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RANDOM GAPS UNDER CH

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Abstract. It is proved that if the Continuum Hypothesis is true, then one random real always produces a destructible (ω_1, ω_1) gap.

1. Introduction

Recall that for a Boolean algebra $(\mathcal{B}, \leq)$, for a pair of types (ψ, θ) a pair (A, B) of subsets of $\mathcal{B}$ is a (ψ, θ) *pregap in* $(\mathcal{B}, \leq)$ if

(a) A and B are linearly ordered by $\leq$,
(b) $(A, \leq)$ has type ψ and $(B, \leq)$ has type θ,
(c) $A \perp B$, i.e. $a \cdot b = 0$ for all $a \in A$ and $b \in B$.

The pregap is a *gap* if moreover

(d) there is no $c \in \mathcal{B}$ such that $a \leq c$ for all $a \in A$, and $b \leq -c$ for all $b \in B$.

Such an element c is said to *interpolate* A and B.

One of the more striking early discoveries in set theory is Hausdorff's construction [Hau36] of an (ω_1, ω_1) gap in the Boolean algebra $(\mathcal{P}(\mathbb{N})/\operatorname{Fin}, \subseteq^*)$, where $\mathbb{N} = \{0, 1, \dots\}$ denotes the set of all nonnegative integers. Recall that for $a, b \subseteq \mathbb{N}$,

(e) $[a] \subset^* [b]$ iff $a \setminus b$ is finite and $b \setminus a$ is infinite,

and $[a] \subseteq^* [b]$ iff $[a] \subset^* [b]$ or $[a] = [b]$. Lifting this relation to $\mathcal{P}(\mathbb{N})$, we get $a \subseteq^* b$ iff $a \setminus b$ is finite. As is usual when working with gaps in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ we identify members of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ with representatives of their equivalence classes. Thus for a pair of ordinals (γ, δ), we take as a (γ, δ) pregap in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$, a pair $(\vec{a}, \vec{b})$ of sequences of subsets of $\mathbb{N}$ such that

(f) $a_\alpha \subset^* a_\beta$ for all $\alpha < \beta < \gamma$,
(g) $b_\alpha \subset^* b_\beta$ for all $\alpha < \beta < \delta$,
(h) $a_\alpha \cap b_\beta$ is finite for all α, β,

and $c \subseteq \mathbb{N}$ interpolates $\vec{a}$ and $\vec{b}$ accordingly. In the case where (A, B) is a *symmetric* pregap, i.e. $\gamma = \delta$, it will be seen to be useful to choose the representatives so that

(i) $a_\alpha \cap b_\alpha = \emptyset$ for all α.

One of the more interesting phenomenon associated with gaps in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is the phenomenon of destructibility. That is, the possibility of interpolating a gap in some forcing extension (and thus rendering it a nongap) while preserving its types. In other words, for a pair (κ, λ) of cardinals, a (κ, λ) pregap (A, B) is *destructible* if there is a poset $\mathcal{P}$ preserving all cardinals less than or equal to $\max\{\kappa, \lambda\}$ which forces a $c \in \mathcal{P}(\mathbb{N})/\operatorname{Fin}$ interpolating A and B.

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It is well known that the existence of a destructible (ω_1, ω_1) gap is undecidable from the ZFC axioms. Kunen [Kun76a] showed that $\mathfrak{m} > \aleph_1$ (i.e. $\mathrm{MA}_{\aleph_1}$) implies that all (ω_1, ω_1) gaps are indestructible (see also [Sch93]). On the other hand the first destructible (ω_1, ω_1) gap was constructed by Laver [Lav79] by forcing with finite approximations to an (ω_1, ω_1) pregap (see also [Sch93]). It was later shown by Todorčević that a destructible (ω_1, ω_1) gap can be constructed from a diamond sequence (see [Dow95]). This begged the question of whether the Continuum Hypothesis alone is sufficient for the existence of a destructible gap. This was answered negatively in [AT97], where it is also shown that the combinatorial principle $(*)$ is consistent with CH.

Theorem 1 (Abraham-Todorčević $(*)$)**.** *All (ω_1, ω_1) gaps are indestructible.*

Furthermore, a destructible (ω_1, ω_1) gap can be constructed from a Cohen real. Indeed the simplest and most elegant construction of a destructible (ω_1, ω_1) gap is the following one due to Todorčević. We write $\mathcal{C}$ for the Cohen poset on $2^{<\mathbb{N}}$, and write $\dot{c}$ for the canonical $\mathcal{C}$-name for the Cohen real in $\mathcal{P}(\mathbb{N}) \cong 2^{\mathbb{N}}$. Operations on $\mathcal{P}(\mathbb{N})$ are extended memberwise to subfamilies of $\mathcal{P}(\mathbb{N})$.

Theorem 2 (Todorčević)**.** *Suppose that $(\vec{a}, \vec{b})$ is an (ω_1, ω_1) gap in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. Then*

$$\mathcal{C} \Vdash (\vec{a} \cap \dot{c}, \vec{b} \cap \dot{c}) \textit{ is a destructible } (\omega_1, \omega_1) \textit{ gap},$$

i.e. $(\vec{a} \cap \dot{c}, \vec{b} \cap \dot{c}) = ((a_\alpha \cap \dot{c} : \alpha < \omega_1), (b_\alpha \cap \dot{c} : \alpha < \omega_1))$.

Proof. See [TF95, Theorem 9.3]. $\square$

This construction can be viewed as the simplest gap introduced by a Cohen real. A question which then naturally arises is *what happens if c is replaced with a random real*? Indeed it is asked in [TF95, Chapter 9] whether the analogue of Theorem 2 is true for a random real, and this was answered negatively in [Hir01] with the following result. We let $\mathcal{R}$ denote the atomless separable measure algebra (i.e. the poset for adding one random real), and let $\dot{r}$ be the canonical $\mathcal{R}$-name for the random real in the Haar measure space $2^{\mathbb{N}}$ identified with $\mathcal{P}(\mathbb{N})$. Note that the combinatorial principle $(\star_c)$ is also consistent with CH, as established in [Hir00a].

Theorem 3 ($\mathfrak{m} > \aleph_1$ or $(*)$ or $(\star_c)$)**.** *For every (ω_1, ω_1) gap $(\vec{a}, \vec{b})$, $\mathcal{R}$ forces that $(\vec{a} \cap \dot{r}, \vec{b} \cap \dot{r})$ is indestructible.*

More generally, adding a random real need not introduce any destructible gaps at all (see [Hir01]):

Theorem 4 ($\mathfrak{m} > \aleph_1$)**.** *$\mathcal{R}$ forces that all (ω_1, ω_1) gaps are indestructible.*

It is an open problem[1] whether this generalizes to arbitrary measure algebras (i.e. the extension by arbitrarily many random reals).

Now while neither CH nor a random real alone suffices to obtain a destructible gap, we prove here that both together do yield a destructible gap. Indeed the purpose of this note is to prove the following theorem.

Theorem 5 (CH)**.** *$\mathcal{R} \Vdash$ there exists a destructible (ω_1, ω_1) gap.*

[1] *Added in proof*: The author has recently obtained a negative answer in [Hir03].

However, we point out that by Theorem 3 this cannot be obtained by considering the simplest random gap (i.e. the random real analogue of the construction in Theorem 2). The destructible gap is obtained by considering the "next simplest" random gap. That is, we prove that assuming CH, there exists a gap $(\vec{a}, \vec{b})$ and an $\mathcal{R}$-name $\dot{s}$ for a subset of $\mathbb{N}$ such that

$$\mathcal{R} \Vdash (\vec{a} \cap \dot{s}, \vec{b} \cap \dot{s}) \text{ is a destructible gap.} \tag{1}$$

We do not however directly construct these objects.

Definition 6. For a function f whose range consists entirely of finite sets, define

$$\mathbb{R}(f) = \prod_{z \in \operatorname{dom}(f)} f(z).$$

For example, for the identity function $i : \mathbb{N}^+ \to \mathbb{N}^+$ ($\mathbb{N}^+ = \{1, 2, \dots\}$ denotes the positive integers), $\mathbb{R}(i)$ consists of all sequences $(k_n : n \in \mathbb{N}^+)$ such that $k_n < n$ for all n. We give $\mathbb{R}(f)$ the product topology, and it is equipped with the product measure $\nu = \prod_{z \in \operatorname{dom}(f)} \nu_z$ where $\nu_z(\{w\}) = 1/|f(z)|$ for all $w \in f(z)$.

Notation. Let $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be fixed. For $a \subseteq \mathbb{N} \times \mathbb{N}$, we let

$$[a] = \bigcup_{(m,n) \in a} \{(m, n)\} \times h(m, n).$$

Note that every element of $\mathbb{R}(h)$ is a subset of $[\mathbb{N} \times \mathbb{N}]$.

Instead of the objects in (1), we fix a function $h : \mathbb{N}^2 \to \mathbb{N}$ and construct a gap $(\vec{a}, \vec{b})$ in $\mathcal{P}([\mathbb{N} \times \mathbb{N}])/\operatorname{Fin}$, i.e. $a_\alpha, b_\alpha \subseteq [\mathbb{N} \times \mathbb{N}]$ for all α, such that

$$\mathcal{R} \Vdash (\vec{a} \cap \dot{r}_{\mathbb{R}(h)}, \vec{b} \cap \dot{r}_{\mathbb{R}(h)}) \text{ is a destructible gap,} \tag{2}$$

where $\dot{r}_{\mathbb{R}(h)}$ denotes an $\mathcal{R}$-name for the random real in the measure space $\mathbb{R}(h)$. To see that the objects in (2) yield those in (1), fix a bijection $g : \mathbb{N} \to [\mathbb{N} \times \mathbb{N}]$. Then g induces a homeomorphism $\Phi_g : \mathcal{P}([\mathbb{N} \times \mathbb{N}]) \to \mathcal{P}(\mathbb{N})$ via

$$\Phi_g(d) = g^{-1}(d). \tag{3}$$

Furthermore it is clear that Φ_g induces an isomorphism $\widetilde{\Phi}_g : \mathcal{P}([\mathbb{N} \times \mathbb{N}])/\operatorname{Fin} \to \mathcal{P}(\mathbb{N})/\operatorname{Fin}$ via $\widetilde{\Phi}_g([d]) = [\Phi_g(d)]$ (here square brackets denote the equivalence class). Therefore $(\Phi''_g(\vec{a}), \Phi''_g(\vec{b}))$ is a pregap in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ and $\Phi_g(\dot{r}_{\mathbb{R}(h)})$ is an $\mathcal{R}$-name for a subset of $\mathbb{N}$ such that

$$(\Phi''_g(\vec{a}) \cap \Phi_g(\dot{r}_{\mathbb{R}(h)}), \Phi''_g(\vec{b}) \cap \Phi_g(\dot{r}_{\mathbb{R}(h)}))$$

is a destructible gap with probability one.

It is instructive to note that this is not the first instance of such a construction involving both CH and a random real in a measure space of the form $\mathbb{R}(h)$. Indeed, in [Hir00b, Example 5.5] there is a CH construction of a maximal *tower* in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ (i.e. a well-ordered subfamily) such that for some $e : \mathbb{N} \to \mathbb{N}$ with fast enough growth, when the random real in $\mathbb{R}(e)$ is transferred to $\mathcal{P}(\mathbb{N})$ via the map analogous to Φ_g it extends this tower (i.e. kills its maximality).

There is another motivation for proving Theorem 5. Many analogies exist between (ω_1, ω_1) gaps and Aronszajn trees (see e.g. [AT97]). For example, Souslin trees are analogous to destructible gaps because any Souslin tree can be destroyed by an ω_1-preserving forcing notion which forces a cofinal branch through the tree. Furthermore, Souslin trees can be constructed from a diamond sequence, and also from a Cohen real. Analogously to Kunen's theorem above, $\mathfrak{m} > \aleph_1$ implies that all Aronszajn trees are special. Thus destructible (ω_1, ω_1) gaps correspond to Souslin trees, gaps correspond to Aronszajn trees and indestructible gaps correspond to special Aronszajn trees. However, in [Hir00a] it is proved that $(\star_c)$ implies that all Aronszajn trees are special in any forcing extension by a measure algebra. Thus by adding one random real to a model satisfying CH and $(\star_c)$, Theorem 5 breaks this analogy between gaps and Aronszajn trees. Indeed, to the author's knowledge this is the first model where all Aronszajn trees are special yet there exists a destructible (ω_1, ω_1) gap. Theorem 5 also answers negatively a question along these lines of Todorčević in [Tod00, p. 253].

2. The main lemma

The following Ramsey theoretic characterization of destructibility is well known. We refer the reader to one of the sources [Woo84], [Tod89], [Sch93], [TF95, Lemma 9.2] for its proof. The results of this section all require that an (ω_1, ω_1) pregap satisfies condition (i). Note that in the following Lemmas 7 and 8, a pregap refers to a pregap in $(\mathcal{P}(S)/\operatorname{Fin}, \subseteq^*)$ for some countable set S.

Lemma 7. *For any (ω_1, ω_1) pregap $(\vec{a}, \vec{b})$ the following are equivalent:*

(a) *$(\vec{a}, \vec{b})$ is destructible.*

(b) *For every uncountable $X \subseteq \omega_1$ there exist $\alpha \neq \beta$ in X such that $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset$.*

(c) *There is a poset with the ccc which interpolates $(\vec{a}, \vec{b})$.*

The following sufficient condition for an $\mathcal{R}$-name for an (ω_1, ω_1) pregap to name a destructible gap with positive probability is from [Hir01]. Note that it has no analogue for the trivial measure algebra, i.e. if there is an uncountable $X \subseteq \omega_1$ such that $a_\alpha \cap b_\beta = \emptyset$ for all $\alpha, \beta \in X$, then it is clear that $(\vec{a}, \vec{b})$ is not a gap. On the other hand, we are going to prove Theorem 5 by constructing an $\mathcal{R}$-name for a gap satisfying this condition.

We use standard notation for Boolean algebras, and thus for a sentence ϕ in the forcing language of a given Boolean algebra, $\|\phi\|$ denotes the Boolean probability that ϕ is true. The minimum and maximum elements of a Boolean algebra are denoted by 0 and 1, respectively. Note that we actually force with the poset $\mathcal{R}^+ = \mathcal{R} \setminus \{0\}$ of positive elements of $\mathcal{R}$. As usual, for each $x \in \mathcal{R}$, $\mathcal{R}_x$ denotes the Boolean subalgebra $\{y \in \mathcal{R} : y \leq x\}$.

Lemma 8. *Let $(\vec{a}, \vec{b})$ be an $\mathcal{R}$-name for an (ω_1, ω_1) pregap. Then for every $x \in \mathcal{R}^+$, if there exists an uncountable $X \subseteq \omega_1$ and a $\rho < 1$ such that*

$$\mu(x \cdot (\|\dot{a}_\alpha \cap \dot{b}_\beta \neq \emptyset\| + \|\dot{a}_\beta \cap \dot{b}_\alpha \neq \emptyset\|)) \leq \rho \cdot \mu(x) \quad \text{for all } \alpha, \beta \in X,$$

then $x \cdot \|(\vec{a}, \vec{b})$ is destructible$\| \neq 0$.

Proof. Given $x \in \mathcal{R}^+$, assume that x forces that $(\vec{a}, \vec{b})$ is indestructible. Suppose we are given an uncountable $X \subseteq \omega_1$ and $\rho < 1$. By Lemma 7 there is an $\mathcal{R}$-name $\dot{Y}$ for an uncountable subset of X such that

$$x \leq \|\forall \alpha, \beta \in \dot{Y}\ \alpha \neq \beta \rightarrow (\dot{a}_\alpha \cap \dot{b}_\beta) \cup (\dot{a}_\beta \cap \dot{b}_\alpha) \neq \emptyset\|. \tag{4}$$

Recursively choose for each $\eta < \omega_1$ a finite antichain $\{y_\eta^0, \dots, y_\eta^{m_\eta - 1}\} \subseteq \mathcal{R}_x^+$ and $\{\alpha(\eta, 0) < \cdots < \alpha(\eta, m_\eta - 1)\} \subseteq \omega_1$ such that

$$\mu\biggl(\sum_{i<m_\eta} y_\eta^i\biggr) > \sqrt{\rho} \cdot \mu(x), \tag{5}$$

$$y_\eta^i \leq \|\alpha(\eta, i) \in \dot{Y}\| \quad \text{for all } i < m_\eta, \tag{6}$$

$$\alpha(\eta, 0) > \alpha(\xi, m_\xi - 1) \quad \text{for all } \xi < \eta. \tag{7}$$

Find a finite $s_\eta \subseteq S$ large enough so that

$$\mu(\|\dot{a}_{\alpha(\eta,i)} \setminus s_\eta \subseteq \dot{a}_{\alpha(\eta,m_\eta-1)}\|) > 1 - \frac{(\sqrt{\rho} - \rho) \cdot \mu(x)}{4m} \quad \text{for all } i < m_\eta, \tag{8}$$

$$\mu(\|\dot{b}_{\alpha(\eta,i)} \setminus s_\eta \subseteq \dot{b}_{\alpha(\eta,m_\eta-1)}\|) > 1 - \frac{(\sqrt{\rho} - \rho) \cdot \mu(x)}{4m} \quad \text{for all } i < m_\eta. \tag{9}$$

By going to an uncountable subsequence we may assume that $m_\eta = m$ and $s_\eta = s$ for all η. Since $\mathcal{R}$ is separable, which means that the metric space $(\mathcal{R}, \nu)$ where $\nu(a, b) = \mu(a \bigtriangleup b)$ is separable, we can find an uncountable $Z \subseteq \omega_1$ such that

$$\mu(y_\xi^i \bigtriangleup y_\eta^i) \leq \frac{(\sqrt{\rho} - \rho) \cdot \mu(x)}{4m} \quad \text{for all } i < m, \text{ for all } \xi, \eta \in Z. \tag{10}$$

Using separability in a similar manner we can further assume that

$$\mu(\|\dot{b}_{\alpha(\xi,i)} \cap s = \dot{b}_{\alpha(\eta,i)} \cap s\|) > 1 - \frac{(\sqrt{\rho} - \rho) \cdot \mu(x)}{4m} \quad \text{for all } i < m. \tag{11}$$

Take any $\xi \neq \eta$ in Z. We claim that

$$\mu(x \cdot (\|\dot{a}_{\alpha(\xi,m-1)} \cap \dot{b}_{\alpha(\eta,m-1)} \neq \emptyset\| + \|\dot{a}_{\alpha(\eta,m-1)} \cap \dot{b}_{\alpha(\xi,m-1)} \neq \emptyset\|)) > \rho \cdot \mu(x). \tag{12}$$

By (4), (6) and (7),

$$y_\xi^i \cdot y_\eta^i \leq \|\dot{a}_{\alpha(\xi,i)} \cap \dot{b}_{\alpha(\eta,i)} \neq \emptyset\| + \|\dot{a}_{\alpha(\eta,i)} \cap \dot{b}_{\alpha(\xi,i)} \neq \emptyset\| \quad \text{for all } i < m. \tag{13}$$

Since by (i) and (11), $\mu(\|(\dot{a}_{\alpha(\xi,i)} \cap \dot{b}_{\alpha(\eta,i)} \cap s) \cup (\dot{a}_{\alpha(\eta,i)} \cap \dot{b}_{\alpha(\xi,i)} \cap s) = \emptyset\|) > 1 - (\sqrt{\rho} - \rho) \cdot \mu(x)/4m$ for all $i < m$, it follows from (8) and (9) that

$$\begin{aligned} &\mu\left(\sum_{i<m} y_\xi^i \cdot y_\eta^i\right) - \frac{3(\sqrt{\rho} - \rho) \cdot \mu(x)}{4} \\ &\qquad < \mu(\|\dot{a}_{\alpha(\xi,m-1)} \cap \dot{b}_{\alpha(\eta,m-1)} \neq \emptyset\| + \|\dot{a}_{\alpha(\eta,m-1)} \cap \dot{b}_{\alpha(\xi,m-1)} \neq \emptyset\|). \end{aligned} \tag{14}$$

However, by (5) and (10), $\mu(\sum_{i<m} y_\xi^i \cdot y_\eta^i) \geq \sqrt{\rho} \cdot \mu(x) - (\sqrt{\rho} - \rho) \cdot \mu(x)/4$, which completes the proof of (12) and hence also the proof of the lemma. $\square$

Lemma 8 points to a new phenomenon. Every (ω_1, ω_1) gap can be *frozen*, which means that there is a poset which forces that the gap is indestructible (see e.g. the references at the beginning of this section). However, setting $x = 1$, the condition in Lemma 8 is clearly upwards absolute for transitive models. Therefore, an $\mathcal{R}$-name for a gap satisfying the hypothesis remains an $\mathcal{R}$-name for a destructible pregap (and possibly a nongap) in any forcing extension, and thus freezing is impossible in this context. Furthermore, using this lemma it is shown in [Hir01] that such an $\mathcal{R}$-name $(\vec{a}, \vec{b})$ for a gap can be destroyed by a poset satisfying property K, i.e. there is a poset with property K which forces that $\|(\vec{a}, \vec{b})$ is a gap$\| \neq 1$. On the other hand, it is easy to see that a poset with property K can never destroy an (ω_1, ω_1) gap.

3. The construction

Preliminarily to the construction we review two notions to be used here. Suppose $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ and $[f]$ is a member of the ultraproduct $\mathcal{R}^{\mathbb{N}}/\mathcal{U}$ with a representative $f \in \mathcal{R}^{\mathbb{N}}$. Then in $V[\mathcal{G}]$, a forcing extension by $\mathcal{R}$, we consider the limit

$$\lim_{x\to\mathcal{G}} \lim_{n\to\mathcal{U}} \frac{\mu(x\cdot f(n))}{\mu(x)}. \tag{15}$$

For a function $g : \mathbb{N} \to \mathbb{R}$, one defines $\lim_{n\to\mathcal{U}} g(n) = w$ if for every open G about w, $\{n : g(n) \in G\} \in \mathcal{U}$. In $V[\mathcal{G}]$, for a function $h : \mathcal{R} \to \mathbb{R}$, one defines $\lim_{x\to\mathcal{G}} h(x) = v$ if for every open H about v, there is a $y \in \mathcal{G}$ such that $h''\{x \in \mathcal{G} : x \leq y\} \subseteq H$. A limit as in (15) first appears in Kunen's paper [Kun76b], and has its origins in Solovay's paper [Sol71] (see also [Jec97], [Kan94, §17]). A basic fact which we use is that

$$y \Vdash \lim_{x\to\dot{\mathcal{G}}} \lim_{n\to\check{\mathcal{U}}} \frac{\mu(x\cdot f(n))}{\mu(x)} > \check{\varepsilon} \quad \text{iff} \quad \lim_{n\to\mathcal{U}} \frac{\mu(z\cdot f(n))}{\mu(z)} > \varepsilon \quad \text{for all } z \leq y, \tag{16}$$

and the corresponding statement with $< \varepsilon$ in place of $> \varepsilon$. We refer the reader to one of these expositions for the proof that the limit in (15) is well defined and always exists, and the proof of fact (16).

The other basic fact is that given a sequence $(x_n : n \in \mathbb{N})$ of elements in $\mathcal{R}$ such that $\mu(x_n) \geq \delta$ for all n, for some $\delta > 0$,

$$\mu(\|\{n : x_n \in \dot{\mathcal{G}}\} \text{ is infinite}\|) \geq \delta. \tag{17}$$

In probabilistic terms this is just another way of saying that for a sequence of events with probability bounded away from zero, there is a positive probability that infinitely many of these events occur. To see that (17) holds, note that $\|\{x_n : n \in \dot{\mathcal{G}}\}$ is infinite$\| = \prod_{n=0}^{\infty}\sum_{m=n}^{\infty} \|x_m \in \dot{\mathcal{G}}\|$.

Fix some function $h : \mathbb{N} \to \mathbb{N}^+$ such that

$$\rho = 2\cdot\sum_{m=0}^{\infty} \frac{1}{h(m)} < 1. \tag{18}$$

Let $\hat{h} : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ be the function where $\hat{h}(m,n) = h(m)$ for all m,n. The square bracket notation from the Introduction will refer to this $\hat{h}$. We write $\dot{r}$ for $\dot{r}_{\mathbb{R}(\hat{h})}$.

Before proceeding, we try to give some motivation for the conditions imposed in our construction below. To ensure that we construct an $\mathcal{R}$-name for a destructible gap, we require that for all α,β, $\|a_\alpha \cap b_\beta \cap \dot{r} \neq \emptyset\| + \|a_\beta \cap b_\alpha \cap \dot{r} \neq \emptyset\|$ has measure at most ρ. Then by Lemma 8, $(a_\alpha \cap \dot{r}, b_\alpha \cap \dot{r} : \alpha < \omega_1)$ will be destructible with positive probability. Thus we also have to ensure that the $\mathcal{R}$-name $(\vec{a}\cap\dot{r}, \vec{b}\cap\dot{r})$ for a pregap is in fact a gap with probability one, which gives the following opposing restriction.

Lemma 9. *Let $(\vec{a},\vec{b})$ be an $\mathcal{R}$-name for a pregap. If $\|(\vec{a},\vec{b})$ is a gap$\| = 1$, then for every uncountable $X \subseteq \omega_1$, there exists $\beta \in X$ and an $\mathcal{R}$-name $\dot{\alpha}$ for an ordinal in X such that $\|\dot{a}_{\dot{\alpha}} \cap \dot{b}_{\check{\beta}} \neq \emptyset\| = 1$.*

Proof. Since $(\dot{a}_\alpha, \dot{b}_\alpha : \alpha \in X)$ is a gap with probability one, the set named by

$$\dot{c} = \bigcup_{\alpha\in X} \dot{a}_\alpha \tag{19}$$

interpolates the gap with probability zero, and therefore there is an $\mathcal{R}$-name $\dot\beta$ for an ordinal in X such that $\|\dot b_{\dot\beta} \cap \dot c$ is infinite$\| = 1$. Thus if $\gamma \in X$ is above the supremum of the set $\{\beta \in X : \|\dot\beta = \check\beta\| \neq 0\}$, then $\|\dot b_{\check\gamma} \cap \dot c$ is infinite$\| = 1$. In particular, $\dot b_\gamma \cap \dot c \neq \emptyset$ with probability one, and therefore there is an $\mathcal{R}$-name $\dot\alpha$ for an element of X such that $\dot a_{\dot\alpha}$ intersects $\dot b_\gamma$ with probability one. □

In particular, by approximating the value of $\dot\alpha$ by a finite antichain, this means that finite-to-one intersections must occur with high probability. In other words on any uncountable $X \subseteq \omega_1$ there will be a finite $\Gamma \subseteq X$ and a $\beta \in X$ such that $\|\bigcup_{\alpha\in\Gamma} a_\alpha \cap b_\beta \cap \dot r \neq \emptyset\|$ has measure arbitrarily close to one. Conditions (iii) and (v) together imply that for all α, β, for every $m \in \mathbb{N}$, $a_\alpha \cap b_\beta \cap [\{m\} \times \mathbb{N}]$ has at most one element, and therefore $\|a_\alpha \cap b_\beta \cap [\{m\} \times \mathbb{N}] \cap \dot r \neq \emptyset\|$ has measure at most $h(m)^{-1}$, guaranteeing that $\|a_\alpha \cap b_\beta \cap \dot r \neq \emptyset\| + \|a_\beta \cap b_\alpha \cap \dot r \neq \emptyset\|$ has measure at most ρ (see (21)). On the other hand, there is nothing preventing the existence of $\alpha_0, \dots, \alpha_{h(m)-1}$ and $n \in \mathbb{N}$ such that $(m, n, i) \in a_{\alpha_i}$ for all $i < h(m)$, and a β such that $\{(m, n)\} \times h(m) \subseteq b_\beta$, in which case $\|\bigcup_{i<h(m)} a_{\alpha_i} \cap b_\beta \cap \dot r \neq \emptyset\| = 1$, because

$$\| |[\{(m,n)\}] \cap \dot r| = 1 \| = 1 \tag{20}$$

for all m, n.

We use CH to get an enumeration $\{(\dot s_\alpha, x_\alpha) : \alpha < \omega_1\}$ of all pairs $(\dot s, x)$ where $\dot s$ is an $\mathcal{R}$-name for a subset of $[\mathbb{N} \times \mathbb{N}]$ and $x \in \mathcal{R}^+$.

We define $a_\beta, b_\beta \subseteq [\mathbb{N} \times \mathbb{N}]$ by recursion on $\beta < \omega_1$ so that:

(i) $a_\beta \cap b_\beta \neq \emptyset$,
(ii) for all $\alpha < \beta$, $a_\alpha \subseteq^* a_\beta$ and $b_\alpha \subseteq^* b_\beta$,
(iii) for all $m, n \in \mathbb{N}$, $a_\beta \cap [\{(m, n)\}]$ has at most one element (i.e. a_β is a function on some subset of $\mathbb{N} \times \mathbb{N}$),
(iv) for all $m \in \mathbb{N}$, $a_\beta \cap [\{m\} \times \mathbb{N}]$ is finite (i.e. $\mathrm{dom}(a_\beta) \in \emptyset \times \mathrm{Fin}$),
(v) for all $m \in \mathbb{N}$, there is at most one $n \in \mathbb{N}$ such that $b_\beta \cap [\{(m, n)\}] \neq \emptyset$ (i.e. $\mathrm{dom}(b_\beta) \in \emptyset \times [\mathbb{N}]^{\leq 1}$),
(vi) for infinitely many $m \in \mathbb{N}$, $b_\beta \cap [\{m\} \times \mathbb{N}] = \emptyset$,
(vii) either
   (a) $x_\beta - \|a_{\beta+1} \cap \dot r \subseteq^* \dot s_\beta\| \neq 0$, or
   (b) $x_\beta - \|b_{\beta+1} \cap \dot r \subseteq^* \dot s_\beta^{\mathsf{c}}\| \neq 0$.

Assuming for the moment that this is possible, let us see why this yields the desired object. By (i) and (ii), $(a_\alpha, b_\alpha : \alpha < \omega_1)$ forms a pregap. And for all $\alpha, \beta < \omega_1$,

$$\begin{aligned} \mu(\|a_\alpha \cap b_\beta \cap \dot r \neq \emptyset\|) &\leq \sum_{m=0}^\infty \mu(\|a_\alpha \cap b_\beta \cap [\{m\} \times \mathbb{N}] \cap \dot r \neq \emptyset\|) \\ &= \sum_{m=0}^\infty \max_{n\in\mathbb{N}} \mu(\|a_\alpha \cap b_\beta \cap [\{(m,n)\}] \cap \dot r \neq \emptyset\|) \\ &\leq \sum_{m=0}^\infty \frac{1}{h(m)} = \frac{\rho}{2}, \end{aligned} \tag{21}$$

where (v) is used for the equality, and (iii) is used to obtain the following inequality. Therefore $\mu(\|a_\alpha \cap b_\beta \cap \dot r \neq \emptyset\| + \|a_\beta \cap b_\alpha \cap \dot r \neq \emptyset\|) \leq \rho$; hence, Lemma 8 with $x = 1$ and $X = \omega_1$ entails that $(\vec a \cap \dot r, \vec b \cap \dot r)$ is destructible with positive probability. It remains to verify that $(\vec a \cap \dot r, \vec b \cap \dot r)$ is a gap with probability one. However, if there were some $x \in \mathcal{R}^+$ and some $\mathcal{R}$-name $\dot{s}$ for a subset of $[\mathbb{N} \times \mathbb{N}]$ for which

$$x \Vdash a_\alpha \cap \dot{r} \subseteq^* \dot{s} \quad \text{and} \quad b_\alpha \cap \dot{r} \subseteq^* \dot{s}^{\mathsf{c}} \quad \text{for all } \alpha < \omega_1, \tag{22}$$

then choosing β such that $(x_\beta, \dot{s}_\beta) = (x, \dot{s})$ would result in a contradiction with (vii). Note that we can conclude that with probability one, both $(a_\alpha \cap \dot{r} : \alpha < \omega_1)$ and $(b_\alpha \cap \dot{r} : \alpha < \omega_1)$ have order type ω_1 in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. For otherwise some $x \in \mathcal{R}^+$ would force that either $a_\alpha \cap \dot{r}$ or $b_\alpha \cap \dot{r}$ is eventually constant modulo Fin, and then x would force that $(\vec{a} \cap \dot{r}, \vec{b} \cap \dot{r})$ is not a gap.

Now it remains to show how to perform such a construction. Fix a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. We start the recursion by setting $a_0 = b_0 = \emptyset$. For a given $\beta < \omega_1$ assume that a_α and b_α have been defined for all $\alpha < \beta$. Suppose that β is a successor. Define $g \in \mathcal{R}^{\mathbb{N} \times \mathbb{N}}$ by

$$g(m,n) = \|[\{(m,n)\}] \cap (\dot{r} \setminus \dot{s}_{\beta-1}) \neq \emptyset\|. \tag{23}$$

First we consider the following case:

Case 1. There exist infinitely many $m \in \mathbb{N}$ such that

$$x_{\beta-1} \cdot \left\| \lim_{z \to \dot{\mathcal{G}}} \lim_{n \to \check{\mathcal{U}}} \frac{\mu(z \cdot g(m,n))}{\mu(z)} > 0 \right\| \text{ has measure greater than } \frac{\mu(x_{\beta-1})}{2}. \tag{24}$$

Let $\{m_j\}_{j=0}^\infty$ be an enumeration without repetition of all $m \in \mathbb{N}$ satisfying (24). We define a sequence $\{u(j)\}_{j=0}^\infty$ such that

$$u(j) \text{ is a finite partial function from } \{m_j\} \times \mathbb{N} \text{ into } h(m_j), \tag{25}$$

$$\operatorname{dom}(u(j)) \cap (\operatorname{dom}(a_{\beta-1}) \cup \operatorname{dom}(b_{\beta-1})) = \emptyset, \tag{26}$$

$$\mu(x_{\beta-1} \cdot \|u(j) \cap (\dot{r} \setminus \dot{s}_{\beta-1}) \neq \emptyset\|) > \mu(x_{\beta-1})/2. \tag{27}$$

To see that this is possible, take $j \in \mathbb{N}$. By (24) there is a $y \leq x_{\beta-1}$ and $\varepsilon > 0$ such that

$$\mu(y) > \frac{\mu(x_{\beta-1})}{2}, \tag{28}$$

$$y \Vdash \lim_{z \to \dot{\mathcal{G}}} \lim_{n \to \check{\mathcal{U}}} \frac{\mu(z \cdot g(m_j,n))}{\mu(z)} > \varepsilon. \tag{29}$$

We choose $n_0 < n_1 < \cdots < n_i < \cdots$ so that

$$(m_j, n_i) \notin \operatorname{dom}(a_{\beta-1}) \cup \operatorname{dom}(b_{\beta-1}), \tag{30}$$

and choose $k_i \in h(m_j)$. Having picked (n_i, k_i), by letting

$$z_i = y \cdot \sum_{i^* \leq i} \|(m_j, n_{i^*}, k_{i^*}) \in \dot{r} \setminus \dot{s}_{\beta-1}\|, \tag{31}$$

we can use (iv), (v) and (29) with (16) to obtain $n_{i+1} > n_i$ satisfying (30) such that

$$\mu((y - z_i) \cdot g(n_{i+1})) > \varepsilon \cdot \mu(y - z_i). \tag{32}$$

Then if we choose the best $k_{i+1} \in h(m_j)$ we must have

$$\mu((y - z_i) \cdot \|(m_j, n_{i+1}, k_{i+1}) \in \dot{r} \setminus \dot{s}_{\beta-1}\|) > \frac{\varepsilon}{h(m_j)} \cdot \mu(y - z_i). \tag{33}$$

Hence at some finite stage, i.e. $i = \lceil \log(1 - \mu(x_{\beta-1})/(2 \cdot \mu(y)))/\log(1 - \varepsilon/h(m_j)) \rceil$, we will have that $\mu(z_i) > \mu(x_{\beta-1})/2$. Then since (30) implies that (26) holds, we are done by putting $u(j) = \{(m_j, n_{i^*}, k_{i^*}) : i^* \leq i\}$.

Now we define a_β and b_β by

$$a_\beta = a_{\beta-1} \cup \bigcup_{j=0}^{\infty} u(j) \quad \text{and} \quad b_\beta = b_{\beta-1}. \tag{34}$$

Condition (26) implies that (i) holds, and obviously (ii) holds. Clearly (25) and (26) imply that condition (iii) is satisfied, and since each $u(j)$ is finite, condition (iv) is also satisfied. Conditions (v) and (vi) automatically hold. And by (17), from (27) it follows that $x_{\beta-1} \cdot \|a_\beta \cap (\dot{r} \setminus \dot{s}_{\beta-1})$ is infinite$\|$ has measure at least $\mu(x_{\beta-1})/2$ and in particular is nonzero, which implies (vii)(a).

The remaining case is

Case 2. There is an $l \in \mathbb{N}$ such that for all $m \geq l$,

$$x_{\beta-1} \cdot \left\| \lim_{z \to \dot{\mathcal{G}}} \lim_{n \to \dot{\mathcal{U}}} \frac{\mu(z \cdot g(m,n))}{\mu(z)} > 0 \right\| \text{ has measure at most } \frac{\mu(x_{\beta-1})}{2}. \tag{35}$$

Let $\{m_j^*\}_{j=0}^\infty$ be an enumeration without repetition of all $m \in \mathbb{N} \setminus l$ such that $b_{\beta-1} \cap [\{m\} \times \mathbb{N}] = \emptyset$. By (20) and (35), for each j there is an $n_j^* \in \mathbb{N}$ such that

$$(m_j^*, n_j^*) \notin \operatorname{dom}(a_{\beta-1}), \tag{36}$$

$$\mu(x_{\beta-1} \cdot \|[\{(m_j^*, n_j^*)\}] \cap \dot{r} \cap \dot{s}_{\beta-1} \neq \emptyset\|) > \frac{\mu(x_{\beta-1})}{3}. \tag{37}$$

Define

$$a_\beta = a_{\beta-1} \quad \text{and} \quad b_\beta = b_{\beta-1} \cup \bigcup_{j=0}^{\infty} [\{(m_{2j}^*, n_{2j}^*)\}]. \tag{38}$$

By (36), condition (i) holds. Conditions (ii)–(vi) are all clearly satisfied. By (17), it follows from (37) that $x_{\beta-1} \cdot \|b_\beta \cap \dot{r} \cap \dot{s}_{\beta-1}$ is infinite$\|$ has measure at least $\mu(x_{\beta-1})/3$, and in particular is nonzero, which implies (vii)(b).

Now suppose that β is a limit. Let $\{\xi_j\}_{j=0}^\infty \subseteq \beta$ be a strictly increasing cofinal sequence. Using (ii) and (vi), recursively choose $m_0 < m_1 < \cdots < m_j < \cdots$ so that

$$\bigcup_{i<j} b_{\xi_i} \cap [\{m_j\} \times \mathbb{N}] = \emptyset. \tag{39}$$

Define

$$a_\beta = \bigcup_{n=0}^{\infty} a_{\xi_n} \setminus \left(\bigcup_{i<n} b_{\xi_i} \cup [n \times \mathbb{N}] \cup \bigcup_{i<n} [\operatorname{dom}(a_{\xi_i} \setminus a_{\xi_n})] \right), \tag{40}$$

$$\begin{aligned} b_\beta = \bigcup_{n=0}^{\infty} b_{\xi_n} \setminus \Bigg(& a_\beta \cup \bigcup_{j=0}^{\infty} [\{m_j\} \times \mathbb{N}] \\ & \cup \bigcup_{i<n} \bigcup \{[\{m\} \times \mathbb{N}] : [\{m\} \times \mathbb{N}] \cap (b_{\xi_i} \setminus b_{\xi_n}) \neq \emptyset\} \Bigg). \end{aligned} \tag{41}$$

Obviously condition (i) is satisfied. For each n, since $a_{\xi_n} \cap b_{\xi_i}$ and $a_{\xi_i} \setminus a_{\xi_n}$ are finite for all $i < n$, by (iv) only finitely many elements are being removed from a_{ξ_n}. Hence for all $\alpha < \beta$, $a_\alpha \subseteq^* a_\beta$. Also note that for each n, since $b_{\xi_n} \cap [\{m_j\} \times \mathbb{N}] = \emptyset$ for all $j > n$, and since $b_{\xi_i} \setminus b_{\xi_n}$ is finite for all $i < n$, by (v) only finitely many elements are being removed from b_{ξ_n}, and hence $b_\alpha \subseteq^* b_\beta$ for all $\alpha < \beta$. Thus condition (ii) holds. Since the last union in (40) guarantees that a_β is a function, condition (iii) is satisfied. Noting that $a_\beta \cap [\{m\} \times \mathbb{N}] \subseteq \bigcup_{i \le m} a_{\xi_i} \cap [\{m\} \times \mathbb{N}]$ one sees that condition (iv) is satisfied. It is clear that the last union of (41) guarantees condition (v). Finally, the set $\{m_0, m_1, \dots\}$ witnesses that condition (vi) holds. The recursion is now complete.

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