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# FRAMINGS OF KNOTS SATISFYING DIFFERENTIAL RELATIONS

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Abstract. This paper introduces the notion of a differential framing relation for knots in a three-dimensional manifold. There is a canonical map from the space of knots that satisfy a framing relation into the space of framed knots. Under reasonable assumptions this canonical map is a weak homotopy equivalence.

## §1. Introduction

We intend to generalize the following theorem from the work of Gluck and Pan.

**Theorem 1.1** ([6], [7]). *Any two smooth simple closed curves in 3–space, each having nowhere vanishing curvature, can be deformed into one another through a one–parameter family of such curves if and only if they have the same knot type and the same self–linking number.*

The hypothesis of nonvanishing curvature is essentially a condition on the derivatives of a space curve, namely, that the velocity and acceleration vectors be linearly independent. Significantly, a curve that satisfies this condition everywhere has a canonical, nowhere vanishing normal vector field, to wit, the normal component of its acceleration. Generalizing this, a framing relation is a condition on the higher order derivatives of curves in a 3–dimensional manifold so that any solution curve possesses a canonical, nowhere vanishing normal vector field. The precise definition of a framing relation as a special type of differential relation in a jet bundle is stated in Definition 2.2 below.

Throughout this paper, $M$ denotes a smooth 3–manifold, $X^{(r)}$ denotes the bundle of $r$–jets of smooth maps of the circle $S^1$ into $M$, and $\mathcal{E}$ denotes the space of smooth embeddings of $S^1$ into $M$ endowed with the $C^\infty$ topology. We may call an element $x \in \mathcal{E}$ a parameterized knot in $M$. A framing of $x$ is a nowhere vanishing normal vector field $\xi$ along $x$. Let $\mathcal{E}_\mathcal{F}$ denote the space of framed knots in $M$, that is, ordered pairs $(x, \xi)$, where $x$ is an embedding of $S^1$ into $M$ and $\xi$ is a framing of $x$. We give $\mathcal{E}_\mathcal{F}$ the $C^\infty$ topology.

If $\mathcal{R} \subset X^{(r)}$ is a framing relation, we let $\mathcal{E}_\mathcal{R} \subset \mathcal{E}$ denote the space of knots which satisfy the relation. The canonical nowhere vanishing normal vector field along a knot $x \in \mathcal{E}_\mathcal{R}$ produces a canonical map $\hat{\nu} : \mathcal{E}_\mathcal{R} \to \mathcal{E}_\mathcal{F}$ by Definition 2.3.

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**Theorem 1.2.** *If $\mathcal{R} \subset X^{(r)}$ is a framing relation that is ample over $X^{(s)}$ for every $2 \leq s \leq r$ and that satisfies the SC condition, then the canonical map $\hat{\nu} : \mathcal{E}_{\mathcal{R}} \to \mathcal{E}_{\mathcal{F}}$ is a weak homotopy equivalence.*

The proof of Theorem 1.2 and the relevant terminology are presented in §2. For the SC condition, see Definition 2.4. Ampleness (Definition 2.1) is a hypothesis in the dense $h$–principle for open ample partial differential relations which is due to Gromov [8]. Gluck and Pan [6], [7] hinted that Theorem 1.1 follows from the $C^1$–dense one–parametric $h$–principle proved in [5], but gave no details. Our proof of Theorem 1.2 realizes this hint, though in a more general setting.

*Remark.* When $M$ is Euclidean 3–space, we define the linking number of a framed knot $(x, \xi) \in \mathcal{E}_{\mathcal{F}}$ to be the linking number of $x$ with any sufficiently small $\epsilon$ displacement, $x + \epsilon\xi$, of $x$ in the direction of $\xi$. Clearly, two framed knots $(x, \xi)$ and $(y, \zeta)$ are in the same connected component of $\mathcal{E}_{\mathcal{F}}$ if and only if $x$ and $y$ have the same knot type and the linking numbers of $(x, \xi)$ and $(y, \zeta)$ are the same. The self–linking number of a knot having nowhere vanishing curvature is its linking number with a curve obtained by small displacement in the direction of the normal component of the acceleration field. Thus, when $\mathcal{R}$ is the nonvanishing curvature relation, Theorem 1.1 is just the statement that $\hat{\nu}_* : \pi_0(\mathcal{E}_{\mathcal{R}}) \to \pi_0(\mathcal{E}_{\mathcal{F}})$ is injective. This explains how Theorem 1.2 generalizes Theorem 1.1.

In §3, we present examples of $r$–th order framing relations for every $r \geq 2$ which satisfy both ampleness and the SC condition. Special attention is devoted to the nonvanishing curvature relation in a 3–dimensional Finsler manifold.

Knots in Euclidean space that admit an osculating sphere at every point are solutions of an ample 3-rd order framing relation that is invariant under Möbius transformations and also satisfies the SC condition. Example 3.4 implies the following analogue of Theorem 1.1.

**Theorem 1.3.** *Any two smooth simple closed curves in 3–space, each admitting an osculating sphere at every point, can be deformed into one another through a one–parameter family of such curves if and only if they have the same knot type and their associated canonical framings have the same linking number.*

It is instructive to consider the family of knots parameterized by $\lambda \in [0, 1]$,

$$\begin{aligned} x(t) &= \cos(t) - \lambda \cos(2t), \\ y(t) &= \sin(t) - \lambda \sin(2t), \\ z(t) &= \lambda \sin(t), \end{aligned}$$

where $t \in S^1$. According to [4, pp. 616–617], these knots have nowhere vanishing curvature except when $\lambda = 1/4$. Also, the self–linking number is 0 when $\lambda \in [0, 1/4)$ and is 1 when $\lambda \in (1/4, 1]$. On the other hand, direct computation implies that these knots admit osculating spheres at every point except when $\lambda = 0$, and that the associated canonical normal framing has linking number 0 when $\lambda \in (0, 1]$. Thus two different framing relations will generally have different spaces of solution knots. Moreover, for knots that satisfy two relations simultaneously, the different associated canonical framings need not be isotopic.

Benham, Lin, and Miller [2] pose a problem concerning knots of nowhere vanishing curvature in Euclidean 3–space. They desire to show that the inclusion of the set of all knots in any fixed isotopy class with nonvanishing curvature and prescribed self–linking number into the set of all knots in that isotopy class is a weak homotopy equivalence. On the contrary, in §4, we apply Theorem 1.2 to prove that this inclusion is not a weak homotopy equivalence. This is remarkable because it contrasts with the comparable situation for knots of fixed writhe [2], [9].

With minor modifications, our results also hold for links in 3–manifolds.

## §2. Framing relations

Consider $X = S^1 \times M$ as the trivial fiber bundle over $S^1$. For a nonnegative integer $r$, let $X^{(r)}$ denote the space of $r$–jets of smooth sections of $X$. Since $X$ is a trivial bundle, $X^{(r)}$ can be identified with the space of $r$–jets of smooth maps of $S^1$ into $M$. The source map $\sigma : X^{(r)} \to S^1$, the target map $\tau_0 : X^{(r)} \to M$, and the tangent map $\tau_1 : X^{(r)} \to TM$ are defined by the formulas $\sigma(u) = t$, $\tau_0(u) = x(t)$ and $\tau_1(u) = \dot{x}(t)$, where $u = j^r_t(x)$ is the $r$–jet at $t \in S^1$ of a map $x : S^1 \to M$. There are natural projection maps $p^r_s : X^{(r)} \to X^{(s)}$ for $0 \leq s \leq r$. Recall that $p^s_{s-1} : X^{(s)} \to X^{(s-1)}$ is an affine bundle with fibers diffeomorphic to $\mathbf{R}^3$, since $M$ is 3–dimensional.

Gromov [8] defines a differential relation in $X^{(r)}$ to be a subset $\mathcal{R} \subset X^{(r)}$. A knot $x \in \mathcal{E}$ satisfies the given differential relation $\mathcal{R} \subset X^{(r)}$ if $j^r_t(x) \in \mathcal{R}$ for all $t \in S^1$. Let $\mathcal{E}_{\mathcal{R}}$ denote the subspace of $\mathcal{E}$ of all knots which satisfy $\mathcal{R}$.

If $\mathcal{R}$ is open in $X^{(r)}$, let $\Gamma(\mathcal{R})$ denote the space of smooth maps $\phi : S^1 \to \mathcal{R}$ such that $\sigma \circ \phi = id_{S^1}$. In this case, if $s \leq r$, the map $p^r_s : \mathcal{R} \to X^{(s)}$ is an open relation over $X^{(s)}$ in the terminology of [8, p. 175] and [12, p. 72]. We define the space of "knots" that formally satisfy $\mathcal{R}$ over $X^{(s)}$ to be the set

$$\mathcal{E}^s_{\mathcal{R}} = \{\phi \in \Gamma(\mathcal{R}) : p^r_s \circ \phi = j^s(x) \quad \text{for some} \quad x \in \mathcal{E}\}$$

topologized as a subspace of $\Gamma(\mathcal{R})$ with the $C^\infty$ topology. Note that $\tau_0 \circ \phi = x$ in this case. Clearly $j^r : \mathcal{E}_{\mathcal{R}} \to \mathcal{E}^r_{\mathcal{R}}$ is a homeomorphism.

For convenience, we state the definition of ampleness in our context.

**Definition 2.1** ([12, p. 78])**.** A differential relation $\mathcal{R} \subset X^{(r)}$ is ample over $X^{(s)}$ $(s \leq r)$ if $Conv(\mathcal{R}_u, \phi) = X^{(s)}_u$ for every pair $(u, \phi) \in X^{(s-1)} \times \mathcal{R}$ with $p^r_{s-1}(\phi) = u$. Here $\mathcal{R}_u = \{\psi \in \mathcal{R} : p^r_{s-1}(\psi) = u\}$, $X^{(s)}_u = \{v \in X^{(s)} : p^s_{s-1}(v) = u\}$, and $Conv(\mathcal{R}_u, \phi)$ is the convex hull, in the fiber $X^{(s)}_u$, of the $p^r_s$ image of the connected component of $\mathcal{R}_u$ that contains $\phi$.

The following proposition is a consequence of the $C^{s-1}$–dense general parametric $h$–principle for ample open relations. A sketch of the proof is given in the Appendix.

**Proposition 2.1.** *Let $2 \leq s \leq r$. If $\mathcal{R} \subset X^{(r)}$ is an open relation which is ample over $X^{(s)}$, then the inclusion map $\mathcal{E}^s_{\mathcal{R}} \to \mathcal{E}^{s-1}_{\mathcal{R}}$ is a weak homotopy equivalence.*

Let $X^{(r)}_0 = \{u \in X^{(r)} : \tau_1(u) \neq 0\}$. Consider the pull–back of the tangent bundle,

$$\tau^*_0(TM) = \{(u, V) \in X^{(1)}_0 \times TM : \tau_0(u) = \pi_M(V)\},$$

coming from the projection map $\tau_0 : X^{(1)}_0 \to M$. It is a 3–dimensional vector bundle over $X^{(1)}_0$, with a canonical line bundle,

$$L = \{(u, V) : V \quad \text{is a multiple of} \quad \tau_1(u)\}.$$

The quotient bundle $N = \tau_0^*(TM)/L$ is a 2–dimensional vector bundle over $X_0^{(1)}$ whose fiber over $u$ consists of the normal vectors to $\tau_1(u)$. If $\tau_0^*(TM)$ possesses a fiber metric, we can identify $N$ with the normal subbundle complementary to $L$. If $(u, V) \in \tau_0^*(TM)$, let $(u, V^\perp)$ denote its projection in $N$.

We next introduce the notion of a framing relation. Examples are given in §3.

**Definition 2.2.** Given a smooth fiber bundle mapping $\nu : X_0^{(r)} \to N$, as fiber bundles over $X_0^{(1)}$, the framing relation induced by $\nu$ is the differential relation

$$\mathcal{R} = \{u \in X_0^{(r)} : \nu(u) \neq 0\} \subset X^{(r)}.$$

Define $\mathcal{F}$ to be the set of nonzero elements of $N$. Then $\mathcal{R} = \nu^{-1}(\mathcal{F})$. Regarding $\mathcal{F}$ as a relation over $X^{(1)}$, framings $(x, \xi) \in \mathcal{E}_\mathcal{F}$ are formal solutions of $\mathcal{F}$ over $X^{(1)}$. This explains the notation $\mathcal{E}_\mathcal{F}$ for the space of framed knots. The $C^\infty$ topology on $\mathcal{E}_\mathcal{F}$ comes from its being a subset of $C^\infty(S^1, \mathcal{F})$.

**Definition 2.3.** The framing relation $\mathcal{R} \subset X^{(r)}$ induced by $\nu$ induces a map $\hat{\nu} : \mathcal{E}_\mathcal{R} \to \mathcal{E}_\mathcal{F}$ defined by $\hat{\nu}(x) = (x, \xi)$ for $x \in \mathcal{E}_\mathcal{R}$, where $\xi(t) = \nu(j_t^r(x))$. If $1 \leq s \leq r$, we use the same notation to denote the map $\hat{\nu} : \mathcal{E}_\mathcal{R}^s \to \mathcal{E}_\mathcal{F}$ defined by $\hat{\nu}(\phi) = (\tau_0 \circ \phi, \nu \circ \phi)$ for each $\phi \in \mathcal{E}_\mathcal{R}^s$.

Let

$$\mathcal{R}^\diamond = \{(u, v) \in \mathcal{R} \times \mathcal{R} : \nu(u) = \nu(v)\}.$$

**Definition 2.4.** We say that a framing relation $\mathcal{R} \subset X^{(r)}$ satisfies the SC condition if (1) $\nu : \mathcal{R} \to \mathcal{F}$ is a submersion onto $\mathcal{F}$, and (2) there exists a smooth homotopy

$$H : \mathcal{R}^\diamond \times [0,1] \to \mathcal{R}$$

such that $H((u, v), 0) = u$, $H((u, v), 1) = v$, and $\nu(H((u, v), t)) = \nu(u) = \nu(v)$ for all $(u, v) \in \mathcal{R}^\diamond$ and $t \in [0, 1]$.

*Remark.* SC is short for submersion/contraction. Observe that condition (2) implies that the fibers of $\nu$ are contractible, since for each fixed $u \in \mathcal{R}$ the map $H((u, -), -)$ is a contraction of the fiber $\nu^{-1}(\nu(u))$ to the point $u$.

The next proposition explains the importance of the SC condition. It is proved in §5.

**Proposition 2.2.** *If the framing relation $\mathcal{R} \subset X^{(r)}$ induced by $\nu$ satisfies the SC condition, then $\hat{\nu} : \mathcal{E}_\mathcal{R}^1 \to \mathcal{E}_\mathcal{F}$ is a weak homotopy equivalence.*

The proof of Theorem 1.2 is achieved by combining Propositions 2.1 and 2.2.

## §3. Examples

A linear connection $\nabla$ on $M$ sets up a diffeomorphism

$$\Lambda : X^{(r)} \to S^1 \times TM \oplus \cdots \oplus TM,$$

where $TM \oplus \cdots \oplus TM$ is the $r$–fold Whitney sum of the tangent bundle $TM$. The elements of $S^1 \times TM \oplus \cdots \oplus TM$ are $r+1$–tuples $(t, u_1, \dots, u_r)$, where $t \in S^1$ and $u_1, \dots, u_r$ all lie in the same tangent space. For $u \in X^{(r)}$, pick a representative curve $x : S^1 \to M$ and set $\Lambda(u) = (t, T, \nabla_T T, \dots, \nabla_T^{r-1} T)$, where $t = \sigma(u)$ is the source of $u$, $T = \dot{x}(t)$ is the tangent vector to $x$ at $t$, and $\nabla_T^k T$ is the $k$–th covariant derivative $(1 \leq k \leq r-1)$ of $\dot{x}$ along $x$ evaluated at $t$. That $\Lambda$ is well defined, and is a diffeomorphism, can be seen from the local coordinate representation of covariant differentiation. Under identification by $\Lambda$, the projection $p^r_s : X^{(r)} \to X^{(s)}$ takes the form $p^r_s(t, u_1, \ldots, u_r) = (t, u_1, \ldots, u_s)$ for $s \leq r$. We utilize this identification in Examples 3.1 and 3.2 and in the Appendix.

**Example 3.1.** Given a linear connection $\nabla$ on $M$ and $r \geq 2$, define a map $\nu = \nu^{(r)}_{\nabla} : X^{(r)}_0 \to N$ by $\nu(u) = \nu(t, u_1, \ldots, u_r) = (t, u_1, u_r^{\perp})$, where $\perp$ denotes the projection normal to $u_1$. Let $\mathcal{R}$ be the framing relation induced by $\nu$. Then

$$\mathcal{R} = \{(t, u_1, \ldots, u_r) : u_1 \wedge u_r \neq 0\}.$$

Hence $\nu$ is a submersion of $\mathcal{R}$ onto $\mathcal{F}$. Obviously,

$$\mathcal{R}^{\diamond} = \{((t, u_1, \ldots, u_r), (t, v_1, \ldots, v_r)) \in \mathcal{R} \times \mathcal{R} : u_1 = v_1 \quad \text{and} \quad u_r \equiv v_r \mod u_1\}.$$

Using the vector space structure in the Whitney sum, the map $H : \mathcal{R}^{\diamond} \times [0,1] \to \mathcal{R}$ defined by

$$H((t, u_1, \ldots, u_r), (t, v_1, \ldots, v_r), s) = (t, u_1 + s(v_1 - u_1), \ldots, u_r + s(v_r - u_r)),$$

satisfies $H((u, v), 0) = u$, $H((u, v), 1) = v$, and $\nu(H((u, v), s)) = \nu(u) = \nu(v)$ for all $(u, v) \in \mathcal{R}^{\diamond}$ and $s \in [0, 1]$. Thus $\mathcal{R}$ satisfies the SC condition.

Fix $\phi = (t, \phi_1, \ldots, \phi_r) \in \mathcal{R}$ and set $u = p^r_{s-1}(\phi) = (t, \phi_1, \ldots, \phi_{s-1})$ for some $2 \leq s \leq r$. Clearly, $\mathcal{R}_u$ consists of elements of the form $(t, \phi_1, \ldots, \phi_{s-1}, \psi_s, \ldots, \psi_r)$ with $\phi_1 \wedge \psi_r \neq 0$. Thus $\mathcal{R}_u$ is always connected, and $Conv(\mathcal{R}_u, \phi) = X^{(s)}_u$. This shows that $\mathcal{R}$ is ample over $X^{(s)}$ for every $2 \leq s \leq r$. Therefore Theorem 1.2 applies to the framing relation induced by $\nu^{(r)}_{\nabla}$.

Note that when $r \geq 3$ this relation is not $Diff(S^1)$-invariant. Thus a parameterized knot that satisfies the relation may fail to do so after reparametrization. The next family of examples are $Diff(S^1)$-invariant.

**Example 3.2.** Suppose $M$ has a Riemannian metric $g = \langle -, - \rangle$ with the Levi–Civita connection $\nabla$. For $u \in X^{(r)}_0$ define $\rho(u)$ to be equal to the $r$–jet of the arclength reparameterization of a representative of $u$. Clearly $\rho(u)$ depends only on $u$ and not the representing curve. For $r \geq 2$, define $\nu^{(r)}_g(u) = (p^r_1(u), \xi(u))$, where $\xi(u)$ is the component of the $(r-1)$–st covariant derivative of $\rho(u)$ which is perpendicular to $\tau_1(u)$. Let $\mathcal{R}$ be the framing relation induced by $\nu^{(r)}_g$. By construction, $\mathcal{R}$ is invariant under reparameterizations and thus is a $Diff(S^1)$ invariant framing relation.

**Lemma 3.1.** *$\rho(t, u_1, \ldots, u_r) = (t, v_1, \ldots, v_r)$, where $v_1 = |u_1|^{-1} u_1$ and where*

$$v_k = |u_1|^{-k} u_k + \alpha_{k-1}(t, u_1, \ldots, u_{k-1}) + \beta_k(t, u_1, \ldots, u_k) u_1$$

*for every $2 \leq k \leq r$. $\alpha_{k-1}$ is a vector valued function that depends only on $(t, u_1, \ldots, u_{k-1})$, and $\beta_k$ is a real valued function that depends only on $(t, u_1, \ldots, u_k)$.*

*Proof.* If $T$ is the tangent vector field along a curve representing the given $r$–jet, then $S = \langle T, T \rangle^{-1/2} T$ is the unit tangent field for the arclength parameter. An induction over $k$ implies

$$\nabla^{k-1}_S S = |T|^{-k} \nabla^{k-1}_T T + \hat{\alpha}_{k-1}(T, \nabla_T T, \ldots, \nabla^{k-2}_T T) + \hat{\beta}_k(T, \nabla_T T, \ldots, \nabla^{k-1}_T T) T,$$

where, for every $k \geq 2$, the vector field $\hat{\alpha}_{k-1}$ depends only on $(T, \nabla_T T, \ldots, \nabla^{k-2}_T T)$, and the function $\hat{\beta}_k$ depends only on $(T, \nabla_T T, \ldots, \nabla^{k-1}_T T)$. $\square$

*Remark.* For $k = 2$ and 3, the computation in Lemma 3.1 produces the formulas

$$v_2 = |u_1|^{-2}u_2 - |u_1|^{-4}\langle u_2, u_1\rangle u_1$$

and

$$v_3 = |u_1|^{-3}u_3 - 3|u_1|^{-5}\langle u_2, u_1\rangle u_2 - |u_1|^{-5}(\langle u_3, u_1\rangle + \langle u_2, u_2\rangle - 4|u_1|^{-2}\langle u_2, u_1\rangle^2)u_1.$$

Let $\perp$ denote the component normal to $u_1$; then by Lemma 3.1,

$$\nu_g^{(r)}(t, u_1, \ldots, u_r) = (t, u_1, |u_1|^{-r}u_r^{\perp} + \alpha_{r-1}^{\perp}(t, u_1, \ldots, u_{r-1})). \tag{3.1}$$

Thus, for fixed $(t, u_1, \ldots, u_{r-1})$, the $r$–th component of $\nu_g^{(r)}$ is a rank 2 affine mapping in $u_r$. Therefore $\nu_g^{(r)}$ is a submersion of $\mathcal{R}$ onto $\mathcal{F}$. This fact also implies that $\mathcal{R}_u$ is connected for every pair $(u, \phi) \in X^{(s-1)} \times \mathcal{R}$ with $p_{s-1}^r(\phi) = u$, because its complement in $(p_{s-1}^r)^{-1}(u)$ has codimension two. Clearly (3.1) shows that $Conv(\mathcal{R}_u, \phi) = X_u^{(s)}$. Therefore $\mathcal{R}$ is ample over $X^{(s)}$ for every $2 \leq s \leq r$.

Write $u = (t, u_1, \ldots, u_r)$, $v = (t, v_1, \ldots, v_r)$, and $w = (t, w_1, \ldots, w_r)$. Then $(u, v) \in \mathcal{R}^{\diamond}$ if and only if $u_1 = v_1$ and

$$|u_1|^{-r}u_r^{\perp} + \alpha_{r-1}^{\perp}(t, u_1, \ldots, u_{r-1}) = |v_1|^{-r}v_r^{\perp} + \alpha_{r-1}^{\perp}(t, v_1, \ldots, v_{r-1}).$$

Next define the homotopy $H$ by

$$H((u, v), s) = w,$$

where $w_1 = u_1 = v_1$, $w_i = u_i + s(v_i - u_i)$ for $i = 2, \ldots, r-1$, decomposing $w_r$ into tangential and normal components, $w_r^{\top} = u_r^{\top} + s(v_r^{\top} - u_r^{\top})$ and

$$w_r^{\perp} = u_r^{\perp} + |u_1|^r\alpha_{r-1}^{\perp}(t, u_1, \ldots, u_{r-1}) - |v_1|^r\alpha_{r-1}^{\perp}(t, w_1, \ldots, w_{r-1}).$$

Note the dependence on $(w_1, \ldots, w_{r-1})$! Therefore $\nu_g^{(r)}$ satisfies the SC condition.

Let us compare Examples 3.1 and 3.2 for $r = 2$, where $\nabla$ is the Levi–Civita connection of $g$. By the remark following Lemma 3.1, $v_2 = |u_1|^{-2}u_2 - |u_1|^{-4}\langle u_2, u_1\rangle u_1$. Thus $v_2^{\perp} = |u_1|^{-2}u_2^{\perp}$, and hence $\nu_g^{(2)}(t, u_1, u_2) = (t, u_1, |u_1|^{-2}u_2^{\perp})$. On the other hand, $\nu_{\nabla}^{(2)}(t, u_1, u_2) = (t, u_1, u_2^{\perp})$. Therefore $\nu_{\nabla}^{(2)}$ and $\nu_g^{(2)}$ induce the same framing relation as subsets of $X^{(2)}$. Although $\nu_{\nabla}^{(2)}$ and $\nu_g^{(2)}$ are different maps, they are homotopic, which implies that they induce homotopic maps $\hat{\nu}_{\nabla}^{(2)}$ and $\hat{\nu}_g^{(2)}$. In Euclidean space, this framing relation is called the freedom relation [8, p. 8] and is satisfied precisely by curves of nonvanishing curvature.

There are several definitions of the curvature of a curve in a Finsler space, but, for any one of these definitions, vanishing curvature means the curve is a geodesic [11, pp. 151–155]. In the next example, we define curves of "nowhere vanishing curvature" in a 3–dimensional Finsler space by means of a 2–nd order framing relation that avoids a specific formula for curvature of a curve. This generalizes Example 3.2 with $r = 2$. Our main reference on Finsler geometry is [11]. For a modern treatment, see [1].

**Example 3.3.** Let $F : TM \to \mathbf{R}$ be a Finsler metric on $M$. Given a curve $x$ in $M$, we can define a 1–form $\omega$ along $x$ by the local coordinate expression

$$\omega = \sum_{i=1}^{3} \left( \frac{d}{dt} \left( \frac{\partial F(x, \dot{x})}{\partial \dot{x}^i} \right) - \frac{\partial F(x, \dot{x})}{\partial x^i} \right) dx^i.$$

Note that $\omega = 0$ is the Euler–Lagrange equation, and so, $x$ is a geodesic if and only if $\omega = 0$. Certainly, the value of $\omega$ at any point along the curve depends only on the 2–jet of the curve at that point. Thus we obtain a map $\omega : X^{(2)} \to T^*M$ that satisfies $\omega(u) \in T^*_{\tau_0(u)}M$ for $u \in X^{(2)}$. Moreover, a short computation using (1.11) and (1.12) in [11, p. 4] shows that

$$\sum_{i=1}^{3} \left( \frac{d}{dt} \left( \frac{\partial F(x,\dot{x})}{\partial \dot{x}^i} \right) - \frac{\partial F(x,\dot{x})}{\partial x^i} \right) \dot{x}^i = 0. \tag{3.2}$$

In other words, $\omega(u)(\tau_1(u)) = 0$. We can regard the metric tensor [11, p. 14] associated to the Finsler metric, which in local coordinates is given by

$$g_{ij}(x,\dot{x}) = \frac{1}{2} \frac{\partial^2 F^2(x,\dot{x})}{\partial \dot{x}^i \partial \dot{x}^j},$$

as a fiber metric on $\tau_0^*(TM)$. We then define $\nu : X_0^{(2)} \to \tau_0^*(TM)$ by setting $\nu(u)$ equal to the vector in $T_{\tau_0(u)}M$ metrically dual to the covector $\omega(u)$ with respect to the inner product $g(\tau_0(u), \tau_1(u))$. By (3.2), the image of $\nu$ lies in $N = L^\perp$, where $L$ is the canonical line bundle in $\tau_0^*(TM)$. Therefore, in local coordinates,

$$\nu(t,x,\dot{x},\ddot{x}) = \left( t, x, \dot{x}, \sum_{i=1}^{3} g^{ij}(x,\dot{x}) \left( \frac{d}{dt} \left( \frac{\partial F(x,\dot{x})}{\partial \dot{x}^i} \right) - \frac{\partial F(x,\dot{x})}{\partial x^i} \right) \frac{\partial}{\partial x^j} \right),$$

where $g^{ij}(x,\dot{x})$ is the inverse matrix of $g_{ij}(x,\dot{x})$.

This formula shows that, fiberwise over $X_0^{(1)}$, $\nu$ is an affine linear map, of rank 2, since the matrix $\frac{\partial^2 F}{\partial \dot{x}^i \partial \dot{x}^j}$ has rank 2 [11, p. 9]. Thus $\nu$ is a submersion of $\mathcal{R}$ onto $\mathcal{F}$. The fibers of $\nu$ are just families of parallel lines in each fiber over $X_0^{(1)}$, so the contractibility condition is satisfied. Furthermore, for each $u \in X_0^{(1)}$, $\mathcal{R}_u$ is connected and its convex hull is all of $X_u^{(2)}$, which shows that $\mathcal{R}$ is ample over $X^{(2)}$. Hence the nonvanishing curvature relation in a 3–dimensional Finsler space satisfies the hypothesis of Theorem 1.2.

Example 3.3 suggests the possibility of defining framing relations for other variational problems on a manifold. We next consider a framing relation arising from osculating spheres to curves in 3–space.

**Example 3.4.** Osculating spheres are a standard construction in the theory of space curves [3, p. 33], [13, p. 25]. The osculating sphere of a space curve at a given point on the curve is the unique sphere to which the curve makes 3rd order contact at the given point. The condition that a space curve has an osculating sphere at a given point is a framing relation that we can describe as follows. Translating the given point to the origin $(0,0,0)$ in $\mathbf{R}^3$, we may assume that the curve has the form

$$\begin{aligned} x(t) &= x_1 t + \frac{x_2}{2}t^2 + \frac{x_3}{6}t^3 + \dots, \\ y(t) &= y_1 t + \frac{y_2}{2}t^2 + \frac{y_3}{6}t^3 + \dots, \\ z(t) &= z_1 t + \frac{z_2}{2}t^2 + \frac{z_3}{6}t^3 + \dots. \end{aligned} \tag{3.3}$$

The equation of a general sphere passing through the origin has the form

$$Ax + By + Cz + D(x^2 + y^2 + z^2) = 0, \tag{3.4}$$

where we explicitly allow the case $D = 0$ when the sphere degenerates to a plane. The curve (3.3) makes 3rd order contact with the sphere (3.4) when $A$,$B$,$C$,$D$ is a nonzero solution of the homogeneous linear system

$$\begin{array}{ccccccccc} Ax_1 & + & By_1 & + & Cz_1 & & & = & 0, \\ \frac{1}{2}Ax_2 & + & \frac{1}{2}By_2 & + & \frac{1}{2}Cz_2 & + & D(x_1^2+y_1^2+z_1^2) & = & 0, \\ \frac{1}{6}Ax_3 & + & \frac{1}{6}By_3 & + & \frac{1}{6}Cz_3 & + & D(x_1x_2+y_1y_2+z_1z_2) & = & 0. \end{array} \tag{3.5}$$

The sphere (3.4) satisfying (3.5) is unique precisely when (3.5) has a 1–dimensional solution set. In other words, the coefficient matrix

$$\begin{bmatrix} x_1 & y_1 & z_1 & 0 \\ \frac{1}{2}x_2 & \frac{1}{2}y_2 & \frac{1}{2}z_2 & x_1^2+y_1^2+z_1^2 \\ \frac{1}{6}x_3 & \frac{1}{6}y_3 & \frac{1}{6}z_3 & x_1x_2+y_1y_2+z_1z_2 \end{bmatrix}$$

has rank 3. Equivalently, the two vectors $(x_1, y_1, z_1)$ and

$$\xi = (x_1^2+y_1^2+z_1^2)(x_3,y_3,z_3) - 3(x_1x_2+y_1y_2+z_1z_2)(x_2,y_2,z_2)$$

are linearly independent. On writing $u_1 = (x_1, y_1, z_1)$, $u_2 = (x_2, y_2, z_2)$ and $u_3 = (x_3, y_3, z_3)$, the remark following Lemma 3.1 shows that $v_3^\perp = |u_1|^{-5}\xi^\perp$. Thus a space curve has an osculating sphere at each point exactly when it satisfies the framing relation in Example 3.2 with $r = 3$ and with $M = \mathbf{R}^3$. Therefore the space of knots in $\mathbf{R}^3$ which admit osculating spheres at each point is weakly homotopic to the space of framed knots in $\mathbf{R}^3$, and Theorem 1.3 is a corollary of Theorem 1.2.

**Example 3.5.** Framing relations need not satisfy the hypotheses of Theorem 1.2. For example, if $V$ is a smooth vector field on $M$, define $\nu_V : X_0^{(2)} \to N$ by setting $\nu_V(u) = (p_1^2(u), V_{\tau_0(u)}^\perp)$. Then $u \in \mathcal{R}$ if and only if $\tau_1(u)$ and $V_{\tau_0(u)}$ are linearly independent. Clearly $\nu_V : \mathcal{R} \to \mathcal{F}$ is not onto. But still the contractibility condition SC(2) holds, and $\mathcal{R}$ is ample over $X^{(2)}$.

## §4. An application

Suppose the framing relation $\mathcal{R} \subset X^{(r)}$ induced by $\nu$ is ample and satisfies the SC condition. Fix a connected component $\mathcal{E}_\mathcal{R}^\circ$ of $\mathcal{E}_\mathcal{R}$. Let $\mathcal{E}_\mathcal{F}^\circ$ and $\mathcal{E}^\circ$ be the corresponding connected components of $\mathcal{E}_\mathcal{F}$ and $\mathcal{E}$ respectively. Let $\iota : \mathcal{E}_\mathcal{R}^\circ \to \mathcal{E}^\circ$ be the inclusion map, $\hat{\nu} : \mathcal{E}_\mathcal{R}^\circ \to \mathcal{E}_\mathcal{F}^\circ$ the restriction of $\hat{\nu}$, and $\varpi : \mathcal{E}_\mathcal{F}^\circ \to \mathcal{E}^\circ$ the projection map $\varpi(x, \xi) = x$. Clearly $\varpi \circ \hat{\nu} = \iota$. By Theorem 1.2, $\hat{\nu}$ is a weak homotopy equivalence. Thus, were $\iota$ a weak homotopy equivalence, then $\varpi$ would be one as well. But ...

**Proposition 4.1.** *If $M$ is parallelizable, then $\varpi : \mathcal{E}_\mathcal{F}^\circ \to \mathcal{E}^\circ$ is not a weak homotopy equivalence.*

*Remark.* Every compact orientable 3–manifold is parallelizable by a theorem of Stiefel.

*Proof.* We exhibit a noncontractible loop of $\mathcal{E}_\mathcal{F}^\circ$ which is mapped by $\varpi$ into a contractible loop in $\mathcal{E}^\circ$. Since $M$ is parallelizable, there exists a global trivialization $\Psi : TM \to M \times \mathbf{R}^3$ of the tangent bundle of $M$. This global trivialization imposes an oriented Riemannian metric on $M$ by requiring that $\Psi$ carries each tangent space of $M$ isometrically onto $\mathbf{R}^3$. Let $\Psi_2 : TM \to \mathbf{R}^3$ be $\Psi$ followed by projection onto $\mathbf{R}^3$.

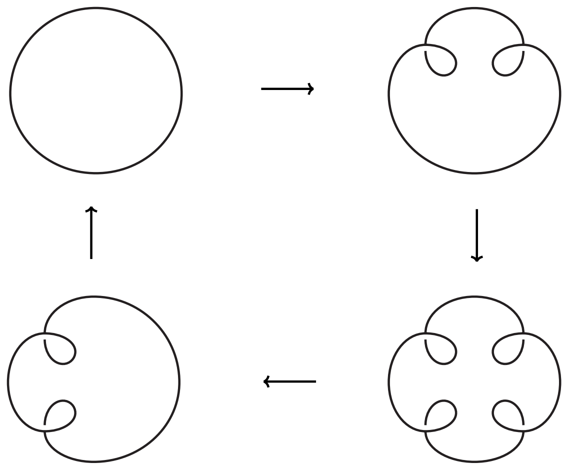

FIGURE 1.

Fix $t_0 \in S^1$. Let $(x, \xi) \in \mathcal{E}^\circ_{\mathcal{F}}$. Certainly the three vectors $\Psi_2(\dot{x}(t_0))$, $\Psi_2(\xi(t_0))$ and their cross product $\Psi_2(\dot{x}(t_0)) \times \Psi_2(\xi(t_0))$ form an oriented basis for $\mathbf{R}^3$. Applying the Gram–Schmidt orthogonalization process to this basis results in a $3 \times 3$ orthogonal matrix, which we denote by $F(x, \xi)$. Clearly, $F : \mathcal{E}^\circ_{\mathcal{F}} \to SO(3)$ is continuous.

Fix $(x, \xi) \in \mathcal{E}^\circ_{\mathcal{F}}$. We construct a loop $(x, \xi^\theta)$, $\theta \in S^1$, in $\mathcal{E}^\circ_{\mathcal{F}}$ by defining $\xi^\theta(t)$ to be the result of rotating $\xi(t)$ in $T_{x(t)}M$ about the axis $\dot{x}(t)$ by a positive angle $\theta \in S^1$ for each $t \in S^1$. This implicitly uses the oriented Riemannian structure on $M$ coming from the parallelization. Clearly, $\varpi(x, \xi^\theta) = x$ is a constant loop and thus contractible in $\mathcal{E}^\circ$. On the other hand, $(x, \xi^\theta)$ is not contractible in $\mathcal{E}^\circ_{\mathcal{F}}$, because $F(x, \xi^\theta)$ is a generator of $\pi_1(SO(3)) \approx \mathbf{Z}_2$. $\square$

Therefore $\iota : \mathcal{E}^\circ_{\mathcal{R}} \to \mathcal{E}^\circ$ is not a weak homotopy equivalence if $M$ is parallelizable. In particular, in case $M = \mathbf{R}^3$ and $\mathcal{R}$ is the nonvanishing curvature relation, then $\mathcal{E}^\circ_{\mathcal{R}}$ is just the set of all knots in a fixed isotopy class with nonvanishing curvature and prescribed self–linking number. In this case, the failure of $\iota$ to be a weak homotopy equivalence gives a negative answer to the problem from [2] described in §1.

**Example 4.1.** Proposition 4.1 implies that each connected component of the space of knots of nonvanishing curvature contains a noncontractible loop that is contractible in the space of knots. Figure 1 represents a concrete realization of such a loop in the component containing the standard trivial knot, i.e., the circle.

*Notes*: (1) During each of the 4 stages, a pair of opposite twists is either created or annihilated. Gluck and Pan [6], [7] show how to do this while maintaining nonvanishing curvature. For example, during the first stage the top arc of the circle is pulled forward, down, back and up.

(2) The Frenet frame attached to the point at the top of the circle undergoes a rotation of 360° about the tangent vector during the first stage and stays fixed during the remaining three stages. Thus the loop is noncontractible in the space of knots of nonvanishing curvature, because the Frenet frame traces out a nontrivial loop in $SO(3)$.

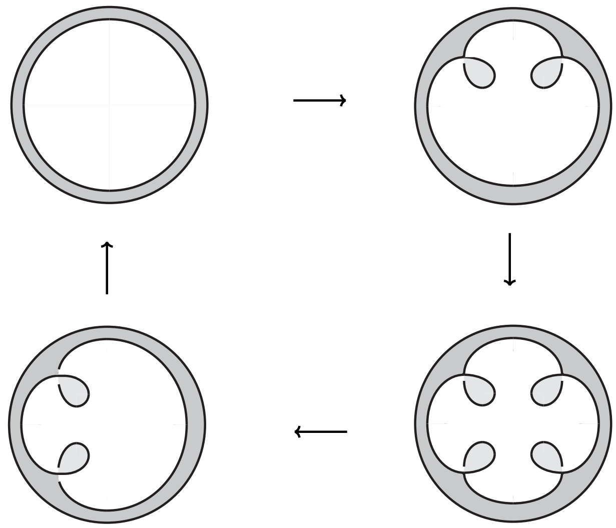

FIGURE 2.

(3) The family of the embedded annuli, represented in Figure 2, detail the contraction of this loop in the space of knots.

## §5. THE SC CONDITION

Assume the framing relation $\mathcal{R} \subset X^{(r)}$ induced by $\nu$ satisfies the SC condition.

**Proposition 5.1.** *Let $P$ be a compact Hausdorff space, and let $Q \subset P$ be a closed subspace which is a neighborhood retract. Let $F : P \to \mathcal{E}_{\mathcal{F}}$ and $\Phi : Q \to \mathcal{E}^1_{\mathcal{R}}$ be continuous maps such that $\hat{\nu} \circ \Phi = F$ on $Q$. Then there exists $\hat{F} : P \to \mathcal{E}^1_{\mathcal{R}}$ such that $\hat{\nu} \circ \hat{F} = F$ and $\hat{F}|Q = \Phi$.*

Proposition 2.2 is a direct consequence of Proposition 5.1. Surjectivity of $\hat{\nu}_* : \pi_k(\mathcal{E}^1_{\mathcal{R}}) \to \pi_k(\mathcal{E}_{\mathcal{F}})$ is obtained by applying Proposition 5.1 with $P = S^k$ and $Q = \{*\}$. Similarly, injectivity of $\hat{\nu}_* : \pi_k(\mathcal{E}^1_{\mathcal{R}}) \to \pi_k(\mathcal{E}_{\mathcal{F}})$ is obtained by applying Proposition 5.1 with $P = S^k \times [0,1]$ and $Q = (S^k \times \{0\}) \cup (\{*\} \times [0,1]) \cup (S^k \times \{1\})$. Thus $\hat{\nu}$ is a weak homotopy equivalence.

The proof of Proposition 5.1 is based on the following two lemmas. The map $F : P \to \mathcal{E}_{\mathcal{F}}$ induces a map $f : S^1 \times P \to \mathcal{F}$ defined by $f(t,p) = (j^1_t x, \xi(t))$, where $F(p) = (x, \xi)$. Note that $f$ is smooth in $t$. Similarly there is a map $\varphi : S^1 \times Q \to \mathcal{R}$ that satisfies $\nu \circ \varphi = f$ on $S^1 \times Q$.

**Lemma 5.2.** *For every $(t,p) \in S^1 \times P$, there exist a neighborhood $U$ of $(t,p)$ and a map $\hat{f} : U \to \mathcal{R}$ such that $\nu \circ \hat{f} = f$ on $U$. Moreover, if $p \in Q$, then $\hat{f}$ can be chosen to agree with $\varphi$ on $U \cap (S^1 \times Q)$.*

*Proof.* Suppose $p \notin Q$. There exists a neighborhood $V$ of $p$ in $P$ which is disjoint from $Q$. Because $\nu$ is a submersion of $\mathcal{R}$ onto $\mathcal{F}$, there exist an open neighborhood $W$ of $f(t,p)$ and a smooth section $\mu : W \to \mathcal{R}$ of $\nu$. Set $U = f^{-1}(W) \cap (S^1 \times V)$ and $\hat{f} = \mu \circ f$. Then $\nu \circ \hat{f} = f$ on $U$.

Suppose $p \in Q$. Let $V$ be a neighborhood of $Q$ in $P$, and $\rho : V \to Q$ a retraction. Set $\hat{\rho} = id_{S^1} \times \rho$. Since $\nu$ is a submersion of $\mathcal{R}$ onto $\mathcal{F}$ and $\nu \circ \varphi(t,p) = f(t,p)$, there exist neighborhoods $W$ of $\varphi(t,p)$ in $\mathcal{R}$ and $W_1$ of $f(t,p)$ in $\mathcal{F}$, and a map $\upsilon : W \to \mathbf{R}^q$, such that the map $\Upsilon = (\nu, \upsilon) : W \to W_1 \times \mathbf{R}^q$ is a diffeomorphism. Set $U = \hat{\rho}^{-1}(\varphi^{-1}(W)) \cap f^{-1}(W_1)$ and $\hat{f} = \Upsilon^{-1} \circ (f, \upsilon \circ \varphi \circ \hat{\rho})$. Then $\nu \circ \hat{f} = f$ on $U$, and $\hat{f}$ agrees with $\varphi$ on $U \cap (S^1 \times Q)$. □

**Lemma 5.3.** *Let $K_1$, $K_2$ be closed subsets of $S^1 \times P$, and let $\hat{f}_1 : U_1 \to \mathcal{R}$, $\hat{f}_2 : U_2 \to \mathcal{R}$ be maps defined on respective open neighborhoods $U_1$, $U_2$ of $K_1$, $K_2$ such that $\nu \circ \hat{f}_1 = f$ on $U_1$ and $\nu \circ \hat{f}_2 = f$ on $U_2$. Then there exists a map $\hat{f} : U \to \mathcal{R}$, defined on some neighborhood $U$ of $K_1 \cup K_2$, such that $\nu \circ \hat{f} = f$ on $U$, $\hat{f} = \hat{f}_1$ on some neighborhood $V_1$ of $K_1$, and $\hat{f} = \hat{f}_2$ on $U_2 \backslash U_1$.*

*Proof.* Pick a neighborhood $V_1$ of $K_1$ with $\overline{V}_1 \subset U_1$. Pick a function $\lambda : S^1 \times P \to [0,1]$, smooth in $t$, such that $\lambda = 0$ on $\overline{V}_1$ and $\lambda = 1$ on some neighborhood $V_2$ of $K_2 \backslash U_1$ with $\overline{V}_2 \subset U_2 \backslash \overline{V}_1$. Set

$$\hat{f}(t,p) = \begin{cases} \hat{f}_1(t,p) & \text{if} \quad (t,p) \in V_1, \\ H(\hat{f}_1(t,p), \hat{f}_2(t,p), \lambda(t,p)) & \text{if} \quad (t,p) \in U_1 \cap U_2, \\ \hat{f}_2(t,p) & \text{if} \quad (t,p) \in V_2, \end{cases}$$

where $H$ is the homotopy in the SC condition. Note that $U = V_1 \cup (U_1 \cap U_2) \cup V_2$ is a neighborhood of $K_1 \cup K_2$. □

To complete the proof of Proposition 5.1, apply Lemma 5.2 to find a finite cover $U_i$ of $S^1 \times P$ and maps $\hat{f}_i : U_i \to \mathcal{R}$ such that $\nu \circ \hat{f}_i = f$ on $U_i$. Induction over the cover using Lemma 5.3 produces a map $\hat{f} : S^1 \times P \to \mathcal{R}$ such that $\nu \circ \hat{f} = f$ and $\hat{f}|S^1 \times Q = \varphi$. For each fixed $p \in P$, $F(p)$ is a framed knot. Thus the curve $t \mapsto \tau(\hat{f}(t,p))$ is an embedding in $M$. Hence $\hat{f}$ induces the required map $\hat{F} : P \to \mathcal{E}^1_{\mathcal{R}}$.

## Appendix

The purpose of this appendix is to sketch the proof of Proposition 2.1. The sketch is based on ideas from [8] used to prove a $C^{s-1}$ dense parametric $h$–principle for open ample partial differential relations. We include this appendix because the ordinary differential relations that we consider do not present the complications of the general theory, and because it was difficult to find a version of the $h$–principle in the literature with the precise hypothesis and conclusion we needed. (See [8], [12].)

Let $\iota$ denote the inclusion of $\mathcal{E}^s_{\mathcal{R}}$ into $\mathcal{E}^{s-1}_{\mathcal{R}}$. To prove Proposition 2.1, we must show that $\iota_* : \pi_k(\mathcal{E}^s_{\mathcal{R}}) \to \pi_k(\mathcal{E}^{s-1}_{\mathcal{R}})$ is an isomorphism for every $k \geq 0$. Surjectivity of $\iota_*$ is obtained by applying the following Proposition A with $P = S^k$ and $Q = \{*\}$, while injectivity is obtained by applying it with $P = S^k \times [0,1]$ and $Q = (S^k \times \{0\}) \cup (\{*\} \times [0,1]) \cup (S^k \times \{1\})$.

**Proposition A.** *Let $r \geq s \geq 2$. Let $\mathcal{R} \subset X^{(r)}$ be open and let $\mathcal{R}$ be ample over $X^{(s)}$. Let $P$ be a compact Hausdorff space and $Q$ a closed subspace of $P$. Suppose $F : P \to \mathcal{E}^{s-1}_{\mathcal{R}}$ with $F|Q : Q \to \mathcal{E}^s_{\mathcal{R}} \subset \mathcal{E}^{s-1}_{\mathcal{R}}$. Then there exists a homotopy $H : P \times [0,1] \to \mathcal{E}^{s-1}_{\mathcal{R}}$ with $H(p,0) = F(p)$ , $H(p,1) \in \mathcal{E}^s_{\mathcal{R}}$, and $H(q,\lambda) = F(q)$ for all $p \in P$, $q \in Q$, and $\lambda \in [0,1]$.*

*Proof.* The hypotheses imply that there is a map $\phi_0 : S^1 \times P \to \mathcal{R}$, smooth in $S^1$, such that $p^r_{s-1}\phi_0 = j^{s-1}_t f_0$, where $f_0 = \tau \circ \phi_0 : S^1 \times P \to M$, and such that $p^r_s \phi_0(t,q) = j^s_t f_0(t,q)$ for all $q \in Q$. Thus the $C^{s-1}$ dense parametric $h$–principle stated below implies Proposition A provided the $C^{s-1}$ neighborhood of $f_0$ is chosen so that the maps $f_\lambda(-,p)$ are embeddings. This can be done, because the maps $f_0(-,p)$ are embeddings and because the embeddings of one manifold into another form an open subset of all maps in the $C^1$ topology [10, p. 37]. □

**The $C^{s-1}$ Dense Parametric $h$–Principle.** *Let $r \geq s \geq 2$. Let $\mathcal{R} \subset X^{(r)}$ be open and let $\mathcal{R}$ be ample over $X^{(s)}$. Let $P$ be a compact Hausdorff space and let $K$ be a closed subset of $S^1 \times P$. Given $\phi_0 : S^1 \times P \to \mathcal{R}$, smooth in $S^1$, such that $p^r_{s-1}\phi_0 = j^{s-1}_t f_0$, where $f_0 = \tau \circ \phi_0 : S^1 \times P \to M$, and such that $p^r_s\phi_0(t,q) = j^s_t f_0(t,q)$ for all $(t,q) \in K$, there exists a homotopy $h : S^1 \times P \times [0,1] \to \mathcal{R}$, smooth in $S^1$, with $h(t,p,0) = \phi_0(t,p)$ for all $(t,p) \in S^1 \times P$, and $h(t,q,\lambda) = \phi_0(t,q)$ for all $(t,q,\lambda) \in K \times [0,1]$.*

*Meanwhile, for all $\lambda \in [0,1]$, $f_\lambda(t,p) = \tau \circ h(t,p,\lambda)$ remains in an arbitrary $C^{s-1}$ neighborhood of $f_0$, $p^r_{s-1}h(t,p,\lambda) = j^{s-1}_t f_\lambda(t,p)$ and $p^r_s h(t,p,1) = j^s_t f_1(t,p)$.*

*Proof.* There are three steps in the proof. In the first step we deform $\phi_0$ without deforming $f_0$ to a $\phi_1$ so that $p^r_{s-1}\phi_1 = j^{s-1}_t f_0$ on $S^1 \times P$ and $p^r_s\phi_1 = j^s_t f_0$ on an open $V \supset K$.

Choose any linear connection $\nabla$ on $M$. As in §3, use $\nabla$ to write

$$X^{(r)} = S^1 \times TM \oplus \cdots \oplus TM.$$

Then $\phi_0 = (\sigma, \phi^1_0, \ldots, \phi^r_0)$ is the component decomposition of $\phi_0$ relative to the decomposition of $X^{(r)}$. Each $\phi^i_0$ is a map $\phi^i_0 : S^1 \times P \to TM$. Since $\mathcal{R} \subset X^{(r)}$ is open and $S^1 \times P$ is compact, there is a fiberwise convex neighborhood $W$ of the zero section in $TM$ such that if $\psi : S^1 \times P \to W \subset TM$, then

$$(t, \phi^1_0, \ldots, \phi^{s-1}_0, \phi^s_0 + \psi, \phi^{s+1}_0, \ldots, \phi^r_0) \in \mathcal{R}.$$

Let $\nabla^{s-1} f_0(t,p)$ denote the $(s-1)$-th order covariant derivative of $t \mapsto f_0(t,p)$ for fixed $p$. Let $V_0 = \{(t,p) : \nabla^{s-1} f_0 - \phi^s_0 \in W\}$ and pick a function $\chi : S^1 \times P \to [0,1]$ which is smooth in $S^1$, whose support is contained in $V_0$, and which is 1 throughout a neighborhood $V \subset V_0$ of $K$. Then

$$\phi_\lambda = (t, \phi^1_0, \ldots, \phi^{s-1}_0, \phi^s_0 + \lambda\chi(\nabla^{s-1} f_0 - \phi^s_0), \phi^{s+1}_0, \ldots, \phi^r_0)$$

has the required properties. This completes the first step.

The second step is embodied in the following local extension lemma.

**Lemma B.** *Let $V \subset S^1 \times P$ be an open neighborhood of the closed set $K \subset S^1 \times P$. Suppose $\phi_0 : S^1 \times P \to \mathcal{R}$ and $f_0 = \tau \circ \phi_0$. Assume that*

(A) $p^r_{s-1}\phi_0 = j^{s-1}_t f_0$ *on* $S^1 \times P$, *and*
(B) $p^r_s\phi_0 = j^s_t f_0$ *on* $V$.

*Also assume $f_0([0,1] \times P_0)) \subset \mathbf{R}^3 \subset M$, where $\mathbf{R}^3$ is a coordinate chart on $M$. Let $L$ be a closed set and $W$ an open set with $L \subset W \subset \overline{W} \subset int([0,1] \times P_0)$.*

*Then there exists a homotopy $\phi_\lambda : S^1 \times P \to \mathcal{R}$, $\lambda \in [0,1]$, of $\phi_0$ such that, on setting $f_\lambda = \tau \circ \phi_\lambda$,*

(A′) $p^r_{s-1}\phi_\lambda = j^{s-1}_t f_\lambda$ *on* $S^1 \times P$,
(B′) $p^r_s\phi_1 = j^s_t f_1$ *on a neighborhood of* $K \cup L$, *and*
(C′) $f_\lambda$ *remains arbitrarily* $C^{s-1}$ *close to* $f_0$ *throughout the homotopy.*

*Proof of the Lemma.* For curves defined on $[0,1]$ in the coordinate chart $\mathbf{R}^3 \subset M$ we can identify $X^{(r)} = [0,1] \times \mathbf{R}^3 \times \cdots \times \mathbf{R}^3$. By ampleness we can find a map $\psi : [0,1] \times P_0 \times [0,1] \to \mathcal{R}$, $\psi(t,p,u) = (t, f(t,p,u), \psi^1(t,p,u), \ldots, \psi^r(t,p,u))$, where $f = \tau \circ \psi$, such that

(1) $\psi(t,p,0) = \phi_0(t,p)$,
(2) $p^r_{s-1}\psi(t,p,u) = p^r_{s-1}\phi_0(t,p)$,

and the curve $u \mapsto \psi(t,p,u)$ strictly surrounds $\partial_t^s f_0(t,p)$. Moreover by (B) one can assume that $\psi^s(t,p,u)$ is approximately $\partial_t^s f_0(t,p)$ on some neighborhood $V' \subset V$ of $K$. Applying the reparameterization lemma [8, p. 169], we may also assume that

(3) $\partial_t^s f_0(t,p) = \int_0^1 \psi^s(t,p,u)du.$

Since $\psi^s(t,p,u)$ is approximately $\partial_t^s f_0(p,t)$ on some neighborhood $V' \subset V$ of $K$ and $\mathcal{R}$ is open, it is possible to deform $\psi$, without upsetting (3), so that we have

(4) $\psi^s(t,p,u) = \partial_t^s f_0(t,p)$ on some open neighborhood $V_0$ of $K$ with $V_0 \subset V'$.

Next choose a function $\chi : S^1 \times P \to [0,1]$ and then a function $\beta : S^1 \times P \to [0,1]$, both smooth in $t$, such that

(5) $supp(\chi) \subset W \backslash K$ and $\chi = 1$ on a neighborhood $W_1$ of $L \backslash V_0$, and
(6) $supp(\beta) \subset V_0$ with $\beta = 1$ on a neighborhood $W_2$ of $L \cap supp(\chi) \backslash W_1$.

(6) is possible because $L \cap supp(\chi) \backslash W_1 \subset V_0$ by (5).

Gromov's rapidly oscillating function $h_\epsilon : [0,1] \to [0,1]$ produces a $C^{s-1}$ arbitrarily close approximation $f_\epsilon^\circ$ to $f_0$ on $[0,1] \times P_0$ as $\epsilon \to 0$ such that

$$\partial_t^s f_\epsilon^\circ(t,p) = \psi^s(t,p,h_\epsilon(t)) \tag{7}$$

on $[0,1] \times P_0$. See [8, p. 171] for the proof. Thus the map $F_\epsilon$ defined on $[0,1] \times P_0$ by

$$F_\epsilon = f_0 + \chi \cdot (f_\epsilon^\circ - f_0) \in \mathbf{R}^3$$

and elsewhere by $F_\epsilon = f_0$ is an arbitrary $C^{s-1}$ close approximation to $f_0$ on $S^1 \times P$ as $\epsilon \to 0$. Moreover, $F_\epsilon$ is $C^s$ close to $f_0$ on $V_0$, which is where $\partial_t^s f_\epsilon^\circ = \partial_t^s f_0$, by (4) and (7). Both these statements follow from the Leibniz rule for the higher derivatives of a product:

$$\partial_t^k F_\epsilon = \partial_t^k f_0 + \sum_{i=0}^k \binom{k}{i} \partial_t^i \chi \cdot \partial_t^{k-i}(f_\epsilon^\circ - f_0).$$

Now define the homotopy $\phi_\lambda$ on $[0,1] \times P_0$ by the formula

$$\phi_\lambda(t,p) = (t, \phi_\lambda^0(t,p), \ldots, \phi_\lambda^r(t,p)),$$

where $\phi_\lambda^i(t,p) = \partial_t^i f_0(t,p) + \lambda(\partial_t^i F_\epsilon(t,p) - \partial_t^i f_0(t,p))$ for $i = 0, \ldots, s-1$,

$$\begin{aligned}\phi_\lambda^s(t,p) = (1 - \beta(t,p)) \cdot \psi^s(t,p,\lambda\chi(t,p)h_\epsilon(t,p)) \\ + \beta(t,p) \cdot (\partial_t^s(f_0(t,p) + \lambda(F_\epsilon(t,p) - f_0(t,p)))),\end{aligned}$$

and $\phi_\lambda^j(t,p) = \psi^j(t,p,\lambda\chi(t,p)h_\epsilon(t,p))$ for $j = s+1, \ldots, r$. Define $\phi_\lambda = \phi_0$ elsewhere. Clearly, $\phi_\lambda = \phi_0$ outside of $supp(\chi)$.

By construction, $\phi_\lambda(t,p)$ is approximately $\psi(t,p,\lambda\chi(t,p)h_\epsilon(t,p)) \in \mathcal{R}$. Thus, because $\mathcal{R}$ is open, $\phi_\lambda \in \mathcal{R}$ provided $\epsilon$ is sufficiently small. To see this, observe that $\phi^i_\lambda(t,p) \approx \partial^i_t f_0(t,p) = \phi^i_0(t,p) = \psi^i(t,p,\lambda\chi(t,p)h_\epsilon(t,p))$ for $i = 0,\ldots,s-1$ by (2) and (A). Off the support of $\beta$, $\phi^s_\lambda(t,p) = \psi^s(t,p,\lambda\chi(t,p)h_\epsilon(t,p))$. Meanwhile, on the support of $\beta$, we have $\partial^s_t F_\epsilon \approx \partial^s_t f_0$ and $\psi^s(t,p,\lambda\chi(t,p)h_\epsilon(t,p)) = \partial^s_t f_0(t,p)$ by (4). Therefore $\phi^s_\lambda(t,p) \approx \psi^s(t,p,\lambda\chi(t,p)h_\epsilon(t,p))$.

By construction $f_\lambda = \tau \circ \phi_\lambda = f_0 + \lambda(F_\epsilon - f_0)$ remains arbitrarily $C^{s-1}$ close to $f_0$ throughout the homotopy, as long as $\epsilon$ is sufficiently small. Moreover, $p^r_{s-1}\phi_\lambda = j^{s-1}_t f_\lambda$ on $S^1 \times P$ for all $\lambda \in [0,1]$. Thus (A′) and (C′) hold.

Finally we check that condition (B′) holds.

(i) Observe that on $V_0\backslash supp(\chi)$ we have $\chi = 0$ and $f_1 = F_\epsilon = f_0$. Thus by (4), if $(t,p) \in V_0\backslash supp(\chi)$, then

$$\begin{aligned}\phi^s_1(t,p) &= (1-\beta(t,p))\cdot\psi^s(t,p,0) + \beta(t,p)\cdot\partial^s_t f_0(t,p)\\ &= (1-\beta(t,p))\cdot\partial^s_t f_0(t,p) + \beta(t,p)\cdot\partial^s_t f_0(t,p)\\ &= \partial^s_t f_0(t,p) = \partial^s_t f_1(t,p).\end{aligned}$$

(ii) On $W_1$, $\chi = 1$ and $f_1 = F_\epsilon = f^\circ_\epsilon$. Thus, by (7), if $(t,p) \in W_1$, then

$$\phi^s_1(t,p) = (1-\beta(t,p))\cdot\psi^s(t,p,h_\epsilon(t,p)) + \beta(t,p)\cdot\partial^s_t F_\epsilon(t,p) = \partial^s_t f_1(t,p).$$

(iii) On $W_2$, $\beta = 1$, and thus $\phi^s_1 = \partial^s_t F_\epsilon = \partial^s_t f_1$. Therefore, (i), (ii), and (iii) imply $p^r_s\phi_1 = j^s_t f_1$ on $V_0\backslash supp(\chi) \cup W_1 \cup W_2$, which is a neighborhood of $K \cup L$. □

For the final step in the proof of the $h$–principle, there exist, by compactness of $S^1 \times P$, finitely many closed sets $L_i$, whose interiors cover $S^1 \times P$, such that $L_i \subset int([a_i,b_i] \times P_i)$ with $f_0([a_i,b_i] \times P_i)$ contained in a coordinate chart on $M$ as in Lemma B. Given a $C^{s-1}$ neighborhood of $f_0$, shrink that neighborhood so that for any $f$ in the shrunken neighborhood, $f([a_i,b_i] \times P_i)$ is contained in the same coordinate chart containing $f_0([a_i,b_i] \times P_i)$ for each $i$. Inductively, having constructed a homotopy to a solution over a neighborhood of $K \cup L_1 \cup \cdots \cup L_{i-1}$, use Lemma B to obtain a homotopy to a solution over $K \cup L_1 \cup \cdots \cup L_i$. The concatenation of these homotopies produces the homotopy which proves the $C^{s-1}$ dense parametric $h$–principle.

□

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