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THE STEEPEST POINT OF THE BOUNDARY LAYERS OF SINGULARLY PERTURBED SEMILINEAR ELLIPTIC PROBLEMS

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Abstract. We consider the nonlinear singularly perturbed problem

$$-\epsilon^2\Delta u = f(u), \;\; u > 0 \quad \text{in } \Omega, u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbf{R}^N$ $(N \geq 2)$ is an appropriately smooth bounded domain and $\epsilon > 0$ is a small parameter. It is known that under some conditions on f, the solution u_ϵ corresponding to ϵ develops boundary layers when $\epsilon \to 0$. We determine the steepest point of the boundary layers on the boundary by establishing an asymptotic formula for the slope of the boundary layers with exact second term.

1. Introduction

We consider the following nonlinear singularly perturbed problem:

$$-\epsilon^2\Delta u = f(u) \quad \text{in } \Omega, \tag{1.1}$$

$$u > 0 \quad \text{in } \Omega, \tag{1.2}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\Omega \subset \mathbf{R}^N$ $(N \geq 2)$ is an appropriately smooth bounded domain satisfying uniform interior and exterior sphere conditions and $\epsilon > 0$ is a small parameter. We assume that f satisfies the following:

(f.1) $f \in C^{1,\gamma}(\mathbf{R})$, where $0 < \gamma < 1$ is a constant.

(f.2) There exists $u_0 > 0$ such that $f(0) = f(u_0) = 0$ and $f(u) > 0$ for $0 < u < u_0$.

(f.3) $f'(0) > 0$ and $f'(u_0) < 0$.

(f.4) If there exist $u_1, u_2 > 0$ such that $u_0 < u_1 < u_2, f(u_1) = f(u_2) = 0, f > 0$ in (u_1, u_2), then there exists $\tilde{u} \in [0, u_2)$ such that $\int_{\tilde{u}}^{u_2} f(s)ds \leq 0$.

The most typical examples of f are $f(u) = u - |u|^{p-1}u$ $(p > 1)$ and $f(u) = \sin u$, which are called logistic equation in population dynamics and equation of simple pendulum, respectively, and have been considered by many authors. In these cases, we know that $u_0 = 1$ (resp. $u_0 = \pi$), $u_\epsilon < 1$ (resp. $u_\epsilon < \pi$) and $u_\epsilon \to 1$ (resp. $u_\epsilon \to \pi$) uniformly on any compact subset in Ω as $\epsilon \to 0$. Indeed, under the conditions (f.1)–(f.4), the following facts hold (cf. [3]):

(p.1) For a given $0 < \epsilon \ll 1$, there exists a unique solution $u_\epsilon \in C^2(\bar{\Omega})$ of (1.1)–(1.3).

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(p.2) $\|u_\epsilon\|_\infty < u_0$.

(p.3) $u_\epsilon \to u_0$ uniformly on any compact subset $K \subset \Omega$ as $\epsilon \to 0$.

Therefore, we see that u_ϵ develops boundary layers as $\epsilon \to 0$. We also refer to [2]–[7], [9] and the references therein. Hence, a natural question raised immediately is to ask where on the boundary the steepest point of the boundary layers is situated, and it is the purpose of this paper to answer this question by establishing the two-term asymptotic formula for the slope of the boundary layers as $\epsilon \to 0$.

Now we state our results. Let $F(u) := \int_0^u f(s)ds$.

Theorem 1.1. *Let $P \in \partial\Omega$ be fixed. Let ν be a unit outer normal to $\partial\Omega$ at P. Then the following asymptotic formula holds as $\epsilon \to 0$:*

$$\frac{\partial u_\epsilon}{\partial \nu}(P) = -\sqrt{2F(u_0)}\epsilon^{-1} + \frac{(N-1)C_0 H(P)}{\sqrt{2F(u_0)}} + o(1), \tag{1.4}$$

where

$$C_0 := \int_0^{u_0} \sqrt{2(F(u_0) - F(s))}ds$$

and $H(P)$ is the mean curvature of $\partial\Omega$ at P.

Therefore, if $P_1 \in \partial\Omega$ is the only point which attains the minimum of the mean curvature of $\partial\Omega$, then P_1 is the steepest point of the boundary layers. We remark that the first term in (1.4) has been obtained in [5] when $f(u) = u - |u|^{p-1}u +$ perturbation. When $\Omega = B_R$ or $A_{a,R}$, the special cases, (1.4) has been obtained in [11] and [12].

2. Proof of Theorem 1.1

We first introduce a diffeomorphism which straightens the boundary portion near a point $P \in \partial\Omega$ (cf. [10]).

Through translation and rotation of the coordinate system, we may assume that P is the origin and the inner normal to $\partial\Omega$ at P is pointing in the direction of the positive x_N axis. Let $x' = (x_1, x_2, \cdots, x_{N-1})$. Then there exists a smooth function $\psi(x')$, defined for $|x'| \ll 1$, which satisfies:

(ψ1) $\psi(0) = 0$,

(ψ2) $\nabla\psi(0) = 0$,

(ψ3) $\partial\Omega \cap M = \{(x', x_N) : x_N = \psi(x')\}$,

(ψ4) $\Omega \cap M = \{(x', x_N) : x_N > \psi(x')\}$,

where M is a neighborhood of $P = 0$. For $y \in \mathbf{R}^N$ near 0, we define a mapping $x = \Phi(y) = (\Phi_1(y), \Phi_2(y), \cdots, \Phi_N(y))$ by

$$\Phi_j(y) = y_j - y_N \frac{\partial \psi}{\partial x_j}(y') \quad (j = 1, 2, \cdots, N-1), \tag{2.1}$$

$$\Phi_N(y) = y_N + \psi(y'). \tag{2.2}$$

Then Φ has the inverse mapping $y = \Phi^{-1}(x)$ for $|x| < b$, where $b > 0$ is a constant. Let $B^+_{b/\epsilon} := \{z \in \mathbf{R}^N : |z| < b/\epsilon, z_N > 0\}$. We put

$$w_\epsilon(z) := u_\epsilon(\Phi(\epsilon z)), \quad z \in B_{b/\epsilon}. \tag{2.3}$$

Then we see from [10, Lemma 4.1] that for $z \in B^+_{b/\epsilon}$, w_ϵ satisfies

$$\sum_{i,j=1}^N (\delta_{ij} + 2\epsilon\psi_{ij} z_N + \alpha_{ij}(z)) \frac{\partial^2 w_\epsilon}{\partial z_i \partial z_j} + \epsilon \sum_{j=1}^N b_j(z) \frac{\partial w_\epsilon}{\partial z_j} + f(w_\epsilon) = 0, \tag{2.4}$$

where

$$\psi_{ij} = \frac{\partial^2 \psi}{\partial x_i \partial x_j}(0), \tag{2.5}$$

$$\psi_{iN} = \psi_{Nj} = 0 \quad \text{for } i, j = 1, \cdots, N, \tag{2.6}$$

$$b_j(z) = -\delta_{jN}(N-1)H(0) + \beta_j(z), \tag{2.7}$$

$$|\alpha_{ij}(z)| \leq C_2 \epsilon^2 |z|^2 \quad \text{for } z \in B^+_{b/\epsilon}, \tag{2.8}$$

$$|\beta_j(z)| \leq C_2 \epsilon |z| \quad \text{for } z \in B^+_{b/\epsilon}. \tag{2.9}$$

To calculate $(\partial u_\epsilon/\partial \nu)(0)$ for $0 < \epsilon \ll 1$, we precisely study the asymptotic behavior of $(\partial w_\epsilon/\partial z_N)(0)$ as $\epsilon \to 0$. Let w be a unique solution of

$$-w''(t) = f(w(t)), \quad t > 0, \tag{2.10}$$

$$w(0) = 0, \tag{2.11}$$

$$w'(0) = \sqrt{2F(u_0)}. \tag{2.12}$$

Then we know from [3] that

$$w'(t) > 0, \quad t > 0, \tag{2.13}$$

$$\lim_{t\to\infty} w(t) = u_0, \quad \lim_{t\to\infty} w'(t) = 0. \tag{2.14}$$

Lemma 2.1. *Let $D := \{z \in \mathbf{R}^N : z_N > 0\}$. Then $w_\epsilon \to w$ in $C^2_{loc}(D)$ as $\epsilon \to 0$.*

Proof. Since $\|w_\epsilon\|_\infty = \|u_\epsilon\|_\infty < u_0$, by a standard regularity argument (cf. [3]), we see that $\{w_\epsilon\}$ is a precompact in C^2_{loc}. Therefore, by choosing a subsequence if necessary, there exists $W \in C^2(D)$ such that $w_\epsilon \to W$ as $\epsilon \to 0$ in $C^2_{loc}(D)$. Then we see from (2.4)–(2.9) that W satisfies

$$-\Delta W = f(W) \quad \text{in } D, \tag{2.15}$$

$$W > 0 \quad \text{in } D, \tag{2.16}$$

$$W = 0 \quad \text{on } \partial D. \tag{2.17}$$

It should be mentioned that (2.16) follows from [3]. Now our aim is to show that $W = w$. To do this, we apply [3, Proposition 2.5]. That is, if we show

$$\lim_{z_N\to\infty} W((z', z_N)) = u_0 \tag{2.18}$$

uniformly for $z' \in \mathbf{R}^{N-1}$, then we obtain $W = w$. Let an arbitrary $\eta \in \mathbf{R}^{N-1}$ be fixed. Furthermore, let an arbitrary $z_N > 2u_0/C_1$ be fixed, where $C_1 > 0$ is a constant defined in (2.19) below. Then there exists $0 < \epsilon_0 \ll 1$ such that $(\eta, z_N) \in B^+_{b/\epsilon}$ for $0 < \epsilon < \epsilon_0$. So $w_\epsilon((\eta, z_N))$ is well defined for $0 < \epsilon < \epsilon_0$. Let an arbitrary $0 < \tau < u_0$ be fixed. By [3, Lemma 3.1], we see that for all $x \in \Omega$,

$$u_\epsilon(x) > \min\{C_1 \mathrm{dist}(x, \partial\Omega)/\epsilon, \tau\}, \tag{2.19}$$

where $C_1 > 0$ is a constant. Since $\nabla\psi(0) = 0$ by ($\psi 2$), we see that $D\Phi(0) = I$, the identity map. Then by Taylor expansion, for $|z| \ll 1$, we have $\Phi(z) = \Phi(0) + D\Phi(0)z + o(|z|)$. Therefore, since $\Phi(0) = 0$,

$$\Phi(\epsilon(\eta, z_N)) = \epsilon(\eta, z_N) + o(\epsilon\sqrt{|\eta|^2 + z_N^2}). \tag{2.20}$$

Let $Q = Q_{\Phi(\epsilon(\eta, z_N))}$ be the nearest point on $\partial\Omega$ from $\Phi(\epsilon(\eta, z_N))$. Then since $Q \to 0$ as $\epsilon \to 0$, we see that $n_Q \to (0, \cdots, 0, 1)$ as $\epsilon \to 0$, where n_Q is the unit inner normal

to $\partial\Omega$ at Q. Therefore, if $0 < \epsilon \ll 1$, then $\mathrm{dist}(\Phi(\epsilon(\eta, z_N)), \partial\Omega) = \epsilon(1 + o(1))z_N$. Then by (2.19), for $0 < \epsilon \ll 1$, we obtain

$$w_\epsilon((\eta, z_N)) \quad = \quad u_\epsilon(\Phi(\epsilon(\eta, z_N))) > \min\{C_1(1 - o(1))z_N, \tau\} > \tau, \tag{2.21}$$

since $z_N > 2u_0/C_1 (> u_0/(C_1(1 - o(1))))$. Now let $\epsilon \to 0$. Then we see that $W((\eta, z_N)) > \tau$ for any $\eta \in \mathbf{R}^{N-1}$ and $z_N > 2u_0/C_1$. This implies (2.18). Then we see that, from any subsequence of $\{w_\epsilon\}$, we can choose a subsequence of $\{w_{\epsilon_j}\}$, which satisfies $\epsilon_j \to 0$ as $j \to \infty$ and $w_{\epsilon_j} \to w$ in $C^2_{loc}(D)$ as $j \to \infty$. This implies that $w_\epsilon \to w$ in $C^2_{loc}(D)$ as $\epsilon \to 0$. Thus the proof is complete. □

Therefore, we find that the top term of $(\partial u_\epsilon/\partial\nu)(0)$ can be obtained from $w'(0)$. That is, by Lemma 2.1 and (2.12), as $\epsilon \to 0$,

$$-\epsilon\frac{\partial u_\epsilon}{\partial\nu}(0) = \frac{\partial w_\epsilon}{\partial z_N}(0) \to \sqrt{2F(u_0)}. \tag{2.22}$$

Next, for $z = (z', z_N) \in B^+_{b/\epsilon}$, we put

$$\phi_\epsilon(z) := \frac{w_\epsilon(z) - w(z_N)}{\epsilon}. \tag{2.23}$$

Then it follows from (2.4) that for $z \in B^+_{b/\epsilon}$,

$$\begin{aligned} &\sum_{i,j=1}^N (\delta_{ij} + 2\epsilon\psi_{ij}z_N + \alpha_{ij}(z)) \left(\frac{\partial^2 w}{\partial z_i\partial z_j} + \epsilon\frac{\partial^2\phi_\epsilon}{\partial z_i\partial z_j}\right) \\ &\quad + \epsilon\sum_{j=1}^N b_j(z)\left(\frac{\partial w}{\partial z_j} + \epsilon\frac{\partial\phi_\epsilon}{\partial z_j}\right) + f(w + \epsilon\phi_\epsilon) = 0. \end{aligned} \tag{2.24}$$

Therefore, by (2.5)–(2.9), for $z \in B^+_{b/\epsilon}$,

$$\begin{aligned} &\sum_{i,j=1}^N (\delta_{ij} + 2\epsilon\psi_{ij}z_N + \alpha_{ij}(z))\frac{\partial^2\phi_\epsilon}{\partial z_i\partial z_j} + \frac{\alpha_{NN}(z)}{\epsilon}w''(z_N) + \epsilon\sum_{j=1}^{N-1}\beta_j(z)\frac{\partial\phi_\epsilon}{\partial z_j} \\ &\quad + (-(N-1)H(0) + \beta_N(z))\left(w'(z_N) + \epsilon\frac{\partial\phi_\epsilon}{\partial z_N}\right) + \frac{f(w + \epsilon\phi_\epsilon) - f(w)}{\epsilon} = 0. \end{aligned} \tag{2.25}$$

Then it is expected that the second term should be derived from the derivative of ϕ_ϵ.

Lemma 2.2. *Let an arbitrary compact set $K \subset \bar{D}$ be fixed. Then there exists a constant $C_K > 0$ such that $\|\phi_\epsilon\|_{\infty,K} := \sup_{z\in K}|\phi_\epsilon(z)| \le C_K$ for any $0 < \epsilon \ll 1$.*

Proof. Without loss of generality, we may assume that $K = \bar{B}^+_R$. Assume that there exists a subsequence of $\{\phi_\epsilon\}$, denoted by $\{\phi_\epsilon\}$ again, such that $\|\phi_\epsilon\|_{\infty,K} \to \infty$ as $\epsilon \to 0$. Since Ω satisfies the uniform interior sphere condition, there exists a constant $0 < \xi_1 \ll 1$ such that the ball $B_{\xi_1}(Q - \xi_1\nu_Q)$ is tangent to $\partial\Omega$ at $Q \in \partial\Omega$ and satisfies $B_{\xi_1}(Q - \xi_1\nu_Q) \subset \Omega$ for all $Q \in \partial\Omega$, where ν_Q is an outward unit vector to Ω at $Q \in \partial\Omega$. Moreover, since Ω satisfies the uniform exterior sphere condition and is bounded, there exist constants $0 < \xi_2 \ll 1$ and $\xi_3 \gg 1$ such that the annulus $\Omega \subset A_{Q,\xi_2,\xi_3} := \{x \in \mathbf{R}^N : \xi_2 < |x - (Q + \xi_2\nu_Q)| < \xi_3\}$ and inner ball $B_{\xi_2}(Q + \xi_2\nu_Q)$ is tangent to $\partial\Omega$ at Q for all $Q \in \partial\Omega$. Let $\phi_{\epsilon,Q}$ and $\psi_{\epsilon,Q}$ be the unique solutions

to (1.1)–(1.3), in which Ω is replaced by $B_{\xi_1}(Q-\xi_1\nu_Q)$ and A_{Q,ξ_2,ξ_3}, respectively. Then it follows from maximum principle that

$$\phi_{\epsilon,Q} < u_\epsilon < \psi_{\epsilon,Q}.$$

This implies that (we write $\nu = \nu_Q$ for short)

$$\frac{\partial \psi_{\epsilon,Q}}{\partial \nu}(Q) < \frac{\partial u_\epsilon}{\partial \nu}(Q) < \frac{\partial \phi_{\epsilon,Q}}{\partial \nu}(Q).$$

Moreover, we know from [12, Theorems 1 and 2] (cf. [11] for the case $f(u) = u - u^p$) that as $\epsilon \to 0$,

$$\begin{aligned}
\frac{\partial \phi_{\epsilon,Q}}{\partial \nu}(Q) &= -\sqrt{2F(u_0)}\epsilon^{-1} + \frac{(N-1)C_0}{\sqrt{2F(u_0)}\xi_1} + O(\epsilon),\\
\frac{\partial \psi_{\epsilon,Q}}{\partial \nu}(Q) &= -\sqrt{2F(u_0)}\epsilon^{-1} - \frac{(N-1)C_0}{\sqrt{2F(u_0)}\xi_2} + O(\epsilon).
\end{aligned}$$

Therefore, we see that for any $Q \in \partial\Omega$ and $0 < \epsilon \ll 1$,

$$\frac{\partial u_\epsilon}{\partial \nu}(Q) = -\sqrt{2F(u_0)}\epsilon^{-1} + O(1). \tag{2.26}$$

Then by this, for $(z', 0) \in \partial \bar{B}_R^+$, we obtain

$$\begin{aligned}
\frac{\partial w_\epsilon}{\partial z_N}(z', 0) &= \epsilon(-\nabla\psi(\Phi((\epsilon z', 0))), 1) \cdot \nabla u_\epsilon(\Phi((\epsilon z', 0)))\\
&= -\epsilon \frac{\partial u_\epsilon}{\partial \nu}((\epsilon z', \psi(\epsilon z')))\\
&= \sqrt{2F(u_0)} + O(\epsilon).
\end{aligned} \tag{2.27}$$

Therefore, by (2.12), (2.23) and (2.27), for $0 < \epsilon \ll 1$,

$$\frac{\partial \phi_\epsilon}{\partial z_N}(z', 0) = \frac{(\sqrt{2F(u_0)} + O(\epsilon)) - \sqrt{2F(u_0)}}{\epsilon} = O(1). \tag{2.28}$$

Now we put $\xi_\epsilon := \phi_\epsilon / \|\phi_\epsilon\|_{\infty,K}$. Then clearly, $\|\xi_\epsilon\|_{\infty,K} = 1$ and by (2.5)–(2.9) and (2.25), we can choose a subsequence of $\{\xi_\epsilon\}$ such that $\xi_\epsilon \to \xi$ in $C^2(K)$ as $\epsilon \to 0$. Moreover, we see from (2.25) and (2.28) that

$$\begin{aligned}
\Delta\xi + f'(w)\xi &= 0 \quad \text{in } K,\\
\xi &= 0 \quad \text{on } \partial K,\\
\frac{\partial \xi}{\partial z_N} &= 0 \quad \text{on } \partial K.
\end{aligned}$$

This implies that there exists $B_\delta^+ \subset K$ such that $\xi \equiv 0$ in B_δ^+. This along with the unique continuation theorem of second order elliptic equation, $\xi \equiv 0$ in K. This is a contradiction. Thus the proof is complete. □

By Lemma 2.2 and (2.25), we see that by choosing a subsequence of $\{\phi_\epsilon\}$ if necessary, there exists $\phi \in C^2(D)$ such that as $\epsilon \to 0$,

$$\phi_\epsilon \to \phi \quad \text{in } C^2_{loc}(D), \tag{2.29}$$

which satisfies

$$\Delta\phi + f'(w)\phi = (N-1)H(0)w'(z_N) \quad \text{in } D, \tag{2.30}$$

$$\phi = 0 \quad \text{on } \partial D. \tag{2.31}$$

We note that ϕ depends on the subsequence of $\{\epsilon\}$ at this stage. However, we see later that ϕ is uniquely determined.

Lemma 2.3. *$\phi(z', z_N) = \phi(z_N)$ for $z \in D$.*

Proof. Let an arbitrary compact subset $K \subset \partial D$ be fixed. We show that by choosing a subsequence of $\{\epsilon\}$ if necessary, there exists a constant $C_2 > 0$ such that for any $z' \in K$, as $\epsilon \to 0$,

$$\frac{\partial \phi_\epsilon}{\partial z_N}(z', 0) \to C_2. \tag{2.32}$$

We note that

$$\begin{aligned} \frac{\partial \phi_\epsilon}{\partial z_N}(z', 0) &= \left[-\frac{\partial u_\epsilon}{\partial \nu}(\epsilon z', \psi(\epsilon z')) + \frac{\partial u_\epsilon}{\partial \nu}(0,0) \right] + \left[-\frac{\partial u_\epsilon}{\partial \nu}(0,0) - \sqrt{2F(u_0)}\epsilon^{-1} \right] \\ &:= I_\epsilon(z') + II_\epsilon. \end{aligned} \tag{2.33}$$

We know from (2.26) that $|II_\epsilon|$ is bounded. Therefore, by choosing a subsequence if necessary, we may assume that $II_\epsilon \to C_2$ as $\epsilon \to 0$. We know that

$$\begin{aligned} I_\epsilon(z') = &-(\nabla \psi(\epsilon z'), -1) \cdot \left(\frac{\partial u_\epsilon}{\partial x_1}(\epsilon z', \psi(\epsilon z')), \cdots, \frac{\partial u_\epsilon}{\partial x_N}(\epsilon z', \psi(\epsilon z')) \right) \\ &+ (0, -1) \cdot \left(\frac{\partial u_\epsilon}{\partial x_1}(0,0), \cdots, \frac{\partial u_\epsilon}{\partial x_N}(0,0) \right). \end{aligned} \tag{2.34}$$

Now we consider the cubes $Q_1 \subset \Omega, \cdots, Q_L \subset \Omega$ satisfying the following properties:

(q.1) The length of the edge of Q_1 is ϵ^{k_0}, where $k_0 > 0$ will be specified later. Q_j $(j = 2, \cdots, L)$ is congruent to Q_1.

(q.2) A vertex q_j of Q_j $(j = 1, \cdots, L)$ is on $\partial\Omega$. In particular, a vertex q_1 of Q_1 coincides with the origin and a vertex q_L of Q_L coincides with $(\epsilon z', \psi(\epsilon z'))$. Moreover, there exists a constant $C > 0$ such that $C^{-1}\epsilon^{k_0} \le |q_{j+1} - q_j| \le C\epsilon^{k_0}$ for $j = 1, \cdots, L-1$.

(q.3) There exist $y_j \in Q_j \cap Q_{j+1}$ $(j = 1, \cdots, L-1)$ and a constant $C > 0$ such that for $j = 1, \cdots, L-1$,

$$C^{-1}\epsilon^{k_0} \le |q_{j+1} - y_j| \le C\epsilon^{k_0}, \;\; C^{-1}\epsilon^{k_0} \le |q_j - y_j| \le C\epsilon^{k_0}.$$

Then by (q.2), we see that there exists a constant $C > 0$ such that $L \le C\epsilon^{1-k_0}$. Moreover, we know from [1, p. 108] that for $p > N$, $t, s \in \bar{Q}_j$ $(j = 1, \cdots, L)$ and $m = 1, \cdots, N$,

$$\left| \frac{\partial u_\epsilon}{\partial x_m}(t) - \frac{\partial u_\epsilon}{\partial x_m}(s) \right| \le C_3 |t - s|^{1-N/p} \|u_\epsilon\|_{2,p,Q_j}. \tag{2.35}$$

We know from [8] that for any $p > 1$ and $0 < \epsilon \ll 1$,

$$\begin{aligned} \|u_\epsilon\|_{2,p,\Omega} &\le C_4 \left(\frac{1}{\epsilon^2} \|f(u_\epsilon)\|_{0,p} + \|u_\epsilon\|_{0,p} \right) \\ &\le C_4 |\Omega|^{1/p} (\max_{0 \le s \le u_0} f(s)\epsilon^{-2} + u_0) \\ &\le 2C_4 |\Omega|^{1/p} \epsilon^{-2}. \end{aligned} \tag{2.36}$$

By definition,

$$\|u_\epsilon\|_{2,p,Q_j}^p = \int_{Q_j} |u_\epsilon|^p dx + \sum_{m=1}^N \int_{Q_j} \left|\frac{\partial u_\epsilon}{\partial x_m}\right|^p dx + \sum_{m,k=1}^N \int_{Q_j} \left|\frac{\partial^2 u_\epsilon}{\partial x_k \partial x_m}\right|^p dx. \tag{2.37}$$

Let $\xi = |u_\epsilon|, |\partial u_\epsilon/\partial x_m|$, or $|\partial^2 u_\epsilon/\partial x_m \partial x_k|$. Let χ_{Q_j} be a characteristic function of Q_j. Then for $\alpha, \beta > 1$ satisfying $1/\alpha + 1/\beta = 1$, by Hölder's inequality,

$$\int_{Q_j} \xi^p dx = \int_\Omega \xi^p \chi_{Q_j} dx \le \|\xi\|_{0,p\alpha}^p |Q_j|^{1/\beta} \le C_5 \epsilon^{k_0 N/\beta} \|\xi\|_{0,p\alpha,\Omega}^p. \tag{2.38}$$

Then we obtain

$$\|u_\epsilon\|_{2,p,Q_j} \le C_6 \epsilon^{k_0 N/(p\beta)} \|u_\epsilon\|_{2,p\alpha,\Omega}. \tag{2.39}$$

Let $\bar\alpha, \bar\beta > 1$ satisfy $1/\bar\alpha + 1/\bar\beta = 1$. Then by the same argument as above, we obtain

$$\|u_\epsilon\|_{2,p\alpha,Q_j} \le C_7 \epsilon^{k_0 N/(p\bar\beta)} \|u_\epsilon\|_{2,p\alpha\bar\alpha,\Omega}. \tag{2.40}$$

Since (2.35), in which p is replaced by $p\alpha\bar\alpha$ is also valid, by (2.36), (2.39) and (2.40), we obtain

$$\begin{aligned} \|u_\epsilon\|_{2,p,Q_j} &\le C_8 \epsilon^{k_0 N/(p\beta) + k_0 N/(p\bar\beta)} \|u_\epsilon\|_{2,p\alpha\bar\alpha,\Omega} \\ &\le C_9 \epsilon^{k_0 N/(p\beta) + k_0 N/(p\bar\beta) - 2}. \end{aligned} \tag{2.41}$$

Then by (q.3), (2.35) and (2.41)

$$\begin{aligned} &\left|\frac{\partial u_\epsilon}{\partial x_m}(\epsilon z', \psi(\epsilon z')) - \frac{\partial u_\epsilon}{\partial x_m}(0,0)\right| \\ &\quad = \left|\frac{\partial u_\epsilon}{\partial x_m}(q_L) - \frac{\partial u_\epsilon}{\partial x_m}(q_1)\right| \\ &\quad \le \sum_{j=1}^{L-1} \left|\frac{\partial u_\epsilon}{\partial x_m}(q_{j+1}) - \frac{\partial u_\epsilon}{\partial x_m}(q_j)\right| \\ &\quad \le \sum_{j=1}^{L-1} \left(\left|\frac{\partial u_\epsilon}{\partial x_m}(q_{j+1}) - \frac{\partial u_\epsilon}{\partial x_m}(y_j)\right| + \left|\frac{\partial u_\epsilon}{\partial x_m}(y_j) - \frac{\partial u_\epsilon}{\partial x_m}(q_j)\right| \right) \\ &\quad \le C_{10} \epsilon^{(k_0 N/p)(1/\beta + 1/\bar\beta - 1) - 1}. \end{aligned} \tag{2.42}$$

Choose $k_0 \gg 1$ and $0 < \delta \ll 1$ and put $p = N + \delta, \beta = \bar\beta = 1 + \delta$ so that $(k_0 N/p)(1/\beta + 1/\bar\beta - 1) - 1 > 0$. Then by ($\psi$2) and (2.42), we see that $I_\epsilon(z') \to 0$ uniformly on K as $\epsilon \to 0$. Therefore, by this and (2.33), we obtain (2.32), and (2.32) implies that $(\partial\phi/\partial z_N)(z', 0) = C_2$ on $K \cap \partial D$. Now let φ be a unique solution to the ODE

$$\varphi''(t) + f'(w(t))\varphi(t) = (N-1)H(0)w'(t), \quad t > 0, \tag{2.43}$$

$$\varphi(0) = 0, \tag{2.44}$$

$$\varphi'(0) = C_2. \tag{2.45}$$

Now put $\eta(z) := \phi(z) - \varphi(z_N)$. Then by (2.30), (2.31) and (2.43)–(2.45), η satisfies

$$\begin{aligned} \Delta\eta + f'(w)\eta &= 0 \quad \text{in } K, \\ \eta &= \frac{\partial \eta}{\partial z_N} = 0 \quad \text{on } K \cap \partial D. \end{aligned}$$

This implies that $\eta \equiv 0$ on K. Then unique continuation theorem of elliptic equation implies $\phi(z) = \varphi(z_N)$. Thus the proof is complete. □

Note that ϕ still depends on the choice of the subsequence of $\{\epsilon\}$ at this stage. To show that ϕ is independent of the choice of the subsequence of $\{\epsilon\}$, we shall prove that $\phi'(0)$ is independent of choice of the subsequence of $\{\epsilon\}$. To do this, we need the following lemma.

Lemma 2.4. $\|\phi\|_\infty \le C$.

Proof. We put $g_\epsilon(z_N) := \phi_\epsilon(0, \cdots, 0, z_N)$. We show that for $0 < \epsilon \ll 1$,

$$\|g_\epsilon\|_{L^\infty(R)} \le C. \tag{2.46}$$

Then by letting $\epsilon \to 0$, we obtain our assertion. The proof is divided into four steps.

Step 1. Let $B_1 := \{x_1^2 + \cdots + (x_N - r_1)^2 < r_1^2\} \subset M \subset \Omega$, where $r_1 > 0$ ($r_1 < b$) is a small constant and M is defined in ($\psi 3$) and ($\psi 4$). Furthermore, let $U_{1,\epsilon}$ be a unique solution to (1.1)–(1.3) in which Ω is replaced by B_1. Since $U_{1,\epsilon}$ is a subsolution to (1.1)–(1.3), we see that for $0 < \epsilon \ll 1$,

$$U_{1,\epsilon}(x) \le u_\epsilon(x), \quad x \in B_1. \tag{2.47}$$

As for the subsolution, we refer to [3, Appendix]. Now for $z_N \in B^+_{b/\epsilon}$, we put

$$\zeta_\epsilon(z_N) := \frac{U_{1,\epsilon}(\Phi(\epsilon(0, \cdots, 0, z_N)) - w(z_N)}{\epsilon} = \frac{U_{1,\epsilon}(\epsilon z_N) - w(z_N)}{\epsilon}. \tag{2.48}$$

By (2.23) and (2.47),

$$g_\epsilon(z_N) \ge \zeta_\epsilon(z_N). \tag{2.49}$$

Step 2. (a) We show that $\|\zeta_\epsilon\|_\infty \le C$ for $0 < \epsilon \ll 1$. Let $\tilde{B}_0 := \{|x| < r_1\}$ and $V_{1,\epsilon} = V_{1,\epsilon}(r)$ $(r = |x|)$ be a unique solution to (1.1)–(1.3), in which Ω is replaced by $\tilde{B}_0$. Then $V_{1,\epsilon}$ satisfies

$$-\epsilon^2 \left(V''_{1,\epsilon}(r) + \frac{N-1}{r} V'_{1,\epsilon}(r) \right) = f(V_{1,\epsilon}(r)), \ \ 0 < r < r_1.$$

Put $s = r_1 - r$. Then clearly, $U_{1,\epsilon}(s) = V_{1,\epsilon}(r)$ and $U_{1,\epsilon}$ satisfies

$$-\epsilon^2 \left\{ U''_{1,\epsilon}(s) - \frac{N-1}{r_1 - s} U'_{1,\epsilon}(s) \right\} = f(U_{1,\epsilon}(s)), \quad 0 < s < r_1. \tag{2.50}$$

By this, (2.10), (2.48) and mean value theorem, we obtain

$$-\zeta''_\epsilon(z_N) + \frac{N-1}{r_1 - \epsilon z_N}(w'(z_N) + \epsilon \zeta'_\epsilon(z_N)) = f'(w(z_N) + \theta_\epsilon \epsilon \zeta_\epsilon(z_N))\zeta_\epsilon(z_N), \tag{2.51}$$

where $0 < \theta_\epsilon := \theta_\epsilon(z_N) < 1$. Let

$$\bar{\zeta}_\epsilon(z) := \frac{U_{1,\epsilon}(\Phi(\epsilon z)) - w(z_N)}{\epsilon}.$$

Then by Lemma 2.2 for the case $\Omega = B_1$, we see that $\|\bar{\zeta}_\epsilon\|_\infty \le C_K$ for any compact subset $K \subset \bar{D}$ and $0 < \epsilon \ll 1$. Since $\zeta_\epsilon(z_N) = \bar{\zeta}_\epsilon(0, z_N)$, we see that $|\zeta_\epsilon(z_N)| \le C_l$ for any $0 \le z_N \le l$.

(b) Now assume that there exists a sequence $\{z_{N,\epsilon}\}$ ($z_{N,\epsilon} \in B^+_{r_1/\epsilon}$) such that $z_{N,\epsilon} \to \infty$ and $\|\zeta_\epsilon\|_\infty = |\zeta_\epsilon(z_{N,\epsilon})| \to \infty$ as $\epsilon \to 0$. Let $0 < \delta_0 < r_1$ be a fixed constant. We know (see Appendix) that for $0 < s < r_1$ and $0 < \epsilon \ll 1$,

$$u_0 - C\exp(-Cs/\epsilon) \le U_{1,\epsilon}(s) \le u_0. \tag{2.52}$$

Moreover, we know that for $t \gg 1$,

$$w(t) = u_0 - O(e^{-C_{10}t}). \tag{2.53}$$

By this and (2.52), we see that if there exists a constant $C_{11} > 0$ such that $\epsilon z_{N,\epsilon} \ge C_{11}$, then as $\epsilon \to 0$,

$$\zeta_\epsilon(z_{N,\epsilon}) = O\left(\frac{e^{-C_{12}/\epsilon}}{\epsilon}\right) \to 0. \tag{2.54}$$

Therefore, $\epsilon z_{N,\epsilon} \to 0$ as $\epsilon \to 0$.

Step 3. Under the assumption of Step 2 (b), we put $\eta_\epsilon(t) := \zeta_\epsilon(z_N - z_{N,\epsilon})/\|\zeta_\epsilon\|_\infty$, where $t = z_N - z_{N,\epsilon}$ and consider (2.51) in the interval $J := (-\delta z_{N,\epsilon}, \delta z_{N,\epsilon})$, where $\delta > 0$ is a small constant. Then by (2.51), for $t \in J$, we have

$$\begin{aligned} -\eta_\epsilon''(t) + \frac{N-1}{r_1 - \epsilon(t + z_{N,\epsilon})}\left\{\frac{w'(t + z_{N,\epsilon})}{\|\zeta_\epsilon(z_N)\|_\infty} + \epsilon\eta_\epsilon'(t)\right\} \\ = f'(W(t) + \theta_\epsilon\epsilon\zeta_\epsilon(t + z_{N,\epsilon}))\eta_\epsilon(t), \end{aligned} \tag{2.55}$$

where $W(t) = w(z_N)$. We show that

$$\epsilon\zeta_\epsilon(t + z_{N,\epsilon}) = \epsilon\zeta_\epsilon(z_N) = U_{1,\epsilon}(\epsilon z_N) - w(z_N) \to 0 \tag{2.56}$$

uniformly in J as $\epsilon \to 0$. Indeed, if $t \in J$, then clearly, we have

$$(1-\delta)z_{N,\epsilon} \le z_N = t + z_{N,\epsilon} \le (1+\delta)z_{N,\epsilon}.$$

This implies that if $t \in J$, then $z_N \to \infty$ as $\epsilon \to 0$ uniformly on J. Then by (2.52) and (2.53), as $\epsilon \to 0$,

$$\begin{aligned} |\epsilon\zeta_\epsilon(t + z_{N,\epsilon})| &= |\epsilon\zeta_\epsilon(\epsilon z_N)| \le |u_0 - U_{1,\epsilon}(\epsilon z_N)| + |u_0 - w(z_N)| \\ &= O(\exp(-Cz_N)) + O(\exp(-Cz_N)) \\ &\to 0. \end{aligned} \tag{2.57}$$

Then by the limiting procedure, we obtain that there exists $\eta \in C^2(\mathbf{R})$ such that

$$\eta''(t) = -f'(u_0)\eta(t), \quad t \in \mathbf{R}, \tag{2.58}$$

$$\|\eta\|_\infty = \eta(0) = 1. \tag{2.59}$$

By solving the first equation, we obtain $\eta(t) = c_1 e^{-\sqrt{-f'(u_0)}t} + c_2 e^{\sqrt{-f'(u_0)}t}$ for $t \in \mathbf{R}$. However, this is impossible, since $\|\eta\|_\infty = 1$. Therefore, $\zeta_\epsilon(z_N)$ is bounded.

Step 4. Finally, let $A_1 := \{a^2 < x_1^2 + \cdots + (x_N + a)^2 < R^2\}$ which satisfies $\Omega \subset A_1$, where $a > 0$ is a small constant and $R > 0$ is a large constant. Furthermore, let $U_{2,\epsilon}$ be a unique solution to (1.1)–(1.3) in which Ω is replaced by A_1. Since $U_{2,\epsilon}$ is a supersolution to (1.1)–(1.3), we see that for $0 < \epsilon \ll 1$,

$$U_{2,\epsilon}(x) \ge u_\epsilon(x), \quad x \in \Omega. \tag{2.60}$$

Now for $z_N \in B^+_{b/\epsilon}$, we put

$$\zeta_{2,\epsilon}(z_N) := \frac{U_{2,\epsilon}(\Phi(\epsilon(0,\cdots,0,z_N)) - w(z_N)}{\epsilon}. \tag{2.61}$$

Then by (2.23) and (2.60),

$$g_\epsilon(z_N) \le \zeta_{2,\epsilon}(z_N). \tag{2.62}$$

Then by the same argument as that just above, we also obtain that $\zeta_{2,\epsilon}(z_N)$ is bounded. Hence, by (2.49) and (2.62), we obtain our assertion. □

By Lemma 2.4 and (2.43), we see that $\|\phi'\|_\infty \le C$.

Lemma 2.5. $\phi'(0) = -(N-1)H(0)C_0/\sqrt{2F(u_0)}$.

Proof. Multiply (2.43) by w' and integrate it over $(0,R)$. Then

$$\int_0^R \phi''(t)w'(t)dt + \int_0^R f'(w(t))w'(t)\phi(t)dt = (N-1)H(0)\int_0^R w'(t)^2dt. \tag{2.63}$$

Then integration by parts yields

$$\phi'(R)w'(R) - \phi'(0)w'(0) + f(w(R))\phi(R) = (N-1)H(0)\int_0^R w'(t)^2dt. \tag{2.64}$$

Let $R \to \infty$. Then by (2.12), (2.14) and (f.2), we obtain

$$\phi'(0) = -\frac{(N-1)H(0)}{\sqrt{2F(u_0)}}\int_0^\infty w'(t)^2dt. \tag{2.65}$$

So the last task for us is to calculate $\int_0^\infty w'(t)^2dt$. Multiply (2.10) by w'. Then we have

$$\frac{d}{dt}\left(\frac{1}{2}w'(t)^2 + F(w(t))\right) \equiv 0. \tag{2.66}$$

This implies that $w'(t)^2/2 + F(w(t)) \equiv$ constant for $t \ge 0$. Then by (2.12) and putting $t = 0$, for $t \ge 0$, we obtain

$$\frac{1}{2}w'(t)^2 + F(w(t)) = \frac{1}{2}w'(0)^2 = F(u_0). \tag{2.67}$$

By (2.13), for $t \ge 0$, we have

$$w'(t) = \sqrt{2(F(u_0) - F(w(t))}. \tag{2.68}$$

By this and (2.14)

$$\begin{aligned} \int_0^\infty w'(t)^2dt &= \int_0^\infty \sqrt{2(F(u_0) - F(w(t))}w'(t)dt \\ &= \int_0^{u_0} \sqrt{2(F(u_0) - F(s)}ds = C_0. \end{aligned} \tag{2.69}$$

This along with (2.65) implies our assertion. □

Since we have

$$\begin{aligned} -\epsilon\frac{\partial u_\epsilon}{\partial \nu}(0) &= \frac{\partial w_\epsilon}{\partial z_N}(0) = w'(0) + \epsilon\frac{\partial \phi_\epsilon}{\partial z_N}(0) \\ &= \sqrt{2F(u_0)} + \epsilon(\phi'(0) + o(1)), \end{aligned} \tag{2.70}$$

this along with Lemma 2.5 implies Theorem 1.1. Thus we get Theorem 1.1. □

3. Appendix

In this section, we prove (2.52) for completeness. We consider the equation

$$\epsilon^2\left(V_{1,\epsilon}''(r)+\frac{N-1}{r}V_{1,\epsilon}'(r)\right)+f(V_{1,\epsilon}(r))=0,\ \ 0<r<r_1, \tag{3.1}$$

$$V_{1,\epsilon}(r)>0,\ \ 0<r<r_1, \tag{3.2}$$

$$V_{1,\epsilon}'(0)=V_{1,\epsilon}(r_1)=0. \tag{3.3}$$

It is known (cf. [5]) that by the change of independent variable

$$v_\epsilon(t):=V_{1,\epsilon}(r),\ \ t:=h(r)=\begin{cases}\frac{1}{N-2}\left[\frac{1}{r_1^{N-2}}-\frac{1}{r^{N-2}}\right] & (N\geq 3),\\ \log\frac{r}{r_1} & (N=2),\end{cases} \tag{3.4}$$

(3.1)–(3.3) is transformed into

$$-\epsilon^2 v''(t) = g(t)f(v(t)),\ \ -\infty<t<0, \tag{3.5}$$

$$v(t) > 0,\ \ -\infty<t<0, \tag{3.6}$$

$$v(0) = v'(-\infty)=0, \tag{3.7}$$

and v_ϵ is a solution to (3.5)–(3.7), where $g(t)=[h^{-1}(t)]^{2(N-1)}$. Let $a>0$ be fixed and consider here the auxiliary problem

$$-\epsilon^2 W''(t) = g(-a)f(W(t)),\ \ -a<t<0, \tag{3.8}$$

$$W(t) > 0,\ \ -a<t<0, \tag{3.9}$$

$$W(-a) = W(0)=0. \tag{3.10}$$

This equation has a unique solution W_ϵ if $0<\epsilon\ll 1$. Then we know from [5] that

$$W_\epsilon(t)\leq v_\epsilon(t),\ \ -a\leq t\leq 0. \tag{3.11}$$

Lemma 3.1. *There exist constants $C_2, C_3>0$ such that for $-a<t<0$ and $0<\epsilon\ll 1$*

$$0<u_0-C_2\exp(C_3t/\epsilon)\leq W_\epsilon(t). \tag{3.12}$$

Proof. Multiply (3.8) by $W_\epsilon'(t)$. Then for $-a\leq t\leq 0$, we obtain

$$(\epsilon^2 W_\epsilon''(t)+g(-a)f(W_\epsilon(t)))W_\epsilon'(t)=0.$$

This implies that

$$\frac{d}{dt}\left\{\frac{1}{2}\epsilon^2 W_\epsilon'(t)^2+g(-a)F(W_\epsilon(t))\right\}\equiv 0.$$

So, by putting $t=-a/2$, we obtain

$$\frac{1}{2}\epsilon^2 W_\epsilon'(t)^2+g(-a)F(W_\epsilon(t))=g(-a)F(\|W_\epsilon\|_\infty). \tag{3.13}$$

Then since $W_\epsilon'(t)<0$ for $-a/2<t<0$, we have

$$-W_\epsilon'(t)=\frac{1}{\epsilon}\sqrt{g(-a)}\sqrt{2F(\|W_\epsilon\|_\infty)-2F(W_\epsilon(t))}. \tag{3.14}$$

Let an arbitrary $0 < \delta \ll 1$ be fixed. Then for $-a/2 < -t_1 < 0$,

$$
\begin{aligned}
t_1 &= \int_{-t_1}^{0} dt \\
&= \int_{-t_1}^{0} \frac{-\epsilon W_\epsilon'(t)}{\sqrt{g(-a)}\sqrt{2F(\|W_\epsilon\|_\infty) - 2F(W_\epsilon(t))}} dt \\
&= \frac{\epsilon}{\sqrt{g(-a)}} \int_0^{W_\epsilon(-t_1)} \frac{1}{\sqrt{2F(\|W_\epsilon\|_\infty) - 2F(s)}} ds \\
&= \frac{\epsilon}{\sqrt{g(-a)}} \int_{u_0 - W_\epsilon(-t_1)}^{u_0} \frac{1}{\sqrt{2F(\|W_\epsilon\|_\infty) - 2F(u_0 - \eta)}} d\eta \\
&= \frac{\epsilon}{\sqrt{g(-a)}} \left[\int_\delta^{u_0} + \int_{u_0 - W_\epsilon(-t_1)}^{\delta} \right].
\end{aligned} \tag{3.15}
$$

Let $0 < C_4 < -f'(u_0)$ be fixed. Then for $u_0 - \|W_\epsilon\|_\infty \le \eta \le \delta$,

$$
m(\eta) := 2F(\|W_\epsilon\|_\infty) - 2F(u_0 - \eta) - C_4\eta^2 + C_4(u_0 - \|W_\epsilon\|_\infty)^2 > 0. \tag{3.16}
$$

Indeed, by (f.3) and Taylor expansion, for $u_0 - \|W_\epsilon\|_\infty \le \eta \le \delta$, we have

$$
\begin{aligned}
m'(\eta) &= 2f(u_0 - \eta) - 2C_4\eta \\
&= 2(-f'(u_0) - C_4)\eta - O(\eta^2) > 0.
\end{aligned} \tag{3.17}
$$

Since $m(u_0 - \|W_\epsilon\|_\infty) = 0$, by (3.17), we obtain (3.16). Then by (3.16), we obtain

$$
\begin{aligned}
t_1 &\le \frac{\epsilon}{\sqrt{g(-a)}} \left[C_\delta + \int_{u_0 - W_\epsilon(-t_1)}^{\delta} \frac{1}{\sqrt{C_4\eta^2 - C_4(u_0 - \|W_\epsilon\|_\infty)^2}} dt \right] \\
&= \frac{\epsilon}{\sqrt{g(-a)}} \left[C_\delta + \frac{1}{\sqrt{C_4}} \log|\delta + \sqrt{\delta^2 - (u_0 - \|W_\epsilon\|_\infty)^2}| \right. \\
&\qquad \left. - \frac{1}{\sqrt{C_4}} \log|(u_0 - W_\epsilon(-t_1))| \right].
\end{aligned}
$$

This implies

$$
\log|u_0 - W_\epsilon(-t_1)| \le \bar{C}_\delta - \frac{\sqrt{C_4 g(-a)}}{\epsilon} t_1. \tag{3.18}
$$

This yields (3.12).

By (3.4), let

$$
r_0 := \begin{cases} \left(\frac{2r_1^{N-2}}{2 + a(N-2)r_1^{N-2}} \right)^{1/(N-2)} & (N \ge 3), \\ r_1 e^{-a/2} \quad (N = 2). \end{cases} \tag{3.19}
$$

Then by this, Lemma 3.1 and a direct calculation, for $r_0 \le r < r_1$, we obtain

$$
u_0 - C_3 \exp(-C_4(r_1 - r)/\epsilon) \le V_{1,\epsilon}(r) < u_0. \tag{3.20}
$$

Then since $V_{1,\epsilon}(r) \ge V_{1,\epsilon}(r_0)$ for $0 \le r \le r_0$, we obtain

$$
\begin{aligned}
&u_0 - C_3 \exp(-C_4(r_1 - r_0)(r_1 - r)/(r_1\epsilon)) \\
&\quad \le u_0 - C_3 \exp(-C_4(r_1 - r_0)/\epsilon) \\
&\quad \le V_{1,\epsilon}(r_0) \le V_{1,\epsilon}(r) < u_0.
\end{aligned} \tag{3.21}
$$

Then by (3.20) and (3.21), it is clear that (3.21) holds for $0 \le r < r_1$ and $0 < \epsilon \ll 1$. Since $U_{1,\epsilon}(s) = V_{1,\epsilon}(r)$ and $s = r_1 - r$, we obtain (2.52). Thus the proof is complete. $\square$

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