LATTICE INVARIANTS AND THE CENTER OF THE GENERIC DIVISION RING

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Abstract. Let $G$ be a finite group, let $M$ be a $ZG$-lattice, and let $F$ be a field of characteristic zero containing primitive $p^h$ roots of 1. Let $F(M)$ be the quotient field of the group algebra of the abelian group $M$. It is well known that if $M$ is quasi-permutation and $G$-faithful, then $F(M)^G$ is stably equivalent to $F(ZG)^G$. Let $C_n$ be the center of the division ring of $n \times n$ generic matrices over $F$. Let $S_n$ be the symmetric group on $n$ symbols. Let $p$ be a prime. We show that there exist a split group extension $G_0$ of $S_p$ by a $p$-elementary group, a $G_0$-faithful quasi-permutation $ZG_0$-lattice $M$, and a one-cocycle $\alpha$ in $\text{Ext}^1_{G_0}(M, F^*)$ such that $C_p$ is stably isomorphic to $F(M)^{G_0}$. This represents a reduction of the problem since we have a quasi-permutation action; however, the twist introduces a new level of complexity. The second result, which is a consequence of the first, is that, if $F$ is algebraically closed, there is a group extension $E$ of $S_p$ by an abelian $p$-group such that $C_p$ is stably equivalent to the invariants of the Noether setting $F(E)$.

INTRODUCTION

Let $C_p$ denote the center of the division ring of $p \times p$ generic matrices over a field $F$. We study the question of whether $C_p$ is stably rational over $F$, when $p$ is a prime and $F$ is a field of characteristic zero containing primitive $p^h$ roots of 1.

The question of rationality of the center of the generic division ring has been studied extensively, in particular for its connection to important problems in other fields such as geometric invariant theory and Brauer groups.

For the span of more than a century, stable rationality of the center has been shown for the primes 2, 3, 5 and 7, and for 4 ([S], [F1], [F2] and [BL]), with rationality proven for 2, 3 and 4. It was also shown that $C_n$ is retract rational over $F$ for all $n$ square-free, [SD1].

Let $G$ be a finite group and let $F$ be a field. Given a $ZG$-lattice $M$, we may form the group algebra $F[M]$ of the abelian group $M$. There is an action of $G$ on its quotient field $F(M)$ via the $G$-action on $M$. If $M = ZG$, then $F(M)^G$ is denoted by $F(G)$, and referred to as the Noether setting of $G$. The main tool for determining whether $F(M)^G$ is stably rational over $F$ is given by work of Endo, Miyata, Lenstra and Swan, and it is as follows. For any finite group $G$ and for any $G$-faithful $ZG$-lattices $M$ and $M'$, the fields $F(M)$ and $F(M')$ are stably isomorphic as $F$-algebras and the isomorphism respects their $G$-actions if and only if $M$ and

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$M'$ are in the same flasque class. In particular, if $F(G)^G$ is stably rational over $F$, then so is $F(M)^G$ for any $G$-faithful quasi-permutation $ZG$-lattice $M$.

Let $S_n$ be the symmetric group on $n$ letters. It was shown in [B1] that $C_n$ is stably isomorphic to the fixed field under the action of $S_n$ of $F(G_n)$, where $G_n$ is a specific $ZS_n$-lattice which we define below. To determine directly whether the invariant field of $F(G_n)$ is stably rational over $F$ turned out to be quite an intractable problem for primes $p$ greater than or equal to 5. One of the reasons for this intractability is that for $p \geq 5$, $G_p$ is not quasi-permutation [BL]. The main results of this article are as follows. We show that there is a split group extension $G'$ of $S_p$ by a $p$-elementary group, a $G'$-faithful quasi-permutation $ZG'$-lattice $M$, and a one-cocycle $\alpha \in \text{Ext}_G^1(M, F^*)$ such that $C_p$ is stably isomorphic to $F_\alpha(M)^{G'}$. Furthermore the Noether setting $F(G')$ has stably rational invariants over the base field. This represents a reduction of the problem since we have a quasi-permutation action; however, the twist introduces a new level of complexity. The second result, which is a consequence of the first, is that, if $F$ is algebraically closed, there is a group extension $E$ of $S_p$ by an abelian $p$-group, such that $C_p$ is stably equivalent to the invariants of the Noether setting $F(E)$. These results are described below.

Section 1 consists mostly of preliminary results and definitions. Let $N$ be the normalizer of a $p$-Sylow subgroup of $S_p$. The starting point is our result in [B1], which says that $G_p$ and $ZS_p \otimes_{ZN} G_p$ are in the same flasque class, implying that $F(G_p)$ and $F(ZS_p \otimes_{ZN} G_p)$ are $S_p$-stably isomorphic. If we let $H$ be a $p$-Sylow subgroup of $S_p$, then $N$ is the semi-direct product of $H$ by a cyclic group $C$, of order $p-1$. We show

**Theorem 1.6.** Let $h$ generate $H$ and let $I_C$ denote the augmentation ideal of $ZC$. The flasque classes of $G_p$ and $ZS_p \otimes_{ZN} I_C \otimes ZH(h-1)^2$ are equal. Consequently, $C_p$ and $F(ZS_p \otimes_{ZN} I_C \otimes ZH(h-1)^2)^{G_p}$ are stably isomorphic over $F$.

The theorem is a consequence of the decomposition of $G_p$ into indecomposable $\hat{Z}_pN$-modules, described in [B2], and it represents the first step in the proof of the main theorem.

The second reduction step is done in Theorem 2.2. Let $U$ be the standard rank $p$ permutation representation of $S_p$, and let $A$ be the kernel of the augmentation map $U$. We show that there exists a transcendency basis for $F(A)$ on which $N$ acts linearly. This basis plays a crucial role in the proof. It is exhibited in Theorem 2.1.

Let $M$ be a $ZG$-lattice and let $\alpha \in \text{Ext}_G^1(M, F^*)$ be a one-cocycle. We have a new $G$-action on $F(M)$ via $\alpha$. Such an action is said to be $\alpha$-twisted, and the corresponding field is denoted by $F_\alpha(M)$. The lattice $ZS_p \otimes_{ZN} I_C \otimes ZH(h-1)^2$ of Theorem 1.6 embeds into the quasi-permutation $ZS_p$-lattice $ZS_p \otimes_{ZH} A$, which allows us to express $C_p$ in terms of $ZS_p \otimes_{ZH} A$.

**Theorem 2.2.** There exists a finite $ZS_p$-module $P$ of exponent $p$, and a $\gamma$-twisted action of $G'$ = $P \rtimes S_p$ on $F(ZS_p \otimes_{ZH} A)$ such that the center, $C_p$, is stably isomorphic to $F_\gamma(ZS_p \otimes_{ZH} A)^{G'}$.

The key point is that the transcendency basis for $F_\gamma(ZS_p \otimes_{ZH} A)$, obtained by lifting the basis of Theorem 2.1 to $S_p$, behaves well with respect to the twisting, as is shown in Proposition 2.5 and Theorem 2.6. This basis allows us to shift the twisting from $P$ to $S_p$, so that the action of $P$ on the field is purely monomial, and

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1The idea for this basis was given to us by William Chin, to whom we are deeply indebted.
the action of $S_p$ on the $F$-span of the basis is $F$-linear. Furthermore, and this is essential to the argument, the monomial action of $P$ is a quasi-permutation action. This is the third reduction step.

**Theorem 2.6.** There exist a $G'$-faithful quasi-permutation $ZG'$-lattice $M$ and a one-cocycle $\beta \in \text{Ext}^{1}_{G'}(M,F^*)$ such that the center $C_p$ is stably isomorphic to $F_{\beta}(M)^{G'}$.

In Proposition 2.7, we show that $M$ is in fact induced from a $p$-Sylow subgroup of $G'$. Most of the results in Section 2 require a fair amount of computations to specifically determine certain group actions. We have chosen to show explicit computations in the simplest setting, and then induce from there.

The fourth and final reduction is done in Theorem 3.3. We show that there is a group extension $E$ of $S_p$ by an abelian $p$-group such that the invariants of the Noether setting $F(E)$ are stably isomorphic to the center.

1.

Let $G$ be a finite group. An equivalence relation is defined in the category $L_{G}$ of $ZG$-lattices as follows. The $ZG$-lattices $M$ and $M_0$ are said to be equivalent if there exist permutation modules $P$ and $P_0$ such that $M \oplus P = M_0 \oplus P_0$. The equivalence class of a $ZG$-lattice $M$ will be denoted by $[M]$. The set of equivalence classes forms an abelian monoid under the direct sum. Lattices equivalent to 0 are said to be stably permutation. These are lattices $M$, for which there exist permutation modules $P$ and $R$ such that $M \oplus P = R$. Lattices whose equivalence class has an inverse are called invertible.

Flasque classes play a crucial role with respect to stable rationality, as will be illustrated by the following results, which are direct consequences of the work of Swan [SR], Lenstra [L] and Endo and Miyata [EM]. These results are included for the reader’s convenience. A detailed study of flasque classes can be found in [CTS].

**Definition.** Let $K$ and $L$ be extension fields of $F$, on which a finite group $G$ acts as a subgroup of their groups of $F$-automorphisms. We say that $L$ and $K$ are $G$-isomorphic ($G$-stably isomorphic) if they are isomorphic (stably isomorphic) as $F$-algebras, and the isomorphism respects their $G$-actions.

**Theorem 1.1 ([B2, Theorem 1.1]).** Let $G$ be a finite group and let $M$ and $M'$ be $G$-faithful $ZG$-lattices. Then $M$ and $M'$ are in the same flasque class if and only if $F(M)$ and $F(M')$ are $G$-stably isomorphic.
Theorem 1.2. If \( G \) is a finite group and \( M \) is a quasi-permutation \( G \)-faithful \( ZG \)-lattice, then \( F(M) \) is \( G \)-stably isomorphic to \( F(ZG) \).

Proof. The proof of \([L]\) Proposition 1.5] holds; the proof also follows directly from Theorem \([L]\) \(\square\)

We will also need the following results.

Lemma 1.3. Let \( G \) be a finite group, and let \( K \) be a field. Let \( L = K(v_1, \ldots, v_n) \) be a rational extension of \( K \), and suppose that there is a \( K \)-linear action of \( G \) on the \( K \)-span of the \( v_i \).

a) If \( G \) acts faithfully on \( K \), then \( L \) and \( K \) are \( G \)-stably isomorphic.

b) If \( G \) acts faithfully on \( L \), then \( L \) is \( G \)-stably isomorphic to \( K(ZG) \).

Proof. a) Speiser’s Lemma, e.g., \([W]\).

b) The proof is basically that of \([SD2] \) Proposition 3.1]. We include it for the reader’s convenience. We have

\[ L(ZG) = K(v_1, \ldots, v_n)(ZG) = K(ZG)(v_1, \ldots, v_n). \]

By a) \( L(ZG) \) is stably isomorphic to \( L \), and \( K(ZG)(v_1, \ldots, v_n) \) is stably isomorphic to \( K(ZG) \). \(\square\)

We now define the \( ZS_n \)-lattice \( G_n \) mentioned in the introduction. Let \( U \) be the standard rank \( p \) permutation representation of \( S_p \), and let \( A \) be the kernel of the augmentation map on \( U \). More precisely, \( U \) is the \( ZS_n \)-lattice with \( Z \)-basis \( \{u_i: 1 \leq i \leq n\} \) with \( S_n \)-action given by \( gu_i = u_{g(i)} \) for all \( g \in S_n \), and \( A \) is defined by the exact sequence

\[ 0 \to A \to U \to Z \to 0, \]

\[ u_i \to 1. \]

Then \( G_n = A \otimes Z A \) has the property that \( F(G_n)^{S_n} \) is stably isomorphic to \( C_n \), the center of the division ring of \( n \times n \) generic matrices over \( F \) \([F1] \) Theorem 3]. Note that \( U \) and \( Z \) are \( ZS_p \)-permutation lattices, and hence \( A \) is quasi-permutation.

Lemma 1.4. If \( M \) is a quasi-permutation \( S_n \)-faithful \( ZS_n \)-lattice, then \( F(M)^{S_n} \) is stably rational over \( F \).

Proof. The proof follows directly from Theorem \([L2] \) and the fact that \( F(U)^{S_n} \) is rational over \( F \), generated by the elementary symmetric functions on the \( u_i \). \(\square\)

Given a finite group \( G \), a \( ZG \)-lattice \( M \), and a field \( L \) on which \( G \) acts as automorphisms, the field \( L(M) \) has a \( G \)-action induced from the action of \( G \) on \( M \) and on \( L \). The reader should note that the action on \( L \) may be trivial. However there exist other \( G \)-actions on \( L(M) \). These actions were found by Saltman \([SD3] \), and called \( \alpha \)-twisted actions. They are defined as follows.

Let \( \alpha \) be in \( \text{Ext}^1_G(M, L^*) \), where \( L^* \) is the multiplicative group of \( L \). Let the equivalence class of

\[ 1 \to L^* \to M' \to M \to 1 \]

in \( \text{Ext}_G(M, L^*) \) be \( \alpha \). Writing \( M \) and \( M' \) as multiplicative abelian groups, we have

\[ M' = \{x \cdot m: x \in L^*, m \in M\} \]
and the $G$-action on $M'$ is given by $g \ast x \cdot m = g(x)d_g(gm) \cdot gm$, where $d : G \to \text{Hom}_\mathbb{Z}(M, L^*)$ is the derivation corresponding to $\alpha$. In particular, for $x = 1$, we have

$$g \ast m = d_g(gm) \cdot gm.$$  

Thus we obtain an $\alpha$-twisted action on $L(M)$. We denote by $L_\alpha(M)$ the field $L(M)$ with the corresponding $G$-action.

Given a finite group $G$ and a $ZG$-lattice $M$ with $Z$-basis $S = \{m_1, \ldots, m_s\}$, the set $S$ will also represent a transcendency basis for $F(M)$ over $F$, when no confusion can arise; in other words, the operation in $M$ will be denoted either by addition or multiplication, according to whether we are viewing $M$ as a $ZG$-lattice or as a subgroup of the multiplicative group of $F(M)$, but the basis elements will be denoted by the same symbols.

Now let $H$ be a subgroup of $G$ of index $r$, and let $\{g_i : i = 1, \ldots, r\}$ be a transversal for $H$ in $G$. Then a $Z$-basis for $ZG \otimes_{ZH} M$ is $\{g_i \otimes m_j : i = 1, \ldots, r, j = 1, \ldots, s\}$. Let $\{f_j = f_j(m_1, \ldots, m_s) : j = 1, \ldots, s\}$ be a transcendency basis for $F(M)$ over $F$, and let $f_{ij} = f_j(g_i \otimes m_1, \ldots, g_i \otimes m_s)$; then $F(ZG \otimes_{ZH} M) = F(f_{ij} : i = 1, \ldots, r, j = 1, \ldots, s)$.

**Notation.** Henceforth we adopt the following notation unless otherwise specified.

- $G = S_p$, the symmetric group on $p$ letters for a prime $p$.
- $H = p$-Sylow subgroup of $G$.
- $N =$ normalizer of $H$ in $G$. Thus $N = H \rtimes C$ is the semi-direct product of $H$ by a cyclic group $C$ of order $p - 1$.
- $a =$ primitive $(p - 1)^{st}$ root of $1$ mod $p$.
- We will let $h$ and $c$ generate $H$ and $C$ respectively, and $c h^{-1} = h^a$.
- For any finite group $G$ and any $ZG$-lattice $M$, $\widehat{M}$ will denote the $p$-adic completion of $M$, and for any prime $q$, $M_q$ will denote the localization of $M$ at $q$.  
- $F =$ field of characteristic zero containing primitive $p^{th}$ roots of $1$.

**Remark 1.5.** Since $ZN/H \cong ZC \cong Z[x]/(x^{p - 1} - 1)$ as $ZN$-lattices, the decomposition of $\widehat{ZN}/H$ into indecomposables is given by

$$\widehat{ZN}/H \cong \bigoplus_{k=1}^{p-1} Z_k,$$

where $Z_k$ is the $\widehat{ZN}$-module of $\widehat{Z}$-rank $1$ on which $H$ acts trivially, and such that $cl = \vartheta^k$, where $\vartheta$ is a primitive $(p - 1)^{st}$ root of $1$ in $\widehat{Z}$ which is congruent to $a$ mod $p$. We also set $X_k = Z_k/pZ_k$. This notation will be used throughout the article.

The lattice $A$ defined above is isomorphic to $ZH(h - 1)$ as a $ZN$-lattice. To see this, it suffices to note that $U \cong ZG/S_{p-1}$, and therefore $\text{Res}_N^Q(U) = ZN/C \cong ZH$. The action of $C$ on $ZH$ is given by $ch^i = h^a$. Note that $A$ is $ZC$-free, since $C$ permutes its $Z$-basis $\{h^i - 1 : i = 1, \ldots, p - 1\}$. We denote by $A'$ the $ZN$-lattice $ZH(h - 1)^2$; from the $ZN$-exact sequence

$$(*) \quad 0 \to A' \to A \to X_1 \to 0$$

where the map $A \to X_1$ is given by $(h - 1) \to 1$, we see that $A'$ is $ZC$-projective since $X_1$ is of order $p$, and hence $A'_q \cong A_q$ for all primes $q$ dividing the order of $C$.  

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Also $A'$ is isomorphic to $A$ as a $ZH$-lattice with the isomorphism given by
$$h^i(h - 1)^2 \to h^i(h - 1).$$
These facts will be used later.

**Definition.** A $ZG$-module is said to be invertible or permutation projective, if it is a direct summand of a permutation $ZG$-module.

**Theorem 1.6.** Let $A'$ denote the $ZN$-lattice $ZH(h - 1)^2$ and let $I_C$ denote the augmentation ideal of $ZC$. Then the flasque classes $\Phi(G_p)$ and $\Phi(ZG \otimes ZN I_C \otimes A')$ are equal, consequently $C_p$ and $F(ZG \otimes ZN I_C \otimes A')^G$ are stably isomorphic over $F$.

**Proof.** By [B2] Theorem 2.2, $\Phi(ZG \otimes ZN A') = [G_p]$. Since $G_p$ is invertible by [BL] Proposition 3, Section 3.1], $\Phi(G_p) = -[G_p]$. We have a $ZN$-exact sequence
$$0 \to A' \to ZC \otimes A' \to I_C \otimes A' \to 0$$
where the map $ZC \otimes A' \to I_C \otimes A'$ is $(c - 1) \otimes 1_{A'}$. This map splits at the prime $p$ since $C$ is of order $p - 1$, and hence $(I_C)_p$ is $Z_p N$-projective. At any prime $q$, we have an injection
$$\text{Ext}_N^1((I_C \otimes A')_q, A'_q) \to \text{Ext}_{N_q}^1((I_C \otimes A')_q, A'_q)$$
by [BK] Corollary 10.2 and Theorem 10.3, Chapter III], where $N_q$ is the $q$-Sylow subgroup of $N$. If $q \neq p$, we may assume that $N_q$ is contained in $C$, and then $\text{Ext}_{N_q}^1((I_C \otimes A')_q, A'_q) = 0$ since $A'$ is $ZC$-projective. Thus the sequence splits and
$$A' \oplus I_C \otimes A' \cong ZC \otimes A'.$$
Therefore
$$\Phi(ZG \otimes ZN A') + \Phi(ZG \otimes ZN I_C \otimes A') = \Phi(ZG \otimes ZN ZC \otimes A').$$
By Remark 1.3, $A' = ZH(h - 1)^2$ is isomorphic to $A$ as a $ZH$-lattice, and hence $\Phi(ZG \otimes ZN ZC \otimes A') = 0$, since $ZG \otimes ZN ZC \otimes A' \cong ZG \otimes ZH A' \cong ZG \otimes ZH A$ which is quasi-permutation. Thus
$$\Phi(G_p) = -[G_p] = -\Phi(ZG \otimes ZN A') = \Phi(ZG \otimes ZN I_C \otimes A').$$
By Theorem 1.1 and [F1] Theorem 3], $C_p$ and $F(ZG \otimes ZN I_C \otimes A')^G$ are stably isomorphic.

**Remark 1.7.** The following technique is outlined in [SD2] and will be used throughout the article. Let $G$ be a finite group and let
$$0 \to A \to B \to C \to 0$$
be a $ZG$-exact sequence where $A$ and $B$ are $ZG$-lattices and $C$ is a finite $ZG$-module of exponent $n$. Let $F$ be a field containing primitive $n^{th}$ roots of 1. Then the field extension
$$F(A) \subset F(B)$$
is a Kummer extension with Galois group $C' = \text{Hom}(C, F^*)$. There is a natural action of $G$ on $C'$ via its action on $C$. Since the $G$-action of $F(A)$ extends to the $G$-action of $F(B)$, we have Galois extensions
$$F(A)^G \subset F(A) \subset F(B)$$
and the Galois group of $F(B)$ over $F(A)^G$ is $C' \rtimes G$. 

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2.

We adopt the following notation. If $x_1, \ldots, x_r$ are commuting indeterminates over $F$, we will denote $F(x_1, \ldots, x_r)$ by $F(x_i)$.

**Theorem 2.1.** There exists a transcendency basis for $F(A)$ on which $N$ acts linearly.

**Proof.** Let $a_i = h^i - h^{i-1}$ for $i = 1, \ldots, p - 1$. Then $\{a_i: i = 1, \ldots, p - 1\}$ is a $\mathbb{Z}$-basis for $A$. Let $\varepsilon$ be a primitive $p^{th}$ root of 1. For $k = 0, \ldots, p - 1$, let $m_k = 1 + \sum_{k=1}^{p-1} \varepsilon^{ki} a_1 a_2 \cdots a_i$ and let $n_k = m_k/m_0$ for $k = 1, \ldots, p - 1$. Since the Van der Monde determinant

$$
\begin{vmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & \varepsilon & \varepsilon^2 & \ldots & \varepsilon^{p-1} \\
1 & \varepsilon^2 & \ldots & \ldots & \varepsilon^{2(p-1)} \\
1 & \ldots & \ldots & \ldots & \ldots \\
1 & \varepsilon^{p-1} & \ldots & \ldots & \varepsilon
\end{vmatrix}
$$

is nonzero, we can express the $m_k$ as $F$-linear combinations of the elements $1, a_1, a_1 a_2, \ldots, a_1 a_2 \cdots a_{p-1}$, and therefore $F(A) = F(m_k)$. Furthermore

$$
\sum_{k=1}^{p-1} n_k = -1 + p/m_0,
$$

hence $F(m_k) = F(n_k)$. Recall that in $N$, $chc^{-1} = h^a$, and note that $ha_i = a_{i+1}$ if $i = 1, \ldots, p - 2$ and $ha_{p-1} = a_1 a_2^{-1} \cdots a_{p-1}$ in multiplicative notation. For $k = 0, \ldots, p - 1$ we have

$$
hm_k = 1 + \varepsilon^k a_2 + \varepsilon^{2k} a_2 a_3 + \cdots + \varepsilon^{(p-2)k} a_2 \cdots a_{p-1} + \varepsilon^{(p-1)k} a_2 \cdots a_{p-1} a_1^{-1} a_2^{-1} \cdots a_{p-1}^{-1}
$$

$$
= 1 + \varepsilon^k a_2 + \varepsilon^{2k} a_2 a_3 + \cdots + \varepsilon^{(p-2)k} a_2 \cdots a_{p-1} + \varepsilon^{(p-1)k} a_1^{-1}
$$

$$
= \varepsilon^{-k} a_1^{-1} m_k.
$$

For $i = 1, \ldots, p - 1$, let $v_i = a_1 a_2 \cdots a_i$. Then in additive notation $v_i$ is $h^i - 1$, and $c(h^i - 1) = h^{mi} - 1$. Hence $cv_i = v_{ia}$. Furthermore

$$
m_k = 1 + \sum_{k=1}^{p-1} \varepsilon^{ki} v_i.
$$

Thus

$$
cm_k = m_{k/a}.
$$

Consequently

$$
hn_k = \varepsilon^{-k} n_k \quad \text{and} \quad cn_k = n_{k/a}
$$

where the indices are computed mod $p$. \hfill $\square$

**Theorem 2.2.** Let $k$ denote the field with $p$ elements, let $P = kG \otimes_{kN} IC \otimes X_{p-2}$, and let $G' = P \rtimes G$. Then there is a twisted action of $P$ on $F(ZG \otimes_{ZN} ZC \otimes A)$ given by a one-cocycle $\gamma \in \text{Ext}_Z^1(ZG \otimes_{ZN} ZC \otimes A, F^*)$, such that the center $C_P$ is stably isomorphic to $F_\gamma(ZG \otimes_{ZN} ZC \otimes A)^{G'}$.

**Proof.** Consider the $ZN$-exact sequence (2) of Remark 1.5, namely

$$
0 \to A' \to A \to X_1 \to 0
$$
where \( h - 1 \rightarrow 1_{X_1} \). Tensoring by \( I_C \) over \( Z \) we get

\[
0 \to I_C \otimes A' \to I_C \otimes A \to I_C \otimes X_1 \to 0.
\]

Consider the \( ZN \)-exact sequence

\[
0 \to A \to ZC \otimes A \to I_C \otimes A \to 0
\]

where the map \( ZC \otimes A \to I_C \otimes A \) is \((1 - c) \otimes 1_A\). As in the proof of Theorem 1.6 \([2]\), \((1 - g) \otimes 1_A\) splits at the prime \( p \) since \((I_C)_p \) is \( Z_p C \)-projective, and it splits at all other primes dividing the order of \( N \) since \( A \) is \( ZC \)-free. Thus the sequence splits, and

\[
ZC \otimes A \cong I_C \otimes A \oplus A.
\]

Adding \( A \) to the last two terms of \((1)\) we get

\[
0 \to I_C \otimes A' \oplus A \to ZC \otimes A \to I_C \otimes X_1 \to 0.
\]

Tensoring this sequence by \( ZG \) over \( ZN \), we obtain

\[
0 \to ZG \otimes_{ZN} I_C \otimes A' \oplus ZG \otimes_{ZN} A \to ZG \otimes_{ZN} ZC \otimes A \to ZG \otimes_{ZN} (I_C \otimes X_1) \to 0.
\]

Clearly \( \text{Hom}(I_C \otimes X_1, F^*) \cong I_C \otimes X_{p - 2} \). Thus \( P \cong \text{Hom}(ZG \otimes_{ZN} I_C \otimes X_1, F^*) \) and by Remark \([1, 7]\), \( F_2(ZG \otimes_{ZN} ZC \otimes A)^G = F(ZG \otimes_{ZN} I_C \otimes A' \oplus ZG \otimes_{ZN} A)^G \) for some \( \gamma \in \text{Ext}^1_G(ZG \otimes_{ZN} ZC \otimes A, F^*). \) Now

\[
\Phi(ZG \otimes_{ZN} I_C \otimes A' \oplus ZG \otimes_{ZN} A) = \Phi(ZG \otimes_{ZN} I_C \otimes A') + \Phi(ZG \otimes_{ZN} A).
\]

But \( \Phi(ZG \otimes_{ZN} A) = 0 \) since the sequence

\[
0 \to ZG \otimes_{ZN} A \to ZG \otimes_{ZN} U \to ZG/N \to 0
\]

implies that \( ZG \otimes_{ZN} A \) is quasi-permutation. \( F_2(ZG \otimes_{ZN} ZC \otimes A)^G \) is stably isomorphic to \( F(ZG \otimes_{ZN} I_C \otimes A')^G \) by Theorem 1.1 which in turn is stably isomorphic to \( C_p \) by Theorem 1.6 \( \Box \)

We will now go back through the process that led us to prove Theorem 2.2 in order to determine precisely the \( G^* \)-action on \( F_1(ZG \otimes_{ZN} ZC \otimes A) \). Note that \( ZG \otimes_{ZN} ZC \otimes A \cong ZG \otimes_{ZN} A \).

Remark 2.3. To simplify notation we let \( X_{p - 2} = Y = \text{Hom}(X_1, F^*) \), and we let \( y \) generate \( Y \). This notation will be used throughout the rest of the article. Recall that \( a \) was defined to be a primitive \((p - 1)^{st} \) root of \( 1 \) mod \( p \), and let \( b = a^{p - 2} = a^{-1} \mod p \). Then \( H \) acts trivially on \( Y \) and \( cy = y^b \). Consider the \( ZN \)-sequence \((*)\) of Remark \([1, 5]\), namely

\[
0 \to A' \to A \to X_1 \to 0.
\]

By Remark \([1, 4]\) we have an action of \( N' = Y \rtimes N \) on \( F_3(A) = F(n_k) \) for some \( \delta \in \text{Ext}^1_{N'}(A, F^*) \), as follows. Let \( \varepsilon \) be a primitive \( p^{th} \) root of \( 1 \), which we fix once and for all; then

\[
y a_k = \varepsilon a_k \quad \text{for all } k.
\]

Thus

\[
y m_k = y \left( 1 + \sum_{k=1}^{p-1} \varepsilon^{k^2} a_1 a_2 \cdots a_i \right) = 1 + \sum_{k=1}^{p-1} \varepsilon^{k^2} \varepsilon^i a_1 a_2 \cdots a_i = m_{k+1}
\]

for \( k = 0, \ldots, p - 1 \), and the indices on the \( m_k \) are computed \( \mod p \). Therefore

\[
y n_k = n_{k+1} n_1^{-1}, \quad \text{for } k = 1, \ldots, p - 2, \quad \text{and } y n_{p-1} = n_1^{-1}, \quad \text{hence } Y \text{ acts on the } Z \text{-span of } \{ n_k : k = 1, \ldots, p - 1 \} \text{ as on the } Z \text{-basis for } I_Y, \text{ the augmentation ideal of the}
group ring $ZY$, where $n_k$ corresponds to $y^k - 1$ in additive notation. The ring $ZY$ has a $ZN'$-lattice structure via the action of $N$ on $Y$. We thus have an isomorphism of $N'$-fields

$$F_\delta(A) = F(n_k) \to F_\gamma(I_Y)$$

for the appropriate $\gamma \in \text{Ext}_{N'}^1(I_Y, F^*)$.

The proof the main theorem uses a generalization of this construction.

**Definition.** Let $K = F(x_1, \ldots, x_n)$ be a rational extension of $F$, and let $G$ be a finite group acting on $K$. The $G$-action on $K$ is said to be monomial if for all

$$i = 1, \ldots, n, \quad gx_i = a_i x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

for some $a_i \in F, \alpha_i \in \mathbb{Z}$, and purely monomial if $a_i = 1$ for all $i$.

Lemma 2.4 and Proposition 2.5 are needed for the proof of Theorem 2.6 in the following sense. They illustrate the techniques used in Theorem 2.6 in a simpler setting. Their purpose is to allow us to write the proof of Theorem 2.6 in an abbreviated and much simpler form.

**Lemma 2.4.** Let $Q = \text{Hom}(IC \otimes X_1, F^*) \cong IC \otimes Y$ and let $N' = Q \rtimes N$. There exists an $N'$-action on $ZC \otimes ZY$, such that $ZC \otimes ZY$ is a $ZN'$-permutation module.

**Proof.** Recall that $Z_{p-2}$ was defined to be the $\hat{Z}N$-module of $\hat{Z}$-rank 1 with trivial $H$-action and such that $c \cdot 1_{\hat{Z}} = \vartheta 1_{\hat{Z}}$, where $\vartheta$ is a primitive $(p - 1)^{st}$ root of 1 in $\hat{Z}$ congruent to $b \mod p$. Also $Y$ was defined to be $Z_{p-2}/pZ_{p-2}$. There is a natural action of $N$ on $ZC \otimes ZY$ via the action of $C$ on $ZC$ and on $Y$, and with trivial $H$-action. Specifically

$$c(c^i \otimes y^j) = c^{i+1} \otimes y^{ib}.$$

Let $k$ be the field of $p$ elements, and let

$$R = kC \otimes Y \cong \frac{\hat{Z}C \otimes Z_{p-2}}{p\hat{Z}C \otimes Z_{p-2}}.$$ 

An additive $\hat{Z}$-basis for $\hat{Z}C \otimes Z_{p-2}$ is $\{c^i \otimes 1_{\hat{Z}}: i = 1, \ldots, p - 1\}$, and $c(c^i \otimes 1_{\hat{Z}}) = c^{i+1} \otimes 1_{\hat{Z}}$.

Let $t_i = i, a_1, \ldots, p - 1$ be a multiplicative generating set for $R$, where $t_i$ corresponds to $c^i \otimes 1_{\hat{Z}} \mod p\hat{Z}C \otimes Z_{p-2}$. Letting $t_i$ generate $T_i$, we may write $R \cong kC \otimes Y \cong T_1 \times \cdots \times T_{p-1}$ as an abelian group, with $c^{t_i} = t_{i+1}^b$. We define an action of $N'' = R \rtimes N$ on $ZC \otimes ZY$ as follows. We set

$$\text{Res}_{R''}^{N''} ZC \otimes ZY \cong ZT_1 \oplus \cdots \oplus ZT_{p-1},$$

$$c^i \otimes y^j \to t_i^j.$$

The action of $R$ is obtained via this isomorphism. More specifically, the set $\{c^i \otimes y^j: i = 1, \ldots, p - 1, j = 1, \ldots, p\}$ is a $Z$-basis for $ZC \otimes ZY$, and we have

$$t_k(c^i \otimes y^j) = c^i \otimes y^j \quad \text{if } k \neq i,$$

$$t_i(c^i \otimes y^j) = c^i \otimes y^{j+1}.$$ 

The action of $C$ is given by $c(c^i \otimes y^j) = c^{i+1} \otimes y^{ib}$. A computation shows that this is in fact a group action, and clearly $N''$ permutes the basis elements. Now

$$Q = IC \otimes Y \cong IC \otimes \frac{Z_{p-2}}{pZ_{p-2}}.$$
Thus $Q$ is the $N$-submodule of $R$ generated by $y_i = t_i t_{p-1}^{-1}$, with
\[ cy_i = y_{i+1} y_1^{-b} \text{ if } i \neq p - 2 \quad \text{and} \quad cy_{p-2} = y_1^{-b}. \]

Letting $Y_i$ be the subgroup of $Q$ generated by $y_i$ we have
\[ Q = Y_1 \times \cdots \times Y_{p-2} \]
as an abelian group. The action of $N'$ on $ZC \otimes ZY$ is just the action of the subgroup of $N''$ generated by the $y_i$ and $N$. Specifically the action of $N'$ on $ZC \otimes ZY$ is given by
\begin{align*}
y_k(c^i \otimes y^j) &= c^i \otimes y^j \text{ if } k \neq i, i \neq p - 1, \\
y_i(c^i \otimes y^j) &= c^i \otimes y^{j+1}, \\
y_i(1 \otimes y^j) &= 1 \otimes y^{j-1}, \\
c(c^i \otimes y^j) &= c^{i+1} \otimes y^j.
\end{align*}

Here $1 = c^{p-1}$ and the powers of $y$ are computed mod $p$. \hfill \Box

**Proposition 2.5.** Let $Q = \text{Hom}(I_C \otimes X_1, F^*) \cong I_C \otimes Y$ and let $N' = Q \times N$. There exist one-cocycles $\gamma \in \text{Ext}_Y^1(ZC \otimes A, F^*)$ and $\beta \in \text{Ext}_Y^1(ZC \otimes I_Y, F^*)$ such that $F_{\gamma}(ZC \otimes A)$ and $F_{\beta}(ZC \otimes I_Y)$ are isomorphic as $N'$-fields.

**Proof.** In the proof of Theorem 2.2 we have seen that there is a twisted action of $N'$ on $F(ZC \otimes A)$. We need to determine precisely what this action is. We keep the notation of Lemma 2.4 and use the following $Z$-basis for $I_C \otimes A$:
\[ \{(c^i - c^{i+1}) \otimes a_j : i = 1, \ldots, p - 2; j = 1, \ldots, p - 1\}. \]

Consider sequence (1) of Theorem 2.2 namely
\[ 0 \to I_C \otimes A' \to I_C \otimes A \to I_C \otimes X_1 \to 0. \]

By Remark 1.7 we have an action of $Q$ on $F(I_C \otimes A)$ given in multiplicative notation by
\[ y_k(c^i \otimes a_j)(c^{i+1} \otimes a_j)^{-1} = \begin{cases} \varepsilon(c^i \otimes a_j)(c^{i+1} \otimes a_j)^{-1} & \text{if } k = i, \\ (c^i \otimes a_j)(c^{i+1} \otimes a_j)^{-1} & \text{otherwise,} \end{cases} \]
for $k = 1, \ldots, p - 2$. The reader should note that $(c^i \otimes a_j)(c^{i+1}a_j)^{-1}$ represents an element of the transcendence basis of $F(I_C \otimes A)$ over $F$.

The action of $N'$ on $ZC \otimes A$ is obtained from sequences (2) and (3) of Theorem 2.2. We have already seen that sequence (2) splits, and we let the splitting map for $A \to ZC \otimes A$ be denoted by $\delta$. Then
\[ f : ZC \otimes A \to I_C \otimes A \oplus A \\
c^i \otimes a_j \mapsto c^i(1 - c) \otimes a_j + \delta(c^i \otimes a_j) \]
is an isomorphism. Note that $f(1 \otimes a_j) = -\sum_{i=1}^{p-1} c^i(1 - c) \otimes a_j + \delta(\otimes a_j)$. In multiplicative notation
\[ f(1 \otimes a_j) = \left( \prod_{i=1}^{p-1} (c^i \otimes a_j)(c^{i+1} \otimes a_j)^{-1} \right)^{-1} \delta(1 \otimes a_j). \]
With this in mind, we see that the $Q$-action on $F(ZC \otimes A)$ given via $f$ is:

$$y_k c^i \otimes a_j = \begin{cases} 
\varepsilon^i c^i \otimes a_j & \text{if } k = i \text{ and } i \neq p - 1, \\
\varepsilon^{-1} c^i \otimes a_j & \text{if } i = p - 1, \\
c^i \otimes a_j & \text{otherwise}.
\end{cases}$$

We now change the transcendency basis for $F(ZC \otimes A)$. For $k = 1, \ldots, p - 1$, let $n_k = n_k(a_1, \ldots, a_{p-1})$ be defined as in Theorem 2.4. Define

$$n_{ik} = n_k(c^i \otimes a_1, \ldots, c^i \otimes a_{p-1})$$

for $i, k = 1, \ldots, p - 1$. Then by an argument similar to that of Remark 2.3 we have the following action of $Q$:

- $y_i$ acts trivially on $\{n_{jk} : i \neq j \text{ and } j \neq p - 1\}$,
- $y_k n_{ik} = n_{ik+1} n_{1i}^{-1}$ if $k \neq p - 1$,
- $y_k n_{p-1} = n_{1i}$,
- $y_k n_{p-1} k = n_{p-1 k - 1} n_{p-1}^{-1}$ if $k \neq 1$,
- $y_k n_{p-1} 1 = n_{p-1}^{-1} - 1 = -1$.

The action of $N$ is given by $hn_{ik} = \varepsilon^{-k} n_{ik}$ and $cn_{ik} = n_{i+1 k k}$. The indexes are computed mod $p - 1$ but powers of $b$ are computed mod $p$. Thus we have an isomorphism of $N'$-fields

$$F_\gamma(ZC \otimes A) \cong F(n_{ik})$$

for some one-cocycle $\gamma \in \Ext^1_{N'}(ZC \otimes A, F^*)$.

On the other hand $ZC \otimes ZY$ is a $ZN'$-lattice by Lemma 2.4. Now $ZC \otimes I_Y$ is the $ZN'$-submodule of $ZC \otimes ZY$ generated by $c^i \otimes (y^k - 1)$, thus the $N'$-action is given in additive notation:

- $y_i$ acts trivially on $\{c^i \otimes (y^k - 1) : i \neq j \text{ and } j \neq p - 1\}$,
- $y_i c^i \otimes (y^k - 1) = c^i \otimes (y^{k+1} - 1) - (c^i \otimes (y^k - 1))$ if $i \neq p - 1$,
- $y_i c^i \otimes (y^{p-1} - 1) = -c^i \otimes (y - 1)$,
- $y_i 1 \otimes (y^k - 1) = 1 \otimes (y^{k-1} - 1) - (1 \otimes (y^{p-1} - 1))$ if $k \neq 1$,
- $y_i 1 \otimes (y - 1) = -1 \otimes (y^{p-1} - 1)$.

Therefore as a $Q$-lattice the $Z$-span of the $n_{ik}$ is isomorphic to $ZC \otimes I_Y$. There is an $N'/H$-isomorphism $f$, from the $Z$-span of the $n_{ik}$ to $ZC \otimes I_Y$, sending $n_{ik}$ to $c^i \otimes (y^k - 1)$ in additive notation. We define a new $N'$-action on $F(ZC \otimes I_Y)$ by having $H$ act on the element of the transcendency basis corresponding to $c^i \otimes (y^k - 1)$ as on $n_{ik}$. The map $f$ clearly induces an $N'$-isomorphism from $F(n_{ik})$ to $F_\beta(ZC \otimes I_Y)$ for the appropriate $\beta \in \Ext^1_{N'}(ZC \otimes I_Y, F^*)$. Thus $F_\gamma(ZC \otimes A)$ and $F_\beta(ZC \otimes I_Y)$ are isomorphic as $N'$-fields.

The crucial point here is that now the action of $Q$ on $F(n_{ik})$ is purely monomial, and the action of $N$ is linear. The proof of the next theorem uses this argument induced up to $G'$.

**Theorem 2.6.** There exist a $G'$-faithful quasi-permutation $ZG'$-lattice $M$ and a one-cocycle $\beta \in \Ext^1_{G'}(M, F^*)$ such that the center $C_p$ is stably isomorphic to $F_\beta(M)^G$.
Proof. For $k = 1, \ldots, p - 1$ let $n_k = n_k(a_1, \ldots, a_{p-1})$ be defined as in Theorem 2.1. Let $S_{p-2}$ be the subgroup of $G$ fixing $p - 1$ and $p$, and let $\{g_i: g_i \in S_{p-2}\}$ be a transversal for $N$ in $G$. This transversal will be fixed throughout the paper. As in Proposition 2.5, let $n_{ijk} = n_k(g_i \otimes c^j \otimes a_1, \ldots, g_i \otimes c^j \otimes a_{p-1})$ for $i = 1, \ldots, (p - 2)!$, $j, k = 1, \ldots, p - 1$. Then

$$F_n(ZG \otimes ZN \otimes ZC \otimes A) = F(n_{ijk})$$

for some $\gamma \in \text{Ext}_G^1(ZG \otimes ZN \otimes ZC \otimes A, F^*)$, and the $G$-action is given by: Let $gg_i = g_ic^j h^l$, where $s, t$ and $l$ depend on $g$ and $g_i$. Then

$$g n_{ijk} = e^{-lk} n_{s, j + t, k}.$$  

Write $P = \sum_{ij} Y_{ij}$, where $Y_{ij} = g_i \otimes Y_j$, and let $y_{ij}$ generate $Y_{ij}$. Then

$y_{st}$ acts trivially on $n_{ijk}$ if $s \neq i$ or $(i = s, t \neq j$ and $k \neq p - 1),

y_{ij} n_{ijk} = n_{ij + 1} n_{ij}^{-1}$ if $k \neq p - 1,

y_{ijn} n_{ijp - 1} = n_{ijp - 1},

y_{ijn} n_{ijp - 1} = n_{ijp - 1} - 1$ if $k \neq 1,

y_{ijn} n_{ijp - 1} = n_{ijp - 1} - 1$.

By an argument similar to that of Proposition 2.5, $P$ acts on the $Z$-span of the $n_{ijk}$ as on the $ZG$-lattice

$$M = ZG \otimes ZN \otimes ZC \otimes I_{Y}$$

with $n_{ijk}$ corresponding to $g_i \otimes c^j \otimes (y^k - 1)$ in additive notation. This induces a $G'$-isomorphism from $F(n_{ijk})$ to $F_{\beta}(M)$ for the appropriate $\beta \in \text{Ext}_G^1(M, F^*)$. \square

Proposition 2.7. Let $PH$ denote the $p$-Sylow subgroup of $G'$. There exists a $Z[PH]$-lattice $I$ such that the center $C_p$ is stably isomorphic to $F_{\beta}(ZG' \otimes Z[PH] I)$ for some $\beta \in \text{Ext}_G^1(ZG' \otimes Z[PH] I, F^*)$.

Proof. The $ZG'$-lattice $ZG \otimes ZN \otimes ZC \otimes ZY$ is a permutation lattice with $Z$-basis

$\{g_i \otimes c^j \otimes y^k: 1 \leq i \leq (p - 2)!$, $1 \leq j \leq p - 1, 1 \leq k \leq p\}$.  

If we let $g_1 = 1_G$, then, for any $i, j, k$, $y^k g_i (1 \otimes 1 \otimes 1) = g_i \otimes c^j \otimes y^k$. Therefore it is a transitive permutation lattice. The stabilizer in $G'$ of $1 \otimes 1 \otimes 1$ is the semi-direct product of the subgroup, $K$, of $P$ generated by $\{y_{ij}, y_{ij}^{-1}: i \neq 1, j \neq k\}$, by $H$. We denote this stabilizer by $KH$. Thus $ZG \otimes ZN \otimes ZC \otimes ZY \cong ZG'/KH$. Let $I$ be the $Z[PH]$-lattice defined by the exact sequence

$$0 \rightarrow I \rightarrow Z[PH]/KH \rightarrow Z \rightarrow 0.$$  

Then it is immediate that $M = ZG \otimes ZN \otimes ZC \otimes I_{Y}$ is isomorphic to $ZG' \otimes Z[PH] I$. \square

3.

In this section we assume the base field $F$ to be algebraically closed of characteristic zero. We will keep all the notation of the previous sections unless otherwise specified. Recall that

$$\hat{Z}/H \cong \bigoplus_{k=1}^{p-1} Z_k,$$

where $Z_k$ is the $\hat{Z}$-module of $\hat{Z}$-rank 1 on which $H$ acts trivially, and such that $c l = \vartheta^k$, where $\vartheta$ is a primitive $(p - 1)^{st}$ root of 1 in $\hat{Z}$ which is congruent to a
mod \( p \). We also set \( X_k = Z_k/pZ_k \). Recall also that we had set \( X_{p-2} = Y \), and to simplify notation we set \( X_{p-1} = X \), so \( X \) is the trivial \( ZN \)-module of \( p \) elements. Finally \( k \) was defined to be the field of \( p \) elements.

**Notation** 3.1. If \( G \) is a finite group, the Noether setting of \( G \) is \( F(ZG) \) and it is denoted by \( F(G) \). For any \( ZG \)-lattice \( S \), \( S \) will denote \( \text{Hom}(S, F^*) \).

The \( ZG \)-lattice \( ZG \otimes_{[PH]} I \) will be denoted by \( M \), as above.

**Theorem 3.2.** The center \( C \) of the division ring of \( p \times p \) generic matrices over an algebraically closed field \( F \) of characteristic zero is stably isomorphic to the Noether setting of the group \( G'' \) defined by

\[
\gamma: 1 \to kG/H \to G'' \to G' \to 1
\]

where \( \gamma \in H^2(G', kG/H) \). Furthermore the sequence

\[
0 \to kG/H \to \tilde{Z}[G'/KH] \to \tilde{M} \oplus \tilde{ZG}'/PH \to 0
\]

is exact and \( \gamma \) is the image of an element \( \beta \in \text{Ext}_{G'}^1(M, F^*) \) under its connecting homomorphism.

**Proof.** Recall that \( P \) was defined to be \( kG \otimes_{KN} I_C \otimes Y \), and that \( G' = P \times G \). By Theorem 2.6 \( C \) is stably isomorphic to \( F_{\beta}(M)G' \). Furthermore, the action of \( P \) on \( F_{\beta}(M) \) is not twisted. We have the \( ZG \)-exact sequence

\[
0 \to M \to ZG'/KH \to ZG'/PH \to 0
\]

from which we obtain

\[
0 \to M \oplus ZG'/PH \to ZG'/KH \to kG'/PH \to 0
\]

by sending \( M \) to \( f(M) \) and \( \bar{y} \) to \( \sum_{j=1}^{p^2} y \otimes y^j \) for \( g \in G'/PH \), where \( y \) generates \( PH/KH \).

Since \( \text{Hom}(kG'/PH, F^*) \cong kG/H \) as \( G' \)-modules, [B] Theorem I-2.1] says that there exists a group \( G'' \) given by the exact sequence

\[
\gamma: 1 \to kG/H \to G'' \to G' \to 1
\]

such that \( F(ZG'/KH)^{G''} = F_{\beta}(M \oplus ZG'/PH)^{G'} \). The class of \( \gamma \in H^2(G', kG/H) \) is the image of \( \beta \) by the connecting homomorphism \( \delta \) of the exact sequence

\[
0 \to kG/H \to \tilde{Z}[G'/KH] \to \tilde{M} \oplus \tilde{ZG}'/PH \to 0.
\]

Furthermore the action of \( G'' \) on \( F(ZG'/KH) \) is linear. Also note that this action is \( G'' \)-faithful since the action of \( G' \) on \( F_{\beta}(M \oplus ZG'/PH) \) is \( G' \)-faithful. By Theorem 1.2 \( F(ZG'/KH)^{G''} \) is stably isomorphic to \( F(G'') \), and by Lemma 1.3 \( F_{\beta}(M \oplus ZG'/PH)^{G''} \) is stably isomorphic to \( F_{\beta}(M)^{G'} \). Thus \( F(G'')^{G''} \) is stably isomorphic to the center \( C_\beta \).

**Theorem 3.3.** There is a group extension \( E \) of \( G \) by an abelian \( p \)-group such that the center \( C_\beta \) of the division ring of \( p \times p \) generic matrices over an algebraically closed field \( F \) of characteristic zero is stably isomorphic to the invariants of the Noether setting \( F(E) \).

**Proof.** Consider the group \( G'' \) of Theorem 3.2. It is defined by the exact sequence

\[
\gamma: 1 \to kG/H \to G'' \to G' \to 1
\]
where \( \gamma \in H^2(G', kG/H) \) is the image of \( \beta \), the one-cocycle from \( G' \) to \( \text{Hom}(M, F^*) \) which gives the twisting of the action of \( G' \) on \( F_\beta(M) \). Now \( \text{Res}_{G'}^G(\beta) = 0 \) since the action of \( P \) on \( F_\beta(M) \) is not twisted. By Theorem 3.2, \( \gamma = \delta(\beta) \) where \( \delta \) is the connecting homomorphism of the exact sequence

\[
0 \to kG/H \to \tilde{Z}[G'/KH] \to M \oplus \tilde{Z}G'/PH \to 0.
\]

Thus \( \text{Res} \gamma = \text{Res}(\delta(\beta)) = \delta(\text{Res}(\beta)) = 0 \), and hence there is a group monomorphism from \( P \) to \( G'' \). Let \( S \) be the subgroup of \( G'' \) generated by \( kG/H \) and the image of \( P \). Then \( S \) is normal and we have a group extension

\[
1 \to S \to G'' \to G \to 1.
\]

The result now follows from Theorem 3.2.

References


[F1] E. Formanek, *The center of the ring of \( 3 \times 3 \) generic matrices*, Linear Multilinear Algebra **7** (1979), 203–212. MR 80h:16019


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