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INVOLUTIONS FIXING $\mathbb{RP}^{\text{odd}} \sqcup P(h,i)$, II

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Abstract. This paper studies the equivariant cobordism classification of all involutions fixing a disjoint union of an odd-dimensional real projective space $\mathbb{RP}^j$ with its normal bundle nonbounding and a Dold manifold $P(h,i)$ with h a positive even and $i > 0$. The complete analysis of the equivariant cobordism classes of such involutions is given except that the upper and lower bounds on the codimension of $P(h,i)$ may not be best possible. In particular, we find that there exist such involutions with nonstandard normal bundle to $P(h,i)$. Together with the results of part I of this title (Trans. Amer. Math. Soc. **354** (2002), 4539–4570), the argument for involutions fixing $\mathbb{RP}^{\text{odd}} \sqcup P(h,i)$ is finished.

1. Introduction

In [Gu], [LL] and [Lü], Guo, Liu and Lü studied the equivariant cobordism classification of all involutions fixing a disjoint union of an odd-dimensional real projective space $\mathbb{RP}^j$ with its normal bundle nonbounding and a Dold manifold $P(h,i) = S^h \times \mathbb{CP}^i/(x,z) \sim (-x,\bar{z})$ with $h > 0$ and $i > 0$, respectively. In [Lü], when h is odd, such involutions are classified completely in some sense; when h is even, classifying such involutions can be reduced to the problem for *even* projective spaces. The objective of this paper is to study the case in which h is even.

Suppose (M^m, T) is a closed manifold with involution fixing a disjoint union of $\mathbb{RP}^j$ with normal bundle ν^{m-j} nonbounding and $P(h,i)$ with normal bundle ν^k, so $m = h + 2i + k$. If j is odd and h is even, following [Lü], one has $k > 0$, and one can write

$$w(\nu^{m-j}) = (1+\alpha)^q \text{ with } q \text{ odd}$$

where $H^*(\mathbb{RP}^j;\mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/(\alpha^{j+1} = 0)$ and $\alpha \in H^1(\mathbb{RP}^j;\mathbb{Z}_2)$ is a generator, and

$$w(\nu^k) = (1+c)^a(1+c+d)^b w(\rho)^\varepsilon$$

where $H^*(P(h,i);\mathbb{Z}_2) = \mathbb{Z}_2[c,d]/(c^{h+1} = d^{i+1} = 0)$, $c \in H^1(P(h,i);\mathbb{Z}_2)$ and $d \in H^2(P(h,i);\mathbb{Z}_2)$ are generators, $\varepsilon = 0$ or 1, and $w(\rho) = 1+$ terms of dimension at least 4 is an exotic class ($\varepsilon = 0$ except for $h = 2, 4$, or 6; see also [St1]). From [Lü, Proposition 1.2], one knows the following fact.

Fact. *If there is an involution (M^m, T) fixing $\mathbb{RP}^j \sqcup P(h,i)$ with j odd and h even and with the fixed component $\mathbb{RP}^j$ with its normal bundle nonbounding, then $m = j + q$.*

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Based upon this fact, consider the involution T_q on $\mathbb{RP}^{j+q}$ defined by

$$T_q([x_0,\dots,x_j,x_{j+1},\dots,x_{j+q}]) = [x_0,\dots,x_j,-x_{j+1},\dots,-x_{j+q}],$$

fixing $\mathbb{RP}^j$ with normal bundle $\nu^q = q\iota$ having $w(\nu^q) = (1+\alpha)^q$ and $\mathbb{RP}^{q-1}$ with normal bundle $\nu^{j+1} = (j+1)\iota$ having $w(\nu^{j+1}) = (1+\alpha)^{j+1}$, where ι denotes the real line bundle. If j is odd and h is even, forming the union

$$(M^m, T) \sqcup (\mathbb{RP}^{j+q}, T_q),$$

then one obtains an involution $(\bar{M}^{j+q}, \bar{T})$ fixing $\mathbb{RP}^{q-1}$ with $w(\nu^{j+1}) = (1+\alpha)^{j+1}$ and $P(h,i)$ with normal bundle ν^k. This means that the existence of (M^m,T) fixing $\mathbb{RP}^j \sqcup P(h,i)$ with j odd and h even depends upon the existence of $(\bar{M}^{j+q}, \bar{T})$ fixing $\mathbb{RP}^{q-1}$ with $w(\nu^{j+1}) = (1+\alpha)^{j+1}$ and $P(h,i)$ with normal bundle ν^k. In [Lü], there is an example in which (M^m,T) fixing $\mathbb{RP}^j \sqcup P(h,i)$ with j odd and h even exists in a very special case. In this paper we will complete the analysis of the cobordism classes of all possible involutions fixing $\mathbb{RP}^j \sqcup P(h,i)$ with j odd and h even.

According to [CF], $\mathbb{RP}(\nu^k), \mathbb{RP}(\nu^q)$ and $\mathbb{RP}(\nu^{j+1})$ are cobordant to each other in $B\mathbb{Z}_2$. Form the class

$$w[r] = \frac{w(\mathbb{RP}(\nu))}{(1+e)^{m-h-2i-r}},$$

where e is the characteristic class of the double cover of $\mathbb{RP}(\nu)$ by the sphere bundle of ν, so that each $w[r]_x$ is a polynomial in $w_y(\mathbb{RP}(\nu))$ and e. Then one has

(1.1)
$$w[r] = \begin{cases} (1+c)^h(1+c+d)^{i+1}\left\{(1+e)^r + (a+b)c(1+e)^{r-1} + \cdots\right\} & \text{on } P(h,i), \\ (1+\alpha)^{j+1}\left\{(1+e)^{h+2i+r-j} + q\alpha(1+e)^{h+2i+r-j-1} + \cdots\right\} & \text{on } \mathbb{RP}^j, \\ (1+\alpha)^q\left\{(1+e)^{h+2i+r-q+1} + \binom{j+1}{2}\alpha^2(1+e)^{h+2i+r-q-1} + \cdots\right\} & \text{on } \mathbb{RP}^{q-1}. \end{cases}$$

One will use the characteristic numbers on $\mathbb{RP}(\nu)$ formed by $w[r]_x$ to eliminate those cases in which involutions do not exist.

From [Lü] one knows that for the case in which h is odd, the exotic class never occurs in $w(\nu^k)$. However, for the case in which h is even, one will see that the exotic class can occur in $w(\nu^k)$. Also, Stong [St2] recently finished the argument for involutions fixing a disjoint union of a point and a Dold manifold. This will help us complete the argument for the case $q=1$. Hence, this paper mainly deals with the case $q>1$.

The main results will be stated in Section 2, and the proofs will be finished in Sections 3.

Throughout this paper, the coefficient group is $\mathbb{Z}_2$; w denotes the total Stiefel-Whitney class and w_s denotes the s-th Stiefel-Whitney class. One says that ν^k is standard if the exotic class cannot occur in $w(\nu^k)$; otherwise, ν^k is nonstandard. Also, one uses the convention that

(1) h is even and j is odd.
(2) $m = j+q = h+2i+k$.

2. Main results

From Section 1, one needs to consider the existence of $(\bar{M}^{j+q}, \bar{T})$ fixing $\mathbb{RP}^{q-1}$ with $w(\nu^{j+1}) = (1+\alpha)^{j+1}$ and $P(h,i)$ with normal bundle ν^k.

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Recently, Stong [St2] gave the complete analysis of equivariant cobordism classes of involutions fixing a disjoint union of a point and a Dold manifold, so that the case $q=1$ can be settled. If $q=1$, by Stong's work one has that

(1) If ν^k is standard, then $(\bar{M}^{j+1},\bar{T})$ fixing $\mathbb{RP}^0 \sqcup P(h,i)$ exists only for $(j+1,h,i,k)=(2^{s+1},2^s-2,1,2^s)$ with $s>1$ and $w(\nu^{2^s})=(1+c)^{2^s-1}(1+c+d)$, and is cobordant to the involution $\underbrace{(\mathbb{RP}^2,T_1)\times\cdots\times(\mathbb{RP}^2,T_1)}_{2^s}$ where $T_1[x_0,x_1,x_2]=[x_0,x_1,-x_2]$.

(2) If ν^k is nonstandard, then all cases in which $(\bar{M}^{j+1},\bar{T})$ exists are

(i) $(j+1,h,i,k)=(2^{s+1},2^s-2,1,2^s)$ with $s=2,3$ and the normal bundle ν^{2^s} is cobordant to the bundle with $1+c^{2^s-2}d$ as its total characteristic class; further, $(\bar{M}^{2^{s+1}},\bar{T})$ is cobordant to $\underbrace{(\mathbb{RP}^2,T_1)\times\cdots\times(\mathbb{RP}^2,T_1)}_{2^s}$.

(ii) $(j+1,h,i,k)=(2^{t+2},2,2^t-1,2^{t+1})$ with $t>1$ and $\nu^{2^{t+1}}$ is cobordant to the bundle with $1+\frac{c^2d}{1+d}$ as the total class; further $(\bar{M}^{2^{t+2}},\bar{T})$ is cobordant to $\underbrace{(\mathbb{RP}^2,T_1)\times\cdots\times(\mathbb{RP}^2,T_1)}_{2^{t+1}}$.

(iii) $(j+1,h,i,k)=(16,4,2,8)$ and $w(\nu^8)=(1+c)^4(1+c+d)^3(1+c^4d^2)$; further, $(\bar{M}^{16},\bar{T})$ is cobordant to

$$(\underbrace{\mathbb{RP}^2\times\cdots\times\mathbb{RP}^2}_{8},\underbrace{T_1\times\cdots\times T_1}_{8})\sharp(P(4,2)\times P(4,2),\text{twist}),$$

where $\sharp$ denotes the equivariant connected sum along a fixed point set. And there may be involutions for $(j+1,h,i)=(8+k,4,2)$ with $k=6,7$ and $w(\nu^k)=(1+c)^4(1+c+d)^3(1+c^4d^2)$, but cobordism methods cannot settle this as pointed out by Stong [St2].

Notice that when $(h,i)=(4,2)$, the Euler characteristic $\chi(P(4,2))=1$ and so

$$\chi(M^{j+1=8+k})=\chi(\mathbb{RP}^j)+\chi(P(4,2))=1$$

and k must be even.

Thus one has

Theorem 2.1. *Suppose (M^{j+1},T) fixes $\mathbb{RP}^j$ with normal bundle ν^1 having $w(\nu^1)=1+\alpha$ and $P(h,i)$ with normal bundle ν^k. Then (M^{j+1},T) exists only when $(j+1,h,i,k)$ and $w(\nu^k)$ are chosen as the four cases stated as above. More precisely,*

(1) *When $(j+1,h,i,k)=(2^{s+1},2^s-2,1,2^s)$ with $s>1$ and $w(\nu^{2^s})=(1+c)^{2^s-1}(1+c+d)$, $(M^{2^{s+1}},T)$ is cobordant to the involution $(\underbrace{\mathbb{RP}^2\times\cdots\times\mathbb{RP}^2}_{2^s},\underbrace{T_1\times\cdots\times T_1}_{2^s})\sqcup(\mathbb{RP}^{2^{s+1}},T_1)$.*

(2) *When $(j+1,h,i,k)=(2^{s+1},2^s-2,1,2^s)$ with $s=2,3$ and the normal bundle ν^{2^s} is cobordant to the bundle with $1+c^{2^s-2}d$ as its total characteristic class, $(M^{2^{s+1}},T)$ is cobordant to $(\underbrace{\mathbb{RP}^2\times\cdots\times\mathbb{RP}^2}_{2^s},\underbrace{T_1\times\cdots\times T_1}_{2^s})\sqcup(\mathbb{RP}^{2^{s+1}},T_1)$.*

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(3) *When* $(j+1, h, i, k) = (2^{t+2}, 2, 2^t - 1, 2^{t+1})$ *with* $t > 1$ *and* $\nu^{2^{t+1}}$ *is cobordant to the bundle with* $1 + \frac{c^2 d}{1+d}$ *as the total class,* $(M^{2^{t+2}}, T)$ *is cobordant to*

$$(\underbrace{\mathbb{RP}^2 \times \cdots \times \mathbb{RP}^2}_{2^{t+1}}, \underbrace{T_1 \times \cdots \times T_1}_{2^{t+1}}) \sqcup (\mathbb{RP}^{2^{t+2}}, T_1).$$

(4) *When* $(j+1, h, i, k) = (16, 4, 2, 8)$ *and* $w(\nu^8) = (1+c)^4(1+c+d)^3(1+c^4d^2)$, (M^{16}, T) *is cobordant to*

$$(\underbrace{\mathbb{RP}^2 \times \cdots \times \mathbb{RP}^2}_{8}, \underbrace{T_1 \times \cdots \times T_1}_{8}) \sharp (P(4,2) \times P(4,2), twist) \sqcup (\mathbb{RP}^{16}, T_1)$$

and there may be an involution for $(j+1, h, i, k) = (14, 4, 2, 6)$ *and* $w(\nu^6) = (1+c)^4(1+c+d)^3(1+c^4d^2)$.

For the case $q > 1$, one will see that there are also involutions with ν^k nonstandard. If ν^k is standard with $q > 1$, the following examples will help us finish the argument.

Now let us construct some involutions fixing $\mathbb{RP}^h \sqcup P(h, i)$ with h even. On page 3 of the paper [Lü], replace j by h to obtain the desired involutions.

First, write $i = 2^u(2v+1)$ and let

$$k_0 = \begin{cases} 2^u + 1 & \text{if } u = 1, \\ 2^u & \text{if } u \neq 1. \end{cases}$$

By the work of Perhger and Stong [PS], there is an involution (N^{i+l}, T_l) with $1 \le l \le k_0$ having fixed point set $* \sqcup \mathbb{RP}^i$, with the normal bundle of $\mathbb{RP}^i$ in N^{i+l} being $\iota \oplus (l-1)\mathbb{R}$, with ι the nontrivial line bundle, where $*$ denotes a point. This is constructed by applying the operation Γ $l-1$ times to the involution $(\mathbb{RP}^{i+1}, T_1)$ defined by

$$T_1([x_0, x_1, \ldots, x_{i+1}]) = [-x_0, x_1, \ldots, x_{i+1}],$$

which fixes $\mathbb{RP}^0 \sqcup \mathbb{RP}^i$ with the normal bundle ι on $\mathbb{RP}^i$, and cobording away various bounding fixed components (see [Ro]).

Then one may obtain the involution $T_{N^{i+l}}$ on

$$P(h, N^{i+l}) = \frac{S^h \times N^{i+l} \times N^{i+l}}{-1 \times \text{twist}}$$

induced by $1 \times T_l \times T_l$, whose fixed data are $\mathbb{RP}^h$ with normal bundle $\nu^{2i+2l} = (i+l)\iota \oplus (i+l)\mathbb{R}$ having $w(\nu^{2i+2l}) = (1+\alpha)^{i+l}$ and $P(h, i)$ with normal bundle $\nu^{2l} = \eta \oplus (l-1)\xi \oplus (l-1)\mathbb{R}$, where ξ induced by ι is a 1-plane bundle over $P(h, i)$, and η is a 2-plane bundle over $P(h, i)$. From [Do] and [Uc], one knows that $w(\xi) = 1 + c$ and $w(\eta) = 1 + c + d$ so $w(\nu^{2l}) = (1+c)^{l-1}(1+c+d)$.

In a similar way to [Lü], one may conclude that for $1 \le l < k_0$, $P(h, N^{i+l})$ bounds, and for $l = k_0$, $P(h, N^{i+k_0})$ does not bound. Also, by applying the operation Γ and its inverse operation Γ^{-1} to $(P(h, N^{i+l}), T_{N^{i+l}})$, one may obtain more involutions, each of which has the same fixed information as $(P(h, N^{i+l}), T_{N^{i+l}})$.

Proposition 2.2. *For* $1 \le l \le k_0$ *with* $i + l$ *even, there exist integers* X_1 *and* X_2 *with* $X_1 < X_2$ *such that* $\Gamma^{x-2l}(P(h, N^{i+l}), T_{N^{i+l}})$ *has the same fixed information as* $(P(h, N^{i+l}), T_{N^{i+l}})$ *for* $x \in [X_1, X_2]$, *but not for* $x \notin [X_1, X_2]$, *where* $X_1 = l + 1$ *if* $l - 1 \le h$ *and* $2 \le X_1 \le h + 2$ *if* $l - 1 > h$, *and* $X_2 \le 2k_0$.

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Proof. If $l-1 \le h$, then the normal bundle to the fixed point set of $(P(h,N^{i+l}), T_{N^{i+l}})$ has only $l-1$ sections, and $w_{l+1}(\nu^{2l}) = c^{l-1}d \neq 0$. Hence, one may apply the inverse operation Γ^{-1} at most $l-1$ times to $(P(h,N^{i+l}), T_{N^{i+l}})$, so that $\Gamma^{(l+1)-2l}(P(h,N^{i+l}), T_{N^{i+l}})$ has the same fixed information as $(P(h,N^{i+l}), T_{N^{i+l}})$, and thus $X_1 = l+1$.

If $l-1 > h$, since stability says every vector bundle over $\mathbb{RP}^h$ is realizable by an h-plane bundle, one has that $(l-1)\xi$ induced from a bundle $(l-1)\iota$ over $\mathbb{RP}^h$ can be realized by an h-plane bundle over $P(h,i)$. Thus, there is an integer $X_1 \le h+2$ such that the normal bundle to the fixed point set of $(P(h,N^{i+l}), T_{N^{i+l}})$ has just $2l-X_1$ sections, and $w_{X_1}(\nu^{2l}) \neq 0$. Furthermore, one can only apply the inverse operation Γ^{-1} at most $2l-X_1$ times to $(P(h,N^{i+l}), T_{N^{i+l}})$ such that the resulting involution $\Gamma^{X_1-2l}(P(h,N^{i+l}), T_{N^{i+l}})$ has the same fixed information as $(P(h,N^{i+l}), T_{N^{i+l}})$.

According to the property of the operation Γ, one can always apply the operation Γ to $(P(h,N^{i+l}), T_{N^{i+l}})$; in particular, there must exist an integer X_2 such that

$$\Gamma^{X_2-2l}(P(h,N^{i+l}), T_{N^{i+l}})$$

has the same fixed information as $(P(h,N^{i+l}), T_{N^{i+l}})$, so for $x \in [X_1, X_2]$,

$$\Gamma^{x-2l}(P(h,N^{i+l}), T_{N^{i+l}})$$

has the same fixed information as $(P(h,N^{i+l}), T_{N^{i+l}})$. Consider $\Gamma^{x-2l}(P(h,N^{i+l}), T_{N^{i+l}})$ with $x \in [X_1, X_2]$; since $i+l$ is odd and h is even, by direct computation, one has

$$w[0]_1 = \begin{cases} c & \text{on } P(h,i), \\ \alpha & \text{on } \mathbb{RP}^h. \end{cases}$$

Dualizing $w[0]_1^h : \mathfrak{N}_{h+2i+x}(B\mathbb{Z}_2) \longrightarrow \mathfrak{N}_{2i+x}(B\mathbb{Z}_2)$, one obtains an involution (D^{2i+x}, E) fixing a point and a complex projective space $\mathbb{CP}^i$ with $1+d$ as its total class of normal bundle, so $2 \le x \le 2k_0$ follows from the arguments in [Ro] and [PS]. □

Now let us state the result for the case $q > 1$.

Theorem 2.3. *Suppose (M^{j+q}, T) fixes $\mathbb{RP}^j$ with normal bundle ν^q having $w(\nu^q) = (1+\alpha)^q$ with odd $q > 1$ and $P(h,i)$ with normal bundle ν^k having $w(\nu^k) = (1+c)^a(1+c+d)^b w(\rho)^\varepsilon$. Let $2^A \le h < 2^{A+1}$ and write $i = 2^u(2v+1)$. Then*

(1) *$q = h+1$ and $j+1 = 2i+k$ so k is even.*

(2) *$i+a$ is odd.*

(3) *$b = 1$ and $a < 2^u$.*

(4) *$j+1 \equiv i+a+1 \mod 2^{A+1}$ and $i+k \equiv a+1 \mod 2^{A+1}$. In particular,*

(*a*) *for $u \le A$, $k = 2^u + a + 1$ and $2^{u+1}(v+1) \equiv 0 \mod 2^{A+1}$;*

(*b*) *for $u > A$, $k \equiv a+1 \mod 2^{A+1}$.*

(5) *(M^{j+q}, T) with nonstandard ν^k exists only for $(h,u,k,a) = (2,1,4,1)$ and $w(\nu^4) = (1+c)(1+c+d)(1+\frac{c^2d}{1+d})$.*

(6) *(M^{j+q}, T) with standard ν^k exists for k in a range $X_1 \le k \le X_2$, and is cobordant to*

$$\Gamma^{k-2a-2}(P(h,N^{i+a+1}), T_{N^{i+a+1}}) \sqcup (\mathbb{RP}^{j+h+1}, T_{h+1})$$

where $2 \le X_1, X_2 \le 2k_0$ and more precisely

(*a*) *for $u \le A$, $X_1 = a+2$ and $X_2 \le 2^u + a + 1$;*

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(*b*) *for* $u > A$, $2 \le X_1 \le h+2$ *and* $X_2 \le 2^{u+1} - (h - \text{common } (h,a))$ *where common* (h,a) *is the common part of the 2-adic expansions of* h *and* a.

Observation. The upper bound of X_2 stated in Theorem 2.3 is attainable in some special cases. For example, take $a = 2^u - 1$; if $u \le A$, then $X_2 = 2^{u+1}$, and if $u > A$, then common $(h,a) = h$ so $X_2 = 2^{u+1}$. Proposition 2.2 shows that l can take the value 2^u, so there exists an involution $(P(h, N^{i+2^u}), T_{N^{i+2^u}})$ which has dimension $h + 2i + 2^{u+1}$ and fixes $\mathbb{RP}^h$ with $w(\nu^{2i+2^{u+1}}) = (1+\alpha)^{i+2^u}$ and $P(h,i)$ with $w(\nu^{2^u}) = (1+c)^{2^u-1}(1+c+d)$, and thus (M^{j+h+1}, T) fixing $\mathbb{RP}^j \sqcup P(h,i)$ exists when $a = 2^u - 1$ and $k = 2^u + a + 1$ and ν^k is standard. For the general case, the proof that the upper bound of X_2 stated as above is attainable seems to be quite difficult. Also, one merely shows that there exists (M^{j+q}, T) with nonstandard ν^k only for $(h,u,k,a) = (2,1,4,1)$ and $w(\nu^4) = (1+c)(1+c+d)(1+\frac{c^2 d}{1+d})$. Since the normal bundle ν^4 is nonstandard, one does not know how to construct an explicit example of such an involution.

3. The proof of Theorem 2.3

This section is devoted to proving Theorem 2.3. Following the notations of Sections 1 and 2, one always assumes in this section that $q > 1$.

Let $2^A \le h < 2^{A+1}$ and $2^B \le i < 2^{B+1}$. Then one may assume that $a < 2^{A+1}$ and $b < 2^{B+1}$, since a (resp. b) is only determined modulo 2^{A+1} (resp. 2^{B+1}). Let $C = \max\{A+1, B+1\}$.

Lemma 3.1. *If* $q > 1$, *then*
(1) $h = q - 1$ *and* $j + 1 = 2i + k$;
(2) $b = 1$;
(3) $i + a$ *is odd;*
(4) *for* $i = 2^u(2v+1)$, *one has*

$$2 \le k \le \begin{cases} 2^{u+1} + 2 & \text{if } u = 1, \\ 2^{u+1} & \text{if } u \ne 1. \end{cases}$$

Proof. From (1.1) one has

$$w[0]_1 = \begin{cases} (i+1+a+b)c & \text{on } P(h,i), \\ e + \alpha & \text{on } \mathbb{RP}^j. \end{cases}$$

Thus one may form the characteristic number for

$$w[0]_1^{q-1} e^{m-1-(q-1)} = (e+\alpha)^{q-1} e^j$$

whose value on $\mathbb{RP}(\nu^q)$ is equal to the coefficient of α^j in $\frac{(1+\alpha)^{q-1}}{(1+\alpha)^q} = \frac{1}{1+\alpha}$ and that coefficient is nonzero. On $\mathbb{RP}(\nu^k)$

$$w[0]_1^{q-1} e^{m-1-(q-1)} = (i+1+a+b)c^{q-1} e^j$$

and the value of this on $\mathbb{RP}(\nu^k)$ must be nonzero. Thus

$$h \ge q - 1 \tag{3.1}$$

and $i + 1 + a + b$ is odd so

$$w[0]_1 = \begin{cases} c & \text{on } P(h,i), \\ e + \alpha & \text{on } \mathbb{RP}^j. \end{cases}$$

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Now one begins with the proof of (1). If $w(\nu^k)$ is standard, then one can write

$$w(\nu^k) = (1+c)^a(1+c+d)^b.$$

Furthermore, one has

$$\begin{aligned} 0 &\neq w[0]_1^{q-1}e^j[\mathbb{RP}(\nu^j)] = \frac{1}{1+\alpha}[\mathbb{RP}^j] \\ &= w[0]_1^{q-1}e^j[\mathbb{RP}(\nu^k)] = \frac{c^{q-1}}{(1+c)^a(1+c+d)^b}[P(h,i)] \\ &= \frac{c^{q-1}}{(1+c)^a}(1+c+d)^{2^C-b}[P(h,i)] \\ &= \binom{2^C-b}{i}\frac{c^{q-1}d^i}{(1+c)^{a+b+i}}[P(h,i)] \end{aligned}$$

so $\binom{2^C-b}{i} \equiv 1 \mod 2$. Further, one has that

$$\begin{aligned} w[0]_1^h e^{m-1-h}[\mathbb{RP}(\nu^k)] &= c^h e^{m-1-h}[\mathbb{RP}(\nu^k)] \\ &= \frac{c^h}{(1+c)^a(1+c+d)^b}[P(h,i)] \\ &= \frac{c^h}{(1+d)^b}[P(h,i)] \\ &= \binom{2^C-b}{i}c^h d^i[P(h,i)] \\ &= 1 \end{aligned}$$

so

$$w[0]_1^h e^{m-1-h}[\mathbb{RP}(\nu^q)] = (e+\alpha)^h e^{m-1-h}[\mathbb{RP}(\nu^q)] = \frac{(1+\alpha)^h}{(1+\alpha)^q}[\mathbb{RP}^j]$$

must be nonzero. Since $h-q < m-q = (j+q)-q = j$, one must have $h-q<0$ so $h \leq q-1$. By (3.1), one concludes that $h = q-1$.

If $w(\nu^k)$ is nonstandard, then h must be 2, 4 or 6 since h is even.

When $h=2$, since $q>1$ is odd and $h \geq q-1$, one has that $q=3$, i.e., $h=q-1$.

When $h=4$, one has the exotic class

$$w(\rho) = 1 + \frac{c^4d^2}{(1+d)^2}$$

by [St1] and from $h = 4 \geq q-1$, the possible values for odd $q>1$ are only 3 or 5.

If $q=3$, the value of $w[0]_1^2e^{m-3}$ on $\mathbb{RP}(\nu^q)$ is nonzero, so the value of this on $\mathbb{RP}(\nu^k)$ must be nonzero, and thus

$$\begin{aligned} 0 &\neq \frac{c^2}{(1+c)^a(1+c+d)^b(1+\frac{c^4d^2}{1+d^2})}[P(4,i)] \\ &= \frac{c^2}{(1+c)^a(1+c+d)^b}\{1+\frac{c^4d^2}{1+d^2}\}[P(4,i)] \\ &= \frac{c^2}{(1+c)^a(1+c+d)^b}[P(h,i)] \text{ (since } c^6=0) \\ &= \binom{2^C-b}{i}\frac{c^2d^i}{(1+c)^{a+b+i}}[P(4,i)] \end{aligned}$$

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so

$$\binom{2^C - b}{i} \equiv 1 \mod 2.$$

Now one has that

$$\begin{aligned} w[0]_1^4 e^{m-5}[\mathbb{RP}(\nu^k)] &= \frac{c^4}{(1+c)^a(1+c+d)^b(1+\frac{c^4d^2}{1+d^2})}[P(4,i)] \\ &= \frac{c^4}{(1+d)^b}[P(4,i)] \\ &= \binom{2^C-b}{i} c^4 d^i [P(4,i)] \\ &= \binom{2^C-b}{i} \\ &= 1 \end{aligned}$$

but

$$w[0]_1^4 e^{m-5}[\mathbb{RP}(\nu^q)] = \frac{(1+\alpha)^4}{(1+\alpha)^3}[\mathbb{RP}^j] = (1+\alpha)[\mathbb{RP}^j] = 0$$

since $j = m - q = m - 3 = 1 + m - h = 1 + 2i + k > 1$. This is a contradiction. Thus one has that when $h = 4$, then q must be 5, i.e., $h = q - 1$.

When $h = 6$, one has the exotic class

$$w(\rho) = 1 + \frac{c^6 d}{(1+d)^4} + \frac{c^4d^2}{(1+d)^2}$$

by [St1] and from the fact $h \geq q-1$, the possible values for odd $q > 1$ are 3, 5 or 7.

If $q = 5$, dualizing $w[0]_1^2$ may reduce this case to the case $h = 4$ with $q = 3$, which does not exist as above, so one obtains that $q = 5$ is impossible.

If $q = 3$, up to cobordism, one may take $\nu^{m-j} = q\iota$, where ι is the nontrivial line bundle over $\mathbb{RP}^j$. Dualizing $w[0]_1^2$, $\mathbb{RP}(3\iota)$ becomes $\mathbb{RP}((3-2)\iota) = \mathbb{RP}(\iota)$ and $\mathbb{RP}(\nu^k)$ becomes $\mathbb{RP}(\nu^k|_{P(4,i)})$. Thus one has that if (M^m, T) fixing $\mathbb{RP}^j$ with $w(\nu^{m-j}) = (1+\alpha)^{q=3}$ and $P(6,i)$ with

$$w(\nu^k) = (1+c)^a(1+c+d)^b\{1 + \frac{c^6d}{(1+d)^4} + \frac{c^4d^2}{(1+d)^2}\}$$

exists, then there is an involution (M^{m-2}, T) fixing $\mathbb{RP}^j$ with $w(\nu^{m-j-2}) = 1+\alpha$ and $P(4,i)$ with normal bundle $\nu^k|_{P(4,i)}$ having

$$w(\nu^k|_{P(4,i)}) = (1+c)^a(1+c+d)^b\{1 + \frac{c^4d^2}{(1+d)^2}\}$$

since $c^6 = 0$ in this case. Note that $m - 2 = j + 3 - 2 = j + 1$.

As in Section 1, consider the involution

$$(\bar{M}^{j+1}, \bar{T}) = (M^{m-2}, T) \sqcup (\mathbb{RP}^{j+1}, T_1);$$

this involution fixes a point $*$ and $P(4,i)$ with normal bundle ν^k having $w(\nu^k) = (1+c)^a(1+c+d)^b\{1 + \frac{c^4d^2}{(1+d)^2}\}$. Since $i + 1 + a + b$ is odd, by direct computation one has

$$w[0]_1 = \begin{cases} c & \text{on } P(4,i), \\ 0 & \text{on } *. \end{cases}$$

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If i is odd, since the characteristic class $e^{m-3=j}$ on $*$ has nonzero value, one has that

$$e^j[\mathbb{RP}(\nu^k)] = \frac{1}{(1+c)^a(1+c+d)^b\{1+\frac{c^4d^2}{(1+d)^2}\}}[P(4,i)]$$

must be nonzero. This forces b to be odd, so on $P(4,i)$, when multiplied by $w[0]_1^4$,

$$w[0] \sim (1+d)^{i+1}(1+\frac{d}{1+e^2}+\cdots)$$

and

$$w[0]_2 \sim d.$$

Thus, one has that

$$w[0]_1^4 w[0]_2^i e^{k-1}[\mathbb{RP}(\nu^k)] = c^4 d^i [P(4,i)] = 1$$

but the value of $w[0]_1^4 w[0]_2^i e^{k-1}$ on $*$ is zero. This is a contradiction. Thus, i must be even so $a+b$ is even.

Dualizing $w[0]_1^4$ for $(\bar{M}^{j+1},\bar{T})$, one obtains an involution (X^{2i+k},T) fixing $P(0,i)$ $=\mathbb{CP}^i$ with $k>0$ and with normal bundle $\nu^k|_{\mathbb{CP}^i}$ having $w(\nu^k|_{\mathbb{CP}^i})=(1+d)^b$. The involutions fixing $\mathbb{CP}^i$ are well known and one has $k=2i$ and $b=i+1$ with (X^{2i+k},T) being cobordant to $(\mathbb{CP}^i\times\mathbb{CP}^i,\text{twist})$.

Returning to the involution $(\bar{M}^{j+1},\bar{T})$, from

$$\begin{aligned} w(\nu^k|_{P(4,i)}) &= (1+c)^a(1+c+d)^b\{1+\frac{c^4d^2}{(1+d)^2}\} \\ &= (1+c)^a(1+c+d)^{i+1}\{1+\frac{c^4d^2}{(1+d)^2}\} \\ &= (1+c)^a(1+c+d)^{i+1}+c^4d^2(1+d)^{i-1} \end{aligned}$$

one has that the coefficient of d^i in $w(\nu^k|_{P(4,i)})$ is

$$\binom{i+1}{i}(1+c)^{a+1}+\binom{i-1}{i-2}c^4=(1+c)^{a+1}+c^4,$$

and since $k=2i$, this must be 1, so $a=3$. Then the coefficient of d^{i-1} in $w(\nu^k|_{P(4,i)})$ is

$$\binom{i+1}{i-1}(1+c)^5+\binom{i-1}{i-3}c^4=\binom{i+1}{i-1}(1+c)+\{\binom{i+1}{i-1}+\binom{i-1}{i-3}\}c^4.$$

Since $\binom{i+1}{i-1}+\binom{i-1}{i-3}=\binom{i+1}{2}+\binom{i-1}{2}=1$, one has that $w_{2i+2}(\nu^k|_{P(4,i)})$ contains the nonzero term c^4d^{i-1} so $k\geq 2i+2$. This is impossible since $k=2i$. Thus one concludes that $q=3$ is impossible, so q must be 7 when $h=6$, i.e., $h=q-1$.

Combining the above arguments, one has that if $q>1$, then $h=q-1$ so $j+1=2i+k$ by the Fact in Section 1. This completes the proof of (1).

(2)–(4) follow from the following arguments. Consider the involution (M^{j+h+1},T) fixing $\mathbb{RP}^j$ with normal bundle $\nu^{h+1}=(h+1)\iota$ and $P(h,i)$ with normal bundle ν^k; dualizing $w[0]_1^h$ gives an involution $(M^{j+1=2i+k},T)$ fixing $\mathbb{RP}^j$ with normal bundle ν^1 having $w(\nu^1)=1+\alpha$ and $P(0,i)=\mathbb{CP}^i$ with normal bundle of $\mathbb{CP}^i$ being $\nu^k|_{\mathbb{CP}^i}$ with $w(\nu^k)=(1+d)^b$. Taking the union of $(\mathbb{RP}^{j+1},T_1)$ and (M^{j+1},T), one obtains an involution $(\bar{M}^{j+1},\bar{T})$ fixing a point and $\mathbb{CP}^i$ with $w(\nu^k)=(1+d)^b$. Royster's

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argument [Ro] for involutions fixing a point and a real projective space also works for fixing a point and a complex projective space to give

$$b = 1.$$

Furthermore, one knows that $a + i$ is odd and the possible values of k by [PS]. Writing $i = 2^u(2v + 1)$ one has

$$2 \leq k \leq \begin{cases} 2^{u+1} + 2 & \text{if } u = 1, \\ 2^{u+1} & \text{if } u \neq 1. \end{cases}$$

This completes the proof. □

Now one divides the argument into the following cases:
(I) ν^k is standard;
(II) ν^k is nonstandard.

In the following discussions, one always assumes that (M^{j+q}, T) fixing $\mathbb{RP}^j \sqcup P(h, i)$ with $q > 1$ satisfies (1)–(4) stated in Lemma 3.1.

(I) The case in which ν^k is standard. One is first concerned with the case in which ν^k is standard. Then one can write

$$w(\nu^k) = (1 + c)^a(1 + c + d).$$

Based upon the examples in Section 2, one has

Lemma 3.2. *If $q > 1$, then $j+1 \equiv a+i+1 \mod 2^{A+1}$ so $i+k \equiv a+1 \mod 2^{A+1}$.*

Proof. Let $x \equiv j+1 \mod 2^{A+1}$ and $y \equiv a+i+1 \mod 2^{A+1}$. Then $x \leq j+1 < m$. One claims that $y < m$. If $i \geq 2^A$, then $m = h + 2i + k > 2i \geq 2^{A+1} > y$. If $i < 2^A$ and $a \geq 2^A$, then $w_{2^A+2}(\nu^k) = \binom{a+1}{2^A+2}c^{2^A+2} + c^{2^A}d \neq 0$ so $k \geq 2^A + 2$ and

$$m = h + 2i + k > h + k > h + 2^A \geq 2^{A+1} > y.$$

If $i < 2^A$ and $a < 2^A$, then $y = a + i + 1$ and $k \geq a + 2$, so

$$m = h + 2i + k > i + k > i + a + 1 = y.$$

By direct compuations, one has

$$w[0]_1 = \begin{cases} \alpha & \text{on } \mathbb{RP}^h, \\ c & \text{on } P(h, i). \end{cases}$$

The argument proceeds as follows.

(1) If $x < y$ then $y - (x + 1) \geq 0$. When $0 \leq y - (x + 1) < h$, one has that

$$\begin{aligned} (w[0]_1 + e)^{y-1}e^{m-y}[\mathbb{RP}(\nu^k)] &= \frac{(1+c)^{y-1}}{(1+c)^a(1+c+d)}[P(h,i)] \\ &= \frac{(1+c)^{y-1}}{(1+c)^{a+1}}\frac{1}{1+\frac{d}{1+c}}[P(h,i)] \\ &= \frac{(1+c)^{y-1}}{(1+c)^{a+i+1}}d^i[P(h,i)] \\ &= \frac{d^i}{1+c}[P(h,i)] \\ &= 1 \end{aligned}$$

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but

$$\begin{aligned}(w[0]_1+e)^{y-1}e^{m-y}[\mathbb{RP}(\nu^{j+1})] &= \frac{(1+\alpha)^{y-1}}{(1+\alpha)^{j+1}}[\mathbb{RP}^h] \\ &= (1+\alpha)^{y-x-1}[\mathbb{RP}^h] \\ &= 0,\end{aligned}$$

which leads to a contradiction. When $h \le y-(x+1) < 2^{A+1}$, one has that

$$\begin{aligned}(w[0]_1+e)^{x-1}e^{m-x}[\mathbb{RP}(\nu^{j+1})] &= \frac{(1+\alpha)^{x-1}}{(1+\alpha)^{j+1}}[\mathbb{RP}^h] \\ &= \frac{(1+\alpha)^{x-1}}{(1+\alpha)^{x}}[\mathbb{RP}^h] \\ &= \frac{1}{1+\alpha}[\mathbb{RP}^h] \\ &= 1\end{aligned}$$

but

$$\begin{aligned}(w[0]_1+e)^{x-1}e^{m-x}[\mathbb{RP}(\nu^{k})] &= \frac{(1+c)^{x-1}}{(1+c)^a(1+c+d)}[P(h,i)] \\ &= \frac{(1+c)^{x-1}}{(1+c)^{a+i+1}}d^i[P(h,i)] \\ &= \frac{d^i}{(1+c)^{y-x+1}}[P(h,i)] \\ &= \binom{2^{A+1}-1-y+x}{h}c^hd^i[P(h,i)] \\ &= 0\end{aligned}$$

since $2^{A+1}-1-y+x = 2^{A+1}-2-(y-x-1) \le 2^{A+1}-2-h < h$.

Thus, $x<y$ is impossible.

(2) If $x>y$, in a similar way to (1), one may obtain that this case is impossible, too.

Combining the above arguments, one has that $x=y$. Furthermore, one has that $i+k \equiv a+1 \mod 2^{A+1}$ by Lemma 3.1(1). □

Now, the argument is divided into the following three cases: (i) $u=0$ (i.e., i is odd); (ii) $u=1$; (iii) $u>1$.

(i) The case $u=0$. If $u=0$, then $i=2v+1$ is odd , so a is even by Lemma 3.1(3). By Lemma 3.1(4), one has that $k=2$ so $a=0$ and $w(\nu^2)=1+c+d$. Also, since the Euler characteristic $\chi(P(h,2v+1)) \equiv 0 \mod 2$, one has

$$\chi(\bar{M}^{j+h+1}) = \chi(\mathbb{RP}^h) + \chi(P(h,2v+1)) \equiv 1 \mod 2.$$

By Lemma 3.2, one has $(1+\alpha)^{2i+2} = (1+\alpha)^{j+1} = (1+\alpha)^{i+1}$ so $(1+\alpha)^{i+1}=1$ and $i+1 \equiv 0 \mod 2^{A+1}$. On the other hand, for the involution $(P(h,N^{i+l}),T_{N^{i+l}})$, take $i=2v+1$ with $i+1\equiv 0 \mod 2^{A+1}$ and $l=1$; then $(P(h,N^{2v+2}),T_{N^{2v+2}})$ has the fixed data $\mathbb{RP}^h$ with $w(\nu^{j+1}) = (1+\alpha)^{i+1} = 1$ and $P(h,2v+1)$ with $w(\nu^2)=1+c+d$, so $(\bar{M}^{j+h+1},\bar{T})$ fixing $\mathbb{RP}^h \sqcup P(h,2v+1)$ exists and it is cobordant to $(P(h,N^{2v+2}),T_{N^{2v+2}})$ with $2v+2\equiv 0 \mod 2^{A+1}$. Thus one has

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Proposition 3.1. *Suppose that* (M^{j+h+1},T) *fixes* $\mathbb{RP}^j$ *with* $w(\nu^{h+1})=(1+\alpha)^{h+1}$ *and* $P(h,2v+1)$ *with* $w(\nu^k)=(1+c)^a(1+c+d)$. *Then* $(k,a)=(2,0)$ *and* $2v+2\equiv 0 \mod 2^{A+1}$. *Further,* (M^{j+h+1},T) *is cobordant to* $(P(h,N^{2v+2}),T_{N^{2v+2}})\sqcup(\mathbb{RP}^{j+h+1},T_{h+1})$.

Note that from Proposition 3.1, $(k,a)=(2,0)$ and $2v+2\equiv 0 \mod 2^{A+1}$ means $a<2^u$ and $i+k\equiv a+1 \mod 2^{A+1}$.

(ii) The case $u=1$. If $u=1$ then $i=4v+2$ is even so

$$\chi(M^{j+h+1})=\chi(\mathbb{RP}^j)+\chi(P(h,4v+2))\equiv 1 \mod 2$$

and k must be even. Further, by Lemma 3.1(4), $2\le k\le 6$. Also, by Lemma 3.1(3), a is odd and by Lemma 3.2, $a\equiv 3 \mod 4$ if $k=2$ or 6 and $a\equiv 1 \mod 4$ if $k=4$.

First, one cannot have $a>8$. For $a\ge 9$, one must have $h\ge 8$ and a must have a power of 2, $8\le 2^A\le h$, in its 2-adic expansion. Then there is at least a nonzero term $w_s(\nu^k)$ with $s>6$ in $w(\nu^k)=(1+c)^a(1+c+d)$. This is impossible since $2\le k\le 6$.

If $a\equiv 3 \mod 4$ (i.e., $a=3$ or 7), from

$$w[0]_1=\begin{cases} c & \text{on } P(h,4v+2),\\ e+\alpha & \text{on } \mathbb{RP}^j,\end{cases}$$

dualizing $w[0]_1^{h-2}$, the problem is reduced to determine the existence of the involution (M^{j+3},T) fixing $\mathbb{RP}^j$ with $w(\nu^3)=(1+\alpha)^3$ and $P(2,4v+2)$ with $w(\nu^k)=(1+c)^a(1+c+d)=(1+c)^3(1+c+d)$. Obviously, $k=2$ is impossible, so k must be 6. Note that $j+1=2i+k=8v+10$ by Lemma 3.1(1). Taking the union $(M^{8v+12},T)\sqcup(\mathbb{RP}^{8v+12},T_3)$, one needs only to determine the existence of the involution $(\bar M^{8v+12},\bar T)$ fixing $\mathbb{RP}^2$ with $w(\nu^{8v+10})=(1+\alpha)^{8v+10}=(1+\alpha)^2$ and $P(2,4v+2)$ with $w(\nu^6)=(1+c)^3(1+c+d)$. From

$$w[4]=\begin{cases} (1+c)^2(1+c+d)^{4v+3}\frac{(1+c+e)^3(1+c+e^2+ce+d)}{1+e} & \text{on } P(2,4v+2),\\ (1+\alpha)^3(1+e)^{8v+6}(1+e^2+\alpha^2) & \text{on } \mathbb{RP}^2,\end{cases}$$

one has that

$$w[4]_1=\begin{cases} c & \text{on } P(2,4v+2),\\ \alpha & \text{on } \mathbb{RP}^2.\end{cases}$$

When multiplied by $w[4]_1$ for $w[4]$, one has that on $P(2,4v+2)$

$$\begin{aligned} w[4] &\sim (1+d)^{4v+2}(1+c+d)(1+e)(1+c+e)(1+c+e^2+ce+d)\\ &\sim (1+d)^{4v}(1+c+e^4+cde+e^2d+ce^4+cde^2+e^4d+cd^2e+e^2d^2\\ &\quad +e^4d^2+cd^3e+e^2d^3+e^4cd^2+e^2cd^3+e^4d^3+d^4+cd^4+cd^4e+e^2d^4)\end{aligned}$$

so

$$w[4]_4\sim e^4+cde+e^2d$$

and

$$w[4]_{8v+6}\sim e^4d^{4v+1}+ced^{4v+2}+e^2d^{4v+2}.$$

On $\mathbb{RP}^2$,

$$w[4]\sim(1+\alpha)(1+e)^{8v+8}$$

so

$$w[4]_4\sim 0$$

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and

$$w[4]_{8v+6} \sim 0.$$

Therefore, one obtains that

$$w[4]_1 w[4]_4 w[4]_{8v+6}[\mathbb{RP}(\nu^{8v+10})] = 0$$

but

$$\begin{aligned}
& w[4]_1 w[4]_4 w[4]_{8v+6}[\mathbb{RP}(\nu^6)] \\
= \;& c(e^4 + cde + e^2 d)(e^4 d^{4v+1} + ced^{4v+2} + e^2 d^{4v+2})[\mathbb{RP}(\nu^6)] \\
= \;& \frac{c(1+cd+d)(d^{4v+1}+cd^{4v+2}+d^{4v+2})}{(1+c)^3(1+c+d)}[P(2,4v+2)] \\
= \;& \frac{cd^{4v+1}(1+cd+d)^2}{(1+c)(1+c+d)}[P(2,4v+2)] \\
= \;& \frac{cd^{4v+1}}{(1+c)(1+c+d)}[P(2,4v+2)] \\
= \;& \frac{cd^{4v+1}}{(1+c)^2(1+\frac{d}{1+c})}[P(2,4v+2)] \\
= \;& c^2 d^{4v+2}[P(2,4v+2)] \\
= \;& 1,
\end{aligned}$$

which gives a contradiction. Thus, there can be no involution $(M^{h+8v+4+k}, T)$ with $a \equiv 3 \mod 4$.

If $a = 5$, then k must be 4 and $h \geq 4$ so $c^4 \neq 0$. Since $w(\nu^k) = (1+c)^5(1+c+d) = (1+c)^6 + (1+c)^5 d$, one has $w_6(\nu^k) = c^4 d \neq 0$, and thus k is more than 4. This is a contradiction. Hence, $a = 5$ is impossible.

If $a = 1$, then k must be 4. In this case, if the involution (M^{h+8v+8}, T) fixing $\mathbb{RP}^{8v+7}$ with $w(\nu^{h+1}) = (1+\alpha)^{h+1}$ and $P(h, 4v+2)$ with $w(\nu^4) = (1+c)(1+c+d)$ exists, then the involution $(\bar{M}^{h+8v+8}, \bar{T})$ fixing $\mathbb{RP}^h$ with $w(\nu^{8v+8}) = (1+\alpha)^{8v+8}$ and $P(h, 4v+2)$ with $w(\nu^4) = (1+c)(1+c+d)$ exists. By Lemma 3.2, one has that $w(\nu^{8v+8}) = (1+\alpha)^{8v+8} = (1+\alpha)^{4v+4}$ so $4v+4 \equiv 0 \mod 2^{A+1}$ and $w(\nu^{8v+8}) = 1$. Now one looks at the involution $(P(h, N^{4v+2+l}), T_{N^{4v+2+l}})$, where l is restricted to $1 \leq l \leq 3$. Take $l = 2$; then $(P(h, N^{4v+4}), T_{N^{4v+4}})$ has the fixed data $\mathbb{RP}^h$ with $w(\nu^{8v+8}) = (1+\alpha)^{4v+4}$ and $P(h, 4v+2)$ with $w(\nu^4) = (1+c)(1+c+d)$, so $(\bar{M}^{h+8v+8}, \bar{T})$ is cobordant to $(P(h, N^{4v+4}), T_{N^{4v+4}})$. Further, one has that (M^{h+8v+8}, T) fixing $\mathbb{RP}^{8v+7}$ with $w(\nu^{h+1}) = (1+\alpha)^{h+1}$ and $P(h, 4v+2)$ with $w(\nu^4) = (1+c)(1+c+d)$ exists, and is cobordant to $(P(h, N^{4v+4}), T_{N^{4v+4}}) \sqcup (\mathbb{RP}^{h+8v+8}, T_{h+1})$.

Combining the arguments, one has

Proposition 3.2. *Suppose (M^{j+h+1}, T) fixes $\mathbb{RP}^j$ with $w(\nu^{h+1}) = (1+\alpha)^{h+1}$ and $P(h, 4v+2)$ with $w(\nu^k) = (1+c)^a(1+c+d)$. Then $(a,k) = (1,4)$ and $4v+4 \equiv 0$ mod 2^{A+1}. Further, (M^{j+h+1}, T) exists, and is cobordant to $(P(h, N^{4v+4}), T_{N^{4v+4}})$ $\sqcup\, (\mathbb{RP}^{h+8v+8}, T_{h+1})$.*

Note that from Proposition 3.2, $(a,k) = (1,4)$ and $4v+4 \equiv 0 \mod 2^{A+1}$ means $a < 2^u$ and $i + k \equiv a + 1 \mod 2^{A+1}$.

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(iii) The case $u > 1$. Since $u > 1$, one has that $i = 2^u(2v+1)$ is even so a is odd and

$$\chi(M^{j+h+1=h+2i+k}) = \chi(\mathbb{RP}^j) + \chi(P(h,i)) \equiv 1 \mod 2.$$

Thus k is even.

Consider the involution $(\bar{M}^{h+2i+k}, \bar{T})$ fixing $\mathbb{RP}^h$ with $w(\nu^{2i+k}) = (1+\alpha)^{2i+k} = (1+\alpha)^{a+i+1}$ and $P(h,i)$ with $w(\nu^k) = (1+c)^a(1+c+d)$. Then one computes the values of $w[0]_1$ and $w[0]_4$. On $P(h,i)$

$$w[0] = (1+c)^h(1+c+d)^{i+1}\{1 + \frac{\binom{a+1}{2}c^2 + d}{1+e^2} + \frac{cd}{(1+e)^3} + \frac{\binom{a+1}{4}c^4 + \binom{a}{2}c^2d}{1+e^4} + \cdots\}$$

so

$$w[0]_1 = c$$

and

$$\begin{aligned} w[0]_4 &= \binom{a+1}{2}c^2e^2 + de^2 + cde + \binom{a+1}{4}c^4 + \binom{a}{2}c^2d + c^2d \\ &\quad + \{\binom{h+i+1}{2}c^2 + d\}\{\binom{a+1}{2}c^2 + d\} + \binom{h+i+1}{4}c^4 + \binom{h+i}{2}c^2d \\ &= \binom{a+1}{2}c^2e^2 + \binom{a+1}{4}c^4 + \binom{h+1}{2}\binom{a+1}{2}c^4 + \binom{h+i+1}{4}c^4 \\ &\quad + de^2 + cde + d^2. \end{aligned}$$

On $\mathbb{RP}^h$,

$$w[0] = (1+\alpha)^{h+1}\{(1+e)^{2i} + \binom{a+i+1}{2}\alpha^2(1+e)^{2i-2} + \binom{a+i+1}{4}\alpha^4(1+e)^{2i-4} + \cdots\}$$

so

$$w[0]_1 = \alpha$$

and

$$w[0]_4 = \binom{a+1}{2}\alpha^2e^2 + \binom{a+i+1}{4}\alpha^4 + \binom{h+1}{2}\binom{a+1}{2}\alpha^4 + \binom{h+1}{4}\alpha^4.$$

Since

$$\binom{a+1}{4} + \binom{a+i+1}{4} + \binom{h+1}{4} + \binom{h+i+1}{4} \equiv 0 \mod 2,$$

one may form the class

$$\begin{aligned} \hat{w}_4 &= \binom{a+1}{2}w[0]_1^2e^2 + \binom{a+i+1}{4}w[0]_1^4 + \binom{h+1}{2}\binom{a+1}{2}w[0]_1^4 + \binom{h+1}{4}w[0]_1^4 \\ &= \begin{cases} de^2 + cde + d^2 & \text{on } P(h,i), \\ 0 & \text{on } \mathbb{RP}^h. \end{cases} \end{aligned}$$

Now suppose that $a \geq 2^u$. Then $2^{A+1} > a \geq 2^u$ so $A \geq u$.

If $a < h$, then one has $k \geq a+2$, so

$$a - 2^u + 2^{u+2}(v+1) = a - 2^u + 2i + 2^{u+1} = a + 2i + 2^u < h + 2i + k = m.$$

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Further, one has

$$\begin{aligned}
& (w[0]_1+e)^{a-2^u}\hat{w}_4^{2^u(v+1)}e^{m-1-a-2i-2^u}[\mathbb{RP}(\nu^k)] \\
= \; & \frac{(1+c)^{a-2^u}(d+cd+d^2)^{2^u(v+1)}}{(1+c)^a(1+c+d)}[P(h,i)] \\
= \; & \frac{d^{2^u(v+1)}(1+c+d)^{2^u(v+1)-1}}{(1+c)^{2^u}}[P(h,i)] \\
= \; & \binom{2^u(v+1)-1}{2^u v}\frac{d^{2^u(2v+1)}}{1+c}[P(h,i)] \\
= \; & 1
\end{aligned}$$

but

$$(w[0]_1+e)^{a-2^u}\hat{w}_4^{2^u(v+1)}e^{m-1-a-2i-2^u}[\mathbb{RP}(\nu^{2i+k})]=0,$$

which is a contradiction.

If $a>h$, then $a\geq 2^A$ so $w_{2^A+2}(\nu^k)=\binom{a+1}{2^A+2}c^{2^A+2}+c^{2^A}d\neq 0$. Thus $2^A+2\leq k\leq 2^{u+1}$. This implies that $u\geq A$ so $u=A$ and

$$(1+c)^{a+i+1}=(1+c)^{a-2^u+1}.$$

Let $x=a-2^u+1$. Then $x\leq 2^u$. Since $\text{common}(h,2^{u+1}-x)\geq 2^u$, one has that $h-\text{common}(h,2^{u+1}-x)\leq h-2^u$. Let $h_0=\min\{h-\text{common}(h,2^{u+1}-x),x-1\}$. Then

$$h_0+2^{u+2}(v+1)\leq h-2^u+2^{u+2}(v+1)=h+2i+2^u<h+2i+k=m$$

and

$$\binom{h_0+2^{u+1}-x}{h}\equiv 1 \mod 2.$$

Furthermore, one has

$$\begin{aligned}
& (w[0]_1+e)^{h_0}\hat{w}_4^{2^u(v+1)}e^{m-1-h_0-2^{u+2}(v+1)}[\mathbb{RP}(\nu^k)] \\
= \; & \frac{(1+c)^{h_0}(d+cd+d^2)^{2^u(v+1)}}{(1+c)^a(1+c+d)}[P(h,i)] \\
= \; & \frac{d^{2^u(v+1)}(1+c+d)^{2^u(v+1)-1}}{(1+c)^{a-h_0}}[P(h,i)] \\
= \; & \binom{2^u(v+1)-1}{2^u v}\frac{d^{2^u(2v+1)}}{(1+c)^{x-h_0}}[P(h,i)] \\
= \; & (1+c)^{h_0+2^{u+1}-x}d^i[P(h,i)] \\
= \; & \binom{h_0+2^{u+1}-x}{h}c^h d^i[P(h,i)] \\
= \; & 1
\end{aligned}$$

but

$$(w[0]_1+e)^{h_0}\hat{w}_4^{2^u(v+1)}e^{m-1-h_0-2^{u+2}(v+1)}[\mathbb{RP}(\nu^{2i+k})]=0.$$

This is a contradiction. One has proved

Lemma 3.3. *If $u>1$ with $q>1$, then $a<2^u$.*

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Note that from Lemmas 3.2 and 3.3, one knows that $i+k \equiv a+1 \mod 2^{A+1}$ and $a < 2^u$ if $u > 1$. For $1 < u \leq A$, since $i+k = 2^{u+1}v + 2^u + k$ and $a+1 \leq 2^u$, one must have

$$k = 2^u + a + 1 \quad \text{and} \quad 2^{u+1}(v+1) \equiv 0 \mod 2^{A+1}.$$

For $u > A$, one has $k \equiv a+1 \mod 2^{A+1}$.

Now by Proposition 2.2 and Lemmas 3.1, 3.2 and 3.3, one sees that if $u > 1$ with $q > 1$, and l is replaced by $a+1$ in $(P(h, N^{i+l}), T_{N^{i+l}})$, when $X_1 \leq k \leq X_2$, then $(\bar{M}^{j+h+1=2i+h+k}, \bar{T})$ fixing $\mathbb{RP}^h$ having $w(\nu^{j+1}) = (1+\alpha)^{i+a+1}$ and $P(h,i)$ having $w(\nu^k) = (1+c)^a(1+c+d)$ exists, and is cobordant to

$$\Gamma^{k-2a-2}(P(h, N^{i+a+1}), T_{N^{i+a+1}}).$$

Thus one has

Proposition 3.3. *Suppose* (M^{j+h+1}, T) *fixes* $\mathbb{RP}^j$ *with* $w(\nu^{h+1}) = (1+\alpha)^{h+1}$ *and* $P(h, 2^u(2v+1))$ *with* $w(\nu^k) = (1+c)^a(1+c+d)$ *and* $u > 1$*. Then*
(1) a *is odd and* $a < 2^u$*;*
(2) k *is even;*
(3) $2^u(2v+1) + k \equiv a+1 \mod 2^{A+1}$.
Further, when k *is restricted to a range* $X_1 \leq k \leq X_2$*,* (M^{j+h+1}, T) *exists and is cobordant to*

$$\Gamma^{k-2a-2}(P(h, N^{2^u(2v+1)+a+1}), T_{N^{2^u(2v+1)+a+1}}) \sqcup (\mathbb{RP}^{j+h+1}, T_{h+1}).$$

Now let us estimate the value X_2. For this, consider

$$\Gamma^{k-2a-2}(P(h, N^{2^u(2v+1)+a+1}), T_{N^{2^u(2v+1)+a+1}})$$

fixing $\mathbb{RP}^h$ having $w(\nu^{2i+k}) = (1+\alpha)^{i+a+1}$ and $P(h, 2^u(2v+1))$ having $w(\nu^k) = (1+c)^a(1+c+d)$ without assuming the condition $2^u(2v+1)+k \equiv a+1 \mod 2^{A+1}$.

When $1 < u \leq A$, one has that $h \geq 2^A \geq 2^u > a$. If $k > 2^u + a + 1$, then

$$h - 2^u + a + 1 + 2^{u+2}(v+1) = h + 2i + 2^u + a + 1 < h + 2i + k.$$

Using the class $\hat{w}_4$ in the proof of Lemma 3.3, one has

$$\begin{aligned} 0 &= (w[0]_1 + e)^{h-2^u+a+1}\hat{w}_4^{2^u(v+1)}e^{k-2^u-a-2}([\mathbb{RP}(\nu^k)] + [\mathbb{RP}(\nu^{2i+k})]) \\ &= (e+c)^{h-2^u+a+1}(de^2 + cde + d^2)^{2^u(v+1)}e^{k-2^u-a-2}[\mathbb{RP}(\nu^k)] + 0 \\ &= (1+c)^{h-2^u+1}d^{2^u(v+1)}(1+c+d)^{2^u(v+1)-1}[P(h,i)] \\ &= \binom{2^u(v+1)-1}{2^u v} d^{2^u(2v+1)}(1+c)^h[P(h,i)] \\ &= c^h d^i [P(h,i)] \\ &= 1, \end{aligned}$$

which is impossible. Thus k must be less than or equal to $2^u + a + 1$, so $X_2 \leq 2^u + a + 1$.

When $A < u$, one has $i+a+1 \equiv a+1 \mod 2^{A+1}$. Let $a_0 = \text{common }(h,a)$. It is easy to see that a_0 is even, and $a_0 \leq h$ and $a_0 < a$; in particular, $\binom{2^u-1-a+a_0}{h} = 1$. If $k > 2^{u+1} - h + a_0$, then

$$a_0 + 2^{u+2}(v+1) = a_0 + 2i + 2^{u+1} < h + 2i + k.$$

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Using the class $\hat{w}_4$ in the proof of Lemma 3.3, one has

$$\begin{aligned} 0 &= (w[0]_1+e)^{a_0}\hat{w}_4^{2^u(v+1)}e^{h+k-a_0-2^{u+1}-1}([\mathbb{RP}(\nu^k)]+[\mathbb{RP}(\nu^{2i+k})]) \\ &= (e+c)^{a_0}(de^2+cde+d^2)^{2^u(v+1)}e^{h+k-a_0-2^{u+1}-1}[\mathbb{RP}(\nu^k)]+0 \\ &= (1+c)^{a_0-a}d^{2^u(v+1)}(1+c+d)^{2^u(v+1)-1}[P(h,i)] \\ &= d^{2^u(2v+1)}(1+c)^{2^u-1-a+a_0}[P(h,i)] \\ &= \binom{2^u-1-a+a_0}{h}c^hd^i[P(h,i)] \\ &= 1, \end{aligned}$$

which is a contradiction. Thus k must be less than or equal to $2^{u+1}-h+a_0 = 2^{u+1}-(h-\text{common }(h,a))$, so $X_2 \le 2^{u+1}-(h-\text{common }(h,a))$.

(II) The case in which ν^k is nonstandard. Now one considers the case in which the exotic class may occur in $w(\nu^k)$. Thus $h=2,4$ or 6. When $h=2$, one knows from [St1] that there are two kinds of possible exotic classes, i.e.,

$$1+\frac{c^2d}{1+d} \text{ with } i\ge 1$$

and

$$1+c^2d^3 \text{ with } i=3.$$

If $i=3$ (i.e., i is odd), then $k=2$ by Lemma 3.1(4), and the exotic class $1+c^2d^3$ cannot occur in $w(\nu^k)$ when $k=2$. Thus, if the exotic class occurs in $w(\nu^k)$, one may write

$$w(\nu^k)=\begin{cases}(1+c)^a(1+c+d)\{1+\frac{c^2d}{1+d}\} & \text{if } h=2,\\ (1+c)^a(1+c+d)\{1+\frac{c^4d^2}{1+d^2}\} & \text{if } h=4,\\ (1+c)^a(1+c+d)\{1+\frac{c^4d^2}{1+d^2}+\frac{c^6d}{1+d^4}\} & \text{if } h=6.\end{cases}$$

First, one has

Lemma 3.4. *If ν^k is nonstandard, then* $(h,u,k,a)=(2,1,4,1)$.

Proof. The argument proceeds as follows.

(A) When $h=4$, from

$$\begin{aligned} w(\nu^k) &= (1+c)^a(1+c+d)\{1+\frac{c^4d^2}{1+d^2}\} \\ &= (1+c)^a(1+c+d)+\frac{c^4d^2}{1+d} \\ &= (1+c)^a(1+c+d)+c^4(d^2+d^3+\cdots+d^i) \end{aligned}$$

one has $w_{2i+4}(\nu^k)\ne 0$ so $k\ge 2i+4$ (note that $i\ge 2$). However, k never exceeds $2i+2$ since

$$2\le k\le \begin{cases}2^{u+1}+2 & \text{if } u=1,\\ 2^{u+1} & \text{if } u\ne 1,\end{cases}$$

by Lemma 3.1(4). Thus, if $h=4$, then there can be no involution with nonstandard ν^k.

(B) When $h=6$, it is easy to see that i must be even. In fact, if i is odd, one can obtain that $k>2$ from the expression of $w(\nu^k)$. However, $k=2$ by Lemma 3.1(4). This is a contradiction. If i is even, then a is odd by Lemma 3.1 and k must

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be even since $\chi(M^{h+2i+k}) = 1$. Now, the argument is divided into two cases: (i) $a \equiv 1 \mod 4$; (ii) $a \equiv 3 \mod 4$.

(i) The case $a \equiv 1 \mod 4$. One has that

$$\begin{aligned}
w(\nu^k) &= (1+c)^a(1+c+d)\{1+\frac{c^4d^2}{1+d^2}+\frac{c^6d}{1+d^4}\} \\
&= (1+c)^a(1+c+d)+(1+c)(1+c+d)\frac{c^4d^2}{1+d^2}+\frac{c^6d}{(1+d)^3} \\
&= (1+c)^a(1+c+d)+(1+c)(1+c+d)\frac{c^4d^2+c^4+c^4}{1+d^2}+\frac{c^6d+c^6+c^6}{(1+d)^3} \\
&= (1+c)^a(1+c+d)+c^4(1+c)(1+c+d)+(1+c)(1+c+d)\frac{c^4}{1+d^2} \\
&\quad +\frac{c^6}{1+d^2}+\frac{c^6}{(1+d)^3} \\
&= (1+c)^a(1+c+d)+c^4(1+c)(1+c+d)+\frac{c^4+c^6+c^4d+c^5d}{1+d^2} \\
&\quad +\frac{c^6}{1+d^2}+\frac{c^6}{(1+d)^3} \\
&= (1+c)^a(1+c+d)+c^4(1+c)(1+c+d) \\
&\quad +\frac{c^5}{1+d^2}+\frac{c^4+c^5}{1+d}+\frac{c^6}{(1+d)^3}
\end{aligned}$$

so $w_{2i+4}(\nu^k) \neq 0$ and $k \geq 2i+4$. However, k never exceeds $2i+2$. Thus, $a \equiv 1 \mod 4$ is impossible.

(ii) The case $a \equiv 3 \mod 4$. One has

$$\begin{aligned}
w(\nu^k) &= (1+c)^a(1+c+d)\{1+\frac{c^4d^2}{1+d^2}+\frac{c^6d}{1+d^4}\} \\
&= (1+c)^a(1+c+d)+(1+c)^3(1+c+d)\frac{c^4d^2}{1+d^2}+\frac{c^6d}{(1+d)^3} \\
&= (1+c)^a(1+c+d)+(1+c)^3(1+c+d)c^4 \\
&\quad +(1+c)^3(1+c+d)\frac{c^4}{1+d^2}+\frac{c^6d}{(1+d)^3} \\
&= (1+c)^a(1+c+d)+(1+c)^3(1+c+d)c^4 \\
&\quad +\frac{c^4+c^4d+c^5d+c^6d}{1+d^2}+\frac{c^6d}{(1+d)^3} \\
&= (1+c)^a(1+c+d)+(1+c)^3(1+c+d)c^4 \\
&\quad +\frac{c^4}{1+d}+\frac{c^5d+c^6d}{1+d^2}+\frac{c^6d}{(1+d)^3}.
\end{aligned}$$

Obviously, one sees that $w_{2i+4}(\nu^k)$ contains the nonzero class c^4d^i so $k \geq 2i+4$, but $k \leq 2i+2$. This is a contradiction. Thus, $a \equiv 3 \mod 4$ is impossible, too.

Thus, if $h = 6$, then there can be no involution with nonstandard ν^k.

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(C) When $h = 2$, one has

(3.2)
$$w(\nu^k) = (1+c)^a(1+c+d)\{1+\frac{c^2d}{1+d}\} = (1+c)^a(1+c+d) + c^2d.$$

Write $i = 2^u(2v+1)$. One proceeds as follows.

If $u = 0$ (i.e., i is odd), then a is even and $k = 2$ by Lemma 3.1. To ensure $k = 2$, one must have $a = 2$, so $w(\nu^2) = (1+c)^2(1+c+d) + c^2d = 1+c+c^2+d$. Consider the involution $(\bar{M}^{2+2i+2}, \bar{T})$ fixing $\mathbb{RP}^2$ with $w(\nu^{2i+2}) = (1+\alpha)^{2i+2} = 1$ and $P(2,i)$ with $w(\nu^2) = 1+c+c^2+d$. From

$$w[0] = \begin{cases} (1+c)^2(1+c+d)^{i+1}\{1+\frac{c}{1+e}+\frac{c^2+d}{1+e^2}+\cdots\} & \text{on } P(2,i), \\ (1+\alpha)^3(1+e)^{2i} & \text{on } \mathbb{RP}^2, \end{cases}$$

one has that

$$w[0]_1 = \begin{cases} c & \text{on } P(2,i), \\ \alpha & \text{on } \mathbb{RP}^2, \end{cases}$$

and

$$w[0]_2 = \begin{cases} ce + c^2 + d + \binom{i+3}{2}c^2 & \text{on } P(2,i), \\ \alpha^2 + e^2 & \text{on } \mathbb{RP}^2, \end{cases}$$

so

$$w[0]_2 + e^2 + w[0]_1^2 = \begin{cases} d + e^2 + ce + \binom{i+3}{2}c^2 & \text{on } P(2,i), \\ 0 & \text{on } \mathbb{RP}^2. \end{cases}$$

Further, the value of the class

$$w[0]_1^2(w[0]_2 + e^2 + w[0]_1^2)^i e$$

on $\mathbb{RP}^2$ is zero, but its value on $P(2,i)$ is nonzero. This is a contrdaction. Thus, $u = 0$ is impossible.

If $u > 0$ (i.e., i is even), then a is odd by Lemma 3.1 and k must be even. In this case,

$$w(\nu^k) = (1+c)^a(1+c+d)\{1+\frac{c^2d}{1+d}\} = 1 + \binom{a+1}{2}c^2 + d + cd + \binom{a}{2}c^2d + c^2d.$$

So

$$w[0] = \begin{cases} (1+c)^2(1+c+d)^{i+1}\{1+\frac{\binom{a+1}{2}c^2+d}{1+e^2}+\frac{cd}{(1+e)^3}+\frac{\binom{a}{2}c^2d+c^2d}{1+e^4}+\cdots\} & \text{on } P(2,i), \\ (1+\alpha)^3\{(1+e)^{2i}+\binom{a+i+1}{2}\alpha^2(1+e)^{2i-2}\} & \text{on } \mathbb{RP}^2. \end{cases}$$

One then has that

$$w[0]_1 = \begin{cases} c & \text{on } P(2,i), \\ \alpha & \text{on } \mathbb{RP}^2, \end{cases}$$

and

$$w[0]_4 = \begin{cases} \binom{a+1}{2}c^2e^2 + e^2d + cde + c^2d + d^2 + \binom{i+1}{2}d^2 & \text{on } P(2,i), \\ \binom{2i}{4}e^4 + \binom{a+i+1}{2}e^2\alpha^2 & \text{on } \mathbb{RP}^2. \end{cases}$$

If $u > 1$, one has

$$w[0]_4 = \begin{cases} \binom{a+1}{2}c^2e^2 + e^2d + cde + c^2d + d^2 & \text{on } P(2,i), \\ \binom{a+1}{2}e^2\alpha^2 & \text{on } \mathbb{RP}^2. \end{cases}$$

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Further, one has that the value of

$$w[0]_1^2 w[0]_4 e^{2i+k-5}$$

on $\mathbb{RP}^2$ is zero, but its value on $P(2,i)$ is nonzero. This leads to a contradiction. Thus, $u>1$ is impossible.

If $u=1$, then $i=4v+2$ is even so $a=1$ or 3 since $i+a$ is odd by Lemma 3.1. Also, one knows from Lemma 3.1(4) that $2\le k\le 6$. From (3.2), obviously, $k=2$ is impossible so $k=4,6$ since k is even by Lemma 3.1. The argument is divided into the following cases.

(1) The case $a=3$. For $a=3$, if $k=4$ then $j+1=2i+k=8v+8\equiv 0 \mod 4$. Thus, $w(\nu^{j+1})=1$ so

$$e^{j+2}[\mathbb{RP}(\nu^{j+1})]=\frac{1}{w(\nu^{j+1})}[\mathbb{RP}^2]=0$$

but

$$\begin{aligned}
e^{j+2}[\mathbb{RP}(\nu^4)] &= \frac{1}{w(\nu^4)}[P(2,4v+2)]\\
&= \frac{1}{(1+c)^3(1+c+d)\{1+\frac{c^2d}{1+d}\}}[P(2,4v+2)]\\
&= \{\frac{1}{(1+c)^3(1+c+d)}+\frac{c^2d}{1+d^2}\}[P(2,4v+2)]\\
&= \frac{1}{1+\frac{d}{1+c}}[P(2,4v+2)]\\
&= \frac{d^{4v+2}}{(1+c)^{4v+2}}[P(2,4v+2)]\\
&= 1.
\end{aligned}$$

This is a contradiction. If $k=6$, then $j+1=2i+k=8v+10$ so $w(\nu^{j+1})=(1+\alpha)^2$. Also, one has that $w(\nu^k)=(1+c)^3(1+c+d)+c^2d=1+d+cd$. Now from

$$w[4]=\begin{cases}(1+c)^2(1+c+d)^{4v+3}\{(1+e)^4+d(1+e)^2+cd(1+e)\} & \text{on } P(2,4v+2),\\ (1+\alpha)^3(1+e)^{8v+6}(1+e^2+\alpha^2) & \text{on } \mathbb{RP}^2,\end{cases}$$

one has that

$$w[4]_1=\begin{cases}c & \text{on } P(2,4v+2),\\ \alpha & \text{on } \mathbb{RP}^2.\end{cases}$$

In a similar way to the case in which ν^k is standard, when multiplied by $w[4]_1$ for $w[4]$, one has that on $P(2,4v+2)$

$$\begin{aligned}
w[4] &\sim (1+d)^{4v+2}(1+c+d)(1+e^4+d+e^2d+cd+cde)\\
&\sim (1+d)^{4v}(1+c+e^4+cde+e^2d+ce^4+cde^2+e^4d+cd^2e+e^2d^2\\
&\quad +e^4d^2+cd^3e+e^2d^3+e^4cd^2+e^2cd^3+e^4d^3+d^4+cd^4+cd^4e+e^2d^4)
\end{aligned}$$

so

$$w[4]_4\sim e^4+cde+e^2d$$

and

$$w[4]_{8v+6}\sim e^4d^{4v+1}+ced^{4v+2}+e^2d^{4v+2}.$$

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On $\mathbb{RP}^2$,

$$w[4] \sim (1+\alpha)(1+e)^{8v+8}$$

so

$$w[4]_4 \sim 0$$

and

$$w[4]_{8v+6} \sim 0.$$

Furthermore, one obtains that the value of

$$w[4]_1 w[4]_4 w[4]_{8v+6}$$

on $[\mathbb{RP}(\nu^{8v+10})]$ is zero, but the value of this on $[\mathbb{RP}(\nu^6)]$ is nonzero, which gives a contradiction. Thus, $a = 3$ is impossible.

(2) The case $a = 1$. If $k = 6$, then $w(\nu^{j+1}) = (1+\alpha)^2$ so

$$e^{8v+11}[\mathbb{RP}(\nu^{j+1})] = \frac{1}{(1+\alpha)^2}[\mathbb{RP}^2] = 1.$$

However, since $a = 1$, one has

$$\begin{aligned} e^{8v+11}[\mathbb{RP}(\nu^6)] &= \frac{1}{(1+c)(1+c+d)(1+\frac{c^2d}{1+d})}[P(2,4v+2)] \\ &= \frac{1}{(1+c)(1+c+d)}[P(2,4v+2)] + \frac{c^2d}{(1+d)^2}[P(2,4v+2)] \\ &= \frac{1}{(1+c)^2(1+\frac{d}{1+c})}[P(2,4v+2)] + 0 \\ &= \frac{d^{4v+2}}{(1+c)^{4v+4}}[P(2,4v+2)] \\ &= d^{4v+2}[P(2,4v+2)] \\ &= 0. \end{aligned}$$

Thus, $k = 6$ is impossible, so k must be 4.

Combining the above arguments, one concludes that if ν^k is nonstandard, then

$$(h,u,k,a) = (2,1,4,1).$$

This completes the proof. $\square$

Note that if ν^k is nonstandard, from Lemma 3.4 one must have $(h,u,k,a) = (2,1,4,1)$, so $a = 1 < 2 = 2^u$ and

$$j+1 = 2i+k = 8v+8 \equiv 4v+4 = i+a+1 \mod 4$$

and

$$i+k \equiv a+1 \mod 4.$$

Proposition 3.4. *The involution (M^{j+h+1},T) fixing $\mathbb{RP}^{j+1} \sqcup P(h,2^u(2v+1))$ with ν^k nonstandard exists if and only if $(h,u,k) = (2,1,4)$ and $w(\nu^4) = (1+c)(1+c+d)(1+\frac{c^2d}{1+d})$.*

Proof. By Lemma 3.4, it suffices to show that the involution (M^{j+h+1},T) fixing $\mathbb{RP}^{j+1} \sqcup P(h,2^u(2v+1))$ with ν^k nonstandard exists if $(h,u,k) = (2,1,4)$ and $w(\nu^4) = (1+c)(1+c+d)(1+\frac{c^2d}{1+d})$. This is equivalent to showing that there exists the involution $(\bar{M}^{j+3},\bar{T})$ fixing $\mathbb{RP}^2$ with normal bundle ν^{j+1} having $w(\nu^{j+1}) = 1$

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and $P(2,4v+2)$ with normal bundle ν^4 having $w(\nu^4)=(1+c)(1+c+d)(1+\frac{c^2d}{1+d})=1+c^2+d+cd+c^2d$.

Now to do that one first needs to show that there is a bundle ν^4 over $P(2,4v+2)$ with

$$w(\nu^4)=(1+c)(1+c+d)(1+\frac{c^2d}{1+d})=(1+c)^3(1+c+d+c^2).$$

For that $(1+c)^3$ is the class of the 2-plane bundle τ (the tangent bundle of $\mathbb{RP}^2$ pulled back to $P(2,4v+2)$) and $(1+c+d+c^2)$ is the class of a 2-plane bundle—the strange tensor product $\tau\otimes\eta$ described in [St1]. Thus $\nu^4=\tau\oplus(\tau\otimes\eta)$.

Next, according to [CF], one also needs to prove that $\mathbb{RP}(\nu^{j+1})$ and $\mathbb{RP}(\nu^4)$ are cobordant in $B\mathbb{Z}_2$, i.e.,

$$w_{\ell_1}\cdots w_{\ell_r}e^{j+2-\ell_1-\cdots-\ell_r}([\mathbb{RP}(\nu^{j+1})]+[\mathbb{RP}(\nu^4)])=0$$

for any $\ell_1,\dots,\ell_r$ with $\ell_1+\cdots+\ell_r\le j+2$, where $w_{\ell_s}=w_{\ell_s}(\mathbb{RP}(\nu))$. Note that $j+1=8v+8$.

First, let us look at the total Stiefel-Whitney classes of $\mathbb{RP}(\nu^4)$ and $\mathbb{RP}(\nu^{j+1})$. On $\mathbb{RP}(\nu^4)$ one has

$$\begin{aligned}
&w(\mathbb{RP}(\nu^4))\\
&\quad=(1+c)^2(1+c+d)^{4v+3}\{(1+e)^4+(c^2+d)(1+e)^2+cd(1+e)+c^2d\}\\
&\quad=(1+c)^2(1+c+d)^{4v+3}\{1+c^2+d+cd+(e^4+c^2e^2+de^2+cde+c^2d)\}.
\end{aligned}$$

According to Borel and Hirzebruch [BH], one knows that

$$e^4+c^2e^2+de^2+cde+c^2d=0$$

so

$$\begin{aligned}
w(\mathbb{RP}(\nu^4)) &= 1+c)^2(1+c+d)^{4v+3}(1+c^2+d+cd)\\
&= 1+c)^2(1+c+d)^{4v+3}(1+c+d+c+c^2+cd)\\
&= (1+c)^3(1+c+d)^{4v+4}\\
&= (1+c)^3(1+d^4)^{v+1}.
\end{aligned}$$

On $\mathbb{RP}(\nu^{j+1})$, since $w(\nu^{j+1})=1$, one has

$$w(\mathbb{RP}(\nu^{j+1}))=(1+\alpha)^3(1+e)^{j+1}=(1+\alpha)^3(1+e^8)^{v+1}$$

with $e^{8v+8}=0$ by [BH].

One sees that if one writes the ℓ-th Stiefel-Whitney class w_ℓ of $\mathbb{RP}(\nu^4)$ as a polynomial

$$p_\ell(c,d),$$

then the ℓ-th Stiefel-Whitney class w_ℓ of $\mathbb{RP}(\nu^{j+1})$ is of the form

$$p_\ell(\alpha,e^2).$$

In particular, it is easy to see that for $\ell\ne 8t,8t+1,8t+2$,

$$p_\ell(c,d)=0,\quad p_\ell(\alpha,e^2)=0,$$

and for $\ell=8t,8t+1,8t+2$

$$p_\ell(c,d)=\binom{v+1}{t}d^{4t},\binom{v+1}{t}cd^{4t},\binom{v+1}{t}c^2d^{4t}$$

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and

$$p_\ell(\alpha, e^2) = \binom{v+1}{t} e^{8t}, \binom{v+1}{t} \alpha e^{8t}, \binom{v+1}{t} \alpha^2 e^{8t}.$$

Since

$$\begin{aligned} \frac{1}{w(\nu^4)} &= \frac{1}{(1+c)\{1+c+d(1+c^2)\}} \\ &= \frac{1}{(1+c)^2\{1+\frac{d(1+c^2)}{1+c}\}} \\ &= \frac{1}{(1+c)^2} \sum_{s=0}^{4v+2} d^s (1+c)^s \\ &= \sum_{s=0}^{4v+2} d^s (1+c)^{s-2}, \end{aligned}$$

one has

$$\frac{d^{4x}}{w(\nu^4)}[P(2,4v+2)] = 0, \quad \frac{cd^{4x}}{w(\nu^4)}[P(2,4v+2)] = 0, \quad \frac{c^2 d^{4x}}{w(\nu^4)}[P(2,4v+2)] = 1.$$

Also,

$$\frac{1}{w(\nu^{j+1})}[\mathbb{RP}^2] = 0, \quad \frac{\alpha}{w(\nu^{j+1})}[\mathbb{RP}^2] = 0, \quad \frac{\alpha^2}{w(\nu^{j+1})}[\mathbb{RP}^2] = 1.$$

Thus, for any characteristic class of

$$w_{\ell_1} \cdots w_{\ell_r}$$

with $\ell_1 + \cdots + \ell_r \leq j+2 = 8v+9$,

$$w_{\ell_1} \cdots w_{\ell_r} e^{j+2-\ell_1-\cdots-\ell_r}([\mathbb{RP}(\nu^{j+1})] + [\mathbb{RP}(\nu^4)]) = 0.$$

This completes the proof. □

Combining the discussions of this section, one completes the proof of Theorem 2.3.

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