}/2[\tau].
\]
One interesting problem is whether the above injection is really an isomorphism. The similar decomposition holds for \( X = (BZ/p)^n \), and the above injection is an isomorphism. (See Lemma 5.6 below.) The case \( X = BO(3) \) is also an isomorphism. Since the direct decomposition of \( BO(3) \cong BSO(3) \times BZ/2 \) is complicated, we only write here that of \( SO(3) \):

\[
H^*(BSO(3); Z/2) \cong Z/2[w_2, w_3] \cong Z/2[c_2, c_3] \{ 1, w_2, w_3 = Q_0 w_2, w_2 w_3 = Q_1 w_2 \} \\
\cong Z/2[c_2, c_3] \{ w_2, Q_0 w_2, Q_1 w_2, c_3 = Q_0 Q_1 w_2 \} \oplus Z/2[c_2]
\]

Since there is the isomorphism \( O(2n + 1) \cong SO(2n + 1) \times Z/2 \), the cohomology of \( BSO(2n + 1) \) is reduced from that of \( BO(2n + 1) \). However, the situation for \( BO(2n) \) is different. In the next section, we will study \( BSO(4) \) for details.

The extraspecial 2-group \( 2^{1+2n}_+ \) is the n-th central product of the dihedral group \( D_8 \) of order 8. It has a central extension

\[
0 \rightarrow Z/2 \rightarrow G \rightarrow V = \bigoplus_{i=0}^{2n} Z/2 \rightarrow 0.
\]

Let \( H^*(BV; Z/2) \cong Z/2[x_1, \ldots, x_{2n}] \). Then Quillen proved \([Q]\)

\[
H^*(BG; Z/2) \cong Z/2[x_1, \ldots, x_{2n}]/(f, Q_0 f, \ldots, Q_{n-2} f) \oplus Z/2[w_{2n}].
\]

Here \( w_{2n} \) is the Stiefel-Whitney class of the real 2^n-dimensional irreducible representation which restricts nonzero on the center, and \( f = \sum_i x_{2i-1} x_{2i} \in H^2(BV; Z/2) \) represents the central extension (4.4).

Letting \( y_i = x_i^2 \) in \( H^*(BG; Z/2) \), we can write \( f^2 = \sum y_{2i-1} y_{2i} \) and

\[
(Q_k - 1 f)^2 = Q_0 Q_k f = \sum y_{2i-1}^{2k} y_{2i} - y_{2i-1} y_{2i}^{2k-1},
\]

\[
Q_k - 1 f = \sum y_{2i-1}^{2k-1} x_{2i} - x_{2i-1} y_{2i}^{2k-1}.
\]

Now we consider the motivic cohomology \( H^{*,*}(BG; Z/2) \) and change \( y_i = \tau^{-1} x_i^2 \).

Since \( f = 0 \in H^{2,2}(BG; Z/2) \), we can see that \( Q_k - 1 f = 0 \) and \( Q_k Q_0 f = 0 \) also in \( H^{*,*}(BG; Z/2) \). However, for general \( n \), \( \sum y_{2i} y_{2i-1} \neq 0 \) in \( H^{*,*}(BG; Z/2) \). Let

\[
A = (Z/2[y_1, \ldots, y_{2n}, c_2])/(Q_0 Q_k f, \ldots, Q_n f) \\
\otimes \Delta(x_1, \ldots, x_{2n}, w_{2n})/(f, Q_0 f, \ldots, Q_{n-2} f)) \otimes Z/2[r].
\]

**Lemma 4.8.** For \( G = 2^{1+2n}_+ \), there is a map \( A \rightarrow H^{*,*}(BG; Z/2) \) which induces the injection \( A/(f^2) \subset h^{*,*}(BG; Z/2) \).

When \( m = 0, 1, -1 \mod 8 \) and \( m > 0 \), we say that \( Spin(m) \) is real type [Q].

When \( Spin(m) \) is real type, from Quillen, we know that \( H^*(BSpin(m); Z/2) \subset H^*(BG; Z/2) \), where \( G = 2^{2k+1}_+ \) and \( h \) is the Hurwitz number (for details see [Q]).

**Corollary 4.9.** Let \( G = Spin(m) \) be real type with Hurwitz number \( h \), and let

\[
A = (Z/2[c_2, c_3, \ldots, c_m, c_{2k}])/(Q_1 Q_0 w_2, \ldots, Q_h Q_0 w_2) \\
\otimes \Delta(w_1, \ldots, w_m, w_{2k})/(w_2, Q_0 w_2, \ldots, Q_h w_2) \otimes Z/2[r].
\]

where \( w_i, i \leq m \) (resp. \( w_{2k} \)) is the Stiefel-Whitney class of the usual \( SO(m) \) representation (resp. of the irreducible 2^k-dimensional spin representation). Then we have a map \( A \rightarrow H^{*,*}(BG; Z/2) \) which induces the injection

\[
A/(c_2) \subset h^{*,*}(BG; Z/2).
\]
We study Spin(7) and the exceptional Lie group G_2. The cohomology of G_2 is given by \( H^*(BG_2; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_4, w_6, w_7] \), where \( w_i \) is the Stiefel-Whitney class of the inclusion \( G_2 \subset SO(7) \). The cohomology \( H^*(BSpin(7); \mathbb{Z}/2) \cong H^*(BG_2; \mathbb{Z}/2) \otimes \mathbb{Z}/2[w_8] \).

**Corollary 4.10.** Let \( A = \mathbb{Z}/2[c, c_1, c_2, c_6, c_7] \otimes \Delta(w_4, w_6, w_7) \otimes \mathbb{Z}/2[\tau] \). Then there is the map \( A \rightarrow H^*(BG_2; \mathbb{Z}/2) \) which induces the injection \( A/(c_2) \subset h^*(BG_2; \mathbb{Z}/2) \).

**Remark.** Similar facts hold for BSpin(7) tensoring \( \mathbb{Z}/2[c_8] \).

The cohomology operations are given by
\[
\begin{align*}
w_4 \xrightarrow{Sq^2} w_6 & \xrightarrow{Sq^4} w_7, \\
w_4w_7 \xrightarrow{w_4} w_6 & \xrightarrow{w_7} w_2, \\
Q_1Q_0(w_4w_6) & = w_2^4, \\
Q_2Q_0(w_4w_6w_7) & = w_7^4.
\end{align*}
\]

**Proposition 4.11.** Let \( w(w_4) = 2, w(w_{(4,0)}) = 2 \) and \( w(w_{(4,6,7)}) = 3 \) with \( t_c(w_{(i_1,\ldots,i_n)}) = w_{i_1} \ldots w_{i_n} \). Then \( h^*(BG_2; \mathbb{Z}/2) \) is a subalgebra of \( \mathbb{Z}/p[\tau] \otimes \mathbb{Z}/2[c_1, c_2, c_6, c_7] \otimes \mathbb{Z}/2\{w_4, S_2w_4, Q_1w_4, Q_2w_4, S_2Q_2w_4, w_{(4,6,7)}\} \).

**Remark.** If \( t_c \otimes \mathbb{Q} \) is epic, then we can take \( w_4 \in h^{4,3}(BG; \mathbb{Z}/2) \), i.e., \( w(w_4) = 2 \).

The kernel \( \ker(t_c)^{2,*} \) is not so big for \( X = BG_2 \). Indeed, it is known \([Y3]\) that \( CH^*(BG_2)/2 \cong \mathbb{Z}/2[c_1, c_2, c_6, c_7]/(rc_6^2, c_2c_7) \), where \( r = 0 \) or \( 1 \).

The cohomology operations are given in \( H^*(B SO(7); \mathbb{Z}/2) \) by
\[
\begin{align*}
Q_1Q_0w_2 & = w_2^3, \\
Q_2Q_0w_2 & = w_5^2, \\
Q_3Q_0w_2 & = w_2w_2^2 + w_5w_2^2 + w_5w_2^2.
\end{align*}
\]

Hence we have \( c_3 = 0, c_5 = 0 \) and \( c_2c_7 = 0 \) in \( CH^*(BG_2)/2 \), but \( c_2 \neq 0 \).

From here we consider the case \( p = odd \). One of the easiest examples is the case \( G = PGL_3 \) and \( p = 3 \). The mod 3 cohomology is given by \([KY], [Ve1]\)
\[
(Z/3[y_2]/\{y_2^2\} \oplus Z/3\{y_3, y_5, y_7\}/\{y_8\}) \otimes Z/3/\{y_{12}\}
\]

It is known that \( y_2^3, y_5^2, y_6^2 \) and \( y_{12} \) are represented by Chern classes. Moreover, \( Q_1Q_0(y_2) = y_8 \). Hence these elements are in the Chow ring; namely,
\[
h^{2,*}(BPGL_3; Z/3) \cong (Z/3[y_2]/\{y_2^2\} \oplus Z/3/\{y_8\}) \otimes Z/3/\{y_{12}\}.
\]

The cohomology operations are given by
\[
(4.7) \quad y_2 \xrightarrow{\beta} y_3 \xrightarrow{\beta} y_7 \xrightarrow{\beta} y_8.
\]

Thus we get \( h^{*,*}(PGL_3; Z/3) \) completely.

**Theorem 4.12.** Letting \( w(y_2) = 2 \), we have the isomorphism
\[
h^{*,*}(BPGL_3; Z/3) \cong (Z/3[y_2]/\{y_2^2\} \oplus Z/3\{y_3\} \oplus Z/3/\{y_8\} \oplus Q(1)\{y_2\}) \otimes Z/3/\{y_{12}, \tau\}.
\]

Next consider the extraspecial \( p \)-group \( G = p^{1+2n} \). When \( n > 2 \), even the cohomology rings \( H^*(BG; Z/p) \) are unknown, while it contains the subring \([TeY1]\)
\[
R = \mathbb{Z}/p[y_1, \ldots, y_{2n}, \gamma^p]/(Q_1Q_0f, \ldots, Q_nQ_0f),
\]
where \( f = \sum_{i=1}^nx_{2i-1}x_{2i} \) for \( \beta x_1 = y_1 \) and \( Q_nQ_0f = \sum y_{2i-1}y_{2i}^p - y_{2i-1}^py_{2i} \). Since \( f = 0 \in H^{2,2}(BG; Z/p) \), we have

**Proposition 4.13.** There is the injection
\[
R \otimes \mathbb{Z}/p[\tau] \hookrightarrow H^{*,*}(BP^{1+2n}; Z/p).
\]
We consider here other arguments for a different but similar group. Let \( p_+^{1+2n} \) be the central product of \( p_+^{1+2n} \) and the circle, i.e. \( p_+^{1+2n} = p_+^{1+2n} \times C S^1 \), identifying \( C \cong \mathbb{Z}/p \subset S^1 \), where \( C \) is the center. Let us write

\[
(4.9) \quad e_A = \prod_{0 \neq (\lambda_1, \lambda_2, \ldots, \lambda_{2n-1})} (\lambda_1 y_1 + \cdots + \lambda_{2n-1} y_{2n-1}).
\]

If we localize by inverting \( e_A \), then the cohomology of \( p_+^{1+2n} \) is expressed easily \( [Y2] \)

\[
(4.10) \quad [e_A^{-1}] H^*(Bp_+^{1+2n}; \mathbb{Z}/p) \cong [e_A^{-1}] R \otimes \Lambda(x_1, x_3, \ldots, x_{2n-1}), \quad \beta(x_i) = y_i.
\]

**Theorem 4.14.** Letting \( w(x_i) = 1 \), we have the ring isomorphism

\[
[e_A^{-1}] H^*(Bp_+^{1+2n}; \mathbb{Z}/p) \cong [e_A^{-1}] R \otimes \mathbb{Z}/p[\tau] \otimes \Lambda(x_1, x_3, \ldots, x_{2n-1}).
\]

**Proof.** There is the splitting abelian subgroup \((\mathbb{Z}/p)^n \cong A \subset p_+^{1+2n}\) such that

\[
h^*(BA; \mathbb{Z}/p) \cong \mathbb{Z}/p[\tau, y_1, y_3, \ldots, y_{2n-1}] \otimes \Lambda(x_1, x_3, \ldots, x_{2n-1}).
\]

Each monomial \( x_1 \cdots x_{i_1} \cdots x_{i_s}, 1 \leq i_1, \ldots, i_s \leq 2n - 1 \), is a \( \mathbb{Z}/p[\tau] \)-module generator in the above cohomology, hence also in the cohomology of \( Bp_+^{1+2n} \).

We consider the case \( n = 1 \) here. Let us write \( E = p_+^{1+2} \) for each odd prime \( p \). The ordinary cohomology is known by Lewis \( [Ly] \), TeY2; namely,

\[
H^*_{\text{even}}(BE)/p \cong (\mathbb{Z}/p[y_1, y_2]/(y_1^2 - y_2^2)) \otimes \mathbb{Z}/p[c_2, \ldots, c_{p-1}] \otimes \mathbb{Z}/p[c_p],
\]

\[
H^*_{\text{odd}}(BE) \cong \mathbb{Z}/p[y_1, y_2, c_p](a_1, a_2)/(y_1 a_2 - y_2 a_1, y_1^2 a_2 - y_2^2 a_1), \quad |a_i| = 3.
\]

It is also known that \( Q_1(a_i) = y_i c_p \) and \( \text{order}(c_p) = p^2 \).

The group \( 2_+^{1+2} \) is the dihedral group \( D_8 \) of order 8. The integral cohomologies are

\[
H^{\text{even}}(BD_8)/2 \cong \mathbb{Z}/2[y_1, y_2, c_2]/(y_1 y_2), \quad H^{\text{odd}}(BD_8) \cong H^{\text{even}}(BD_8)/2\{e\}
\]

where \( c_2 = w_2 \), \( e = (x_1 + x_2)w_2 \) in \( H^*(BD_8; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, x_2, w_2]/(x_1 x_2) \) and \( Q_1 e = (y_1 + y_2) c_2, \text{order}(c_2) = 4 \).

**Theorem 4.15.** For all primes \( p \), we have the isomorphisms

\[
h'^{\ast, \ast}(Bp_+^{1+2}; \mathbb{Z}/p) \cong \{1, \partial_{p}^{-1}\}(H^*_{\text{even}}(Bp_+^{1+2}))/p - \{\partial_{p}^{-1}\}) \otimes \mathbb{Z}/p[\tau],
\]

where \( w(H^{\text{even}}(Bp_+^{1+2})) = 0 \), \( w(H^{\text{odd}}(Bp_+^{1+2})) = 1 \) and \( w(\partial_{p}^{-1}(x)) = w(x) + 1 \).

**Proof.** We will prove this only for odd primes, since the proof for \( p = 2 \) is similar. Since all elements in \( H^{\text{even}}(BE) \) are generated by Chern classes, we have the isomorphism \( h^{2\ast, \ast}(BG; \mathbb{Z}/p) \cong H^{\text{even}}(BE)/p\). We know \( H^{\text{odd}}(BE; \mathbb{Z}/p) \) is generated as an \( H^{\text{even}}(BE)/p \)-module by two elements \( a_1, a_2 \) such that \( Q_1 a_i = y_i c_p \).

The mod \( p \) cohomology is written additively, \( H^*(BE; \mathbb{Z}/p) \cong \{1, \partial_{p}^{-1}\} H^*(BE)/p \).

Here \( \partial_{p} \) is the (higher) Bockstein operator. All elements in \( H^{\text{odd}}(BE) \) are just \( p \)-torsion, and we can take \( a_i' \in H^2(BE; \mathbb{Z}/p) \) such that \( \delta(a_i') = a_i \). Thus we take \( a_i' \in H^{2+1}(BE; \mathbb{Z}/p) \) so that \( a_i \in H^{2+1}(BE; \mathbb{Z}/p) \).

Next consider elements \( x = \partial_{p}^{-1}(y), \ y \in H^{\text{even}}(BE)/p \). If \( y \in (\text{Ideal}(y_1, y_2)) \), then \( \partial_{p}^{-1}(y) = \sum x_i b_i \) for \( b_i \in H^{\text{even}}(BE)/p \), and hence we can take \( w(\partial_{p}^{-1}(y)) = 1 \).

For other elements \( y = c_i e_p^i, 2 \leq i \leq p - 1 \), it is known \( [Ly] \) that \( c_i = \text{Cor}_{M}(u^i) \).
where $P$ notation. Since $BP$ realization map.

we get $w(\partial_p^{-1}(y)) = 1$. The element $y = c^n_p$ is $p^2$-torsion in $H^*(BE; \mathbb{Z}/p)$. Note that $\text{Cor}_{BP}(w^n) = pc^n_p + k$ with $k \in \text{Ideal}(y_1, y_2)$. Thus $y \in H^{2*}(BE; \mathbb{Z}/p)$ is also $p^2$-torsion. Then we can take $w(\partial_p^{-1}(y)) = 1$. This completes the proof.  

We easily compute the following results.

**Corollary 4.16.** For each prime $p$, there is an isomorphism

$$h^{*,*}(BP_{+}^{1+2}; \mathbb{Z}/p) \cong \mathbb{Z}/p[\tau] \otimes (\mathbb{Z}/p[1] \oplus Q'(0)G_0 \oplus Q(0)G_0 \oplus Q(1)G_1),$$

where $Q'(0) = \Lambda(\beta_{p^2})$, $\beta_{p^2}$ is the $p^2$-torsion Bockstein operator, and if $p = 2$, then

$$G_0' \cong \mathbb{Z}/2[c_1, c_2], \quad \beta_{12}(c_1) = c_2,$$

$$G_0 \cong \mathbb{Z}/2[y_1, y_2] \{ x_1, x_2 \} \oplus \mathbb{Z}/2[y_1, y_2] \{ x_1, x_2 \},$$

and if $p$ is an odd prime, then

$$G_1' \cong \mathbb{Z}/2[\{ c_1, c_2 \}].$$

5. **BP-theory and Ker $t^2_{\mathcal{C}}$**

In this section, we always assume $k = \mathbb{C}$. Even this case it seems difficult to know Ker $t_{\mathcal{C}}$. For Chow rings $CH^*(X)$, Totaro found a good way to get nonzero elements in Ker $t_{\mathcal{C}}$. Let $MU^*(\_)$ (resp. $BP^*(\_)$) be the complex cobordism theory (resp. Brown-Peterson theory) with the coefficient ring $MU^* = MU^*(pt) = \mathbb{Z}[x_1, \ldots, |x_i| = -2i$ (resp. $BP^* = \mathbb{Z}[v_1, \ldots, |v_i| = -2(p^i - 1)$). The Thom map induces $\rho : MU^*(X(\mathbb{C})) \otimes_{MU^*} \mathbb{Z} \to H^*(X(\mathbb{C}); \mathbb{Z})$. Totaro constructed [10] the map

$$c : CH^*(X) \to MU^*(X(\mathbb{C})) \otimes_{MU^*} \mathbb{Z}$$

such that the composition $\rho \circ c$ is the usual cycle map $c = t^2_{\mathcal{C}}$, which is also the realization map.

In this section, hereafter, $X$ is just a topological space, e.g., $X(\mathbb{C})$, to simplify the notation. Since $BP^*(X) \cong BP^* \otimes_{MU^*(n)} MU^*(X)(n)$, the similar fact holds for $BP$-theory. Let $P(n)^* = BP^*(/p, v_1, \ldots, v_{n-1})$, e.g., $P(0)^* = BP^*$, $P(1)^* = BP^*/p$ and $P(\infty)^* = Z/p$. Then there are cohomology theories $P(n)^*(-)$ with the coefficient $P(n)^*(/p) \cong P(n)^*$, e.g., $P(0)^*(X) = BP^*(X)$, $P(1)^*(X) = BP^*(X; \mathbb{Z}/p)$ and $P(\infty)^*(X) = H^*(X; \mathbb{Z}/p)$. Hence there are maps of cohomology theories

$$c_p : CH^*(-)/p \to BP^*(-) \otimes_{BP^*} \mathbb{Z}/p \to \cdots \to P(n)^*(-) \otimes_{P(n)^*} \mathbb{Z}/p \to \cdots \to H^*(-; \mathbb{Z}/p)$$

such that the composition is the cycle map $c_p = t_{\mathcal{C}}$. The Morava $K$-theory is defined by $K(n)^*(X) = P(n)^*(X) \otimes_{P(n)^*} K(n)^*$, where $K(n)^* = \mathbb{Z}/p[v_n, v_n^{-1}]$. In
In any case, we can take $0$ for some fine group $(\text{resp. } pH)$. All nonzero differentials are of the form $\partial$.

Let $X$ be a permanent cycle. This element $\sim$.

**Lemma 5.1.** Let $H^*(X)$ have no higher $p$-torsion. Suppose (5.2) holds, and $A = \bigoplus_{n=-1} P(n+1)^* \hat{G}_n$ in (5.3). Then there is the injection $H^*(X; \mathbb{Z}/p) \hookrightarrow \bigoplus_n A_n^G_n$ with $A_n^G_n = \hat{G}_n$.

**Proof.** Let $H$ be a $\mathbb{Z}/p$-module generated by elements $g(n,s_1,\ldots,s_m)$ discussed above. Define the map $j: H \hookrightarrow \bigoplus_n A_n^G_n$ by $j^*(g(n,s_1,\ldots,s_m)) = Q_{s_1}^{-1} \ldots Q_{s_m}^{-1} (\tilde{g}) = \tilde{g} = \ldots = Q_0 g_{n-s_m,\ldots,s_1}$, where $Q_i = Q_{i+1}$.

Suppose $x \in H^*(X)$ is not a permanent cycle. Then by the assumption (5.3), $x$ is not a permanent cycle. Hence $d_{2p-1}(x) \neq 0$ for some $t$, and so $Q_t(x) \neq 0$. Let $t$ be the largest number such that $Q_{i} \neq 0$ for all $j$, we know $\tilde{g}$ is a permanent cycle. This element $\tilde{g} \in E_{2p-1}^0$ generates a $P(N+1)^*$-module for $N = \max(i_0,\ldots,i_1,i)$. This means $x = (Q_{i} \ldots Q_{i_1}^{i_1-1} \tilde{g}) \in H$. \hfill $\square$
Let us write $Q(i, n) = \Lambda(Q_i, \ldots, Q_n)$, so that $Q(0, n) = Q(n)$.

**Lemma 5.2.** Let $H^*(X)_{(p)}$ have no higher $p$-torsion.

1. If (5.2) is satisfied and, in (5.3),
   
   $$A = \bigoplus_{n=1} P(n + 1)^* \tilde{G}_n$$
   
   and $B \cong \bigoplus_{s=0} B^P(p, v_s, \ldots, v_s) \tilde{K}_s$,

   then we have the isomorphisms

   $$H^*(X)/p \cong (\tilde{G}_1 \oplus \tilde{G}_0 \oplus \bigoplus_{n=1} Q(1, n)G_n' - \bigoplus_{s=0} (Q(s)K'_s - \tilde{K}_s'))$$

   $$H^*(X; \mathbb{Z}/p) \cong \left( \bigoplus_{n=1} Q(n)G_n - \bigoplus_{s=0} (Q(s)K_s - \tilde{K}_s) \right)$$

   with $Q_0 \ldots Q_nG_n = \tilde{G}_n$, $Q_0G_n = G'_n$ and $Q_0 \ldots Q_sK_s = \tilde{K}_s$, $Q_0K_s = K'_s$.

2. If $Q_0 \ldots Q_nG_n \in \Im(p)$ and the degrees of $\tilde{K}_s$ and $G_n$ are even, then the converse also holds.

**Proof.** (1) Let $0 \neq x \in \tilde{K}_s$. Since $x$ is not a permanent cycle, $d_{2p^i-1}(x) \neq 0$ and $Q_i(x) \neq 0$. Since $\{p, \ldots, v_s\} \tilde{K}_s$ are permanent cycles, we know $Q_i(x) \in E_{2p^i-1}^{i*}$ is a $P(s + 1)^*$-module, that is, $i = s + 1$ by the Landweber invariant prime ideal theorem, and

   $$\bigoplus_{n=1} Q(n)G_n \supset Q(s)K_s.$$ 

   Since $v_i x$ generates a free $B^P$-module, $x \notin \Im(Q_j)$ for all $j$. Hence we get the injection

   $$H^*(X; \mathbb{Z}/p) \hookrightarrow \bigoplus_{n=1} Q(n)G_n - (Q(s)K_s - \tilde{K}_s).$$

   Let $x = Q_{i_1} \ldots Q_{i_s}g_n$ be in the right-hand side of the above injection, and such that $0 \neq Q_i(x) \in H^*(X; \mathbb{Z}/p)$ but $x \notin H^*(X; \mathbb{Z}/p)$. If $Q_i(x)$ is not a permanent cycle, then $v_i Q_i(x)$ is permanent, so $Q_i(x)$ must be in $\tilde{K}_s$ and hence $x \in Q(s)K_s$; this is a contradiction. Otherwise $Q_i(x) = \tilde{g}_n$ generates a $P(n)^*$-module and $Q_i(x)$ must be in $\Im(Q_j)$ for all $j \leq n$. Hence $x \in H^*(X; \mathbb{Z}/p)$.

(2) By induction on $i$, we assume $E_{2p^i-1}^{i*} \cong C(i) \oplus D(i)$, where

   $$C(i) = P(i)^*(\bigoplus_{i \leq n} Q(i, n)Q_{i-1} \ldots Q_0G_n - \bigoplus_{i \leq s} Q(i, n)Q_{i-1} \ldots Q_0K_s) \oplus \bigoplus_{i-1 \leq s} B^P\tilde{K}_s,$$

   $$D(i) = \bigoplus_{n \leq i-1} P(n + 1)^* \tilde{G}_n \oplus \bigoplus_{s \leq i-2} B^P(p, \ldots, v_s) \tilde{K}_s.$$ 

Here elements of $\tilde{K}_s$ and $D(i)$ are even dimensional. Hence all odd dimensional elements generate free $P(i)^*$-modules. Note that if $i > j$, then there are no nontrivial maps from $P(i)^*$-modules to free $P(j)^*$-modules. We also note that there is no possibility that $d_t(v_kx) = v_y$ for $x \in \tilde{K}_s$ and $y \in E_t^{odd,*}$, $t < 2p^j - 1$. Indeed there is the map $i^*$ of spectral sequences from that for $B^P(X)$ to that for $P(i)^*(X)$; in the last spectral sequence $E_{2p^i-1}^{i*} \cong P(i)^* \otimes H^*(X; \mathbb{Z}/p)$ and $i^*(v_iy) \neq 0$. Hence the next nonzero differential must be of the form $d_{2p^i-1}(x) = v_i \otimes Q_i(x)$. Therefore we have

   $$E_{2p^i}^{i*} \cong C(i + 1) \oplus D(i) \oplus P(i + 1)Q_1 \ldots Q_0G_i \oplus B^P(p, \ldots, v_i) \tilde{K}_{i-1}.$$ 

The last term is computed from $Q_i\tilde{K}_{i-1} \neq 0$ and $\text{Ker} d_{2p^i-1}B^P\{\tilde{K}_{i-1}\} = B^P\{p, \ldots, v_{i-1}\} \tilde{K}_{i-1}$, since $Q_i\tilde{K}_{i-1}$ is $P(i)^*$-free in $E_{2p^i-1}^{i*}$. 

The classifying spaces of groups $BO(n), SO(4), G_2, Spin(m), m \leq 9$ for $p = 2$ and $PGL_3, F_4$ for $p = 3$, and $(\mathbb{Z}/p)^n$ satisfy the assumption of the above lemma. However $SO(6)$ does not satisfy the above lemma.

We will show that the isomorphism (1) in Lemma 5.2 approximates $h^{**}(X; \mathbb{Z}/p)$. Let $Ih^{**}(X)$ be a $\mathbb{Z}/p[\tau]$-submodule of $h^{**}(X; \mathbb{Z}/p)$ generated by image from $h^{**}(X)/p$. The following theorem is almost immediate.

**Theorem 5.3.** Suppose that (1) in Lemma 5.2 holds. Then we have the injection

$$Ih^{**} \hookrightarrow ((G_1/p \bigoplus_{n=1}^\infty Q(1, n)G_n') - \bigoplus_{s=1}^\infty Q(1, s)K_s, -(K_s')) \otimes \mathbb{Z}/p[\tau],$$

$$h^{**}(X; \mathbb{Z}/p) \hookrightarrow \bigoplus_{n=1}^\infty Q(n)G_n - \bigoplus_{s=1}^\infty Q(s)K_s \otimes \mathbb{Z}/p[\tau],$$

with $w(G_n) = n + 1$, $w(G_n') = n$. Moreover, if some $BP^*$-module generator in Ideal$(p, \ldots, v_1)K_s \subset E^*_G$ is represented by transfer of a Chern class, then $\text{Ker}(\tau^{**})$ contains a nonzero element.

The $P(m)^*(-)$ version of above facts also holds, if we consider the spectral sequence

$$E^*_G = H^*(X; \mathbb{Z}/p) \otimes P(m)^* \Rightarrow P(m)^*(X).$$

(5.3)' Let $E^*_G = A \oplus B$, where $A$ (resp. $B$) is the $P(m)^*$-module generated by generators in $E^*_G$ (resp. in $E^*_G$ minus $\text{Ker}(\tau^{**})$).

**Lemma 5.4.** (1) If (5.2) is satisfied and, in (5.3)',

$$A \cong \bigoplus_{n=-1}^\infty P(m + n + 1)^*\mathcal{G}_n(m), \quad B \cong \bigoplus_{s=0}^\infty P(m)^*\{v_m, \ldots, v_s\}K_s(m),$$

then we have the isomorphism

$$H^*(X; \mathbb{Z}/p) \cong (\bigoplus_{n=-1}^\infty Q(n, n + m)G_n(m)) - \bigoplus_{s=0}^\infty Q(m, m + s)K_s(m) - \mathcal{K}_s(m))$$

with $Q_m Q_{m+n}G_n(m) = \mathcal{G}_n(m)$ and $Q_m Q_{m+n}K_s(m) = \mathcal{K}_s(m)$.

(2) If $Q_m Q_{m+n}G_n(m) \in \text{Im}(p)$ and $|K_s(m)| = 0$, then the converse also holds.

The $P(m)^*$-versions also hold for $G = (\mathbb{Z}/p)^n, BO(n), BSO(4), p^{1+2}$. One application for the above lemma is the following.

**Corollary 5.5.** Let $H^*(X; \mathbb{Z}/p)$ (resp. $H^*(Y; \mathbb{Z}/p)$) have the decomposition of Lemma 5.2 (1) (resp. Lemma 5.4 (1) for all $m \geq 0$). Then $H^*(X \times Y; \mathbb{Z}/p)$ also has decomposition similar to that of Lemma 5.2 (1).

**Proof.** We get the following isomorphism:

$$Q(n - 1)G_{n-1} \otimes H^*(Y; \mathbb{Z}/p) \cong Q(n - 1)G_{n-1} \otimes (Q(n, n + k)G_k(n) - \bigoplus Q(n, n + t)K_t(n) - \mathcal{K}_t(n))$$

$$\cong (Q(n + k)G_{n-1} \otimes G_k(n)) - (Q(n + t)G_{n-1} \otimes K_t(n) - Q(n - 1)G_{n-1} \otimes \mathcal{K}_t(n)),$$

since each $Q_i$ is derivative.

**Lemma 5.6.** If $H^*(X; \mathbb{Z}/p) \cong \bigoplus Q(n)G_n$, then $H^*(X \times BZ/p) \cong \bigoplus Q(n)G'_{n}$, where

$$G'_{n} \cong \mathbb{Z}/p[y]/(y^{p^n})G_n \oplus \mathbb{Z}/p[y]G_{n-1}\{x\},$$
Proof. Since we have the decomposition
\[ H^*(B\mathbb{Z}/p;\mathbb{Z}/p) \cong \mathbb{Z}/p[y]/(y^n) \oplus \mathbb{Z}/p[y]Q(n,n)\{x\}, \]
we get the lemma. \[ \square \]

When \( X = (B\mathbb{Z}/p)^n \), inductively we get the decomposition \( H^*((B\mathbb{Z}/p)^n;\mathbb{Z}/p) \cong \bigoplus Q(n)G_n \). Hence \( B = 0 \) and
\[ \text{gr}BP^*(X) \cong \bigoplus P(n+1)^*G_n, \quad H^*(X;\mathbb{Z}/p) \cong \bigoplus Q(n)G_n \oplus \mathbb{Z}/p[\tau]. \]

Of course these are given in (3.9). The similar facts also hold for \( X = BO(n) \). Moreover, W. S. Wilson proved [RWY] that
\[ BP^*(BO(n)) \cong BP^*[c_1, \ldots, c_n]/(c_1 - c_1^*, \ldots, c_n - c_n^*), \]
where \( c_i^* \) is the complex conjugate of the Chern class of the usual complex representation. The cohomology \( h^*(BO(n)) \) is studied in (4.2).

Next consider the case \( X = BSO(4) \). The mod 2 cohomology is \( H^*(X;\mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3]. \) The cohomology operation acts as
\[ Q_0w_2 = w_3, \ Q_1w_3 = w_3^2, \ Q_1w_4 = w_4w_3, \ Q_1Q_2w_4 = w_3^2 w_4. \]
The integral cohomology is written as
\[ H^*(X)_{(2)} \cong Z(2)[w_2^2, w_4] \otimes (Z(2)\{1\} \oplus \mathbb{Z}/2[w_3]\{w_3\}). \]
In the Atiyah-Hirzebruch spectral sequence, nonzero differentials are \( d_{2i+1-1}(x) = v_i \otimes Q_i(x) \) for \( i = 1, 2 \). We can compute
\[ E_8^{*, *} \cong E_8^{*, *}_g \cong Z(2)[c_2] \otimes (BP^*[c_4]\{1, 2w_4\} \oplus P(2)^*[c_3]\{c_3\} \oplus P(3)^*[c_3, c_4]\{c_3 c_4\}), \]
\[ BP^*(X) \otimes \mathbb{Z}(2) \cong Z(2)[c_2, c_4] \otimes (Z(2)\{1, 2w_4\} \oplus \mathbb{Z}/2[c_3]\{c_3\}). \]

Hence the assumption of (1) in Lemma 5.2 is satisfied by
\[ \tilde{G}^{-1}_1 \cong \mathbb{Z}/2[c_2, c_4], \quad \tilde{G}_1 = \mathbb{Z}/2[c_2, c_3]\{c_3\}, \quad \tilde{G}_2 = \mathbb{Z}/2[c_2, c_3, c_4]\{c_3 c_4\}, \quad \tilde{K}_0 = \mathbb{Z}/2[c_2, c_4]\{2w_4\}. \]

Therefore we get

**Proposition 5.7.** Let \( w(w_4) = 2 \). Then the bidegree \( \mathbb{Z}/2[\tau] \)-module \( I h^*(BSO(4)) \)
(resp. \( h^*(BO(4);\mathbb{Z}/2) \)) is isomorphic to a bidegree \( \mathbb{Z}/2[\tau] \)-submodule of
\[ \mathbb{Z}/2[\tau, c_2] \otimes (Z/2[c_4]\{1\} \oplus Z/2[c_3] \otimes Q(1,1)\{w_3\} \oplus Z/2[c_3, c_4] \otimes Q(1,2)\{w_4\}) \]
(resp. \( \mathbb{Z}/2[\tau, c_2] \otimes (Z/2[c_4]\{1\} \oplus Z/2[c_3] \otimes Q(1)\{w_2\} \oplus Z/2[c_3, c_4] \otimes (Q(2) - \mathbb{Z}/p)\{a\}) \),
where \( Q_0a = w_4 \).

**Remark.** If \( w_4 \in H^{4,3}(BSO(4)) \), then \( Ih^*(BSO(4)) \) is isomorphic to the \( \mathbb{Z}/2[\tau] \)-module in the above proposition.

**Remark.** For this case, we have \( K_0 = \mathbb{Z}/2[c_2]\{a\} \) and \( Q_0K_0 = K'_{0} \) in Lemma 5.2. Indeed, \( Q_0a = w_4 \). However, \( w_4 \notin \text{Im}(Q_0) \) in \( h^*(BSO(4);\mathbb{Z}/2) \), because \( a \) itself does not exist in \( h^*(BSO(4);\mathbb{Z}/2) \).
We know that the element corresponding to $2w_4$ is represented by a Chern class $c'_2$ of some representation, and this means the Totaro’s cycle map $\tilde{d}$ is epic. Indeed, Totaro and Pandharipande showed that this map is isomorphic, namely,

$$CH^*(BSO(4))_{(2)} \cong Z_{(2)}[c_2, c_3, c_4, c'_2]/(2c_3, 3c'_2, c'_2^2 - 4c_4).$$

Next consider the $P(1)^*$-version for $BSO(4)$. By using the computations of $Q_iw_j$ and the Atiyah-Hirzebruch spectral sequence, we can prove that

$$\text{gr}P(1)^*(BSO(4)) \cong P(1)^*[c_4]\{1, v_1w_2w_4\} \oplus P(2)^*[c_3] \oplus P(3)^*[c_3^2] \oplus P(4)^*[c_3, c_4][c_3^2c'_2].$$

We have another decomposition of $H^*(BSO(4); \mathbb{Z}/2)$.

**Proposition 5.8.**

$$H^*(BSO(4); \mathbb{Z}/2) \cong \mathbb{Z}/2[c_4] \oplus Q(1, 1)[w_2] \oplus \mathbb{Z}/2[c_3] \oplus (Q(1, 2)[w_2, w_4]) \oplus \mathbb{Z}/2[c_4] \oplus (Q(1, 2)[c_1w_4]) \oplus \mathbb{Z}/2[c_3, c_4] \oplus (Q(1, 3)[Q^{-1}w_2w_4] - \{Q^{-1}w_2w_4\}).$$

We consider the relation between $\text{gr}BP^*(X)$ and $\text{gr}P(1)^*(X)$. When $X = BSO(4)$, it is known [KY] that $K(n)^{odd}(X) = 0$, and hence

$$P(m)^*(X) \cong P(m)^* \otimes_{BP^*} BP^*(X).$$

Therefore no $P(m)^*(X)$ is $v_m$-torsion. Of course we have already seen that for the $grBP^*(-)$-versions the above facts do not hold. If there is a relation $p_0 + v_1a_1 + v_2a_2 + \ldots = 0 \in BP^*(X)$, then it is known [KY] that there is $y \in H^*(X; \mathbb{Z}/p)$ such that $Q_i(y) = \rho(a_i)$, where $\rho : BP^*(X) \to H^*(X; \mathbb{Z}/p)$ is the Thom map. In $H^*(BSO(4); \mathbb{Z}/2)$, we see that

$$Q_0(w_2w_3) = c_3, \quad Q_1(w_2w_3) = 0, \quad Q_2(w_2w_3) = c_2^2.$$

Hence we have the relation $2c_3 + v_2c_2^2 + \ldots = 0 \in BP^*(BSO(2))$. This shows that $c_2^2$ is $P(2)^*$-free in $grBP^*(BSO(4))$, but it is a $P(3)^*$-free module in

$$\text{gr}P(1)^*(BSO(4)) = \text{gr}(BP^*(BSO(4))/2).$$

We also see that for $x = c_3w_3w_4 + c_4w_2w_3$

$$Q_0(x) = c_3c_4, \quad Q_1(x) = Q_2(x) = 0, \quad Q_3(x) = c_3^2c_4^2.$$

This means that $2c_3c_4 + v_3c_3^2c_4^2 + \ldots = 0 \in BP^*(BSO(4))$. Hence $c_3^2c_4^2$ is a $P(3)^*$-free module in $grBP^*(BSO(4))$ but is a $P(4)^*$-free module in $gr(BP^*(BSO(4))/2)$.

Next consider the case $X = BSO(6)$. In this case the assumption (5.3) is not satisfied. In fact, Inoue computed [I]

$$grBP^*(BSO(6)) \cong \bigoplus_{n=-1}^{4} P(n + 1)^*\tilde{G}_n \oplus P(2)^*/(v_2^2)\tilde{G}_1' \oplus BP^*[2]K_0.$$  

(For details, see [I].) In particular, he showed that

$$d_5(2w_6) = v_1^2w_6w_5, \quad d_{11}(v_1 \otimes w_6w_5) = v_2^2w_6^2w_5^2.$$  

However, even this case we can show that

$$H^*(BSO(6); \mathbb{Z}/2) \subset \bigoplus Q(n)G_n \oplus Q(1)G'_1.$$  

Moreover, R. Field [F] announced that

$$CH^*(BSO(2n)) \cong Z_{(2)}[c_2, \ldots, c_{2n}, y_n]/(2c_{odd}, c_{odd}y_n, y_n^2 - (-1)^n2^{2n-2}c_{2n}).$$
with $deg(y_n) = 2n$. Hence Ideal$(y_n) \subset \text{Ker}(t_c)$. However, $y_n$ is not represented by a Chern class of any representation for $n > 2$. We also note that $BP^*(BSO(2n))$ are not known for $n > 3$.

The cases $X = BG_2, BSpin(7)$ are quite similar to the case $X = BSO(4)$. Indeed, $CH^*(BG_2)/2$ and $h^{*,*}(BG_2; \mathbb{Z}/2)$ have been discussed in §4, and
\[ grBP^*(BG_2) \cong Z_2[c_4, c_6] \otimes (BP^*[1, 2w_4] \oplus P(3)^*[c_7]). \]
The infinite term of the spectral sequence for $BP^*(BSpin(7))$ is computed by
\[ Z_2[c_4, c_6] \otimes (BP^*[c_8][1, 2w_4, 2w_8, 2w_4w_8, v_1w_8] \oplus P(3)^*[c_7] \oplus P(4)^*[c_7, c_8](c_8)). \]

Therefore we obtain

**Corollary 5.9.** Let $w(w_8) = 2$. Then the cohomology $Ih^{*,*}(BSpin(7))$ (resp. $h^{*,*}(BSpin(7); \mathbb{Z}/2)$) is isomorphic to a $\mathbb{Z}/2[\tau]$-submodule of $\mathbb{Z}/2[\tau, c_4, c_6] \otimes A$ (resp. $\mathbb{Z}/2[\tau, c_4, c_6] \otimes B$), where
\[ A = \mathbb{Z}/2[c_8] \oplus \mathbb{Z}/2[c_7] \oplus Q(1, 2)[w_4] \oplus \mathbb{Z}/2[c_7, c_8] \oplus (Q(1, 3) - \mathbb{Z}/p)[b], \]
\[ B = (\mathbb{Z}/2[c_8] \oplus \mathbb{Z}/2[c_7] \oplus Q(2)[v_1w_4]) \oplus \mathbb{Z}/2[c_7, c_8] \oplus (Q(3) - Q(1) + Q_0Q_1 - Q_2)[c] \]
with $Q_1b = w_8, Q_0a = w_4, Q_1Q_0c = w_8, Q_2Q_0c = w_4w_8$.

The algebra $BP^*(BSpin(7)) \otimes_{BP^*} Z_2$ is isomorphic to
\[ Z_2[c_4, c_6, c_8] \otimes (Z_2[c_1, 2w_4, 2w_8, 2w_4w_8] \oplus \mathbb{Z}/2[v_1w_8] \oplus \mathbb{Z}/2[c_7](c_7)). \]
It is known that $2w_2, 2w_8, 2w_4w_8$ are represented by Chern classes but $v_1w_8$ is not. However, Totaro has shown that the cycle map $\tilde{c}$ is epic for this case also (see [ScY, Y3]).

**Corollary 5.10.** There is an epimorphism
\[ CH^*(BSpin(7)) \to Z_2[c_4, c_6, c_8] \otimes (Z_2[c_1, c_2, c_4, c_6] \oplus \mathbb{Z}/2[c_7](c_7)), \]
where $c_i$ is the $i$-th Chern class of complexification of the spin representation $\Delta$ and $\xi_3$ is a 6-dimensional element which is not represented by Chern classes. Thus $c_2, c_4, c_6$ are in $\text{Ker}(\rho_2)$ and $\xi_3 \in \text{Ker}(\rho)$.

Next we consider the case $p = odd$. The cases $PGL_3$ and $p^{1+2}$ are easy, and $Ih^{*,*}(BG)$ are given. For example, for $E = p^{1+2}$
\[ grBP^*(BE) \cong BP^* \otimes H^{even}(BE)/(v_1Q, H^{odd}(BE)). \]
Finally we consider the case $G = F_4, p = 3$, whose Chow ring is still unknown. The mod 3 cohomology of $F_4$ is isomorphic to $H^*(BF_4; \mathbb{Z}/3) \cong C \otimes D$ [Ted] with $D = \mathbb{Z}/3[x_{36}, x_{48}]$ and
\[ C = \mathbb{Z}/3[x_4, x_8] \oplus \{1, x_{20}, x_{20}^2\} \oplus \mathbb{Z}/3[x_{26}] \oplus \Delta(x_9) \oplus \{1, x_{20}, x_{21}, x_{25}\}, \]
where two terms of $C$ have the intersection $\{1, x_{20}\}$. Then we can prove [KY]
\[ grBP^*(BF_4) \cong D \otimes (BP^*[3x_4] + BP^* E \oplus P(3)^*[x_{26}]) \]
with $E = \mathbb{Z}/3[x_4, x_8]\{ab|a, b \in \{x_4, x_8, x_{20}\}\}$. Therefore we obtain

**Corollary 5.11.** Let $w(E) = 0$ and $w(x_4) = 2$. Then $Ih^{*,*}(BF_4)$ is a $\mathbb{Z}/3[\tau]$-submodule of
\[ D \otimes (\mathbb{Z}/3[1] \oplus E \oplus \mathbb{Z}/3[x_{26}] \oplus Q(1, 2)[x_4]) \otimes \mathbb{Z}/3[\tau]. \]
The element $3x_4$ can be proved to be represented by a Chern class, and $x_{26} = Q_2Q_1x_4$. The element $x_{36}$ is also represented by a Chern class, and $P^3x_{36} = x_{48}$. If we can prove that $E/3 \subset \text{Im}(cl_p)$ and $x \in H^{4,3}(BF_4, \mathbb{Z}/3)$, then the above module is just $H^{*+}(BF_4)$ for $p = 3$.

Let $G$ be a simply connected Lie group. Then $H^3(G; \mathbb{Z}) \cong \mathbb{Z}$ and $H^4(G; \mathbb{Z}) \cong 0$. Suppose that $H^*(G; \mathbb{Z})$ has $p$-torsion. Then it is known that there is an element $x' \in H^3(G; \mathbb{Z})$ such that $0 \neq Q_1x' \in H^{2p+2}(G; \mathbb{Z}/p)$. Taking the classifying space, we get an element $x \in H^4(BG; \mathbb{Z})$ such that $Q_1x \neq 0$ in $H^{2p+2}(BG; \mathbb{Z}/p)$. By Totaro [To2] it is known that $CH^*(BG) \otimes \mathbb{Q} \cong H^*(BG) \otimes \mathbb{Q}$. Hence there is an $s \geq 1$ such that $p^sx_4 \in H^4(BG)$ is in $\text{Im}(cl)$. Thus there is a nonzero element $c \in CH^2(BG)/p$ with $t_C^{2*+}(c) = 0$. For the groups $G_2$ or Spin(7) for $p = 2$ and $G = F_4$ for $p = 3$, we can take $s = 1$, since $px_4$ is represented by the second Chern class $c_2$.

**Proposition 5.12.** Let $p = 2$, 3 or 5. There is a classifying space $BG$ such that for all $m, n$ with $3 \leq n + 1 < m \leq 2n$, the kernel $\text{Ker}(t_C^{m,n})$ is nonzero.

**Proof.** Let $G = G_2, p = 2, G = F_4, p = 3$ or $G = E_8, p = 5$. Recall that $(B\mathbb{Z}/p)^n$ satisfies the Künneth formula for all spaces. For $\mathbb{Z}/p[U]$-module generators $x \in H^{*+}(B\mathbb{Z}/p)^\infty; \mathbb{Z}/p)$, the elements $xc$ are all nonzero and all in $\text{Ker} t_C$.

### 6. Homotopy category

From the category $\text{Sp}$, Voevodsky constructed [Vo1], [Vo2], [MoVo] the ($\mathbb{A}^1$, algebraic) homotopy category $\text{Hot}$ and the stable homotopy category $\text{SHot}$. There are two different types of spheres in $\text{Sp}$:

$$S^1 = \mathbb{A}^1/\{0, 1\} \quad \text{and} \quad S^1_t = \mathbb{A}^1 - \{0\}.$$  

The Tate object is $T = \mathbb{A}^1/(\mathbb{A}^1 - 0) \cong \mathbb{P}^1 \cong S^1_t \wedge S^1_t$ in $\text{Hot}$. The category $\text{SHot}$ is defined by $T$ as the suspension, e.g., $E = \{E_i\}$, $E_i \in \text{Spt}$ is a spectrum if there is a map $T \wedge E_i \to E_{i+1}$.

Let $\Sigma_T^\infty$ be the functor from $\text{Sp}$ to $T$-spectra that takes $X$ to $\{T^i \wedge X\}$. If $E$ is a $T$-spectrum, then the motivic (generalized) cohomology $E^{*+}(-)$ is defined by

$$E^{m,n}(X) = \text{Hom}_{\text{SHot}}(\Sigma_T^\infty(X), S^{m-n}_t \wedge E),$$

$$E_{m, n}(X) = \text{Hom}_{\text{SHot}}(S^{m-n}_t \wedge S^n_t, \Sigma_T^\infty(X) \wedge E),$$

where $\text{Hom}_{\text{SHot}}(-, -)$ is the homomorphism defined on $\text{SHot}$.

The realization map $t_C$ is originally defined as the functor $t_C : X \to X(\mathbb{C})$ from $\text{Hot}$ to the category of homotopy spaces. Note that this induces

$$t_C : E^{m,n}(X) \to (t_CE)^m(X(\mathbb{C})).$$

The spectrum for the ordinary motivic cohomology is defined as follows. Let $L(X; R)$ for $R = \mathbb{Z}$ or $\mathbb{Z}/p$ be the presheaf sending a connected $U$ to the free $R$-module generated by the set of all closed irreducible $W \subset U \times X$ such that the projection $W \to U$ is finite and surjective. The Eilenberg-MacLane spectrum is defined as

$$K(R(n), 2n) = L(\mathbb{A}^n; R)/L(\mathbb{A}^n - \{0\}; R).$$
Voevodsky proved that $K(R(n), 2n)$ is the $\Omega$-spectrum for the suspension $T$, namely, $K(R(n), 2n) \cong \Omega_T K(R(n+1), 2n+2)$ in $Hot$. Define also, for $m < 2n$,

$$K(R(n), m) = \Omega_{SI}^{2n-m}(R(n), 2n).$$

Thus the ordinary motivic cohomology is defined by

$$H^{m,n}(X; R) = \text{Hom}_{Hot}(X, K(R(n), m)).$$

**Question 6.1.** Let $k \subseteq \mathbb{C}$, and let $0 \neq \tau_n \in H^{n,n}(K(\mathbb{Z}/(n), n); \mathbb{Z}/p)$ (resp. $\tau_{n+1} \in H^{n+1,n}(K(\mathbb{Z}/(n+1), n+1); \mathbb{Z}/p)$) be the fundamental class (representing the identity map). Then are there isomorphisms

$$h^{2*}(K(\mathbb{Z}/p(n), n); \mathbb{Z}/p) \cong \mathbb{Z}/p[Q_{i-1}, \ldots, Q_0 \tau_n | 0 < i < \ldots < n-1],$$

$$h^{2*}(K(\mathbb{Z}/(p), n+1); \mathbb{Z}/p) \cong \mathbb{Z}/p[Q_{i-1}, \ldots, Q_1 \tau_{n+1} | 0 < i < \ldots < n-1].$$

It is well known that the dual $A_{ps}$ of the (topological) Steenrod algebra $A^*_p$ is isomorphic to $\mathbb{Z}/p[\xi_1, \ldots] \otimes \Lambda(\tau_0, \ldots)$, $|\xi_i| = 2(p^2 - 1)$, $|\tau_i| = 2p^2 - 1$. Let $P^j \in A^*_p$ (resp. $Q^j \in A^*_p$) be the dual of $\xi_i^1$ ... (resp. $\tau_0^1, \ldots$, $i_k = 0$ or 1), so that $A^*_p \cong \mathbb{Z}/p[P^j, Q^j]$. Note that $Q^j = \pm Q_0^i, \ldots$. Define $m(J) = \sum_{k=1} j_k$ and $m(I) = \sum_{k=0} j_k$. Then it is also known [Ta] that

$$H^*(K(\mathbb{Z}/p(n), n); \mathbb{Z}/p) \cong \mathbb{Z}/p[P^j, Q^j, \tau_n | m(I) + 2m(J) < n + i_0].$$

On the other hand, suppose that $Q^j P^j \tau_n \in H^{m,n}(K(\mathbb{Z}/p(n), n); \mathbb{Z}/p)$ for $m \geq 2n$, i.e., $w(Q^j P^j \tau) \leq 0$. Since $w(P^j) = 0$ and $w(Q^j) = -1$, we see that

$$0 \geq w(Q^j P^j \tau_n) = n - m(I).$$

This implies $m(J) = 0$, $m(I) = n$ and $i_0 \not= 0$. Hence we know that $Q^j P^j \tau$ is the form of the ring generator of the polynomial in the above question.

**Remark.** Let us write the above as $A = \mathbb{Z}/p[Q_{i-1}, \ldots Q_0 \tau_n | 0 < i < \ldots < n-1]$. By Tamanoi [Ta], the image $p_0(K(\mathbb{Z}/p, n)) = A \subseteq H^*(K(\mathbb{Z}/p, n); \mathbb{Z}/p)$. Moreover, there is [RWY] the isomorphism $BP^*(K(\mathbb{Z}/p, n)) \otimes_{BP^\ast} \mathbb{Z}/p \cong A$.

### 7. Algebraic cobordism

Let $BGL$ denote the infinite Grassmannian, the union of $GL_N(\infty)$ over $N$. The corresponding generalized cohomology theory is the algebraic $K$-theory. The motivic cobordism theory $MGL^\ast(-)$ is the generalized cohomology theory defined by the Thom spectrum $MGL = \{Th(E_n \to GL_n)\}_n$ identifying $Th(E \oplus O) \cong T \wedge Th(E)$ and $E_n \oplus O \to E_n$ for the trivial line bundle $O$. It is known (Hu-Kříž [HK], Vezzosi [Ve2]) that

$$MGL^\ast((\mathbb{P}^\infty)^n) \cong MGL^\ast(pt)[y_1, \ldots, y_n],$$

$$MGL^\ast(BGL_n) \cong MGL^\ast(pt)[c_1, \ldots, c_n],$$

where the $c_i$ are identified with the elementary symmetric polynomials in the $y_i$’s. Hence the Chern classes are also defined in $MGL^\ast(-)(BG)$. The realization maps

$$t^\ast_{\mathbb{C}} : MGL^\ast(BG)(p) \to MU^\ast(BG)(p)$$

are epic for $G = O(n), SO(4), G_2$ for $p = 2$ and $p \geq 2$ for all primes, because the $MU^\ast(BG)(p)$ are generated by Chern classes.
For a smooth scheme $X$ over $k \subset \mathbb{C}$, Levine and Morel [LM1, LM2] constructed the algebraic cobordism theory $\Omega^*(X)$ such that there are natural maps

$$\rho_H : \Omega^*(X) \to H^{2*}(X), \quad \rho_{MGL} : \Omega^*(X) \to MGL^{2*}(X)$$

with $\rho_H = \rho_{(MGL,H)} \rho_{MGL}$ for the algebraic Thom map $\rho_{(MGL,H)} : MGL^{2*}(X) \to H^{2*}(X)$. Moreover, they proved that

$$\rho_H \otimes_{\Omega^*} \mathbb{Z} : \Omega^*(X) \otimes_{\Omega^*} \mathbb{Z} \cong H^{2*}(X), \quad \rho^2_{MGL} : \Omega^*(pt) \cong MU^{2*}(pt).$$

This implies the motivic version of the Totaro cycle map $\tilde{c}$:

$$\rho_{MGL}(\rho_H \otimes_{\Omega^*} \mathbb{Z})^{-1} : CH^*(X) \to MGL^{2*}(X) \otimes_{MGL^{2*}} \mathbb{Z},$$

and moreover $\tilde{t}_C^{2*} \rho_{MGL}(\rho_H \otimes_{\Omega^*} \mathbb{Z})^{-1}$ is the Totaro cycle map $\tilde{c}$. Thus the Thom map $\rho_{(MGL,H)} : MGL^{2*}(X) \to H^{2*}(X)$ is always epic.

For groups $G = (\mathbb{Z}/p)^n, (O(n)$, we can easily prove that

$$\Omega^*(BG) \cong MU^*(BG).$$

Hence in these cases $MGL^{2*}(BG)$ contains $MU^*(BG)$ as a splitting subring.

**Corollary 7.1.** Let $\tilde{c}_p : CH^*(BG)/p \to MU^*(BG) \otimes_{MU^*} \mathbb{Z}/p$ be epic. Then $t_\mathbb{Z}^{2*} : MGL^{2*}(X)/p \to MU^*(BG)/p$ is epic, and $\text{Im} \rho_{MGL,H} \subset \mathbb{Z}/p \otimes H^{2*}(X; \mathbb{Z}/p)$, where $\rho_{(MGL,H)} : MGL^{2*}(X) \to H^{2*}(X; \mathbb{Z}/p)$ is the induced Thom map.

The modified cycle maps $\tilde{c}$ are epic also for the groups $Spin(7)$ for $p = 2$ and $PGL_3$ for $p = 3$.

By the Thom isomorphism, we get $MGL^{2*} (BGL) \cong MGL^{2*} (GL)$. This means that the Steenrod algebra of $MGL^{2*}(-)$ is generated as an $MGL^{2*}(pt)$-module by the Landweber-Novikov operation $S_\alpha$:

$$MGL^{2*}(MGL) \cong MGL^{2*}(pt) \{ S_\alpha | \alpha = (i_1, \ldots, i_n), \; i_j \geq 0 \}.$$ 

Here $S_\alpha : MGL^{2*}(X) \to MGL^{2*+2|\alpha|+|\alpha|}(X)$ and $|\alpha| = \sum_i i_k k$. These operations satisfy the Cartan formula

$$S_\alpha(xy) = \sum_{\alpha = \beta + \gamma} S_\beta(x)S_\gamma(y),$$

and $S_\alpha MU^*(pt)$ is the usual Landweber-Novikov operation.

Križ, Hu and Vezzosi construct algebraic Brown-Peterson theory $ABP^{2*}(-)$ by using a modified Quillen argument. Here we note that we can also construct algebraic BP-theory by using the technique of Novikov (5.4 in [N]). Recall that $MU^*(pt) \cong \mathbb{Z}(x_1, \ldots, x_i)$, $|x_i| = -2i$. Define

$$\Delta_i = \sum_{q \geq 1} \left( x_i / S_{\Delta_i} (x_i) \right)^{q-1} S_{q\Delta_i},$$

where $\Delta_i = (0, \ldots , 0, 1, 0, \ldots , 0)$ (1 in $i$-th place). Note that $\Delta_i (x_i) = S_{\Delta_i} (x_i) = 1$ if $i \neq p^j - 1$. Then we can easily prove that $\pi_i = 1 - x_i \Delta_i$ is a multiplicative projection such that $\pi_i (x_j) = (1 - \delta_{ij} x_j)$. Essentially composing (for details, see p. 587 in [N]) the $\pi_i$, for all $i \neq p^j - 1$, we get the multiplicative projection $\Phi : MGL(p) \to MGL(p)$ such that

$$\Phi(x_i) = \begin{cases} 
  x_i & \text{if } i = p^j - 1 \text{ for some } j, \\
  0 & \text{otherwise}. 
\end{cases}$$
Define the algebraic Brown-Peterson spectrum by $\Phi MGL = ABP$. Of course $t_c(ABP) = BP$

**Theorem 7.2.** Identify $BP^* = MU^*_p/(x_i | i \neq p^n - 1)$. Then

$$ABP^*(X) \cong BP^* \otimes_{MU^*_p} MGL^*(X)_p.$$ 

**Proof.** Since $\pi_x(a) = (1 - x_i \Delta_a)a = a \mod(x_i)$, we get $\Phi(a) = a \mod(x_i | i \neq p^n - 1)$ for all $a \in MGL^*(X)$. The isomorphism is proved, since $ABP^*(X) \subset MGL^*(X)_p$ by the property $\Phi^2 = \Phi$. \hfill \Box

Since $ABP^*(pt) \cong BP^* \otimes_{MU^*_p} MGL^*(pt)$, we can write the above isomorphism as

$$ABP^*(X) \cong ABP^* \otimes_{MGL^*(pt)} MGL^*(X)_p.$$ 

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