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OVERPARTITIONS

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Abstract. We discuss a generalization of partitions, called overpartitions, which have proven useful in several combinatorial studies of basic hypergeometric series. After showing how a number of finite products occurring in q-series have natural interpretations in terms of overpartitions, we present an introduction to their rich structure as revealed by q-series identities.

1. Introduction

A partition of n is a non-increasing sequence of natural numbers whose sum is n. The desire to discover and prove theorems about partitions has been a driving force behind the recent renaissance of basic hypergeometric series. However, it is still not clear how to interpret most q-series identities in a natural way as statements about partitions, and even fewer are deducible using combinatorial properties of partitions. While some progress has been made by considering parts in different congruence classes or by studying statistics on partitions (see [1], [2], [23], for instance), it seems to be most fruitful to employ the perspective of certain direct products of partitions, which we call overpartitions.

An overpartition of n is a non-increasing sequence of natural numbers whose sum is n in which the first occurrence (equivalently, the final occurrence) of a number may be overlined. We denote the number of overpartitions of n by $\overline{p}(n)$. Since the overlined parts form a partition into distinct parts and the non-overlined parts form an ordinary partition, we have the generating function

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + \dots. \tag{1.1}$$

For example, the 14 overpartitions of 4 are

$$4, \overline{4}, 3+1, \overline{3}+1, 3+\overline{1}, \overline{3}+\overline{1}, 2+2, \overline{2}+2, 2+1+1,$$
$$\overline{2}+1+1, 2+\overline{1}+1, \overline{2}+\overline{1}+1, 1+1+1+1, \overline{1}+1+1+1.$$

These objects are natural combinatorial structures associated with the q-binomial theorem, Heine's transformation, and Lebesgue's identity (see [20] for a summary with references). In [18], they formed the basis for an algorithmic approach to the combinatorics of basic hypergeometric series. More recently, overpartitions have been found at the heart of bijective proofs of Ramanujan's ${}_1\psi_1$ summation and the q-Gauss summation [14], [15]. It should come as no surprise, then, that the theory of basic hypergeometric series contains a wealth of information about overpartitions

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and that many theorems and techniques for ordinary partitions have analogues for overpartitions. The following pages are intended as an introduction to the structure of overpartitions revealed by q-series identities.

We begin in the next section by studying the equivalence of the objects generated by the summations in

$$\sum_{n=0}^{\infty} \frac{(-a;q)_n q^n}{(q;q)_n} = \frac{(-aq;q)_\infty}{(q;q)_\infty} \tag{1.2}$$

and

$$\sum_{n=0}^{\infty} \frac{(-1/a;q)_n a^n q^{n(n+1)/2}}{(q;q)_n^2} = \frac{(-aq;q)_\infty}{(q;q)_\infty}. \tag{1.3}$$

Here and throughout we employ the standard q-series notation

$$(a_1, ..., a_j; q)_\infty = \prod_{k=0}^{\infty} (1 - a_1 q^k) \cdots (1 - a_j q^k),$$

$$(a_1, ..., a_j; q)_n = \frac{(a_1, ..., a_j; q)_\infty}{(a_1 q^n, ..., a_j q^n; q)_\infty}.$$

It turns out that the left side of (1.2) counts overpartitions according to the number of parts (see Proposition 2.1), while the left side of (1.3) represents overpartitions using generalized Frobenius partitions. Recall that a Frobenius partition [9] of n is a two-rowed array

$$\begin{pmatrix} a_1 & a_2 & ... & a_k \\ b_1 & b_2 & ... & b_k \end{pmatrix}$$

where $\sum a_i$ is a partition taken from a set A, $\sum b_i$ is a partition taken from a set B, and $k + \sum (a_i + b_i) = n$. The number of such Frobenius partitions of n is denoted by $p_{A,B}(n)$. We shall exhibit a bijection which proves

Theorem 1.1. *Let Q be the set of partitions into distinct nonnegative parts, $\mathcal{O}$ the set of overpartitions into nonnegative parts. There is a one-to-one correspondence between overpartitions λ of n and Frobenius partitions ν counted by $p_{Q,\mathcal{O}}(n)$ in which the number of overlined parts in λ is equal to the number of non-overlined parts in the bottom row of ν.*

In addition to providing a useful representation of overpartitions, the bijection implies q-series identities like

Corollary 1.2.

$$\sum_{k=0}^{n} \frac{(-1/a;q)_k c^k a^k q^{k(k+1)/2}}{(cq;q)_k} \begin{bmatrix} n \\ k \end{bmatrix} = \frac{(-acq;q)_n}{(cq;q)_n}, \tag{1.4}$$

and

Corollary 1.3.

$$\frac{(-bq;q)_\infty}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{z^n q^{n(n+1)/2}}{(-bq;q)_n} = \frac{(-1/z, -zq;q)_\infty}{(b/z;q)_\infty}. \tag{1.5}$$

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These are the q-Chu-Vandermonde summation [17, p. 236, (II.7)] and a limiting case of Ramanujan's ${}_1\psi_1$ summation [17, p. 239, (II.29)], respectively. Here we have used the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}. \tag{1.6}$$

We shall also discover a family of generating functions for column-restricted Frobenius partitions.

Corollary 1.4. *Let $F_S(n)$ denote the number of Frobenius partitions counted by $p_{Q,\mathcal{O}}(n)$ with the restriction that non-overlined parts can only occur under parts from the set S. Then, for any set of nonnegative integers S we have*

$$\sum_{n=0}^{\infty} F_S(n)q^n = \frac{\prod_{s \in S}(1+q^{s+1})}{(q;q)_\infty}.$$

In §3 we look at a graphical representation for overpartitions and its implications. This representation can be utilized to give straightforward proofs of identities like the Rogers-Fine identity [16, p. 15, (14.1)],

$$\sum_{n=0}^{\infty} \frac{(-a;q)_n(tq)^n}{(bq;q)_n} = \sum_{n=0}^{\infty} \frac{(-a;q)_n(-atq/b;q)_n(1+atq^{2n+1})(bt)^n q^{n^2+n}}{(bq;q)_n(tq;q)_{n+1}}. \tag{1.7}$$

The remainder of the paper is devoted to the deduction of facts about overpartitions from identities in the theory of basic hypergeometric series. For instance, it will be natural in this context to define the *rank* of an overpartition as one less than the largest part minus the number of overlined parts less than the largest part. We shall see:

Theorem 1.5. *Let $D(n)$ denote the number of overpartitions with even rank minus the number with odd rank. Then $D(n)$ is equal to 0 if and only if $n \equiv 2 \pmod 4$. On the other hand, $D(n) = 2k$ has infinitely many solutions for any $k \in \mathbb{Z}$.*

Theorem 1.5 is similar in flavor to results obtained by Andrews, Dyson and Hickerson [10, Thm. 4] for partitions into distinct parts. We also give an analogue for overpartitions of Andrews' generalization of the Rogers-Ramanujan identities [6]. The combinatorial interpretation is in terms of a Durfee square dissection of the associated partition of a partition into distinct nonnegative parts (see §4.3 for the definitions).

Theorem 1.6. *The number of overpartitions with parts not divisible by k is equal to the number of overpartitions whose Frobenius representation has a top row with at most $k-2$ Durfee squares in its associated partition.*

As a final example, we apply identities involving partial theta functions to relate sums of squares to Frobenius overpartitions, i.e., Frobenius partitions counted by $P_{\mathcal{O},\mathcal{O}}(n)$.

Theorem 1.7. *Let $D_5(n)$ denote the number of Frobenius overpartitions of n in which the sum of the largest parts in the top and bottom row is odd minus those for which it is even. If $r_2(n)$ denotes the number of representations of n as the sum of two squares, then*

$$D_5(n) = (-1)^n r_2(n). \tag{1.8}$$

These and other theorems on overpartitions are established in §4.

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2. The two representations

From the definition and the generating function (1.1) it follows that overpartitions can be viewed through a number of different lenses. For instance, the number of overpartitions of n is the number of partitions of n in which one part of each odd size may be tagged, or the number of partitions λ of n weighted by $2^{\mu(\lambda)}$, where $\mu(\lambda)$ denotes the number of different part sizes occurring in the partition. From the perspective of q-series, the most natural representations correspond to (1.2) and (1.3). Before proving Theorem 1.1, which describes the correspondence between these two representations, we recall the bijection that establishes the generating function for the number of overpartitions with exactly k parts.

Proposition 2.1. *Let $\overline{p}_{k,l,m}(n)$ denote the number of overpartitions of n into k parts with l overlined parts and rank m (as defined in the introduction). Then*

$$\sum_{l,m,n=0}^{\infty} \overline{p}_{k,l,m}(n) a^l b^m z^k q^n = \frac{(-a;q)_k (zq)^k}{(bq;q)_k}. \tag{2.1}$$

Proof. The function $(zq)^k/(bq;q)_k$ generates a partition λ into k positive (non-overlined) parts, where the exponent on z keeps track of the number of parts and the exponent on b records the largest part minus 1. Note that since there are not yet any overlined parts, this is the same as the rank. Now $(-a;q)_k$ generates a partition $\mu = \mu_1 + \cdots + \mu_j$ into distinct nonnegative parts less than k, with the exponent on a tracking the number of parts. For each of these μ_i beginning with the largest, we add 1 to the first μ_i parts of λ and then overline the $(\mu_i + 1)$th part of λ. Here the parts of λ are written in non-increasing order. This operation leaves the rank invariant and counts one overlined part for each part of μ. For example, if $k = 5$, $\lambda = 8 + 4 + 4 + 2 + 1$, and $\mu = 4 + 3 + 0$, then we have

$$\begin{aligned} (8+4+4+2+1, 4+3+0) &\iff (9+5+5+3+\overline{1}, 3+0) \\ &\iff (10+6+6+\overline{3}+\overline{1}, 0) \\ &\iff (\overline{10}+6+6+\overline{3}+\overline{1}). \end{aligned}$$

The result is obviously an overpartition and the process is easily inverted. □

The mapping above (with $b = 1$) was considered in [18], where it was noted that by summing over the nonnegative integers k, we obtain the q-binomial theorem

$$\sum_{k=0}^{\infty} \frac{(-a;q)_k (zq)^k}{(q;q)_k} = \frac{(-azq;q)_\infty}{(zq;q)_\infty}. \tag{2.2}$$

Proof of Theorem 1.1. From Proposition 2.1 one deduces that the equality of the series in equations (1.2) and (1.3) is equivalent to the statement of the theorem. The bijection below explicitly gives the correspondence. In the case where the overpartition λ has no overlined parts, it reduces to the usual mapping between a partition and its Frobenius symbol [9].

We use the notion of a *hook*. Given a positive integer a and a nonnegative integer b, $h(a,b)$ is the hook that corresponds to the partition $(a, 1, \ldots, 1)$ where there are b ones. Combining a hook $h(a,b)$ and a partition α is possible if and only if $a > \alpha_1$ and $b \geq l(\alpha)$, where $l(\alpha)$ denotes the number of parts of α. The result of the union is $\beta = h(a,b) \cup \alpha$ with $\beta_1 = a$, $l(\beta) = b + 1$ and $\beta_i = \alpha_{i-1} + 1$ for $i > 1$.

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Now take a Frobenius partition ν counted by $p_{Q,\mathcal{O}}(n)$, increase the entries on the top row by 1 and initialize α and β to the empty object, ϵ. Beginning with the rightmost column of ν, we proceed to the left, building α into an ordinary partition and β into a partition into distinct parts. At the i^{th} column, if b_i is overlined, then we add the hook $h(a_i, b_i)$ to α. Otherwise, we add the part b_i to α' (the conjugate of α) and the part a_i to β. Joining the parts of α together with the parts of β gives the overpartition λ. An example is given below starting with $\nu = \left(\begin{smallmatrix} 7 & 5 & 4 & 2 & 0 \\ 6 & \bar{4} & 4 & 3 & \bar{1} \end{smallmatrix}\right)$.

$$
\begin{array}{lll}
\nu & \alpha & \beta \\[1em]
\begin{pmatrix} 8 & 6 & 5 & 3 & 1 \\ 6 & \bar{4} & 4 & 3 & \bar{1} \end{pmatrix} & \epsilon & \epsilon \\
\begin{pmatrix} 8 & 6 & 5 & 3 \\ 6 & \bar{4} & 4 & 3 \end{pmatrix} & (1,1) & \epsilon \\
\begin{pmatrix} 8 & 6 & 5 \\ 6 & \bar{4} & 4 \end{pmatrix} & (2,2,1) & (3) \\
\begin{pmatrix} 8 & 6 \\ 6 & \bar{4} \end{pmatrix} & (3,3,2,1) & (5,3) \\
\begin{pmatrix} 8 \\ 6 \end{pmatrix} & (6,4,4,3,2) & (5,3) \\
\epsilon & (7,5,5,4,3,1) & (8,5,3)
\end{array}
$$

We get $\lambda = (\bar{8}, 7, \bar{5}, 5, 5, 4, \bar{3}, 3, 1)$. The reverse bijection is easily described. Given α and β, we set the Frobenius partition equal to ϵ. We proceed until α and β are empty, at each step adding a column to the Frobenius partition according to the following rule: If $\beta_1 \geq \alpha_1$, then add the column $\left(\begin{smallmatrix} \beta_1 \\ l(\alpha) \end{smallmatrix}\right)$ and decrease the parts of α by 1 and delete the largest part of β. Otherwise add the column $\left(\frac{\alpha_1}{l(\alpha)-1}\right)$ and delete the hook $h(\alpha_1, l(\alpha)-1)$ from α. Finally, decrease by 1 the entries of the top row. For example, using this recipe one easily traces the pair (α, β) above back to the Frobenius partition. $\square$

Proof of Corollary 1.2. Notice that the number of parts in the overpartition λ corresponds to the sum of the number of columns and the rank of the bottom row in the Frobenius partition ν. Here we have defined the rank of an overpartition into nonnegative parts to be the largest part minus the number of overlined parts less than the largest part. Moreover, if λ has parts at most n, then ν has a top row with parts at most $n-1$. Since the term $q^{k(k+1)/2}\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ is the generating function for a partition into exactly k distinct parts $\leq n$, the bijection implies (1.4). $\square$

Proof of Corollary 1.3. The coefficient of z^0 on the right side of (1.5) can be interpreted as the generating function for Frobenius partitions counted by $p_{Q,\mathcal{O}}$ where the power of b tracks the number of non-overlined parts in the bottom row. By the bijection above, this generating function is $(-bq;q)_\infty/(q;q)_\infty$. Arguing as in [14] or [15], this is sufficient to prove the identity in full generality. $\square$

Proof of Corollary 1.4. This follows easily from the bijection described above by noticing that the overlined parts of the overpartition, which are the parts of β, are always mapped to a position above a non-overlined part in the bottom row of the corresponding Frobenius partition. $\square$

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3. A graphical representation

If we think of an overpartition as a partition in which the *final* occurrence of a part may be overlined, then such a partition corresponds to an ordinary Ferrers diagram in which the corners may be colored. Conjugating, or reading the columns of the diagram, gives another overpartition. For example, the overpartition $(9, \overline{7}, 5, \overline{4}, 4, \overline{2}, 1, 1, 1)$ is represented in Fig. 1. Its conjugate is in Fig. 2 and it is $(9, \overline{6}, \overline{5}, 5, 3, \overline{2}, 2, 1, 1)$.

Observe that conjugating maps the number of non-overlined parts to the largest part minus the number of overlined parts. This number is the rank if the largest part is overlined and one more than the rank otherwise, which proves an overpartition-theoretic analogue of a theorem of Fine on partitions into distinct parts [16, p. 47, Eq. (24.6)]:

Proposition 3.1. *For $n, m \geq 1$, the number of overpartitions of n with rank m or $m+1$ is equal to twice the number of overpartitions of n with exactly $m+1$ non-overlined parts.*

As with ordinary partitions, there are many natural statistics associated with such a diagram, and one may readily write down q-series identities by counting overpartitions in different ways. We highlight this for the Durfee rectangle:

Proof of the Rogers-Fine identity (1.7). Proposition 2.1 tells us that the coefficient of $[q^n t^k a^l b^m]$ is the number of overpartitions of n with k parts, l overlined parts and rank m. We shall prove that the coefficient of $[q^n t^k a^l b^m]$ in

$$\frac{(-a;q)_d(-atq/b;q)_d(1+atq^{2d+1})(bt)^d q^{d^2+d}}{(bq;q)_d(tq;q)_{d+1}}$$

is the number of overpartitions of n with Durfee rectangle size d, k parts, l overlined parts and rank m. Let us first recall that the Durfee rectangle is the largest $(d+1)\times d$ rectangle that can be placed on the Ferrers diagram [3]. For example, the size of the Durfee rectangle of the overpartition on Fig. 1 is 3 and of the overpartition on Fig. 2 is 4.

We will interpret the formula "piece by piece":

$$\frac{\overbrace{(-a;q)_d}^{II}\overbrace{(-atq/b;q)_d}^{III}\overbrace{(1+atq^{2d+1})}^{IV}\overbrace{(bt)^d q^{d^2+d}}^{I}}{\underbrace{(bq;q)_d}_{II}\underbrace{(tq;q)_{d+1}}_{III}}.$$

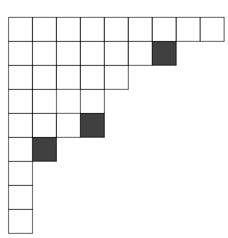

Figure 1. Overpartition

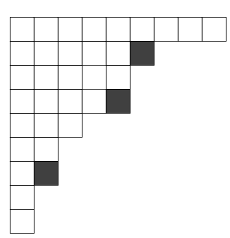

Figure 2. Conjugate

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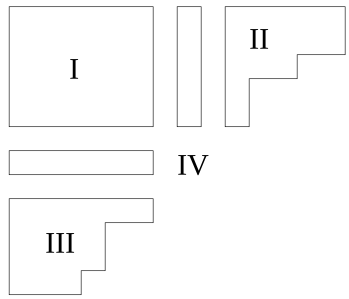

FIGURE 3. Construction

We construct the overpartition as follows:

- Piece I: the Durfee rectangle $(d+1)\times d$. It is obviously an overpartition of d^2+d into d parts, with rank d and 0 overlined parts.
- Piece II: an overpartition into d nonnegative parts. This overpartition is put at the right of the Durfee rectangle. The number of overlined parts (resp. rank) of that overpartition is then added to the number of overlined parts (resp. rank) of the Durfee rectangle.
- Piece III: an overpartition into parts at most $d+1$ where each part increases the number of parts by 1 and the part $d+1$, if it occurs, cannot be overlined. This overpartition is put under the Durfee rectangle. Each overlined part decreases the rank by 1 and each part increases the number of parts by 1.
- Piece IV: Allows a possible overlined part of size $d+1$ under the Durfee rectangle, and in that case increases the first d parts by 1.

The construction is summarized in Fig. 3. Let us give an example with $d=4$, which is illustrated in Fig. 4. We start with the overpartition $\pi=\epsilon$.

- Piece I: $(5,5,5,5)$. $\pi=(5,5,5,5)$
- Piece II: $(\overline{5},4,\overline{3},\overline{0})$. $\pi=(\overline{10},9,\overline{8},\overline{5})$.
- Piece III: $(5,5,\overline{4},4,\overline{3})$. $\pi=(\overline{10},9,\overline{8},\overline{5},5,5,\overline{4},4,\overline{3})$.
- Piece IV: (9). $\pi=(\overline{11},10,\overline{9},\overline{6},\overline{5},5,5,\overline{4},4,\overline{3})$.

$\square$

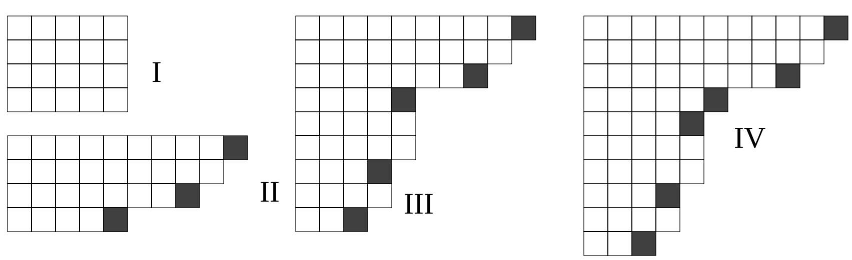

FIGURE 4. Example

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4. Overpartitions and q-series

Armed with the interpretation of certain finite products in terms of overpartitions, we can now have a field day reading off theorems from q-series identities. The point is that basic products correspond to overpartition-theoretic functions in the following way (we include ordinary partitions for comparison):

$$\begin{aligned} \frac{1}{(q;q)_k} &\iff p(n), \\ \frac{(-1;q)_k q^k}{(q;q)_k} &\iff \overline{p}(n), \\ \frac{(-1;q)_k q^{k(k+1)/2}}{(q;q)_k^2} &\iff p_{Q,\mathcal{O}}(n) \iff \overline{p}(n), \\ \frac{(-1;q)_k^2 q^k}{(q;q)_k^2} &\iff p_{\mathcal{O},\mathcal{O}}(n). \end{aligned}$$

4.1. **Overpartitions and divisor functions.** By observing that

$$1/(1-zq^n) = (zq;q)_{n-1}/(zq;q)_n$$

is a generating function for overpartitions, we can easily make connections with divisor series. Even the simplest cases reveal what is surprising behavior for such elementary combinatorial functions.

Theorem 4.1. *Let n have the factorization $2^x p_1^{y_1} ... p_j^{y_j}$, where the p_i are distinct odd primes. Then the number of overpartitions of n with even rank minus the number with odd rank is equal to*

$$2(1-x)(y_1+1)\cdots(y_j+1). \tag{4.1}$$

Proof. From Proposition (2.1) we have that

$$\sum_{n=0}^{\infty} \frac{(-1;q)_n q^n}{(zq;q)_n} = \sum_{m,n=0}^{\infty} \overline{p}(m,n) z^m q^n, \tag{4.2}$$

where $\overline{p}(m,n)$ denotes the number of overpartitions of n with rank m. Set $z=-1$ and observe that $\sum \frac{2q^n}{1+q^n}$ is the generating function for twice the number of odd divisors minus twice the number of even divisors of a natural number, which is in turn expressed by (4.1). □

Proof of Theorem 1.5. This is an obvious corollary, since the expression in (4.1) can be made into any even integer and is 0 only when $x=1$. □

Another simple case relates the ordinary divisor function to the *co-rank* of an overpartition. The co-rank is the number of overlined parts less than the largest part.

Theorem 4.2. *Let n have the factorization above and let $D_1(n)$ be the number of overpartitions with even co-rank minus the number with odd co-rank. Then*

$$D_1(n) = 2(1+x)(y_1+1)\cdots(y_j+1). \tag{4.3}$$

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Proof. From the arguments in Proposition 2.1, we see that

$$\sum_{n=1}^{\infty} \frac{2(-zq;q)_{n-1}q^n}{(q;q)_n} = \sum_{\substack{m\geq 0\\ n\geq 1}}^{\infty} \overline{p}_1(m,n)z^m q^n, \tag{4.4}$$

where $\overline{p}_1(m,n)$ denotes the number of overpartitions of n with co-rank m. Set $z=-1$ and notice that (4.3) is the number of divisors of n. $\square$

As before, we can easily deduce some curious facts. We cite

Corollary 4.3. *$D_1(n)$ is always positive and is infinitely often equal to $2k$ for any natural number k.*

It is worth pointing out that the above theorems can also be deduced using a simple involution on overpartitions. Let m be the smallest part size. If $\overline{m}$ is in the partition, then take off the overline; otherwise overline the first occurrence of m. This involution changes the parity of the rank and of the co-rank except when the partition has only one part size. The theorems follow.

We close with an analogue for overpartitions of a theorem of Uchimura on partitions into distinct parts [12], [21].

Theorem 4.4. *The sum of all overpartitions of n weighted by $(-1)^{k-1}m$, where m is the smallest part and k is the number of parts, is equal to twice the number of odd divisors of n.*

Proof. Set $b=-1$ and $c=q$ in the q-Gauss summation [17, p. 236, (II.8)],

$$\sum_{n=0}^{\infty} \frac{(a,b;q)_n(c/ab)^n}{(c,q;q)_n} = \frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty}. \tag{4.5}$$

Then take $\frac{d}{da}$ of both sides and set $a=1$. This yields

$$\sum_{n=1}^{\infty} \frac{(-1;q)_n(-1)^{n-1}q^n}{(q;q)_{n-1}(1-q^n)^2} = \sum_{n=1}^{\infty} \frac{2q^n}{(1-q^{2n})}.$$

The right side generates odd divisors, while expanding $q^n/(1-q^n)^2 = (q^n+2q^{2n}+\cdots)$ and appealing to Proposition 2.1 show that the left side generates the weighted count of overpartitions. $\square$

We shall return to the relationship between overpartitions and divisor functions when more intricate examples are treated in §4.3 in the context of Bailey chains and again in §4.4 in the context of partial theta functions.

4.2. **Overpartitions and theta series.** Here we use (1.7) to reveal that certain generating functions for overpartitions are given by theta-type series. The example below is related to the *perimeter* of an overpartition, which is defined to be the largest part plus the number of parts.

Theorem 4.5. *Let $D_2(m,n)$ denote the number of overpartitions of n with perimeter m having largest part even minus the number having largest part odd. Then we have*

$$D_2(m,n) = \begin{cases} 2(-1)^n, & m=2k \text{ and } n=k^2, \\ 0, & \text{otherwise.} \end{cases} \tag{4.6}$$

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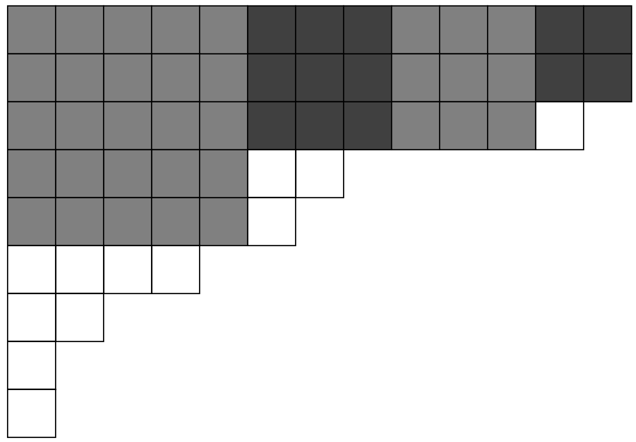

FIGURE 5. Consecutive Durfee squares

Proof. In the Rogers-Fine identity (1.7) let $t = -z, a = b = zq$, multiply both sides by $-2qz^2/(1-zq)$, and shift the summation to get

$$\sum_{n=1}^{\infty} \frac{2(-zq)_{n-1}(-q)^n z^{n+1}}{(zq;q)_n} = 2\sum_{n=1}^{\infty}(-1)^n z^{2n} q^{n^2}. \tag{4.7}$$

Now the left side generates the nonempty overpartitions with an even number of parts minus the nonempty overpartitions with an odd number of parts, where the exponent on z is equal to 1 plus the number of parts plus the rank plus the co-rank. This is easily seen to be equal to the perimeter. $\square$

A combinatorial proof of this result is essentially contained in [13].

4.3. **Overpartitions and the Bailey chain.** Now we turn to significantly deeper results from the theory of basic hypergeometric series. We give two samples of how overpartitions fit nicely into the theory of Bailey chains by applying Andrews' multiple series generalization of Watson's transformation [4]. As usual, the combinatorics is in terms of the Durfee dissection of a partition [6]. We recall that the Ferrers diagram of a partition λ has a largest upper-left justified square called the Durfee square. Since there is a partition to the right of this square, we identify its Durfee square as the second Durfee square of the partition λ. Continuing in this way, we obtain a sequence of successive squares. For example, for the partition $(13, 13, 12, 7, 6, 4, 2, 1, 1)$, we get the sequence $(5, 3, 3, 2)$. See Fig. 5. We also recall that the associated partition of a partition into distinct nonnegative parts is obtained by writing the parts in increasing order and then removing $j-1$ from the jth part.

Proof of Theorem 1.6. We apply Andrews' multiple series transformation [4] with all variables besides c_k, a, and q tending to infinity to find that for any natural

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number k we have

$$\sum_{n_{k-1}\geq\cdots\geq n_1\geq 0}\frac{(c_k;q)_{n_{k-1}}q^{n_{k-1}(n_{k-1}+1)/2+n_{k-2}^2+\cdots+n_1^2}a^{n_{k-1}+\cdots+n_1}}{(q;q)_{n_{k-1}-n_{k-2}}\cdots(q;q)_{n_2-n_1}(q;q)_{n_1}(-c_k)^{n_{k-1}}} \tag{4.8}$$
$$= \frac{(aq/c_k;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(1-aq^{2n})(a,c_k;q)_n a^{kn}q^{kn^2}}{(1-a)(q,aq/c_k;q)_n c_k^n}.$$

If we set $a=1$ and $c_k=-1$, then the right side becomes a theta series which sums (by Jacobi's triple product identity [17, p. 239, Eq. II.28]) to

$$\frac{(-q;q)_\infty(q^k;q^k)_\infty}{(-q^k;q^k)_\infty(q;q)_\infty}.$$

This is the generating function for overpartitions with parts not divisible by k. The $(k-1)$-fold summation on the left becomes

$$\sum_{n_{k-1}\geq\cdots\geq n_1\geq 0} q^{n_{k-1}}\times\frac{q^{n_{k-1}(n_{k-1}-1)/2+n_{k-2}^2+\cdots+n_1^2}(q;q)_{n_{k-1}}}{(q;q)_{n_{k-1}-n_{k-2}}\cdots(q;q)_{n_2-n_1}(q;q)_{n_1}}\times\frac{(-1;q)_{n_{k-1}}}{(q;q)_{n_{k-1}}},$$

which we interpret as a generating function for Frobenius partitions counted by $p_{Q,\mathcal{O}}(n)$. First, the factor $q^{n_{k-1}}$ will count the number of columns. Next, by Proposition 2.1, the quotient

$$\frac{(-1;q)_{n_{k-1}}}{(q;q)_{n_{k-1}}}$$

generates an overpartition into exactly n_{k-1} nonnegative parts for the bottom row. Finally, the top row is generated by the rest of the summand, which, as detailed in [5, pp. 54–55], is the generating function for partitions into n_{k-1} distinct nonnegative parts whose associated partition has at most $k-2$ Durfee squares. □

Now we consider a different application of (4.8) which links a weighted count of the overpartitions in Theorem 1.6 to a generalization of a divisor function which arose from a study of certain types of identities appearing in Ramanujan's lost notebook [11]. Namely, let $m_k(n)$ denote the number of k-middle divisors of n, that is, the number of divisors of n that occur in the interval $[\sqrt{n/k},\sqrt{kn})$.

Theorem 4.6. *Let $D_4^{\pm}(k,n)$ denote the number of overpartitions whose Frobenius representations have a top row with at most $k-2$ Durfee squares in the associated partition, where the number of columns plus the co-rank of the bottom row is even (odd). Then*

$$D_4^-(k,n)-D_4^+(k,n)=2m_k(n). \tag{4.9}$$

Proof. Set $a=1$ in (4.8), differentiate with respect to c_k, set $c_k=1$, and multiply both sides by 2. The result is

$$\sum_{\substack{n_{k-1}\geq\cdots\geq n_1\geq 0\\ n_{k-1}>0}}\frac{-2(q;q)_{n_{k-1}-1}(-1)^{n_{k-1}}q^{n_{k-1}(n_{k-1}+1)/2+n_{k-2}^2+\cdots+n_1^2}}{(q;q)_{n_{k-1}-n_{k-2}}\cdots(q;q)_{n_2-n_1}(q;q)_{n_1}} \tag{4.10}$$
$$=2\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}-\frac{(1+q^n)q^{kn^2}}{1-q^n}.$$

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The multiple sum counts the same overpartitions (in the Frobenius representation) as in the previous theorem but is weighted by $(-1)^t$, where t is 1 plus the number of columns plus the co-rank of the bottom row. As discussed in [11], the right side generates the k-middle divisors. $\square$

4.4. **Frobenius overpartitions and partial theta functions.** Identities involving partial theta functions appeared in Ramanujan's lost notebook and have been extensively studied [7], [22]. Consider, for example, the following infinite product representations for two sums involving partial theta products $(aq, q/a; q)_n$ [19]:

$$\frac{4a}{(1+a)^2} + \sum_{n=1}^{\infty} \frac{(-1;q)_n^2 q^n}{(aq,q/a;q)_n} = \frac{4a(q^2;q^2)_\infty}{(1+a)^2(q;q^2)_\infty(q,aq,q/a;q)_\infty}, \tag{4.11}$$

$$\frac{a}{(1-a)^2} + \sum_{n=1}^{\infty} \frac{(q;q)_{n-1}^2 q^n}{(aq,q/a;q)_n} = \frac{a(q;q)_\infty^2}{(1-a)^2(aq,q/a;q)_\infty}. \tag{4.12}$$

By now these are easily recognizable as statements about Frobenius overpartitions. We highlight just one special case, where we again encounter divisor functions.

Proof of Theorem 1.7. Setting $a = -1$ in (4.12) we obtain

$$1 - 4\sum_{n=1}^{\infty} \frac{(q;q)_{n-1}^2 q^n}{(-q;q)_n^2} = \frac{(q;q)_\infty^2}{(-q;q)_\infty^2}.$$

The sum on the left side generates nonempty Frobenius overpartitions weighted by the parity of the sum of the ranks and co-ranks in the top and bottom row. This is also the parity of the sum of the largest parts in the top and bottom row. The weight one modular form on the right-hand side is the generating function for the number of representations of a natural number n as the sum of 2 squares, weighted by $(-1)^n$. This is known to be $4(-1)^n(d_1(n) - d_3(n))$, where d_i denotes the number of divisors of n that are congruent to i modulo 4 [17]. $\square$

5. Concluding remarks

We have hopefully demonstrated that overpartitions provide a natural setting for the interpretation of q-series identities. We have confined ourselves to an introductory sample, but without a doubt there is much more to be learned about these objects in this context. It is hoped, in addition, that the ongoing revelation of their rich structure will continue to assist in the discovery of simple bijective proofs for identities involving basic hypergeometric series. Finally, it will be interesting to see whether overpartitions have a natural place, as partitions do, in subjects like Representation Theory, Number Theory, Lie Algebras, and Mathematical Physics.

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