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COMPACT COMPOSITION OPERATORS ON BESOV SPACES

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ABSTRACT. We give a Carleson measure characterization of the compact composition operators on Besov spaces. We use this characterization to show that every compact composition operator on a Besov space is compact on the Bloch space. Finally we give conditions that guarantee that the converse holds.

1. INTRODUCTION

Let ϕ be a holomorphic self-map of the unit disc U. Associate to ϕ the composition operator C_ϕ defined by

$$C_\phi f = f \circ \phi \,,$$

for f holomorphic on U. It maps holomorphic functions f to holomorphic functions.

The problem of boundedness and compactness of C_ϕ has been studied in many function spaces. The first setting was in the Hardy space H^2, the space of functions holomorphic on U with square summable power series coefficients (see [14]).

Madigan and Matheson (see [10]) gave a characterization of the compact composition operators on the Bloch space $\mathcal{B}$.

In this paper we study compact composition operators on Besov spaces, B_p, $1 < p < \infty$. We will define and discuss properties of these spaces in section 2. In section 3, using Nevanlinna type counting functions, we give a Carleson measure characterization of the compact operators C_ϕ on Besov spaces.

Let α_λ $(\lambda \in U)$ be the basic conformal automorphism defined by

$$\alpha_\lambda(z) = \frac{\lambda - z}{1 - \overline{\lambda} z}.$$

We prove the following theorem.

Theorem 3.5. *Let $1 < p \le q < \infty$. Then the following are equivalent:*

(1) *$C_\phi : B_p \to B_q$ is a compact operator.*
(2) *$N_q(w, \phi) dA(w)$ is a vanishing q-Carleson measure.*
(3) *$||C_\phi \alpha_\lambda||_{B_q} \to 0$, as $|\lambda| \to 1$.*

In section 4 we give another characterization of the compact composition operators on the Bloch space.

We prove the following theorem.

Theorem 4.1. *Let ϕ be a holomorphic self-map of U. Let $X = B_p$ $(1 < p < \infty)$ or $\mathcal{B}$. Then $C_\phi : X \to \mathcal{B}$ is a compact operator if and only if*

$$||C_\phi \alpha_\lambda||_{\mathcal{B}} \to 0, \;\; as \; |\lambda| \to 1.$$

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The two theorems above show that the compactness of C_ϕ on a Besov space, and its upper limit the Bloch space, depend on the behavior of the norm of the image under C_ϕ of the conformal automorphisms α_λ, for $|\lambda| \to 1$. An immediate corollary of the two theorems above is that if C_ϕ is compact on a Besov space, then it is compact on every Besov space with a larger index, and it is compact on the Bloch space. The converse holds if we suppose that C_ϕ is bounded on a Besov space with a smaller index (see Proposition 4.5). This is the case, for example, when ϕ is univalent and the Besov space is strictly larger than the Dirichlet space (see Theorem 4.4).

2. Preliminaries

The one-to-one holomorphic functions that map U onto itself, denoted by G, that have the form $e^{it}\alpha_\lambda$ (t a real number, $\lambda \in U$), are called the *Möbius transformations.* It is easy to check that the inverse of α_λ under composition is α_λ for $z \in U$. Also,

$$|\alpha'_\lambda(z)| = \frac{1-|\lambda|^2}{|1-\overline{\lambda}z|^2}$$

and

$$1-|\alpha_\lambda(z)|^2 = \frac{(1-|\lambda|^2)(1-|z|^2)}{|1-\overline{\lambda}z|^2} = (1-|z|^2)|\alpha'_\lambda(z)| \tag{1}$$

for $\lambda, z \in U$.

The *Bloch space* $\mathcal{B}$ of U is the space of holomorphic functions f on U such that

$$||f||_{\mathcal{B}} = \sup_{z\in U}(1-|z|^2)|f'(z)| < \infty.$$

It is easy to see that $|f(0)| + ||f||_{\mathcal{B}}$ defines a norm that makes the Bloch space a Banach space. Using (1) it is easy to see that $\mathcal{B}$ is invariant under Möbius transformations, that is, if $f \in \mathcal{B}$, then $f \circ \phi \in \mathcal{B}$, for all $\phi \in G$. In fact,

$$||f\circ\phi||_{\mathcal{B}} = ||f||_{\mathcal{B}}.$$

For $1 < p < \infty$, the *Besov space* B_p is defined to be the space of holomorphic functions f on U such that

$$\begin{aligned} ||f||^p_{B_p} &= \int_U |f'(z)|^p(1-|z|^2)^{p-2}dA(z) \\ &= \int_U |f'(z)|^p(1-|z|^2)^p d\lambda(z) < \infty \end{aligned}$$

where $d\lambda(z)$ is the Möbius invariant measure on U, namely

$$d\lambda(z) = \frac{1}{(1-|z|^2)^2}dA(z).$$

It is easy to see that $|f(0)| + ||f||_{B_p}$ is a norm on B_p that makes it a Banach space.

Here are some examples of functions belonging to a Besov space. Holland and Walsh show in [8, Theorem 1] that if $1 < p < \infty$ and $\gamma < \frac{1}{q}$ (q is such that $\frac{1}{p}+\frac{1}{q} = 1$), then $\left(\log\frac{2}{1-z}\right)^\gamma \in B_p$. Other examples of functions in B_p ($1 < p < \infty$) are provided by gap series. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^{c_n},$$

where (c_n) is a sequence of integers satisfying

$$\frac{c_{n+1}}{c_n} \geq c > 1, \tag{2}$$

where c is a constant and n is a nonnegative integer. A description of Besov spaces that Peller gives in [11, p. 450] easily yields that $f \in B_p$ if and only if $\sum_{k=0}^{\infty} c_k |a_k|^p < \infty$.

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be a holomorphic function on U. The *Dirichlet space*, $\mathcal{D}$, is the collection of functions f holomorphic on U for which

$$||f||_{\mathcal{D}}^2 \overset{\text{def.}}{=} \sum_{n=1}^{\infty} n|a_n|^2 < \infty.$$

The Dirichlet space becomes a Hilbert space with norm $\left(|f(0)|^2 + ||f||_{\mathcal{D}}^2\right)^{\frac{1}{2}}$. It is easy to see, using polar coordinates, that $f \in \mathcal{D}$ if and only if

$$\int_U |f'(z)|^2 dA(z) < \infty.$$

Thus, the Besov-2 space is the Dirichlet space and $B_2 = \mathcal{D} \subset H^2$.

Throughout this paper const. denotes a positive and finite constant which may change from one occurrence to the next, but will not depend on the functions involved. Unlike the Hardy and Bergman spaces a Besov space with a smaller index lies inside all Besov spaces with a larger index. Moreover, they are all subspaces of the space of vanishing mean oscillation $VMOA$, and therefore they are all subspaces of the Bloch space. That is, $B_p \subset B_q \subset \mathcal{B}$, for $1 < p < q$, and

$$||f||_{\mathcal{B}} \leq \text{const.}\, ||f||_{B_q} \leq \text{const.}\, ||f||_{B_p},$$

for any $f \in B_p$.

By making a change of variables and using basic properties of Möbius transformations we can easily see that, for $1 < p < \infty$, the Besov space B_p is invariant under Möbius transformations; that is, $||f \circ \alpha_\lambda||_{B_p} = ||f||_{B_p}$, for all $\lambda \in U$.

3. Carleson measures and compact composition operators on Besov spaces

Shapiro solved the compactness problem for composition operators on Hardy spaces in [14] using the Nevanlinna counting function

$$N_\phi(w) = \sum_{\phi(z)=w} -\log|z|.$$

The following theorem is proved there:

Theorem A. *Let ϕ be a holomorphic function on U. Then C_ϕ is a compact operator on H^2 if and only if*

$$\lim_{|w|\to 1} \frac{N_\phi(w)}{-\log|w|} = 0.$$

Madigan and Matheson characterize compact composition operators in the Bloch space in [10].

The following theorem is proved there:

Theorem B. *Let ϕ be a holomorphic function on U. Then C_ϕ is a compact operator on $\mathcal{B}$ if and only if*

$$\lim_{|\phi(z)|\to 1} \frac{|\phi'(z)|(1-|z|^2)}{1-|\phi(z)|^2} = 0.$$

Next we define a Nevanlinna type counting function and use it to characterize the compact composition operators on Besov spaces.

Definition 3.1. The counting function for the p-Besov space is

$$N_p(w,\phi) = \sum_{\phi(z)=w} \left\{|\phi'(z)|(1-|z|^2)\right\}^{p-2}$$

for $w \in U,\ p > 1$.

The above counting functions come up in the change of variables formula in the respective spaces as follows:

First, for $f \in B_p$ and $p > 1$,

$$\begin{aligned} ||C_\phi f||_{B_p}^p &= \int_U |(f\circ\phi)'(z)|^p (1-|z|^2)^{p-2} dA(z) \\ &= \int_U |f'(\phi(z))|^p |\phi'(z)|^p (1-|z|^2)^{p-2} dA(z). \end{aligned} \tag{3}$$

By making a non-univalent change of variables as done in [15, p. 186] we see that

$$||C_\phi f||_{B_p}^p = \int_U |f'(w)|^p N_p(w,\phi) dA(w). \tag{4}$$

Now consider the restriction of C_ϕ to B_p. Then C_ϕ is a bounded operator if and only if there is a positive constant c such that

$$||C_\phi f||_{B_p}^p \le c||f||_{B_p}^p$$

for all $f \in B_p$ or, equivalently by (4),

$$\int_U |f'(w)|^p N_p(w,\phi) dA(w) \le c||f||_{B_p}^P$$

for all $f \in B_p$. This leads, as in [2], to the definition of Carleson type measures. Since we are interested in characterizing the compact composition operators, we will also talk about vanishing Carleson measures. We would like to use the following operator-theoretic wisdom:

If a "big-oh" condition characterizes the boundedness of an operator, then the corresponding "little-oh" condition should characterize the compactness of the operator.

Definition 3.2. Let μ be a positive measure on U, and let $X = B_p$ $(1 < p < \infty)$ or $\mathcal{B}$. Then μ is an (X,p)-Carleson measure if there is a constant $A > 0$ so that

$$\int_U |f'(w)|^p d\mu(w) \le A||f||_X^p,$$

for all $f \in X$.

In view of (4) above we see that C_ϕ is a bounded operator on B_p if and only if the measure $N_p(w,\phi)dA(w)$ is a (B_p,p)-Carleson measure.

Arazy, Fisher, and Peetre gave the following characterization of (B_p,p)-Carleson measures in [2, Theorem 13] (the equivalence of (1) and (2) was given by Cima and Wogen in [4]).

Theorem C. *For $1<p<\infty$, the following are equivalent:*

(1) *μ is a (B_p,p)-Carleson measure.*
(2) *There exists a constant $A>0$ such that*

$$\mu(S(h,\theta)) \le Ah^p$$

for all $\theta \in [0,2\pi)$, all $h \in (0,1)$.
(3) *There exists a constant $B>0$ such that*

$$\int_U |\alpha'_\lambda(z)|^p d\mu(z) \le B$$

for all $\lambda \in U$.

Hence Theorem C yields:

Theorem D. *Let ϕ be a holomorphic function on U. Then C_ϕ is a bounded operator on B_p $(1<p<\infty)$ if and only if*

$$\sup_{\lambda\in U} ||C_\phi \alpha_\lambda||_{B_p} < \infty\,.$$

We prove a similar theorem for compact composition operators on Besov spaces.

Definition 3.3. For $1<p<\infty$, μ is called a vanishing p-Carleson measure if

$$\lim_{h\to 0} \sup_{\theta\in[0,2\pi)} \frac{\mu(S(h,\theta))}{h^p} = 0\,.$$

Note. It is easy to see that if μ is a vanishing p-Carleson measure, then it is a (B_p,p)-Carleson measure.

The proposition below characterizes vanishing p-Carleson measures. The proof is similar to the one for Carleson measures on H^2 $(p=1)$, as given by Garnett in [7, p. 239] and by Chee in [3].

Proposition 3.4. *For $1<p<\infty$, the following are equivalent:*

(1) *μ is a vanishing p-Carleson measure.*
(2) *$\int_U |\alpha'_\lambda(w)|^p d\mu(w) \to 0$, as $|\lambda| \to 1$.*

Proof. First, suppose that (2) holds. Then, given an $\epsilon>0$ there is a $\delta>0$ such that for $1-\delta<|\lambda|<1$,

$$\int_U |\alpha'_\lambda(w)|^p d\mu(w) < \epsilon.$$

Fix $\epsilon > 0$ and let $\delta > 0$ be as above. Consider any $0 < h < \delta$, $\theta \in [0, 2\pi)$, and let $\lambda = (1-h)e^{i\theta}$ and $w \in S(h,\theta)$. Then,

$$\begin{aligned}
|\alpha_\lambda'(w)| &= \frac{1-|\lambda|^2}{|1-\overline{\lambda}w|^2} \\
&= \frac{1-(1-h)^2}{|1-(1-h)e^{-i\theta}w|^2} \\
&= \frac{h(2-h)}{|e^{i\theta}-(1-h)w|^2} \\
&\geq \frac{h(2-h)}{(|e^{i\theta}-w|+|w-(1-h)w|)^2} \\
&\geq \frac{h(2-h)}{(h+h|w|)^2} \\
&= \frac{2-h}{h(1+|w|)^2} \\
&\geq \frac{1}{4h}\,.
\end{aligned}$$

Hence, $w \in S(h,\theta)$ implies that $|\alpha_\lambda'(w)|^p \geq \frac{1}{4^p h^p}$. Then by our hypothesis,

$$\epsilon > \int_U |\alpha_\lambda'(w)|^p d\mu \geq \int_{S(h,\theta)} |\alpha_\lambda'(w)|^p d\mu \geq \frac{1}{4^p h^p} \mu(S(h,\theta)).$$

This proves (1).

Conversely, suppose that (1) holds. Then, given an $\epsilon > 0$ there is a $\delta > 0$ such that for any $0 < h < \delta$ and any $\theta \in [0, 2\pi)$,

$$\mu(S(h,\theta)) < \epsilon h^p\,. \tag{5}$$

Fix $\epsilon > 0$, and let δ be as above. Fix $h_0 < \delta$ such that (5) holds. Also, fix $\lambda = |\lambda|e^{i\theta} \in U$ with $|\lambda| > 1 - \frac{h_0}{4}$. We will show that for λ large,

$$\int_U |\alpha_\lambda'(w)|^p d\mu(w) < \epsilon\,.$$

Let $E = \left\{w \in U : |e^{i\theta} - |\lambda| w| \geq \frac{h_0}{4}\right\}$. Then for each $\lambda \in U$,

$$\int_U |\alpha_\lambda'(w)|^p d\mu(w) = \int_E |\alpha_\lambda'(w)|^p d\mu(w) + \int_{E^c} |\alpha_\lambda'(w)|^p d\mu(w). \tag{6}$$

We will estimate each of the integrals above.

First if $w \in E$,

$$|\alpha_\lambda'(w)|^p = \left(\frac{1-|\lambda|^2}{|e^{i\theta}-|\lambda|w|^2}\right)^p \leq \left(4^2\,\frac{1-|\lambda|^2}{h_o^2}\right)^p < \epsilon \tag{7}$$

for λ large. Therefore (7) yields that for λ large,

$$\int_E |\alpha_\lambda'(w)|^p d\mu(w) < \epsilon\mu(E) \leq \mu(U)\epsilon < \text{const. } \epsilon\,. \tag{8}$$

Let $n_0 = n_0(\lambda)$ be the smallest positive integer such that

$$2^{n_0}(1-|\lambda|) < h_0 \leq 2^{n_0+1}(1-|\lambda|)\,. \tag{9}$$

We will show that

$$E^c \subset S(2^{n_0}(1-|\lambda|),\theta) \subset S(h_0,\theta)\,. \tag{10}$$

Let $w \in E^c$. Then,

$$\begin{aligned} |w-e^{i\theta}| &= |w-e^{i\theta}+|\lambda|w-|\lambda|w| \\ &\le |w-|\lambda|w|+|e^{i\theta}-|\lambda|w| \\ &< 1-|\lambda|+\frac{h_0}{4} \\ &< 1-|\lambda|+2^{n_0-1}(1-|\lambda|) \\ &\le 2^{n_0}(1-|\lambda|)\,. \end{aligned}$$

This proves that $E^c \subset S(2^{n_0}(1-|\lambda|),\theta)$. Next let $w \in S(2^{n_0}(1-|\lambda|),\theta)$. Then, by (9),

$$|w-e^{i\theta}| \le 2^{n_0}(1-|\lambda|) < h_0\,.$$

Hence $S(2^{n_0}(1-|\lambda|,\theta)) \subset S(h_0,\theta)$. Thus, (10) is proved.

Let $E_k = S(2^k(1-|\lambda|),\theta)$, $k=0,1,...,n_0$. It is clear that

$$E_0 \subset E_1 \subset \cdots \subset E_{n_0} \subset S(h_0,\theta).$$

Then,

$$\begin{aligned} \int_{E^c} |\alpha'_\lambda(w)|^p d\mu(w) &\le \int_{S(2^{n_0}(1-|\lambda|),\theta)} |\alpha'_\lambda(w)|^p d\mu(w) \\ &= \int_{E_0} + \int_{E_1\setminus E_0} + \cdots + \int_{E_{n_0}\setminus E_{n_0-1}} |\alpha'_\lambda(w)|^p d\mu(w)\,. \end{aligned} \tag{11}$$

We will estimate each of the integrals above.

First, if $w \in E_0$, then $|w-e^{i\theta}| < 1-|\lambda|$ and

$$|\alpha'_\lambda(w)| \le \frac{1-|\lambda|^2}{(1-|\lambda|)^2} \le \frac{2}{1-|\lambda|}\,.$$

Since $1-|\lambda| < h_0 < \delta$, (5) yields

$$\begin{aligned} \int_{E_0} |\alpha'_\lambda(w)|^p d\mu(w) &\le \frac{2^p}{(1-|\lambda|)^p}\mu(E_0) \\ &\le \text{const. } \epsilon\,. \end{aligned}$$

Next if $w \in E_k \setminus E_{k-1}$ for some $k = 2,3,...,n_0$, then

$$\begin{aligned} |\alpha'_\lambda(w)| = \frac{1-|\lambda|^2}{|e^{i\theta}-|\lambda|w|^2} &\le \frac{1-|\lambda|^2}{(|w-e^{i\theta}|-|w|(1-|\lambda|))^2} \\ &\le \frac{\text{const.}}{1-|\lambda|}\,\frac{1}{4^k}\,. \end{aligned} \tag{12}$$

Hence, (5), (9), and (12) yield

$$\begin{aligned}\int_{E_k \setminus E_{k-1}} |\alpha'_\lambda(w)|^p d\mu(w) &\leq \frac{\text{const.}}{(1-|\lambda|)^p} \frac{1}{4^{kp}} \mu(E_k) \\ &\leq \frac{\text{const.}}{(1-|\lambda|)^p} \frac{1}{4^{kp}} \epsilon\, 2^{kp}(1-|\lambda|)^p \\ &= \text{const.} \frac{1}{2^{kp}} \epsilon\,. \qquad (13)\end{aligned}$$

Therefore (8), (11), (12), and (13) imply that

$$\int_U |\alpha'_\lambda(w)|^p d\mu(w) \quad < \quad \text{const.}\ \epsilon + \left(\sum_{k=0}^{n_0} \frac{1}{2^{kp}}\right) \text{const.}\ \epsilon \quad < \quad \text{const.}\ \epsilon$$

for λ large. This proves (2). □

Note. In the proof above it was essential that

$$\sum_{k=0}^{\infty} \frac{1}{2^{kp}} < \infty,$$

since n_0 depends on λ.

Theorem 3.5. *Let $1 < p \leq q < \infty$. Then the following are equivalent:*

(1) *$C_\phi : B_p \to B_q$ is a compact operator.*
(2) *$N_q(w, \phi) dA(w)$ is a vanishing q-Carleson measure.*
(3) *$||C_\phi \alpha_\lambda||_{B_q} \to 0$, as $|\lambda| \to 1$.*

In the proof of the theorem above we will need the following lemmas.

Lemma 3.6. *Let $X = B_p$ $(1 < p < \infty)$ or $\mathcal{B}$. Then*

(1) *Every bounded sequence (f_n) in X is uniformly bounded on compact sets.*
(2) *For any sequence (f_n) on X such that $||f_n||_X \to 0$, $f_n - f_n(0) \to 0$ uniformly on compact sets.*

Proof. In [17, p. 82] is shown that a Bloch function can grow at most as fast as $\log \frac{1}{1-|z|}$, that is,

$$\begin{aligned}|f_n(z) - f_n(0)| &\leq \text{const.}\, ||f_n||_{\mathcal{B}} \ \log \frac{1}{1-|z|} \\ &\leq \text{const.}\, ||f_n||_X \ \log \frac{1}{1-|z|}\,.\end{aligned}$$

Hence the result follows. □

Lemma 3.7. *Let X, Y be two Banach spaces of analytic functions on U. Suppose*

(1) *the point evaluation functionals on X are continuous;*
(2) *the closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets;*
(3) *$T : X \to Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.*

Then T is a compact operator if and only if given a bounded sequence (f_n) in X such that $f_n \to 0$ uniformly on compact sets, then the sequence (Tf_n) converges to zero in the norm of Y.

Proof. First, suppose that T is a compact operator and let (f_n) be a bounded sequence in X such that $f_n \to 0$ uniformly on compact sets, as $n \to \infty$.

For the rest of this proof let $|.|_Y$ denote the norm of Y. If the conclusion is false, then there exists an $\epsilon > 0$ and a subsequence $n_1 < n_2 < n_3 < ...$ such that

$$|Tf_{n_j}|_Y \geq \epsilon, \text{ for all } j = 1, 2, 3, ... \tag{14}$$

Since (f_n) is a bounded sequence and T a compact operator, we can find a further subsequence $n_{j_1} < n_{j_2} < ...$ and $f \in Y$ such that

$$|Tf_{n_{j_k}} - f|_Y \to 0\,, \tag{15}$$

as $k \to \infty$. By (1), point evaluation functionals are continuous; therefore, for any $z \in U$,

$$|(Tf_{n_{j_k}} - f)(z)| \leq \text{const.}\, |Tf_{n_{j_k}} - f|_Y. \tag{16}$$

Hence (15) and (16) yield that

$$Tf_{n_{j_k}} - f \to 0 \tag{17}$$

uniformly on compact sets. Moreover, since $f_{n_{j_k}} \to 0$ uniformly on compact sets, (3) yields that $Tf_{n_{j_k}} \to 0$ uniformly on compact sets. Thus by (17), $f = 0$. Hence (15) yields $|Tf_{n_{j_k}}|_Y \to 0$ as $k \to \infty$, which contradicts (14). Therefore we must have $|Tf_n|_Y \to 0$, as $n \to \infty$.

Conversely, let (f_n) be a bounded sequence in X. We will show that the sequence (Tf_n) has a norm convergent subsequence. Without loss of generality, (f_n) belongs to the unit ball of X. By (2) there is a subsequence $n_1 < n_2 < ...$ such that $f_{n_j} \to f$ uniformly on compact sets, for some $f \in X$. Hence, by our hypothesis, $|Tf_{n_j} - Tf|_Y \to 0$, as $j \to \infty$. This finishes the proof of the lemma. $\square$

Note. $(\Rightarrow)$ Only uses (1) and (3). $(\Leftarrow)$ Only uses (2).

Lemma 3.8. *Let $X, Y = B_p$ $(1 < p < \infty)$ or $\mathcal{B}$. Then $C_\phi : X \to Y$ is a compact operator if and only if for any bounded sequence (f_n) in X with $f_n \to 0$ uniformly on compact sets as $n \to \infty$, $||C_\phi f_n||_Y \to 0$ as $n \to \infty$.*

Proof. We will show that (1), (2), (3) of Lemma 3.7 hold for our spaces. By Lemma 3.6 it is easy to see that (1) and (3) hold. To show that (2) holds, let (f_n) be a sequence in the closed unit ball of X . Then by Lemma 3.6, (f_n) is uniformly bounded on compact sets. Therefore, by Montel's Theorem ([5, p. 153]), there is a subsequence $n_1 < n_2 < ...$ such that $f_{n_k} \to g$ uniformly on compact sets, for some $g \in H(U)$. Thus we only need to show that $g \in X$.

(a) If $X = B_p$ $(1 < p < \infty)$, then

$$\begin{aligned}
\int_U |g'(z)|^p (1-|z|^2)^{p-2} dA(z) &= \int_U \lim_{k\to\infty} |f'_{n_k}(z)|^p (1-|z|^2)^{p-2} dA(z) \\
&\leq \liminf_{k\to\infty} \int_U |f'_{n_k}(z)|^p (1-|z|^2)^{p-2} \\
&= \liminf_{k\to\infty} ||f_{n_k}||^p_{B_p} < \infty
\end{aligned}$$

by Fatou's Theorem and our hypothesis.

(b) If $X = \mathcal{B}$,

$$|g'(z)|(1-|z|^2) = \lim_{k\to\infty} |f'_{n_k}(z)|(1-|z|^2) \leq \lim_{k\to\infty} ||f_{n_k}||_{\mathcal{B}} < \infty$$

by our hypothesis. Therefore Lemma 3.7 yields that $C_\phi : X \to Y$ is a compact operator if and only if for any bounded sequence (f_n) in X with $f_n \to 0$ uniformly on compact sets as $n \to \infty$, $|f_n(\phi(0))| + ||C_\phi f_n||_Y \to 0$, as $n \to \infty$, which is clearly equivalent to the statement of this lemma.

This completes the proof of the lemma. □

Note. An immediate consequence of Lemma 3.8 is that if ϕ is a holomorphic self-map of U such that $\sup_{z\in U} |\phi(z)| < 1$, then C_ϕ is a compact operator on every Besov space.

Proof of Theorem 3.5. By (4),

$$||C_\phi \alpha_\lambda||^q_{B_q} = \int_U |\alpha'_\lambda(w)|^q N_q(w,\phi)dA(w)\ .$$

Thus Proposition 3.4 yields (2) $\Leftrightarrow$ (3).

Next we show that (1) $\Rightarrow$ (3). We assume that $C_\phi : B_p \to B_q$ is a compact operator. Note that $\{\alpha_\lambda : \lambda \in U\}$ is a bounded set in B_p since

$$||\alpha_\lambda||_{B_p} = ||z \circ \alpha_\lambda||_{B_p} = ||z||_{B_p},$$

and the norm of α_λ in B_p is

$$|\alpha_\lambda(0)| + ||\alpha_\lambda||_{B_p} < 1 + ||z||_{B_p} < \infty.$$

Also $\alpha_\lambda - \lambda \to 0$, as $|\lambda| \to 1$, uniformly on compact sets since

$$|\alpha_\lambda(z) - \lambda| = |z|\frac{1-|\lambda|^2}{|1-\bar{\lambda}z|}\ .$$

Hence, by Lemma 3.8, $||C_\phi(\alpha_\lambda - \lambda)||_{B_q} \to 0$, as $|\lambda| \to 1$. Therefore $||C_\phi \alpha_\lambda||_{B_q} \to 0$, as $|\lambda| \to 1$.

Finally, let us show that (2) $\Rightarrow$ (1). Let (f_n) be a bounded sequence in B_p that converges to 0 uniformly on compact sets. Then the mean value property for the holomorphic function f'_n yields that

$$f'_n(w) = \frac{4}{\pi(1-|w|)^2} \int_{|w-z|<\frac{1-|w|}{2}} f'_n(z)dA(z). \tag{18}$$

Therefore by Jensen's inequality ([13, Theorem 3.3, p. 62] and (18)),

$$|f'_n(w)|^q \leq \frac{4}{\pi(1-|w|)^2} \int_{|w-z|<\frac{1-|w|}{2}} |f'_n(z)|^q dA(z). \tag{19}$$

Then by (19) and Fubini's Theorem ([13, Theorem 8.8, p. 164]),

$$\begin{aligned} ||C_\phi f_n||^q_{B_q} &= \int_U |f'_n(w)|^q N_q(w,\phi)dA(w) \\ &\leq \int_U \frac{4}{\pi(1-|w|)^2} \left(\int_{|w-z|<\frac{1-|w|}{2}} |f'_n(z)|^q dA(z) \right) N_q(w,\phi)dA(w) \\ &= \frac{4}{\pi} \int_U |f'_n(z)|^q \left(\int_U \frac{1}{(1-|w|)^2} \chi_{\{z:|w-z|<\frac{1-|w|}{2}\}}(z) N_q(w,\phi)dA(w) \right) dA(z). \end{aligned} \tag{20}$$

Note that if $|w-z| < \frac{1-|w|}{2}$, then $w \in S(2(1-|z|), \theta)$, where $z = |z|e^{i\theta}$, since

$$|w - e^{i\theta}| \leq |z-w| + |e^{i\theta} - z| < \frac{1-|w|}{2} + |\frac{z}{|z|} - z| < 2(1-|z|).$$

Moreover, if $|w-z| < \frac{1-|w|}{2}$, then $\frac{1}{(1-|w|)^2} \leq \text{const.} \frac{1}{(1-|z|)^2}$. Therefore (20) yields

(21)

$$\begin{aligned} ||C_\phi f_n||_{B_q}^q &\leq \text{const.} \int_U \frac{|f_n'(z)|^q}{(1-|z|)^2} \left(\int_{S(2(1-|z|),\theta)} N_q(w,\phi) dA(w) \right) dA(z) \\ &= \text{const.} \left(\int_{|z|>1-\frac{\delta}{2}} + \int_{|z|\leq 1-\frac{\delta}{2}} \frac{|f_n'(z)|^q}{(1-|z|)^2} \left(\int_{S(2(1-|z|,\theta)} N_q(w,\phi) dA(w) \right) dA(z) \right) \\ &= \text{const.}(I + II)\,, \end{aligned}$$

for any $0 < \delta < 1$.

Fix $\epsilon > 0$ and let $\delta > 0$ be such that for any $\theta \in [0, 2\pi]$ and any $h < \delta$,

$$\int_{S(h,\theta)} N_q(w,\phi) dA(w) < \epsilon h^q. \tag{22}$$

By (21) and (22),

$$\begin{aligned} I &\leq 2^q \epsilon \int_{|z|>1-\frac{\delta}{2}} \frac{|f_n'(z)|^q}{(1-|z|^2)^2} (1-|z|^2)^q dA(z) \\ &\leq \text{const.}\, \epsilon ||f_n||_{B_q}^q < \text{const.}\ \epsilon. \end{aligned} \tag{23}$$

By (21),

$$\begin{aligned} II &\leq \text{const.} \int_{|z|\leq 1-\frac{\delta}{2}} |f_n'(z)|^q \left(\int_U N_q(w,\phi) dA(w) \right) dA(z) \\ &= \int_{|z|\leq 1-\frac{\delta}{2}} |f_n'(z)|^q ||\phi||_{B_q}^q dA(z) < \text{const.}\ \epsilon \end{aligned} \tag{24}$$

for n large enough, since $f_n' \to 0$ uniformly on compact sets. Combining (21), (23) and (24) we obtain that $||C_\phi f_n||_{B_q} <$ const. ϵ for n large enough. Therefore $||C_\phi f_n||_{B_q} \to 0$ as $n \to \infty$, and Lemma 3.8 yields that $C_\phi : B_p \to B_q$ is a compact operator. This finishes the proof of Theorem 3.5. □

4. Besov space compactness of $\mathbf{C}_\phi$ versus Bloch compactness of $\mathbf{C}_\phi$

Recall the characterization of compact composition operators on the Bloch space that Madigan and Matheson give in [10, Theorem 2].

Theorem B. *Let ϕ be a holomorphic self-map of U. Then C_ϕ is a compact operator on $\mathcal{B}$ if and only if*

$$\lim_{|\phi(z)|\to 1} \frac{|\phi'(z)|(1-|z|^2)}{1-|\phi(z)|^2} = 0.$$

Next we give another characterization of compact composition operators on the Bloch space.

Theorem 4.1. *Let ϕ be a holomorphic self-map of U. Let $X = B_p$ $(1 < p < \infty)$ or $\mathcal{B}$. Then $C_\phi : X \to \mathcal{B}$ is a compact operator if and only if*

$$||C_\phi \alpha_\lambda||_{\mathcal{B}} \to 0, \ \textit{as} \ |\lambda| \to 1.$$

Proof. First, suppose that $C_\phi : X \to \mathcal{B}$ is a compact operator. Then $\{\alpha_\lambda : \lambda \in U\}$ is a bounded set in X, and $\alpha_\lambda - \lambda \to 0$ uniformly on compact sets as $|\lambda| \to 1$. Thus by Lemma 3.8,

$$\lim_{|\lambda| \to 1} ||C_\phi \alpha_\lambda||_{\mathcal{B}} = 0.$$

Conversely, suppose that $\lim ||C_\phi \alpha_\lambda||_{\mathcal{B}} = 0$, as $|\lambda| \to 1$. Let (f_n) be a bounded sequence in X such that $f_n \to 0$ uniformly on compact sets, as $n \to \infty$. We will show that

$$\lim_{n \to \infty} ||C_\phi f_n||_{\mathcal{B}} = 0.$$

Let $\epsilon > 0$ be given and fix $0 < \delta < 1$ such that if $|\lambda| > \delta$, then $||C_\phi \alpha_\lambda||_{\mathcal{B}} < \epsilon$. Hence for any $z_0 \in U$ such that $|\phi(z_0)| > \delta$, $||C_\phi \alpha_{\phi(z_0)}||_{\mathcal{B}} < \epsilon$. In particular,

$$|\alpha'_{\phi(z_0)}(\phi(z_0))| \, |\phi'(z_0)|(1 - |z_0|^2) < \epsilon,$$

that is,

$$\frac{|\phi'(z_0)|}{1 - |\phi(z_0)|^2}(1 - |z_0|^2) < \epsilon \,. \tag{25}$$

Then (25) yields that for any $n \in N$ and $z_0 \in U$ such that $|\phi(z_0)| > \delta$,

$$\begin{aligned} |(f_n \circ \phi)'(z_0)|(1 - |z_0|^2) &= |f_n'(\phi(z_0))| \, |\phi'(z_0)|(1 - |z_0|^2) \\ &< |f_n'(\phi(z_0))|(1 - |\phi(z_0)|^2) \, \epsilon \\ &\leq ||f_n||_{\mathcal{B}} \, \epsilon \\ &\leq ||f_n||_X \, \epsilon < \text{const.} \, \epsilon \,. \end{aligned} \tag{26}$$

Since the set $A = \{w : |w| \leq \delta\}$ is a compact subset of U and $f_n' \to 0$ uniformly on compact sets,

$$\sup_{w \in A} |f_n'(w)| \to 0, \ \text{as} \ n \to \infty \,.$$

Therefore we may choose n_0 large enough so that $|f_n'(\phi(z))| < \epsilon$, for any $n \geq n_0$ and any $z \in U$ such that $|\phi(z)| \leq \delta$. Then, for all such z,

$$\begin{aligned} |(f_n \circ \phi)'(z)|(1 - |z|^2) &= |f_n'(\phi(z))| \, |\phi'(z)|(1 - |z|^2) \\ &< \epsilon |\phi'(z)|(1 - |z|^2) \\ &< ||\phi||_{\mathcal{B}} \, \epsilon \,, \end{aligned} \tag{27}$$

where $n \geq n_0$. Thus, (26) and (27) yield

$$||f_n \circ \phi||_{\mathcal{B}} < \text{const.} \ \epsilon, \ \text{for} \ \ n \geq n_0. \tag{28}$$

Thus (28) yields that $||C_\phi f_n||_{\mathcal{B}} \to 0$ as $n \to \infty$. Hence by Lemma 3.8, $C_\phi : X \to \mathcal{B}$ is a compact operator. □

Notes. (a) It is easy to see that the proof of Theorem 4.1 yields that

$$\lim_{|\lambda| \to 1} ||C_\phi \alpha_\lambda||_{\mathcal{B}} = 0$$

if and only if

$$\lim_{|\phi(z)| \to 1} \frac{|\phi'(z)|(1 - |z|^2)}{1 - |\phi(z)|^2} = 0 \,.$$

Therefore we obtain another proof of Theorem B.

(b) The above theorem is valid for any Banach subspace X of the Bloch space such that the point evaluation functionals on X are continuous and the closed unit ball of X is compact in the topology of uniform convergence on compact sets.

An immediate consequence of Theorem 4.1 along with Lemma 3.8 and Theorem 3.5 is the following proposition.

Proposition 4.2. *Let $1 < p \leq q \leq \infty$. Then: If $C_\phi : B_p \to B_q$ is a compact operator, then so is $C_\phi : \mathcal{B} \to \mathcal{B}$.*

The following proposition gives a sufficient condition for a composition operator to be compact on a Besov space.

Proposition 4.3. *Let $1 < p \leq q < \infty$. If*

$$\lim_{|w|\to 1} \frac{N_q(w,\phi)}{(1-|w|^2)^{q-2}} = 0,$$

then $C_\phi : B_p \to B_q$ is a compact operator.

Proof. Let (f_n) be a bounded sequence in B_p such that $f_n \to 0$ uniformly on compact sets as $n \to \infty$. Let $\epsilon > 0$ be given and fix $\delta > 0$ such that if $1-\delta < |w| < 1$, then

$$N_q(w,\phi) < \epsilon(1-|w|^2)^{q-2}. \tag{29}$$

By (4),

$$\begin{aligned} ||C_\phi f_n||_{B_q}^q &= \int_U |f_n'(w)|^q N_q(w,\phi)\,dA(w) \\ &= \int_{1-\delta<|w|<1} + \int_{|w|\leq 1-\delta} |f_n'(w)|^q N_q(w,\phi)\,dA(w) \\ &= I + II. \end{aligned} \tag{30}$$

By (29),

$$\begin{aligned} I &< \epsilon \int_{1-\delta<|w|<1} |f_n'(w)|^q (1-|w|^2)^{q-2}\,dA(w) \\ &< \epsilon ||f_n||_{B_q}^q < \epsilon \text{ const. } (f_n \text{ is bounded in } B_p). \end{aligned} \tag{31}$$

Since $|f_n'|^q \to 0$ uniformly on $\{w \in U : |w| \leq 1-\delta\}$, we can find a positive integer n_0 such that

$$II \leq \epsilon \int_{|w|<1-\delta} N_q(w,\phi)\,dA(w) < \epsilon \text{ const.}, \tag{32}$$

for $n \geq n_0$, since

$$\int_{|w|<1-\delta} N_q(w,\phi)\,dA(w) \leq ||\phi||_{B_q} < \infty.$$

By (30), (31), and (32), $||C_\phi f_n||_{B_q} < \epsilon$ const. for $n \geq n_0$. Therefore, $||C_\phi f_n||_{B_q} \to 0$ as $n \to \infty$. Hence Lemma 3.8 yields that $C_\phi : B_p \to B_q$ is a compact operator. $\square$

The following theorem and proposition give conditions under which compactness in the Bloch space is equivalent to compactness from a Besov space to some larger Besov space.

Theorem 4.4. *Let ϕ be a univalent holomorphic self-map of U. Then, for $q > 2$, $C_\phi : B_q \to B_q$ is a compact operator if and only if $C_\phi : \mathcal{B} \to \mathcal{B}$ is a compact operator.*

Proof. First, suppose that C_ϕ is a compact operator on the Bloch space. The sufficient condition of Besov space compactness in Proposition 4.3 for a univalent function is

$$\lim_{|w|\to 1} \left\{ \frac{|\phi'(\phi^{-1}(w))|(1-|\phi^{-1}(w)|^2}{1-|w|^2} \right\}^{q-2} = 0$$

or, equivalently,

$$\lim_{|\phi(z)|\to 1} \frac{|\phi'(z)|(1-|z|^2)}{1-|\phi(z)|^2} = 0,$$

which is a compactness condition for the composition operator on the Bloch space (Theorem B). Hence, by our assumption, $C_\phi : B_q \to B_q$ is a compact operator.

The converse follows from Proposition 4.2. This finishes the proof of the theorem. □

Note. Theorem 4.4 is not valid when $q = 2$. There exists a univalent holomorphic self-map of U such that C_ϕ is compact on the Bloch space but not on the Dirichlet space. To describe such an example we will need some preliminaries. First, we need the Koebe Distortion Theorem (see [15, p. 156]), which asserts that if ϕ is a univalent function on U, then for any $z \in U$,

$$\delta_{\phi(U)}(\phi(z)) \sim |\phi'(z)|(1-|z|^2),$$

where $\delta_{\phi(U)}(\phi(z))$ is the Euclidean distance from $\phi(z)$ to $\partial\phi(U)$. Thus the Madigan and Matheson condition of Bloch compactness for a univalent ϕ is equivalent to

$$\lim_{|\phi(z)|\to 1} \frac{\delta_{\phi(U)}(\phi(z))}{1-|\phi(z)|^2} = 0\,. \tag{33}$$

Let $D(0,\alpha)$ denote the disc centered at 0 of radius α. A *nontangential approach region* Ω_α $(0 < \alpha < 1)$ in U, with vertex $\zeta \in \partial U$ is the convex hull of $D(0,\alpha) \cup \{\zeta\}$ minus the point ζ.

If ψ is a univalent holomorphic self-map of U such that $\psi(U) = \Omega_\alpha$ $(0 < \alpha < 1)$, then $\inf_{z\in U} \frac{\delta_{\psi(U)}(\psi(z))}{1-|\psi(z)|^2} > 0$.

Thus by (33), C_ψ is not compact on $\mathcal{B}$. But if we delete certain circular arcs from Ω_α, then for the Riemann map ϕ from U onto the induced domain G, C_ϕ is compact on $\mathcal{B}$. Let $L_n = \{z \in \Omega_\alpha : |z-1| \le \frac{1}{2^n}\}$ $(n \ge 1)$. Then $L_1 \supset L_2 \supset L_3 \supset \cdots$. Remove from $L_n \setminus L_{n+1}$ $(n \ge 1)$ arcs centered at 1, with one end point at $\partial\Omega_\alpha$, in such a way that the successive radii are less than $\frac{1}{3^n}$ apart, and the distance of each arc to $\partial\Omega_\alpha$ is less than $\frac{1}{3^n}$. Then the distance from each $z \in L_n \setminus L_{n+1}$ to the boundary of the induced subdomain, G_n, of $L_n \setminus L_{n+1}$ is less than $\frac{1}{3^n}$.

Let $G = \bigcup_{n\ge 1} G_n$. Then, as $|z| \to 1$, $\delta_G(z) = o(1-|z|)$. Therefore by (33), C_ϕ is compact on $\mathcal{B}$. Moreover, C_ϕ is not compact on the Dirichlet space. This follows from Theorem 3.5.

The theorem above is a special case of the following proposition. We show that if C_ϕ is bounded on some Besov space, then the compactness of C_ϕ on larger Besov spaces is equivalent to the compactness of C_ϕ on the Bloch space. This result is similar to the compactness of C_ϕ on weighted Dirichlet spaces $\mathcal{D}_\alpha$ ($\alpha > -1$). These are spaces of holomorphic functions f on U such that

$$|f(0)|^2 + \int_U |f'(z)|^2(1-|z|^2)^\alpha dA(z) < \infty.$$

MacCluer and Shapiro showed in [9, Main Theorem, p. 893] that if C_ϕ is bounded on some weighted Dirichlet space $\mathcal{D}_\alpha$, then the compactness of C_ϕ on larger weighted Dirichlet spaces is equivalent to ϕ having no angular derivative at each point of ∂U.

Proposition 4.5. *Let $1 < r < q$, $1 < p \le q$. Suppose that $C_\phi : B_r \to B_r$ is a bounded operator. Then $C_\phi : B_p \to B_q$ is a compact operator if and only if $C_\phi : \mathcal{B} \to \mathcal{B}$ is a compact operator.*

Proof. First, suppose that C_ϕ is a compact operator on the Bloch space. For any $\lambda \in U$,

(34)

$$\begin{aligned} &||C_\phi \alpha_\lambda||^q_{B_q} \\ &\quad = \int_U |\alpha'_\lambda(\phi(z))|^q |\phi'(z)|^q (1-|z|^2)^{q-2} dA(z) \\ &\quad = \int_U |\alpha'_\lambda(\phi(z))|^r |\phi'(z)|^r (1-|z|^2)^{r-2} \left(|\alpha'_\lambda(\phi(z))|\,|\phi'(z)|(1-|z|^2)\right)^{q-r} dA(z) \\ &\quad \le ||C_\phi \alpha_\lambda||^{q-r}_{\mathcal{B}} ||C_\phi \alpha_\lambda||^r_{B_r} \\ &\quad \le \text{const.}\, ||C_\phi \alpha_\lambda||^{q-r}_{\mathcal{B}} \\ &\qquad (\text{by Theorem } D \text{ and since } C_\phi : B_r \to B_r \text{ is bounded}). \end{aligned}$$

Therefore (34) and Theorem 4.1 yield that $||C_\phi \alpha_\lambda||_{B_q} \to 0$ as $|\lambda| \to 1$. Thus by Theorem 3.5, $C_\phi : B_p \to B_q$ is a compact operator. The converse follows from Proposition 4.2. This finishes the proof of the proposition. □

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