TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 356, Number 1, Pages 119–133
S 0002-9947(03)03415-9
Article electronically published on August 25, 2003

BACKWARD STABILITY FOR POLYNOMIAL MAPS WITH LOCALLY CONNECTED JULIA SETS

ALEXANDER BLOKH AND LEX OVERSTEEGEN

ABSTRACT. We study topological dynamics on *unshielded* planar continua with weak expanding properties at cycles for which we prove that the absence of wandering continua implies backward stability. Then we deduce from this that a polynomial f with a locally connected Julia set is backward stable outside any neighborhood of its attracting and neutral cycles. For a conformal measure μ this easily implies that one of the following holds: 1. for μ-a.e. $x \in J(f)$, $\omega(x) = J(f)$; 2. for μ-a.e. $x \in J(f)$, $\omega(x) = \omega(c(x))$ for a critical point $c(x)$ depending on x.

1. INTRODUCTION

In the present paper we study the problem of backward stability which is an extension of the Lyapunov (forward) stability to the backward dynamics of a non-invertible map. Essentially, this notion was first introduced by Fatou who showed that $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is backward stable at points not belonging to the limit sets of critical points. Other facts concerning backward stability which follow from classical results (in particular, from the description of the local dynamics at periodic points given, e.g., in [CG]) are that $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is not backward stable at any parabolic periodic point which lies in the Julia set. Obviously, f is not backward stable at attracting periodic points. Thus, the well-known obstacle for the backward stability of a polynomial at a point is that the point could be an attracting or neutral periodic point.

One can talk about backward stability in a more general setting not requiring any analytical or smooth properties of the map. In fact, this is exactly what we do in this paper. Our main result is that continuous maps of so-called *unshielded* plane continua without wandering subcontinua and with some weak expanding properties are backward stable. The main tools are new and developed by combining ideas from continuum theory and dynamics.

As a corollary we show that polynomial maps with locally connected Julia sets are backward stable at points which are neither attracting nor neutral. Thus, we remove

Received by the editors October 10, 2001.
2000 *Mathematics Subject Classification.* Primary 37F10; Secondary 37E25.
Key words and phrases. Complex dynamics, locally connected, Julia set, backward stability, conformal measure.
The first author was partially supported by NSF Grant DMS-9970363 and the second author by NSF grant DMS-0072626.

the restriction from [BL1] (see also [BL3]) where we considered polynomials without neutral or attracting cycles and show that for polynomials with locally connected Julia sets, classical obstacles are the *only* obstacles for backward stability. This is done by methods different from those in [BL1]. We also prove backward stability for laminations. Finally, we use our results as well as some standard arguments and results of [BM] to describe Milnor primitive attractors for conformal measures of polynomials with locally connected Julia sets which serves also as a motivation for our research. In fact, the problem of describing attractors (in wider terms, of describing typical limit sets) of polynomial or rational maps has been considered in a number of papers [Bar, BMO1, BMO2, BM, GPS, Lyu, McM, Pra], and it is not hard to see that it can be solved for conformal measures of maps which are backward stable on their Julia set. Notice that in [BM] an approach applicable to continuous maps of compact metric spaces was discovered (of which the backward stable maps are a particular case) which allows one to describe primitive attractors of *graph-critical* rational functions, i.e. such rational functions whose critical points belong to an invariant graph.

Let us give a short survey of more recent results concerning backward stability. Mañé [Ma] showed that $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is backward stable at non-parabolic and non-attracting points not belonging to the limit sets of *recurrent* critical points. In [P] the backward stability was proven for Collet-Eckmann rational maps of $\widehat{\mathbb{C}}$. In [L] the formal definition of backward stability was given, and then this property was verified for polynomials with one critical point and connected locally connected Julia set (if a Julia set is locally connected, then it is connected, see e.g. [CG]; thus from now on we will simply talk about polynomials with locally connected Julia sets), but without neutral or attracting cycles. In [BL1] the backward stability was verified for polynomials with a locally connected Julia set and an arbitrary number of critical points, but still without neutral or attracting cycles. In fact this was obtained in [BL1] as one of the corollaries of the fact that polynomials with locally connected Julia sets have no wandering subcontinua in the Julia set; the argument used a version of Yoccoz' puzzle construction. Since in the presence of attracting cycles the construction from [BL1] does not apply, we developed in this paper a different approach which combines ideas from continuum theory and topological dynamics, and is applicable to a wide class of continuous maps.

Everywhere below when speaking of convergence of sets we mean convergence in the sense of Hausdorff metric.

We would like to express our gratitude to G. Levin for valuable discussions of the results of this paper.

2. Topological properties of unshielded continua

In what follows X always denotes a compact metric space. Also, we often rely upon well-known facts from topology without referring the reader to any specific source. For our purpose the book [Kurat] can be used as a complete reference. Mainly we study *finitely Suslinian* continua and their subclass (Lemma 2.9) formed by *unshielded* continua.

Definition 2.1. A metric space X is *finitely Suslinian* if for each $\varepsilon > 0$, any collection of pairwise disjoint continua of diameter larger than ε is finite. A metric space Y is *hereditarily locally connected* if any subcontinuum of Y is locally connected.

Theorem 2.2 ([Kurat])**.** *The following claims hold.*

(1) *A finitely Suslinian continuum is hereditarily locally connected.*
(2) *A locally connected continuum is arcwise connected.*

The main objects of study in this paper are *unshielded* continua.

Definition 2.3. A plane continuum X is called *unshielded* if there exists a complementary domain U such that $\overline{U} \supset X$.

The terminology is explained by the fact that any point of X can be "seen" from U. We are interested in studying locally connected unshielded continua. We will also need the following definition.

Definition 2.4. A θ*-curve* is a continuum which is the union of three arcs having the same endpoints and having pairwise no other points in common (so a θ-curve is homeomorphic to the letter θ).

An unshielded continuum contains no θ-curve. On the other hand, a locally connected continuum containing no θ-curve is homeomorphic to an unshielded continuum ([Kurat, vol. 2, p. 328]). Thus, the following so-called θ*-curve* Theorem holds ([Kurat, vol. 2, p. 329]).

Theorem 2.5 (θ-curve Theorem)**.** *Any connected subset of a locally connected unshielded continuum is arcwise connected. Also, a locally connected unshielded continuum is hereditarily locally connected.*

We need the following standard definitions and notations.

Definition 2.6. A connected set is said to be *regular* at a point a if there exists a basis of neighborhoods at a with finite boundaries, and simply *regular* if it is regular at all points. Also, an *arc* is a homeomorphic image of the interval $[0, 1]$. If the endpoints of an arc are a, b (and if it causes no confusion), we will use the notation $[a, b]$. *Open* arcs are arcs without endpoints; they are denoted by (a, b). A *path* is a continuous image of the interval $[0, 1]$. Finally, by $\mathrm{diam}(A)$ we denote the diameter of a set A.

The following useful lemma follows easily from the results of [Why42].

Lemma 2.7. *A locally connected unshielded continuum K is regular.*

Proof. By Corollary 4.3 from [Why42], p. 97, a continuum is regular if and only if any two distinct points are separated in it by a finite set. Let us show that this the case for the continuum K.

Choose an arc $L = [x, y]$. Suppose that there exists a point $z \in (x, y)$ which neither belongs to a simple closed curve with non-trivial intersection with L nor is a vertex of any triod contained in K. Then z separates x and y in K as desired. Otherwise all points of (x, y) are either vertices of sub-triods of K or belong to non-degenerate intersections of simple closed curves and L. Since there are only countably many vertices in sub-triods of K (see, e.g., [BL2]) then there exists a simple closed curve R whose intersection with L is non-degenerate. Observe that since K is unshielded, R does not have non-degenerate intersections with other simple closed curves. Therefore we can find points $a \in R \cap L$ and $b \in R \setminus L$ which are not vertices of sub-triods of K. It follows that x and y are separated in K by a and b which completes the proof. $\square$

Now we can prove the following lemma.

Lemma 2.8. *Let X be an unshielded continuum and let K_n be a sequence of arcs in X converging to a continuum K. Suppose that $L = [x, y] \subset K$, $x \neq y$, is an arc such that X is locally arcwise connected at x and y. Then there is a non-degenerate arc L' and an infinite subsequence n_m such that $L' \subset K_{n_m}$ for all m.*

Proof. Choose arcs $L_n = [x, x_n]$, $R_n = [y, y_n]$ and $[x_n, y_n]$ such that $[x_n, y_n] \subset K_n$, $J_n = L_n \cup [x_n, y_n] \cup R_n$ is an arc, $\lim L_n = x$ and $\lim R_n = y$. Let $X' = K \cup_n J_n$; then $X' \subset X$ is an unshielded continuum. Hence X' does not contain a θ-curve. In particular for any simple closed curve $S \subset X'$ and any arc $I \subset X'$, $S \cap I$ is connected. We may assume that there exists a non-degenerate arc $A \subset L \setminus \cup_n [L_n \cup R_n]$. If $A \subset J_n$ for all n, $A \subset [x_n, y_n] \subset K_n$ and we are done. Hence assume, without loss of generality, that there exists a point $a \in A \setminus J_1$. Then there exists a bounded complementary region D of $X_1 = L \cup J_1$ such that a is contained in $\partial D = S$. Since X_1 is unshielded and locally connected, S is a simple closed curve. Then $A_1 = S \cap L$ is a non-degenerate arc containing a. Let $I_1 = \overline{S \setminus A}$ be the closure of the complementary arc. Since X' contains no θ-curve, any pair of distinct points $p, q \in S$ such that $p \in I_1$ and $q \in A_1$ cuts each J_n between x and y. Hence either A_1 or I_1 is contained in infinitely many J_n. □

Lemma 2.8 implies Lemma 2.9 given here without proof.

Lemma 2.9. *Suppose that a continuum X is locally connected and unshielded. Then in any sequence of continua K_n with diam$(K_n) \not\to 0$ there exists a subsequence K''_n such that the following holds:*

1. *continua K''_n converge to a continuum K' in the Hausdorff sense;*
2. *all continua K''_n contain some arc $L \subset K'$.*

In particular, X is finitely Suslinian.

The first claim of the following lemma is well known while the second one is rather easy to prove, so we put it here without proof.

Lemma 2.10. *Let U_i be a sequence of pairwise disjoint complementary domains of a planar locally connected continuum X. Then diam$(U_i) \to 0$. Moreover, if X is unshielded, then for any two $i \neq j$ the intersection $\partial U_i \cap \partial U_j$ consists of at most one point.*

It is not difficult to suggest an example which shows that not all finitely Suslinian plane continua have the "big intersection" property established for unshielded continua in Lemma 2.9 (and thus not all of them are unshielded). Indeed, consider a sequence of arcs L_i defined as follows:

(1) all L_i are contained in the closed upper half-plane $\{(x, y)|y \geq 0\}$ and have endpoints $a = (0, 0)$ and $b = (1, 0)$;

(2) every L_i intersects the interval I in the x-axis with endpoints a and b only at the set of points $A_i = \{(k/2^{-i}, 0)|k = 0, 1, \dots, 2^i\}$;

(3) every arc L_i intersects any vertical line $x = t, 0 \leq t \leq 1$, at exactly one point and does not intersect other vertical lines at all;

(4) $\lim L_i = [a, b]$, for any two $i < j$ the arc L_i is located non-strictly above the arc L_j and $L_i \cap L_j = A_i$.

Consider the plane continuum $K = I \cup (\bigcup_{i=1}^{\infty} L_i)$; it is easy to see that it is finitely Suslinian. On the other hand, the arcs L_i together with the interval I give us an example of a sequence of continua without the "big intersection" property of Lemma 2.9 and K is clearly not unshielded.

3. Dynamical properties of self-mappings of unshielded continua

We need a few definitions; everywhere below $f : X \to X$ is a continuous map of a compact metric space (X, d) into itself.

Definition 3.1. A map f satisfies the *Contraction Principle* if for a continuum $I \subset X$ with $\liminf \operatorname{diam}(f^n(I)) = 0$ we have $\lim \operatorname{diam}(f^n(I)) = 0$.

The Contraction Principle comes up in one-dimensional dynamics (e.g., [B, BLyu, BM]) and is used in [BLyu, BM] in connection with the study of attractors. Yet in those papers f is a map on an interval or a graph while here we deal with a more general setting.

Definition 3.2. A set K is *wandering* if for all $n \neq m \geq 0$, $f^n(K) \cap f^n(K) = \emptyset$. A map f has *Property A* if for a non-wandering continuum $K \subset X$, which is never mapped into a point, $\liminf \operatorname{diam}(f^n(K)) > 0$.

Definition 3.3. A map f is *expanding* if for some ε the map f restricted onto any ε-ball B centered at a point z is a homeomorphism onto a neighborhood of $f(z)$ such that for any $x, y \in B$ we have $d(f(x), f(y)) > d(x, y)$.

Examples of expanding maps are angle-multiplying circle maps and subshifts of finite type. Because of the compactness if f is expanding, then f is no more than N-to-1 for some N; also, for any $\varepsilon/2$-ball B there are at most N homeomorphic preimages of B on each of which f expands the distance.

We study finite-to-one factors of expanding maps. It is easy to give an example of such a factor which is not finite-to-one: suppose that there exists a compact invariant set $R \subset X$ with infinitely many points outside R mapped into R, identify all points of R and consider the corresponding factor space X' and the quotient map $h : X \to X'$. The map h carries f down onto a factor map $g : X' \to X'$ which is not finite-to-one. To avoid these complications we consider the following class of factor maps.

Definition 3.4. Let $f : X \to X$ be semiconjugate by a map $h : X \to X'$ to a map $g : X' \to X'$. If for any $x' \in X'$ the map f maps $h^{-1}(x')$ *onto* $h^{-1}(g(x'))$, then we say that g is a *full* factor of f.

Lemma 3.5. *Let a map $g : X' \to X'$ be a full factor of a finite-to-one open map $f : X \to X$. Then g is a finite-to-one open map.*

Proof. By way of contradiction assume that there is a point $x' \in X'$ which has infinitely many g-preimages $y'_1, y'_2, \dots$. Then since by the assumption each of the sets $h^{-1}(y'_i)$ maps onto $h^{-1}(x')$, points of $h^{-1}(x')$ must have infinitely many f-preimages, a contradiction.

Also, let us show that g is an open map. Indeed, otherwise there are points y' and $x' = g(y') \in X'$ such that small neighborhoods of y' have images whose interiors do not contain x'. Thus, there exists a neighborhood U' of y' and a sequence of points $z_i' \to x'$ which do not belong to $g(U')$. Choose points $z_i \in h^{-1}(z_i')$; we may assume that $z_i \to z$ for some $z \in h^{-1}(x')$. Then since g is a full factor there is a point $t \in h^{-1}(y')$ such that $f(t) = z$. Choose a neighborhood V of t such that $h(V) \subset U'$. Since f is open the f-image of V contains a neighborhood W of z. Since W contains z_i for big i, V contains points t_i such that $f(t_i) = z_i$. Then $g(h(t_i)) = z_i'$ and since $h(t_i) \in U'$ we conclude that $z_i' \in g(U')$, a contradiction. $\square$

To state the next lemma we need the following new definitions.

Definition 3.6. An f-fixed point x has *Property B* if there exists a sequence of integers m_i and a basis $\{U_i\}$ of neighborhoods at x such that for every i and every $z \in \partial U_i$ we can find $j, 1 \le j \le m_i$, with $f^j(z) \notin U_i$. A periodic point x is *mildly repelling* if for every M such that $f^M(x) = x$, the point x has Property B as a fixed point of f^M.

Clearly, repelling periodic points in the usual sense are mildly repelling. Later on we will use the notion of mildly repelling periodic points to give alternative proofs to some of our results.

We also need several fairly standard definitions given here for the sake of completeness.

Definition 3.7. A point c is said to be a *critical* point of a continuous map $f : X \to X$ if f is not injective on any neighborhood of x. Also, a map $f : X \to Y$ is said to be *light* provided $f^{-1}(y)$ is 0-dimensional for each point $y \in Y$.

Observe that by Lemma 2.7 locally connected unshielded continua are regular. Now we can state an important technical lemma.

Lemma 3.8. *Let $f : X \to X$ be an expanding map semiconjugate by a map $h : X \to X'$ to its full factor map $g : X' \to X'$, where both X, X' are metric compacta. Then g is light and has Property A as well as the following properties:*

1. *for any periodic orbit P of g there exists a neighborhood U of P such that every point of $U \setminus P$ exits U at some moment in the future;*
2. *for any point $x' \in X'$ such that $\omega_g(x') = P'$ is a periodic orbit, there exists a number n such that $g^n(x') \in P'$.*
3. *if X' is a regular compact set (e.g., if it is an unshielded locally connected continuum), then all its periodic points are mildly repelling.*

Proof. In the situation of the lemma the map g does not have to be expanding in terms of the metric on X'. In fact, g can even have critical points. A relevant example is the locally connected Julia set $J(f)$ of a polynomial f. Then $f|J(f)$ is a full factor map of the appropriate angle multiplication map of the circle. However, unless $J(f)$ is homeomorphic to the circle, it contains critical points of f and therefore $f|J(f)$ is not expanding.

Still, it turns out that to a lesser extent local expansion of f is inherited by g. The tool which allows us to observe this is the function d' introduced below. First let us introduce an extension of the metric d defined on the compact subsets of X as follows: $d(L, M) = \{\min d(u, v) : u \in L, v \in M\}$. Now, given two points $x', y' \in X'$ we define $d'(x', y')$ as follows: $d'(x', y') = d(h^{-1}(x'), h^{-1}(y'))$. The function d' is not a metric because it does not satisfy the triangle inequality. Indeed, imagine points x', y' and z' such that the minimal distances from $h^{-1}(x')$ and $h^{-1}(z')$ to $h^{-1}(y')$ are close to 0, but $h^{-1}(x')$ and $h^{-1}(z')$ are located on distinct "sides" of $h^{-1}(y')$ so that $h^{-1}(y')$ "stretches" between $h^{-1}(x')$ and $h^{-1}(z')$, making the minimal distance between $h^{-1}(x')$ and $h^{-1}(z')$ rather big. In other words, an inequality which can be established is that $d'(x', y') + d'(y', z') + \operatorname{diam}(h^{-1}(y')) \geq d'(x', z')$, but this is only a version of the triangle inequality which involves a "correction" equal to $\operatorname{diam}(h^{-1}(y'))$ and not the triangle inequality itself.

However, the function d' has some nice properties. First of all, it is lower semicontinuous as a function on $X' \times X'$. Indeed, let $x'_n \to x', y'_n \to y'$ and $\liminf d'(x'_n, y'_n) = \delta$. Choose points $x_n \in h^{-1}(x'_n)$ and $y_n \in h^{-1}(y'_n)$ so that $d'(x'_n, y'_n) = d(x_n, y_n)$. Refining these sequences, we may assume that $x_n \to x \in h^{-1}(x')$, $y_n \to y \in h^{-1}(y')$ and $d(x, y) = \lim d(x_n, y_n) = \delta$. By the definition of the function d' this implies that $d'(x', y') \leq \delta$, as desired. In other words, we show that if $x'_n \to x'$ and $y'_n \to y'$, then $\liminf d'(x'_n, y'_n) \geq d'(x', y')$. Also, since $d'(x'', y'') = 0$ implies $x'' = y''$, we see that in particular if $x''_n \to x'', y''_n \to y''$ and $d'(x''_n, y''_n) \to 0$, then $x'' = y''$.

Moreover, observe that d' may indeed decrease in the situation similar to the one described in the previous paragraph, where we illustrate the fact that the triangle inequality may not hold for d': just imagine that points x'_n, y'_n approach the same limit point z' but from the "opposite sides", so that $d'(x'_n, y'_n)$ stays big while for their limit point $z' = \lim x'_n = \lim y'_n$ we of course have $d'(z', z') = 0$. It is also just as easy to see that if $x'_n \to x'$, then $d'(x'_n, x') \to 0$.

For us the most important property of d' is that the map g is *d'-expanding*: every $x' \in X'$ has a neighborhood U' such that $d'(g(x'), g(y')) > d'(x', y')$ for every $y' \in U', y' \neq x'$. Since this is in fact a property of compact subsets of X, we prove it as a statement for such subsets. Namely, we prove that if $T = h^{-1}(x')$, then there exists a neighborhood U of x' such that for any set $R \subset h^{-1}(U')$ disjoint from T we have $d(f(R), f(T)) > d(R, T)$. If $R = h^{-1}(y')$ we get $d'(g(x'), g(y')) > d'(x', y')$ for any $y' \in U', y' \neq x'$.

Indeed, otherwise there are neighborhoods U'_n of x' and sets $R_n \subset h^{-1}(U'_n)$ disjoint from T with $\operatorname{diam}(U'_n) \to 0$ and $d(R_n, T) \geq d(f(R_n), f(T))$. Thus, there are points $y_n \in R_n, x_n \in T$ such that for them $d(f(y_n), f(x_n)) \leq d(R_n, T)$. Choosing a subsequence we may assume that $y_n \to y \in T$ and $x_n \to x \in T$; since $\operatorname{diam}(U'_n) \to 0$ then $d(R_n, T) \to 0$ and $f(y) = f(x) = z$. Clearly, $x \neq y$ since otherwise x_n, y_n will be close and we will have $d(f(y_n), f(x_n)) > d(y_n, x_n)$ which contradicts $d(f(y_n), f(x_n)) \leq d(R_n, T)$. Now, for a small neighborhood W of z there is a neighborhood S of x and a neighborhood V of y such that f homeomorphically maps S and V onto W. By Lemma 3.5 there are finitely many g-preimages of $g(x')$ denoted here $x'_1, x'_2, \dots, x'_k$. If W is small, then S and V contain no points of $h^{-1}(x'_1), \dots, h^{-1}(x'_k)$ because S and V contain points $x \in T, y \in T$, respectively, and the sets $h^{-1}(x'_1), \dots, h^{-1}(x'_k)$ are disjoint from T.

If $Q = W \cap f(T)$, then f maps $S \cap T$ and $V \cap T$ onto Q homeomorphically (since V contains no points of $h^{-1}(x'_1), \dots, h^{-1}(x'_k)$), so we can choose points $b_n \in V \cap T$ with $f(b_n) = f(x_n)$, and it is clear that $b_n \to y$. Since f is locally expanding, $d(f(y_n), f(b_n)) > d(y_n, b_n) \geq d(R_n, T)$. On the other hand by the choice of points y_n, x_n we have $d(f(y_n), f(b_n)) = d(f(y_n), f(x_n)) \leq d(R_n, T)$, a contradiction which proves that g is d'-expanding.

This immediately implies that g is light. Indeed, otherwise there exists a continuum $Z' \subset X'$ such that $g(Z')$ is a singleton. However as follows from the fact that g is d'-expanding, every point $z' \in X'$ has a neighborhood U' such that for any $u' \in U', u' \neq z'$ we have $g(z') \neq g(u')$, a contradiction.

Let us show that g has Property A. If $K \subset X'$ is a non-wandering continuum with $\liminf \operatorname{diam}(g^n(K)) = 0$, then we may assume that for some $n > 0$, $g^n(K) \cap K \neq \emptyset$; also observe that all images of K are non-degenerate because g is finite-to-one. Then $g^{rn}(K) \cap g^{(r+1)n}(K) \neq \emptyset$ for all $r \geq 0$ and $\liminf_{r\to\infty} \operatorname{diam}(g^{rn}(K)) = 0$. Let $a' \in X'$ be the limit of a subsequence of sets $g^{nr_i}(K)$ chosen in such a way that $\lim \operatorname{diam}(g^{nr_i}(K)) = 0$. From the continuity and the fact that $g^{rn}(K) \cap g^{(r+1)n}(K) \neq \emptyset$ it follows that $g^n(a') = a'$. Since g is d'-expanding and continuous we can choose a small neighborhood W of a' with $d'(g^n(x'), g^n(a')) > d'(x', a')$ for any $x' \neq a', x' \in W$. Then the properties of the function d' established above imply that for any sufficiently small ε the following two facts hold: a) $d'(y', a') > \varepsilon$ if $y' \notin W$, and b) the closure of the set $B_\varepsilon = \{z' : d'(z', a') < \varepsilon\}$ is contained in W. Let us use these facts to prove that the situation described above is impossible.

Since K is not degenerate, we may assume that $d'(z'_0, a') \geq \varepsilon$ for some $z'_0 \in K$. Let us then show that for every m there exists a point $z'_m \in g^{mn}(K)$ such that $d'(z'_m, a') \geq \varepsilon$. Indeed, otherwise there exists the least m such that for every point $z' \in g^{mn}(K)$ we have $d'(z', a') < \varepsilon$. Since m is the least such number there exists $z'_{m-1} \in g^{n(m-1)}(K)$ with $d'(z'_{m-1}, a') \geq \varepsilon$. If $z'_{m-1} \in W$, then by the choice of W we see that $d'(g^n(z'_{m-1}), a') \geq \varepsilon$ which is impossible. Hence $z'_{m-1} \notin W$. Now, $g^{(m-1)n}(K)$ intersects $g^{mn}(K) \subset B_\varepsilon$ and therefore contains points $z' \in B_\varepsilon \subset \overline{B}_\varepsilon$. By the previous paragraph $X' \setminus W$ and $\overline{B}_\varepsilon$ are disjoint, and since $g^{(m-1)n}(K)$ is connected and contains points of both these closed and disjoint sets, it must contain points of the complement of their union. Clearly, if y' is one of such points then $y' \in W$ and $d'(y', a') \geq \varepsilon$, hence $d'(g^n(y'), a') \geq \varepsilon$, a contradiction with the assumption that for every point $z' \in g^{mn}(K)$ we have $d'(z', a') < \varepsilon$. Thus, g has Property A.

Let us now prove that for any periodic orbit P' there exists a neighborhood U' such that any point $x' \in U' \setminus P'$ exits U' at some moment. Indeed, we may assume that $P' = \{a'\}$ is a fixed point; setting $h^{-1}(a') = T$ we get $f(T) = T$. If U' is a small neighborhood of a', then for any $x' \in \overline{U'} \setminus P'$ we have $d'(g(x'), a') > d'(x', a')$. Suppose that a point x' does not exit U'. Then the sequence $\{d'(g^n(x'), a')\}$ is increasing. Choose a limit point $y' = \lim g^{n_i}(x')$ (clearly, $y' \in \overline{U'}$). We may assume that the sets $h^{-1}(g^{n_i}(x')) = R_i$ converge to a set $R \subset h^{-1}(y')$. Then $\lim d'(g^n(x'), a') = \lim d'(g^{n_i}(x'), a') = d(R, T) = \varepsilon > 0$ and $d'(g^n(x'), a') < \varepsilon$ for any n. By the claim proven during the establishment of the fact that the map g is d'-expanding, we have that if U' is small enough, then $d(f(R), f(T)) > d(R, T)$. Hence the fact that $f(T) = T$ implies that $\varepsilon = \lim\ d'(g^{n_i+1}(x'), a') = d(f(R), T) > d(R, T) = \varepsilon$, a contradiction. Obviously, the existence of a desired neighborhood U' of a periodic orbit P' implies also that if $\omega(x') = P'$, then x' has to be eventually mapped into P' since otherwise it will have limit points outside U'.

It remains to prove claim (3) of the lemma. We may assume that $P' = \{a'\}$ is a g^M-fixed point. Choose a basis of neighborhoods U_i of a so that the boundary ∂U_i of each U_i is finite (this is possible since X' is regular). Since g^M is a full factor of f^M and f^M is expanding, we see by the claim (1) that there exists a small neighborhood W of a' such that for every point $z \in W, z \neq a'$ there exists a number

$k(z)$ such that $g^{k(z)M}(z) \notin W$. Assuming that $U_i \subset W$, we can then choose such numbers for all points of ∂U_i and then choose their maximum m_i. Then for each point $z \in \partial U_i$ there exists a number $k(z) < m_i$ such that $g^{k(z)M}(z) \notin U_i$ which shows that indeed a' is mildly repelling. □

The next lemma easily follows from the obtained results and can serve as an introduction into our study of dynamical properties of maps of continua.

Lemma 3.9. *If X is finitely Suslinian and $f : X \to X$ is a map with Property A, then f satisfies the Contraction Principle.*

Proof. Let I be a continuum with $\liminf \operatorname{diam}(f^n(I)) = 0$. By Property A either I maps into a point under some power of f or I is wandering. In the former case $\lim \operatorname{diam}(f^n(X)) = 0$; in the latter case $\lim \operatorname{diam}(f^n(X)) = 0$ because X is finitely Suslinian. □

To state Theorem 3.11 we need the following definition.

Definition 3.10. A continuous map $f : X \to X$ of a metric space is said to be *backward stable* at a point x if for any δ there exists ε such that for any connected set K with $\operatorname{diam}(K) \le \varepsilon$ contained in the ε-ball centered at x, any $n \ge 0$ and any component M of $f^{-n}(K)$, $\operatorname{diam}(M) \le \delta$; it is *backward stable* if it is backward stable at all points.

If X is compact, then f is *backward stable* if and only if for any δ there is ε such that for any continuum K with $\operatorname{diam}(K) \le \varepsilon$, any $n \ge 0$ and any component M of $f^{-n}(K)$, $\operatorname{diam}(M) \le \delta$. Thus, if f is not backward stable, then there exists a sequence of continua K_n and a sequence of positive integers m_n with $\operatorname{diam}(f^{m_n}(K_n)) \to 0$ while $\operatorname{diam}(K_n) \ge \varepsilon$ for some $\varepsilon > 0$.

Observe the following simple cases in which a map f cannot be backward stable: a) f is not light, and b) X is finitely Suslinian and there exist wandering continua. Thus, any result establishing backward stability for certain maps should have the necessary assumptions corresponding to the cases a) and b). It turns out that in certain cases they are not only necessary but also sufficient.

Theorem 3.11. *Let X be a locally connected and unshielded continuum, and $f : X \to X$ be a light map with Property A and with no wandering continua. Then f is backward stable. In particular, suppose that X is a locally connected and unshielded continuum, and $f : X \to X$ is a full factor of an expanding map. In this case if f has no wandering continua, then f is backward stable.*

Proof. Suppose that f is not backward stable. Then there exists a sequence of continua K_n and a sequence of positive integers m_n such that $\operatorname{diam}(f^{m_n}(K_n)) \to 0$ while $\operatorname{diam}(K_n) \ge \varepsilon$ for some $\varepsilon > 0$. By Lemma 2.9 there exists a continuum S contained in infinitely many K_n which obviously implies that $\operatorname{diam}(f^{m_n}(S)) \to 0$. Then by Property A either S is wandering or otherwise it eventually maps into a point which may be assumed to be periodic. Since f does not collapse continua into points and has no wandering continua then this situation is impossible, and f is backward stable. The particular case of a full factor of an expanding map follows now from what has been proven and Lemma 3.8. □

An alternative approach to proving that maps have Property A relies upon the notion of a mildly repelling periodic point and can be used for a direct proof of the fact that polynomials are backward stable on their Julia sets (see the end of Section 4). For the sake of completeness we prove below an important lemma which represents this alternative approach.

Lemma 3.12. *Let $f : X \to X$ be a map of a metric compact space X into itself such that each periodic point of f is mildly repelling. Then f has Property A.*

Proof. Suppose that K is a non-wandering continuum which is never mapped into a point. Then $K \cap f^M(K) \neq \emptyset$ for some M. If $\liminf \operatorname{diam}(f^n(K)) = 0$, then by continuity $\liminf \operatorname{diam}(f^{Mn}(K)) = 0$ which again by continuity and compactness implies that there exists a point a such that $f^M(a) = a$ and $f^{Mn_r}(K) \to \{a\}$ along a subsequence n_r. Since a is mildly repelling it implies that it has Property B as a fixed point of $F = f^M$ (see Definition 3.6 where Property B is introduced). Denote by m_i the sequence of integers and by U_i the basis of neighborhoods at a which exist according to Property B.

Choose a small neighborhood U_i of a so that $K \not\subset U_i$, and consider the set A of all integers l such that $F^l(K) \not\subset U_i$; observe that $A \neq \emptyset$ because $0 \in A$. Let us prove that in fact A is infinite and, moreover, that gaps in A are no greater than m_i. Indeed, if $l \in A$, then $F^l(K) \not\subset U_i$. If $F^{l+1}(K) \not\subset U_i$, then we are done because $l+1$ is the next element of A. Suppose that $F^{l+1}(K) \subset U_i$. Since $F^l(K)$ and $F^{l+1}(K)$ are non-disjoint, this implies that $F^l(K)$ cannot be contained in the complement of U_i. Hence $F^l(K)$ contains points of $X \setminus U_i$ as well as points of U_i which implies that $F^l(K)$ contains a point $z \in \partial U_i$. Therefore there exists an integer $s < m_i$ such that $F^s(z) \notin U_i$ which implies that $l + s \in A$. In other words, the gaps in A are no longer than m_i. Clearly, this contradicts the existence of the sequence n_r such that $f^{Mn_r}(K) \to \{a\}$ and thus completes the proof. □

4. Applications to laminations and complex dynamics

The main purpose of this section is to apply our results to complex dynamics. One of the major tools used in this area is laminations, so we devote the first half of this section to them. The notion was introduced in [Do], [McM], [Th] (see also [BL1], [BL3]). Consider an equivalence relation $\sim$ on the unit circle S^1 with the following properties:

(E1) $\sim$ is *closed*: the graph of $\sim$ is a closed set in $S^1 \times S^1$;

(E2) $\sim$ defines a *lamination*, i.e., it is *unlinked*: if $t_1 \sim t_2 \in S^1$ and $t_3 \sim t_4 \in S^1$, but $t_2 \not\sim t_3$; then $\{t_1, t_2\}$ is contained in one component of $S^1 \setminus \{t_3, t_4\}$.

(E3) each equivalence class of $\sim$ is totally disconnected.

Call $\sim$ a *closed lamination.* We assume that it is *non-degenerate*, i.e. has a class of more than one point. Equivalence classes of $\sim$ are called *($\sim$-)classes*.

Fix an integer $d > 1$ and denote by $\sigma_d = \sigma : S^1 \to S^1$ the map $\sigma(z) = z^d$. The relation $\sim$ is called *(σ-)invariant* iff:

(D1) $\sim$ is *forward invariant*: for a class g, the set $\sigma(g)$ is a class, too;

(D2) $\sim$ is *backward invariant*: for a class g, its preimage $\sigma^{-1}(g) = \{x \in \mathbb{T} : \sigma(x) \in g\}$ is split into classes;

(D3) for a class g, the map $\sigma : g \to \sigma(g)$ is an orientation preserving covering map.

Consider invariant closed non-degenerate laminations; denote by $\mathbb{D}$ the unit disk. Define an extension $\simeq$ of $\sim$ onto $\mathbb{C}\setminus\overline{\mathbb{D}}$ by declaring that a point in $\mathbb{C}\setminus\overline{\mathbb{D}}$ is equivalent only to itself. Let $p : \mathbb{C} \setminus \mathbb{D} \to (\mathbb{C} \setminus \mathbb{D})/\simeq$ be the factor map and denote $p(S^1)$ by J. Then J is a locally connected unshielded continuum, and since the map σ acts on S^1 and the relation $\sim$ is σ-invariant, we can consider a factor map $f : J \to J$. The theorem below describes some properties of the map f.

Theorem 4.1. *Let $\sim$ be an invariant lamination and $f : J \to J$ be the corresponding factor map. Then f is backward stable and also has the following properties:*

(1) *any periodic orbit P has a neighborhood U such that any point $x \in U \setminus P$ eventually exits U;*
(2) *any point whose ω-limit set is a cycle must be eventually mapped into that cycle.*
(3) *all cycles of f are mildly repelling.*

Proof. By the definition f is a full factor map of a expanding map σ. Hence by Lemma 3.8 f has Property A, and by Lemma 3.5 f is a finite-to-one map (so it does not collapse continua into points). Also, by [BL1] (see also [BL2]) f has no wandering continua. Hence by Theorem 3.11 f is backward stable. Properties (1), (2) and (3) follow from Lemma 3.8. □

Observe that, as was shown in [BL1] (see also [BL2]), there are laminations which have infinite periodic classes and therefore are not polynomial (as is shown in these papers, such classes are then necessarily (pre)critical). In terms of the properties of the quotient space $J = p(S^1)$ it means that it may contain points x such that the number of components of the set $J \setminus \{x\}$ is infinite. Still, as was mentioned above, J is a regular space, and so by Lemma 3.8 all periodic points of the factor map $f : J \to J$ are mildly repelling.

We would like to point out that the approach relying upon mildly repelling periodic points yields an alternative proof of Theorem 4.1. Namely, claims (1), (2) and (3) of this theorem do not depend on its main claim about backward stability of f. Now, in the proof of Theorem 4.1 we refer to Lemma 3.8 to show that f has Property A. However, instead of that we could refer to claim (3) of Theorem 4.1 and Lemma 3.12.

A polynomial f with locally connected $J(f)$ defines a lamination whose factor map is conjugate to $f|J(f)$. So, by Theorem 4.1 $f|J(f)$ is backward stable. Yet we can prove a bit more.

Theorem 4.2. *Let a polynomial f have a locally connected Julia set. Then f is backward stable at any point which is not an attracting or neutral periodic.*

Proof. Assume that x is neither an attracting nor a neutral periodic point and yet f is not backward stable at x. Then by the results of Fatou we may assume that $x \in J(f)$. Since a polynomial with a Cremer point has a non-locally connected Julia set [Sul], then f has no Cremer points and $J(f)$ contains only parabolic or repelling periodic points. A critical point $c \notin J(f)$ with $\omega(c) \cap J(f) \neq \emptyset$ must converge to parabolic cycles. Thus, since x is not parabolic, we may assume that for some $\varepsilon > 0$ the only critical points whose forward iterates may be ε-close to x are the critical points in $J(f)$.

Since by the assumption f is not backward stable at x, then there exist a number $\delta > 0$, a sequence of connected sets B_i converging to x, a sequence of numbers n_i and components C_i of $f^{-n_i}(B_i)$ with $\operatorname{diam}(C_i) > \delta$ for all i (then clearly $n_i \to \infty$). Since $J(f)$ is locally connected, we can enlarge B_i so that it becomes a connected neighborhood of x whose intersection with $J(f)$ is connected and has diameter ε_i, $\varepsilon_i \to 0$, $\varepsilon_i < \varepsilon$ for any i.

By the choice of ε the critical points ever mapped into B_i belong to $J(f)$. Thus $C_i \cap J(f)$ is connected. Since $f|J(f)$ is backward stable, $\operatorname{diam}(C_i \cap J(f)) \to 0$ as $i \to \infty$. Denote the Fatou set $\mathbb{C} \setminus J(f)$ by $F(f)$ and show that for big i there exists a component M_i of $C_i \cap F(f)$ with $\operatorname{diam}(M_i) > \delta/3$. Pick points $y, z \in C_i$ with $d(y, z) > \delta$. Then if necessary find $y' \in J(f) \cap C_i, z' \in J(f) \cap C_i$ such that y, y' belong to the closure of a component R_i of $C_i \cap F(f)$, and z, z' belong to the closure of a component S_i of $C_i \cap F(f)$. By the triangle inequality $d(y, z) \leq d(y, y') + d(y', z') + d(z', z)$. Since $\operatorname{diam}(C_i \cap J(f)) \to 0$ we may assume that $d(y', z') < \delta/3$. Thus, at least one of the distances $d(y, y'), d(z', z)$ is greater than $\delta/3$, and the corresponding component of $C_i \cap F(f)$ (R_i or S_i) has diameter greater than $\delta/3$, as desired.

By Lemma 2.10 there are finitely many Fatou domains with diameters greater than $\delta/3$. By [Sul] all Fatou domains are eventually periodic. Hence after refining our sequence and passing to a power of f we may assume that all M_i are contained in the same invariant Fatou domain V. Let us prove that this is impossible.

Let $f^{n_i}(M_i) = K_i$; then $\operatorname{diam}(K_i) < \varepsilon_i, K_i \to x$ and so $x \in \partial V$. Assume that $M_i \to M$. Then $M \subset \partial V$. Indeed, otherwise there is $z \in M \cap V$. If V contains an attracting point a (or $\overline{V}$ contains a parabolic point a), then a neighborhood W of z is attracted by a. Since $M_i \cap W \neq \emptyset$ for big i, then $K_i = f^{n_i}(M_i)$ contain points close to a which is impossible since $K_i \to x$ and $x \neq a$. Also, if V is a Siegel domain, then W stays positively distant from ∂V, a contradiction to $K_i \to x \in \partial V$.

Let us transfer $f|V$ to the uniformization plane. This yields the appropriate map $g : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ which is semiconjugate to $f|\overline{V}$ by $\phi : \overline{\mathbb{D}} \to \overline{V}$ (the map ϕ is well-defined on the closed unit disk $\overline{\mathbb{D}}$ because the boundary ∂V of V is locally connected). Consider sets $\phi^{-1}(M_i) = M'_i \subset \mathbb{D}$ and $\phi^{-1}(K_i) = K'_i \subset \mathbb{D}$. Assume that $M'_i \to M'$ with $\phi(M') = M$ (so, $M' \subset S^1$) and $K'_i \to K'$ with $\phi(K') = \{x\}$. Then K' is a point. Indeed, otherwise $K' \subset S^1$ is an arc. Since the map $g|S^1$ is conjugate to an irrational rotation or an appropriate power z^n, after finitely many iterations of g the union of images of K' will be the entire S^1, a contradiction to $\phi(K')$ being a point. So, $K' = \{y\}$ is a point, and $y \in S^1$.

Consider all the cases for the domain V and show that the described dynamics is impossible in any of them. If V is a Siegel domain (and g is an irrational rotation) it is impossible. If V contains an attracting fixed point, then so does $g|\mathbb{D}$, some power of g is expanding on S^1 and hence in a small annulus A around S^1. Replacing g by its power and using continuity we may assume that g itself is expanding on A. Now, since all points inside $\mathbb{D}$ are attracted under g to the attracting g-fixed point we can find an open invariant set $W \supset \mathbb{D} \setminus A$ whose closure is s-distant from y with $s > 0$.

Let i be such that $M'_i \subset A$ and the maximal distance between points of K'_i and y is less than s. Then all iterates $M'_i, g(M'_i), \ldots, g^{n_i}(M'_i) = K'_i$ avoid W since otherwise there are points of K'_i belonging to W and thus more than s-distant from y, a contradiction. So, sets $M'_i, g(M'_i), \ldots, g^{n_i}(M'_i)$ are contained in A. Since

$\operatorname{diam}(M_i') \geq \operatorname{diam}(M')/2 > 0$ for big i and $\operatorname{diam}(K_i') \to 0$, this is a contradiction to $g|A$ being expanding.

The remaining case is when $\overline{V}$ contains a parabolic fixed point b. Then $g|\overline{\mathbb{D}}$ contains a parabolic fixed point $a \neq y$ with $\phi(a) = b$. On the other hand, M' is an arc in S^1 and there exists N such that $g^N(M) = S^1$. It is known [CG] that V, and hence $\mathbb{D}$, contain a connected attractive petal U, such that $a \in U$ and $\cap g^n(\overline{U}) = \{a\}$. If M_i' approximates M' well, then $g^N(M_i')$ approximates S^1 well. Moreover, tripling N we may assume that it winds around the circle at least twice still approximating it well, and so $g^N(M_i')$ intersects the attracting petal U of a. Since $\bigcap_{l\geq 0} g^l(\overline{U}) = \{a\}$ then $K_i' = g^{n_i}(M_i') = (g^{n_i - N}(g^N(M_i'))$ is non-disjoint from $g^{n_i-N}(U) \to a$, and since $K_i' \to y$ we get $y = a$, a contradiction. $\square$

In the rest of this paper we apply backward stability to Milnor attractors [Mi] of polynomials with locally connected Julia sets. The connection between the limit sets of typical points (with respect to Lebesgue or a conformal measure) and the behavior of pull-backs of small disks has been understood since early Lyubich's papers [Lyu], where the small scale - large scale passing with bounded distortion was used as a tool. Since then it has become clear that for typical points either the large scale covering takes place or almost every point has forward images well approximated by the appropriate critical images. However one can only conclude that in the latter non-ergodic case typical points have limit sets contained in the limit sets of critical points which is in general insufficient to determine what limit sets they have, and in particular to determine whether there are finitely many of them.

It is easy to see that backward stability is a sufficient condition to show that in the non-ergodic case the limit sets of typical points coincide with the limit sets of critical points. In fact, by the results of [BM] (see also [BL2]) a weaker than backward stability property would suffice. This property has a topologically dynamical nature and may help establish results similar to Theorem 4.3 for maps without backward stability.

Thus, the standard arguments or results of [BM] (see also [BL2]) show that the following result follows from Theorem 4.2.

Theorem 4.3. *For a polynomial f with a locally connected Julia set $J(f)$ and a conformal measure μ, one of the following holds:*

1. *For μ-almost every $x \in J(f)$, $\omega(x) = J(f)$.*
2. *For μ-almost every $x \in J(f)$, $\omega(x) = \omega(c(x))$ for some critical point $c(x)$ depending on x.*

The limit sets assumed on sets of positive measure are called *primitive attractors*. Hence, Theorem 4.3 describes the primitive attractors in the sense of a conformal measure of polynomials with locally connected Julia sets. Let us point out here that by [Lyu] primitive attractors are contained in the union of limit sets of critical points. However this does not imply the description of primitive attractors, nor does it imply that there are finitely many of them. As far as the authors know these problems are solved only for polynomials with locally connected Julia sets (Theorem 4.3) or so-called *graph-critical* rational functions, that is, rational functions where all critical points are contained in an invariant graph [BM], and are not solved in general.

Finally, we outline an alternative proof of the backward stability of the restriction $f|J(f)$ of a polynomial on its locally connected Julia set. Clearly, $J(f)$ is unshielded because $J(f)$ is the boundary of the basin of attraction of infinity. If $J(f)$ is locally connected, then periodic points of f in $J(f)$ are either repelling or parabolic (since if there is a Cremer periodic point, then $J(f)$ is not locally connected [Sul]). It follows from [CG] that repelling and parabolic points are mildly repelling in $J(f)$. Hence by Lemma 3.12 $f|J(f)$ has Property A. Moreover, by [BL1] in this case there are no wandering subcontinua in $J(f)$ and clearly no non-degenerate continua are mapped into points by f. Hence by Theorem 3.11 $f|J(f)$ is backward stable.

References

[Bar] J. A. Barnes, *Conservative exact rational maps of the sphere*, Journal of Math. Analysis and Appl. **230** (1999), pp. 350–374. MR **2000d:**37048

[B] A. Blokh, *The "spectral" decomposition for one-dimensional maps*, Dynamics Reported **4** (1995), pp. 1–59. MR **96e:**58087

[BL1] A. Blokh and G. Levin, *Growing trees, laminations and the dynamics on the Julia set*, IHES Preprint IHES/M/99/77 (1999), pp. 1–40.

[BL2] A. Blokh and G. Levin, *On dynamics of vertices of locally connected polynomial Julia sets* **130** (2002), pp. 3219–3230. MR **2003c:**37066

[BL3] A. Blokh, G. Levin, *An inequality for laminations, Julia sets and "growing trees"*, Erg. Th. and Dyn. Sys. **22** (2002), pp. 63–97.

[BLyu] A. Blokh and M. Lyubich, *Measurable dynamics of S-unimodal maps of the interval*, Ann. Sci. Ecole Norm. Sup. (4) **24** (1991), pp. 545–573. MR **93f:**58132

[BMO1] A. Blokh, J. Mayer and L. Oversteegen, *Recurrent critical points and typical limit sets of rational maps*, Proc. Amer. Math. Soc. **127** (1999), pp. 1215–1229. MR **99f:**58169

[BMO2] A. Blokh, J. Mayer, L. Oversteegen, *Recurrent critical points and typical limit sets for conformal measures*, Topology and its Appl. **108** (2000), pp. 233–244. MR **99f:**58169

[BM] A. Blokh, M. Misiurewicz, *Attractors for graph critical rational maps*, Trans. Amer. Math. Soc. **354** (2002), pp. 3639–3661.

[CG] L. Carleson and T. W. Gamelin, *Complex dynamics*, Universitext: Tracts in Mathematics, Springer-Verlag, New York, NY (1993). MR **94h:**30033

[DMNUrb] M. Denker, R. D. Mauldin, N. Nitecki, and M. Urbański, *Conformal measures for rational functions revisited*, Fund. Math. **157** (1998), pp. 161-173. MR **99j:**58122

[DU91] M. Denker and M. Urbański, *On the existence of conformal measures*, Trans. Amer. Math. Soc. **328** (1991), pp. 563–587. MR **92k:**58155

[Do] A. Douady, *Descriptions of compact sets in* $\mathbb{C}$, in: Topological methods in modern mathematics, Publish or Perish, Houston, TX (1993), pp. 429-465. MR **94g:**58185

[GPS] P. Grzegorczyk, F. Przytycki, and W. Szlenk, *On iterations of Misiurewicz's rational maps on the Riemann sphere*, Annales de l'Inst. H. Poincaré **53** (1990), pp. 431–444. MR **92d:**30017

[Kurat] K. Kuratowski, *Topology*, vol. 2, Academic Press, New York (1968). MR **41:**4467

[L] G. Levin, *On backward stability of holomorphic dynamical systems*, Fundamenta Mathematicae **158** (1998), pp. 97–107. MR **99j:**58171

[Lyu] M. Yu. Lyubich, *Typical behavior of trajectories of the rational mappings of a sphere*, Soviet Math. Dokl. **27:1** (1983), pp. 22–25, originally published in Dokl. Akad. Nauk SSSR **268:1** (1983), pp. 29–32. MR **84f:**30036

[Ma] R. Mañé, *On a theorem of Fatou*, Bol. Soc. Bras. Mat. **24** (1993), pp. 1–11. MR **94g:**58188

[McM] C. T. McMullen, *Complex dynamics and renormalization*, Annals of Mathematical Studies, no. 135, Princeton Univ. Press, Princeton, NJ (1994). MR **96b:**58097

[Mi] J. Milnor, *On the concept of attractor*, Comm. Math. Phys. **99** (1985), pp. 177–195 (Correction and remarks: vol. **102** (1985), pp. 517–519). MR **87i:**58109a

[Pra] E. Prado, *Ergodicity of conformal measures for unimodal polynomials*, Tech. Report 6, SUNY–Stony Brook, 1996, Institute for Mathematical Sciences.

[P] F. Przytycki, *Conical limit set and Poincaré exponent for iterations of rational functions*, Trans. Amer. Math. Soc. **351** (1999), pp. 2081–2099. MR **99h:**58110

[Sul] D. Sullivan, *Conformal dynamical systems*, in: Geometric Dynamics, Lecture Notes in Mathematics, **1007**, Springer-Verlag, Berlin (1983) pp. 725–752. MR **85m:**58112

[Th] W. Thurston, *The combinatorics of iterated rational maps*, Preprint (1985).

[Why42] G. T. Whyburn, *Analytic topology*, **28**, AMS Coll. Publications, Providence, RI (1942). MR **4:**86b

Department of Mathematics, University of Alabama at Birmingham, Birmingham, Alabama 35294-1170

E-mail address: `ablokh@math.uab.edu`

Department of Mathematics, University of Alabama at Birmingham, Birmingham, Alabama 35294-1170

E-mail address: `overstee@math.uab.edu`