UNITS IN SOME FAMILIES OF ALGEBRAIC NUMBER FIELDS

L. YA. VULAKH

ABSTRACT. Multi-dimensional continued fractions associated with $GL_n(\mathbb{Z})$ are introduced and applied to find systems of fundamental units in some families of totally real fields and fields with signature (2,1).

1. Introduction

Let $F$ be an algebraic number field of degree $n$. There exist exactly $n$ field embeddings of $F$ in $\mathbb{C}$. Let $s$ be the number of embeddings of $F$ whose images lie in $\mathbb{R}$, and let $2t$ be the number of non-real complex embeddings so that $n = s + 2t$. The pair $(s, t)$ is said to be the signature of $F$. Let $\mathcal{O}_F$ be the ring of integers of the field $F$. A unit in $F$ is an invertible element of $\mathcal{O}_F$. The set of units in $F$ forms a multiplicative group which will be denoted by $\mathbb{Z}_F^\times$. In 1840 P. G. Lejeune-Dirichlet determined the structure of the group $\mathbb{Z}_F^\times$. He showed that $\mathbb{Z}_F^\times$ is a finitely generated Abelian group of rank $r = s + t - 1$, i.e. $\mathbb{Z}_F^\times$ is isomorphic to $\mu_F \times \mathbb{Z}^r$, where $\mu_F$ is a finite cyclic group. $\mu_F$ is called the torsion subgroup of $\mathbb{Z}_F^\times$. Thus, there exist units $\epsilon_1, ..., \epsilon_r$ such that every element of $\mathbb{Z}_F^\times$ can be written in a unique way as $\zeta^{n_1} \epsilon_1^{n_1} ... \epsilon_r^{n_r}$, where $n_i \in \mathbb{Z}$ and $\zeta$ is a root of unity in $F$. Such a set $\{\epsilon_1, ..., \epsilon_r\}$ is called a system of fundamental units of $F$. Finding a system of fundamental units of $F$ is one of the main computational problems of algebraic number theory (see e.g. [6], p. 217). Much work has been done to solve this problem for certain classes of algebraic number fields (see e.g. [22]). In the case of the real quadratic fields, the continued fraction algorithm provides a very efficient method for solving this problem (see e.g. [22], p. 119). This approach goes back to L. Euler, who applied continued fractions to solve Pell’s equation $x^2 - dy^2 = \pm 1$. (If a square-free positive integer $d \equiv 2$ or $3 \pmod{4}$ and $x, y$ is an integral solution of this equation, then $x + \sqrt{d}y$ is a unit in the real quadratic field $Q(\sqrt{d})$. Moreover, any unit in $Q(\sqrt{d})$ can be obtained this way.) Many attempts have been made to develop a similar algorithm that would find a system of fundamental units in other algebraic number fields. In the case of a cubic field, one of the most successful such algorithms was introduced by G. F. Voronoi [28]. A review of the multi-dimensional continued fraction algorithms and their properties that were known by 1980 can be found in [2].

In [30] and [31], a continued fraction algorithm associated with a discrete group acting in a hyperbolic space was defined. The purpose of the present paper is to extend this definition to the case of the group $\Gamma = GL_n(\mathbb{Z})/\{\pm 1\}$ acting on
\( \mathcal{P} = SL_n(\mathbb{R})/SO_n(\mathbb{R}) \) and apply it to the problem of finding a system of fundamental units in an algebraic number field \( F \). The symmetric space \( \mathcal{P} \) can be identified with the set of definite quadratic forms in \( n \) real variables with the leading coefficient one. \( \mathcal{P} \) with the metric \( ds^2 = \text{trace}((X^{-1}dX)^2) \), where \( X = (x_{ij}) \in \mathcal{P} \) and \( dX = (dx_{ij}) \), is a Riemannian manifold (see e.g. [13] or [27]).

Assume that \( g \in GL_n(\mathbb{R}) \). Let \( ga_i = \lambda_i a_i, i = 1, \ldots, n \), so that \( a_i \) is an eigenvector of \( g \) corresponding to its eigenvalue \( \lambda_i \). For simplicity, assume that all the eigenvalues of \( g \) are distinct. Let \( P = (a_1, \ldots, a_n) \) be the matrix with columns \( a_1, \ldots, a_n \). The set of points in \( \mathcal{P} \) fixed by \( g \) will be called the \( g \)-orbit of \( P \) of \( g \). The axis \( L_P \) of \( g \) depends only on eigenvectors of \( g \), i.e. on \( P \), but not on its eigenvalues (see Section 3). \( L_P \) is a totally geodesic submanifold of \( \mathcal{P} \) of dimension \( s + t - 1 \), where \( s \) is the number of real and \( 2t \) the number of non-real complex eigenvalues of \( g \), so that \( n = s + 2t \). If \( t = 0 \), then \( L_P \) is an \( (n - 1) \)-flat in \( \mathcal{P} \) (see e.g. [13]).

Let \( (1, \omega_2, \ldots, \omega_n) \) be a \( \mathbb{Z} \)-basis of the ring of integers \( \mathbb{Z}_F \) of a number field \( F \) of degree \( n \). Let \( a_1 = (1, \omega_2, \ldots, \omega_n)^T \). Let \( g \in \mathbb{Z}_F \). Then \( g\omega_i = \sum m_{ij}\omega_j \) or \( g\omega_i = M_i a_1 \), where \( \omega_1 = 1, m_{ij} \in \mathbb{Z} \) and \( M_i = (m_{ij}) \) is a square matrix of order \( n \). Let \( \sigma_i \) be the \( n \) distinct embeddings of \( F \) in \( \mathbb{C} \). Let \( a_i = \sigma_i(a_1) \) and \( \gamma_i = \sigma_i(\gamma) \), where \( \gamma_1 = \gamma \). Then \( \gamma_i a_i = M_i a_i \) for \( i = 1, \ldots, n \). Thus, \( a_i \) is an eigenvector of \( M_i \) corresponding to its eigenvalue \( \gamma_i \). It is clear that the map \( \gamma \mapsto M_i \) is an isomorphism of the ring of integers \( \mathbb{Z}_F \) and the commutative ring of \( \mathbb{Z} \)-integral square matrices of order \( n \) with the common axis \( L_P \). The norm of \( \gamma \) equals \( \text{det}(M_i) \), so that \( \gamma \) is a unit in \( \mathbb{Z}_F \) if and only if \( M_i \in \text{GL}_n(\mathbb{Z}) \). The torsion-free subgroup \( \Gamma_L \) of the stabilizer of \( L_P \) is isomorphic to \( \mathbb{Z}_F/\mu_P \). Thus, the problem of finding a system of fundamental units of \( F \) is equivalent to the problem of finding a set of generators of \( \Gamma_L \). The [multi-dimensional continued fraction] Algorithm II introduced in this paper can be used to solve the latter problem. Here, a set of generators of \( \Gamma_L \) and therefore a system of fundamental units is found in some families of fields \( F \) of degree \( n \leq 4 \).

In Section 2, the notion of the height of a point in \( \mathcal{P} \) is introduced. Let \( w = (1,0,\ldots,0)^T \) and \( W = w w^T \). In what follows, the point \( W \), which belongs to the boundary of \( \mathcal{P} \), is analogous to the point \( \infty \) in the upper half-space model \( H^{n+1} = \{(z,t) : z \in \mathbb{R}^n, t > 0 \} \) of the \( (n+1) \)-dimensional hyperbolic space (see [30] and [31]). The set \( K = K(w) \) in \( \mathcal{P} \) is defined so that, for every point \( X \in \mathcal{P} \), the points in the \( \Gamma \)-orbit of \( X \) with the largest height belong to \( K(w) \). The images \( K[g] \) of \( K \), \( g \in \Gamma \), under the action of \( \Gamma \) form the \( K \)-tessellation of \( \mathcal{P} \). The \( K \)-tessellation of \( \mathcal{P} \) is \( \Gamma \)-invariant.

If \( L_P \cap K[g] \neq \emptyset \), \( g \in \Gamma \), then the vector \( u = g^{-1}w \in \mathbb{Z}^n \) is called a convergent of \( L_P \). In Section 3, it is shown that if \( u \) is a convergent of \( L_P \), then \( | \langle a_1, u \rangle \cdots \langle a_n, u \rangle/\det P | \), where \( \langle, \rangle \) denotes the dot product in \( \mathbb{R}^n \), is small (Theorem 7). Algorithm II, which is introduced in Section 3, can be applied to find the sets \( R(g^{-1}w) = L_P \cap K[g] \neq \emptyset \), which form a tessellation of \( L_P \), and the set of convergents of \( L_P \).

If \( g \in \Gamma \), then there are only finitely many sets \( R(u) \) which are not congruent modulo the action of \( \Gamma \). The union of non-congruent sets \( R(u) \) forms a fundamental domain of \( \Gamma_L \). Assume that the characteristic polynomial \( p(x) \) of \( g \) is irreducible. Let \( p(\epsilon) = 0 \). In Section 4, the problem of finding a system of fundamental units in \( F \) is solved for some families of totally real fields \( F = \mathbb{Q}(\epsilon) \) by reducing it, as explained above, to the problem of finding a set of generators of \( \Gamma_L \). In Examples
1-4, new proofs of certain known results are given. The new results obtained in Examples 5 and 6 can be presented as follows.

**Theorem 1.** Assume that $t > 3$ is a positive integer. Let $f(x) = x^4 + tx^3 - x^2 - tx + \alpha = x(x^2 - 1)(x + t) + \alpha$, $\alpha = \pm 1$. Let $f(\epsilon) = 0$. Assume that $\{1, \epsilon, \epsilon^2, \epsilon^3\}$ is a \(\mathbb{Z}\)-basis of the maximal order \(\mathcal{O}_F\) of the totally real quadratic field $F = \mathbb{Q}(\epsilon)$. Then $\mathcal{O}_F^\times/\{\pm 1\} = \langle \epsilon - 1, \epsilon, \epsilon + 1 \rangle$.

Note that the Galois group of $F$ is $D_4$ if $\alpha = 1$, and it is $S_4$ if $\alpha = -1$.

In Section 5, the problem of finding a system of fundamental units is solved for some families of fields with signature $(2, 1)$. The following theorems are proved in Examples 7, 8, and 9 respectively.

**Theorem 2.** Let $f(x) = x^4 + tx^3 + x^2 + tx + 1 = x(x^2 + 1)(x + t) + 1$, where $t \in \mathbb{Z}$. Let $f(\epsilon) = 0$. Assume that $\eta = \epsilon + \epsilon^{-1} = (-t \pm \sqrt{t^2 + 4})/2$ is a fundamental unit of the quadratic subfield $K = \mathbb{Q}((\sqrt{d})$, $d = t^2 + 4$, of the dihedral quartic field $F = \mathbb{Q}(\epsilon)$ with signature $(2, 1)$. Assume that $4t^2 - 9$ is square-free. Then $\{1, \epsilon, \epsilon^2, \epsilon^3\}$ is a \(\mathbb{Z}\)-basis of the maximal order $\mathcal{O}_F$ of the field $F$, and $\mathcal{O}_F^\times/\{\pm 1\} = \langle \epsilon, \eta \rangle$.

**Theorem 3** (cf. [20]). Let $t \geq 4$ be an integer. Let $f(x) = x^4 + tx^3 + x^2 + tx + 1 = x(x^2 + 1)(x + t) - 1$. Let $\epsilon$ be a real root of $f(x)$. Assume that the discriminant of the quartic field $F = \mathbb{Q}(\epsilon)$ with signature $(2, 1)$ is square-free. Then $\{1, \epsilon, \epsilon^2, \epsilon^3\}$ is a \(\mathbb{Z}\)-basis of the maximal order $\mathcal{O}_F$ of the field $F$, and $\mathcal{O}_F^\times/\{\pm 1\} = \langle \epsilon + t \rangle$.

**Theorem 4.** Let $\alpha = \pm 1$. Let $t \geq s$ be positive integers and let $\eta = \frac{1}{2}(t + \sqrt{d})$, where $d = t^2 + 4\alpha$. Let $f(x) = f_1(x)f_2(x) = x^4 + stx^3 + (t - \alpha s^2)x^2 + s(t^2 + 2\alpha)x - \alpha$, where $f_1(x) = (x^2 + sx - 1/\eta)$, $f_2(x) = (x^2 - sx/\eta + \eta)$. Let $f_1(\epsilon) = 0$. Then $\eta \in F = \mathbb{Q}(\epsilon)$. Assume that $\eta$ is a fundamental unit of the quadratic subfield $K = \mathbb{Q}(\sqrt{d})$ of $F$ and that $\Delta = 4s^2t^3 + 12\alpha s^2 t - s^4 + 16\alpha$ is square-free. Denote $p(x) = (s + (at - s^2)x + \alpha x^3)/(ts^2 + 1)$. Then $\{1, \epsilon, \epsilon^2, p(\epsilon)\}$ is a \(\mathbb{Z}\)-basis of the maximal order $\mathcal{O}_F$ of the field $F$, the discriminant of $F$ is $d^2\Delta$, and $\mathcal{O}_F^\times/\{\pm 1\} = \langle \epsilon, \eta \rangle$.

Families of cyclic quartic fields are considered in [15] and [35]. In [33], a one-dimensional version of Algorithm II is applied to find fundamental units in a two-parameter family of complex cubic fields.

The author thanks the referee for the useful suggestions which led to a significant improvement of this work.

## 2. Fundamental Domains and $K$-tessellation

Let $n \geq 2$ be a positive integer. Let $V_n$ be the space of symmetric $n \times n$ real matrices. The dimension of $V_n$ is $N = n(n+1)/2$. The action of $g \in G = GL(n, \mathbb{R})$ on $X \in V_n$ is given by

$$X \mapsto X[g] = g^TXg.$$ 

For a subset $S$ of $V_n$, denote $S[g] = \{X[g] \in V_n : X \in S\}$.

The one-dimensional subspaces of $V_n$ form the real projective space $V$ of dimension $N - 1$, so that for any fixed nonzero $X \in V_n$, all the vectors $kX \in V_n$, $0 \neq k \in \mathbb{R}$, represent one point in $V$. Denote by $\mathcal{P} \subset V$ the set of (positive) definite elements of $V$ and by $\mathcal{B}$ the boundary of $\mathcal{P}$ ($\mathcal{B}$ can be identified with the non-negative elements of $V$ of rank less than $n$). The group $G$ preserves both $\mathcal{P}$ and $\mathcal{B}$, as does its arithmetic subgroup $GL(n, \mathbb{Z})$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
The space \( V_n \) (and \( V \)) can be also identified with the set of quadratic forms \( A[x] = x^T A x \), \( A \in V_n \), \( x \in \mathbb{R}^n \). With each point \( a = (a_1, ..., a_n)^T \in \mathbb{R}^n \), we associate the matrix \( A = aa^T \in \mathcal{B} \) and the quadratic form

\[
A[x] = (a, x)^2 = (a_1 x_1 + ... + a_n x_n)^2
\]

of rank one. For \( g \in G \), we have \( (ga, x) = a^T g^T x = (a, g^T x) \).

Let \( w = (1, 0, ..., 0)^T \) and \( W = ww^T \). Then \( \langle w, x \rangle^2 = x_1^2 \) and \( W[g] = U = uu^T \), where \( u = g^T w \).

Denote by \( G_\infty \) and \( \Gamma_\infty \) the stabilizers of \( w \) in \( G \) and \( \Gamma = GL(n, \mathbb{Z})/\{ \pm 1 \} \) respectively. Then

\[
G_\infty = \{ g \in G : gw = w \} = \{ g \in G : g_1 = w \},
\]

where \( g_1 \) is the first column of \( g \). Thus, \( g \in G_\infty \) if and only if \( W[g^T] = W \).

We shall say that \( A \in V \) is extremal if \( |A[x]| \geq |A[w]| = a_{11}^2 \) for any \( x \in \mathbb{Z}^n \), \( x \neq (0, ..., 0) \). Let \( \mathcal{A}_n = \{ X \in V : X[w] \neq 0 \} \). It is clear that \( \mathcal{P} \subset \mathcal{A}_n \). For \( X \in \mathcal{A}_n \), we shall say that

\[
ht(X) = |\det(X)|^{1/n}/|X[w]|
\]

is the height of \( X \) and, for a subset \( S \) of \( V \), we define the height of \( S \) as

\[
ht(S) = \max ht(X), \quad X \in S.
\]

The elements of \( \mathcal{A}_n \) will be normalized so that \( X[w] = 1 \). For a fixed \( g \in \Gamma \), the set \( \{ X \in \mathcal{A}_n : |X[gw]| < 1 \} \) is called the \( g \)-strip (cf. \[32\], [29], where this definition is introduced for \( n = 2 \)). It is clear that the \( gh \)-strip coincides with the \( g \)-strip for any \( h \in \Gamma_\infty \). The plane

\[
L^+(gw) = L^+(g) = \{ X \in \mathcal{A}_n : X[gw] = 1 \}
\]

is the boundary of the \( g \)-strip which cuts \( \mathcal{P} \). Let \( \mathcal{R}_w \) be the set of all extremal points of \( V \). Denote

\[
K = K(w) = \mathcal{P} \cap \mathcal{R}_w.
\]

(In the notation of \[1\], p.148, \( K \) is the dual core of \( K_{\text{pert.}} \).) Note that \( K \subset \mathcal{A}_n \) is bounded by the planes \( L^+(g) \). If \( h \in \Gamma_\infty \), then \( X[hw] = X[w] \) and, therefore, \( ht(X[h]) = ht(X) \). Thus,

\[
K[h] = K, \quad h \in \Gamma_\infty.
\]

Let \( q(x) \) be an indefinite quadratic form in \( n > 2 \) variables. By Margulis’ theorem \[19\], if \( q(x) \) is not a multiple of a quadratic form with integer coefficients, then the infimum of \( |q(x)| \) taken over all \( x \in \mathbb{Z}^n \), \( x \neq (0, ..., 0) \), is equal to zero. Hence, all the points of \( \mathcal{R}_w - \mathcal{P} \) are rational if \( n > 2 \). By Meyer’s theorem (see e.g. \[5\]), if the coefficients of \( q(x) \) are rational and \( n = 5 \), then there is \( x_0 \in \mathbb{Z}^5 \), \( x_0 \neq (0, ..., 0) \), such that \( q(x_0) = 0 \). Thus, \( \mathcal{R}_w - \mathcal{P} \) is empty and \( K = \mathcal{R}_w \) if \( n > 4 \).

Let \( D \) be any of the fundamental domains of \( \Gamma \) obtained by Minkowski, Korkine and Zolotarev (see e.g. \[24\], p.13), or Grenier \[16\]. For \( X \in D \) we have \( X[w] = \inf \tilde{X}[gw] \), \( g \in \Gamma \), in any of these cases. Hence, \( \bigcup D[g] = K \), the union being taken over all \( g \in \Gamma_\infty \). Note that the fundamental domain described in \[16\] coincides with the domain found by Korkine and Zolotarev in 1873 (see \[18\] or \[24\]). Unless a point \( X \in \mathcal{P} \) is integral, like point \( I \) in Example 1 below, in order to prove that \( X \) is extremal, we shall show that \( X[h] \) is Minkowski reduced for some \( h \in \Gamma_\infty \).
The main features of our approach to the problem of finding a system of fundamental units in an algebraic number field can be seen in the following simple example.

Example 1. Let \( \Gamma = GL_2(\mathbb{Z}) \). Let \( X = (x_{ij}) \in V \). A point \( X \) lies in \( \mathcal{B} \) if and only if \( \det(X) = x_{11}x_{22} - x_{12}^2 = 0 \). Thus, when \( n = 2 \), \( \mathcal{B} \) is a conic in the projective plane \( V \), and \( \mathcal{P} \), which consists of the points \( X \) with \( \det(X) > 0 \), is the Klein model of the hyperbolic plane. Let \( f(x) = x^2 - tx - 1 \), where \( t \in \mathbb{Z} \). Let \( f(\epsilon) = 0 \). Assume that either \( t^2 + 4 \) or \( t^2/4 + 1 \) is a square-free integer. Then \( \{1, \epsilon\} \) is a \( \mathbb{Z} \)-basis of the maximal order \( \mathcal{O}_F \) of the field \( F = \mathbb{Q}(\epsilon) \). Let

\[
E = \begin{bmatrix} 0 & 1 \\ 1 & t \end{bmatrix}.
\]

Then \( f(x) \) is a characteristic polynomial of \( E \); and \( a_1 = (1, \epsilon)^T \) and \( a_2 = (1, -1/\epsilon)^T \) are eigenvectors of \( E \) corresponding to its eigenvalues \( \epsilon \) and \(-1/\epsilon \) respectively. Let \( A_i = a_ia_i^T \). Let \( L_P \) be the axis of \( E \). Then \( L_P \) is the interval \( \mu A_1 + (1 - \mu) A_2 \), \( 0 < \mu < 1 \), which is a geodesic in \( \mathcal{P} \). The identity matrix \( I \) is the intersection of \( L_P \) with \( L^\infty(E) \), and the interval \( (I, I[E]) \) is a fundamental domain of \( \Gamma_L \) on \( L_P \). Thus, \( \Gamma_L = \langle E \rangle \) and, therefore, \( \mathcal{O}_F/\langle \pm 1 \rangle = \langle \epsilon \rangle \).

The period length of the corresponding continued fraction is one (see Remark 2 at the end of Section 3). More examples of this type can be found in [30] and [31].

Similar families of algebraic number fields of degree three and four are considered in Sections 4 and 5.

By [2], \( K[h_g] = K[g] \) for any \( g \in \Gamma \) and \( h \in \Gamma_\infty \). Thus, the sets \( K[g] \) are parameterized by the classes \( \Gamma_\infty \backslash \Gamma \) or by primitive vectors \( u = g^{-1}h^{-1}w = g^{-1}w \), so that \( \pm u \) represent the same \( K[g] \). The sets \( K[g], g \in \Gamma_\infty \backslash \Gamma \), form a tessellation of \( \mathcal{P} \) which will be called the \( K \)-tessellation. It is clear that the \( K \)-tessellation of \( \mathcal{P} \) is \( \Gamma \)-invariant. The vertices of \( K \) are called the perfect forms (see e.g. [1] or [24]). The perfect forms are known for \( n \leq 7 \).

3. Axes of elements of \( G \)

Let \( g \in G \). Let \( ga_i = \lambda_i a_i \), \( i = 1, ..., n \), where, for simplicity, we assume that \( \lambda_i \neq \lambda_j \) if \( i \neq j \). Here \( a_i \) is an eigenvector of \( g \) corresponding to its eigenvalue \( \lambda_i \). Assume that \( \langle a_i, w \rangle \neq 0 \), \( i = 1, ..., n \). Then we can choose \( a_i \) so that

\[
\langle a_i, w \rangle = 1, \quad i = 1, ..., n.
\]

g \in \Gamma is said to be irreducible if its characteristic polynomial is irreducible over \( \mathbb{Q} \). If \( g \in \Gamma \) is irreducible, then all its eigenvalues are distinct. Assume that \( \lambda_1, \lambda_2, ..., \lambda_s \) are real and \( \lambda_s+1, \lambda_s+2, ..., \lambda_s+t, \bar{\lambda}_s+2, ..., \bar{\lambda}_s+t \) are non-real complex eigenvalues of \( g \), so that \( n = s + 2t \). Let \( \lambda_k \neq \pm 1 \). Let

\[
P = (a_1, ..., a_s, a_{s+1}, \bar{a}_{s+1}, ..., a_{s+t}, \bar{a}_{s+t})
\]

be the matrix with columns \( a_1, ..., a_s, a_{s+1}, \bar{a}_{s+1}, ..., a_{s+t}, \bar{a}_{s+t} \) and let

\[
H = \text{diag}(\lambda_1, ..., \lambda_s, \lambda_s+1, \bar{\lambda}_s+1, ..., \lambda_s+t, \bar{\lambda}_s+t).
\]

Then \( g = PHP^{-1} \). For \( k > s \), let \( \alpha_k = \alpha_k + i\beta_k \), where \( \alpha_k, \beta_k \in \mathbb{R}^n \). Then

\[
det P = (2i)^s det(a_1, ..., a_s, \beta_{s+1}, \alpha_{s+1}, ..., \beta_{s+t}, \alpha_{s+t}).
\]
The totally geodesic submanifold $L_P$ of $\mathcal{P}$ fixed by $g = PHP^{-1}$ will be called the axis of $g$. The dimension of $L_P$ is $s + t - 1$. It can be identified with the set of quadratic forms in $A_n$

\begin{equation}
q[x] = \sum_{i=1}^{s} \mu_i \langle x, a_i \rangle^2 + \sum_{i=1}^{t} \mu_{s+i} \langle x, a_{s+i} \rangle^2, \quad \mu_i \geq 0, \quad \sum_{i=1}^{s+t} \mu_i = 1.
\end{equation}

or

\begin{equation}
q[x] = \sum_{i=1}^{s} \mu_i \langle x, a_i \rangle^2 + \sum_{i=1}^{t} \mu_{s+i}(\langle x, a_{s+i} \rangle^2 + \langle x, \beta_{s+i} \rangle^2).
\end{equation}

Hence

$$\det q = (-4)^{-t} \mu_1 \ldots \mu_s \mu_{s+2}^{2} \ldots \mu_{s+t}^{2} (\det P)^2.$$ 

It follows from [3] that $L_P$ is the axis of $h \in G$ if and only if $a_i$, $i = 1, \ldots, n$, are eigenvectors of $h$. Hence, the axis of $g$ depends only on its set of eigenvectors, i.e. on $P$, but not on the eigenvalues of $g$. A point $q \in L_P$ can be also represented as

$$q = \sum_{i=1}^{s+t} \mu_i A_i, \quad A_i = a_i a_i^T, \quad i \leq s, \quad A_{s+i} = \alpha_{s+i} a_{s+i}^T + \beta_{s+i} \beta_{s+i}^T.$$ 

Thus, $L_P$ is the simplex with vertices $A_i$, $i = 1, \ldots, s + t$. All the faces of $L_P$ belong to $B$. Note that $L_P[g^T] = L_P$. The curve $q[g^v] = (\sum \mu_i |\lambda_i|^{2v} A_i) / \sum \mu_i |\lambda_i|^{2v}$ is a geodesic in $L_P$ through $q$. If $|\lambda_1| < |\lambda_2| < \ldots < |\lambda_{s+t}|$, then $q[g^v] \to A_1$ as the real parameter $v \to -\infty$, and $q[g^v] \to A_{s+t}$ as $v \to \infty$. Note that if $t = 0$, then $L_P$ is an $(n - 1)$-flat in $\mathcal{P}$ and the stabilizer of $L_P$ in $G$ is a maximal commutative subgroup of $G$ (see e.g. [13]).

Denote $K(g^{-1}w) = K[g]$ and

\begin{equation}
R(g^{-1}w) = K[g] \cap L_P \neq \emptyset, \quad g \in \Gamma \backslash \Gamma.
\end{equation}

The sets $R(u)$, $u = g^{-1}w$, form a tessellation of $L_P$, which is invariant modulo the action of $\Gamma$ since the $K$-tessellation of $\mathcal{P}$ is $\Gamma$-invariant. We say that this tessellation is period if there are only a finite number of non-congruent sets $R(u)$ modulo the action of Stab($L_P, \Gamma$). In that case, the union of all non-congruent sets $R(u)$ is a fundamental domain of Stab($L_P, \Gamma$). The number of non-congruent sets $R(u)$ in the tessellation of $L_P$ will be called the period length.

Denote $\gamma_n = 1/h_n$, $h_n = \inf \text{ht}(A)$, $A \in K(w)$. Hermite’s constant $\gamma_n$ is known for $n \leq 8$ (see e.g. [3]). Let $C_n = \gamma_n/n$. Then

$$C_2 = 1/\sqrt{3}, \quad C_3 = 2^{1/3}/3, \quad C_4 = 1/\sqrt{8}, \quad C_5 = 8^{1/5}/5,$$

$$C_6 = 1/3^{7/6}, \quad C_7 = 64^{1/7}/7, \quad C_8 = 1/4,$$

and for large $n$ (see [3]),

$$\left\lvert \frac{1}{2\pi e} \right\rvert \leq C_n \leq \frac{1.744}{2\pi e}.$$

Let

$$N_P(x) = \langle x, a_1 \rangle \ldots \langle x, a_n \rangle = \langle x, a_1 \rangle \ldots \langle x, a_s \rangle |\langle x, a_{s+1} \rangle \ldots \langle x, a_{s+t} \rangle|^2,$$

where $\langle x, a_i \rangle = x^T a_i$. Define

$$\nu(L_P) = \inf \frac{N_P(gw)}{\det P},$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
where the infimum is taken over all \( g \in \Gamma \). It is clear that \( \nu(L_P) = \nu(L_{MP}[h]) \) for any \( h \in \Gamma \), and \( M = \text{diag}(\mu_1, \ldots, \mu_s, \mu_{s+1}, \pi_{s+1}, \ldots, \mu_{s+t}, \pi_{s+t}) \), where \( \mu_1, \ldots, \mu_s \in \mathbb{R} \), \( \mu_{s+1}, \ldots, \mu_{s+t} \in \mathbb{C} \) and \( \mu_1 \ldots \mu_{s+t} \neq 0 \). The projective invariant \( \nu(L_P) \) is well known in the geometry of numbers (see e.g. \cite{4, 17}). In particular, for \( n = s = 3 \), it was shown by Davenport \cite{9, 10, 11} that \( \nu(L_P) \leq 1/7 \), where the equality holds only if \( ga_i = (1, \alpha_i, \alpha_i^2) \), \( i = 1, 2, 3 \), the \( \alpha_i \) being the roots of \( \alpha^3 + \alpha^2 - 2\alpha - 1 = 0 \), for some \( g \in \Gamma \). For other matrices \( P \), \( \nu(L_P) \leq 1/9 \). Swinnerton-Dyer \cite{26} found the consequent seventeen values of \( \nu(L_P) \).

A point \( q_m \in L_P \) is said to be the sum of \( L_P \) if \( |\det(q_m)| = \max |\det(q)| \), the maximum being taken over all \( q \in L_P \). It is clear that if \( R = L_P \cap K(w) \neq \emptyset \), then \( q_m \in R \).

**Lemma 5.** Let \( L_P \) be the totally geodesic manifold fixed by \( g \in G \) and defined by \( P \), where \( ga_i = \lambda_i a_i \). Let \( P = (a_1, \ldots, a_n) \) be the matrix with columns \( a_1, \ldots, a_n \). Then

\[
q_m[x] = \frac{1}{n} \sum_{i=1}^{s} \langle x, a_i \rangle^2 + \frac{2}{n} \sum_{i=1}^{t} \det(a_{s+i})
\]

is the sum of \( L_P \),

\[
ht(L_P) = \frac{1}{n} \left| \det P \right|^{2/n} \left| N_P(w) \right|, \\
\nu(L_P) = \inf(nht(L_P(g)))^{-n/2}, \quad g \in \Gamma.
\]

**Proof.** The maximum of \( m = \mu_1 \ldots \mu_s \mu_{s+1}^2 \ldots \mu_{s+t}^2 \) subject to \( \mu_i \geq 0 \), \( \sum_{i=1}^{s+t} \mu_i = 1 \), is attained when \( \mu_k = 1/n \), \( k \leq s \), and \( \mu_k = 2/n \), \( k > s \). Thus, \( \max m = 4^n n^{-n} \), \( (3) \) holds, and \( \max \det(q) = \det(q_m) = |\det P|^2 n^{-n} \). By \( (3) \), \( N_P(w) = q[w] = 1 \). Hence

\[
ht(L_P) = |\det(q_m)|^{1/n} = \frac{1}{n} \left| \det P \right|^{2/n} \left| N_P(w) \right|, \\
\nu(L_P) = \inf(nht(gL_P))^{-n/2}, \quad g \in \Gamma, \quad \text{as required.}
\]

It follows that, for any \( L_P \), \( \text{ht}(X) = |\det(X)|^{1/n} \to 0 \) as \( X \in L_P \) approaches the boundary of \( L_P \). Note that \( nq_m[x] \) is the form size \( (M_x) \) from \cite{9}, p. 169.

In Example 1 above, by Lemma 5, \( q_m = (A_1 + A_2)/2 = (1 + I[\bar{E}])/2 \) and \( \det(q_m) = i^2/4 + 1 \) and \( \text{ht}(L_P) = \sqrt{i^2/4 + 1} \).

Assume that \( L_P \cap K(w) = \emptyset \). Let \( q_m \) be the sum of \( L_P \). Since \( q_m \notin K(w) \), there is \( g \in \Gamma \) such that \( \text{ht}(L_P(g)) \geq \text{ht}(q_m) > \text{ht}(q_m) = \text{ht}(L_P) \). We have obtained the following.

**Lemma 6.** Let \( L_P \) be the totally geodesic manifold fixed by \( g \in G \) and defined by \( P \), where \( ga_i = \lambda_i a_i \). Then

\[
\nu(L_P) = \inf(nht(L_P[g_j]))^{-n/2}, \quad L_P \cap K(g_j w) \neq \emptyset, \quad g_j \in \Gamma.
\]

By Lemma 6,

\[
\nu(L_P) < (nh_n)^{-n/2} = C_n^{-n/2}.
\]

For \( n = 2 \), \( \nu(L_P) < C_2 = 1/\sqrt{3} \). sup \( \nu(L_P) = 1/\sqrt{5} \) is a well-known approximation constant. It is attained only if \( ga_1 = (1, (1 \pm \sqrt{5})/2) \) for some \( g \in \Gamma \). For other matrices \( P \), \( \nu(L_P) \leq 1/\sqrt{8} \) (cf. Example 1 above, where, for \( t = 1 \) and \( 2 \), the
Theorem 7. We have proved the following.

the connection with respect to this face (see [33], p. 1310, and Example 1).

$= 2$ or $3$, the floor of $n$ consists of numbers $(9 - 4/m^2)^{-1/2}$, where $m$ runs through the set of all positive integers such that $(m, m_1, m_2)$ is a solution of the Diophantine equation $m^2 + m_1^2 + m_2^2 = 3mm_1m_2$. This result was obtained by Markov in 1879 (see e.g. [17], Section. 43).

When $n = 3$, we have $\nu(L_P) < \sqrt{2/27} = 0.2722$. This estimate does not depend on the dimension of $L_P$. It was shown by Davenport [12] that $\nu(L_P) \leq 1/\sqrt{23} = 0.2085$ when $s = t = 1$, where the equality holds only if $ga_i = (1, \alpha_i, \alpha_i^2)$, $i = 1, 2, 3$, the $\alpha_i$ being the roots of $\alpha^3 - \alpha - 1 = 0$, for some $g \in \Gamma$. As mentioned above, $\nu(L_P) \leq 1/7 = 0.1429$ when $\dim L_P = 2$.

Assume that $L_P \cap K(gw) \neq \emptyset$, where $g \in \Gamma$. Since $L_P[g] \cap K(w) \neq \emptyset$, by Lemma 5,

$$\text{ht}(L_P[g]) = \text{ht}(L_{g^TP}) = \frac{1}{n} \left( \frac{\det P}{N_{g^TP}(w)} \right)^{2/n} > h_n = 1/\gamma_n.$$ 

But $N_{g^TP}(x) = \langle x, g^T a_1 \rangle ... \langle x, g^T a_n \rangle = \langle gx, a_1 \rangle ... \langle gx, a_n \rangle$. Hence $N_{g^TP}(w) = N_P(gw)$.

A vector $gw \in \mathbb{Z}^n$ such that $L_P \cap K(gw) \neq \emptyset$ will be called a convergent of $L_P$. We have proved the following.

Theorem 7. If a vector $u$ is a convergent of $L_P$ (that is, if $L_P \cap K(u) \neq \emptyset$), then

$$|N_P(u)| < C_n^{n/2} |\det P|,$$

where $C_n = \gamma_n/n$ and $\gamma_n$ is Hermite’s constant. Hence if $L_P$ cuts infinitely many sets $K(u)$, then this inequality has infinitely many solutions in $u \in \mathbb{Z}^n$.

A component of the boundary of a set $R(u)$ of codimension one will be called a face of $R(u)$. We shall say that sets $R(u)$ and $R(u')$ are neighbors if they have a common face, and if the sets $R(u)$ and $R(u')$ are neighbors then the convergents $u$ and $u'$ are neighbors. The following lemma can be used to find the faces of $R = L_P \cap K(w) \neq \emptyset$ (see e.g. Example 4 below).

Lemma 8. Let $L_P$ be the axis of $g \in \Gamma$. Assume that $R = L_P \cap K(w) \neq \emptyset$. Let $R_i = L_P \cap K(u_i)$, $i = 1, 2, ..., \phi_i$ be the neighbors of $R$, so that $R$ and $R_i$ have a common face $\phi_i$. Then $\phi_i \subset L^+(u_i)$.

Proof. Assume that $K(w)$ and $K(gw)$ have a common face and that $X \in K(w) \cap K(gw)$. Then $X[g] \in K(w)$. Hence $X[w] = X[gw] = 1$ and $X \in L^+(gw)$. Thus, the common face of $K(w)$ and $K(gw)$ lies in $L^+(gw)$.

Algorithm II below can be used to enumerate all the sets $R(u)$ which form the tessellation of $L_P$. Our ability to apply this algorithm is based on the assumption that one can find all the faces of $R = L_P \cap K(w)$ for any $L_P$. Then all the faces of any $R_i = L_P \cap K(u_i) \neq \emptyset$, where $u_i = gw$ and $g_i \in \Gamma$, can be found since $R_i[g_i] = L_P[g_i] \cap K(w)$. It is clear that $g_i$ is not unique. Algorithm II will make the choice of $g_i$ more specific.

Let $D$ be a fundamental domain of $\Gamma$. The floor of $D$ consists of the faces of $D$ which do not contain the point $W$. Let $\mathcal{F}_D = \{\phi_1, ..., \phi_m\}$ be the set of faces in the floor of $D$. Let $S_D = \{S_1, ..., S_m\}$, $S_i \in \Gamma$, where the face $\phi_i \subset L^+(S_i^{-1})$. When $n = 2$ or $3$, the floor of $D$ consists of only one face, and one can choose $S_1$ to be the reflection with respect to this face (see [33], p. 1310, and Example 1).
As mentioned above, the tessellation of $L_P$ is invariant with respect to the action of $\Gamma$. Hence, we can assume that $L_P \cap K(w) \neq \emptyset$. Indeed, if $L_P \cap K(w) = \emptyset$, take a point $X \in L_P$ and find $h \in \Gamma$ such that $X[h] \in K(w)$. (Any of the reduction algorithms (see e.g. [8] for references) can be used to find such an $h$.) Then $L_P[h] \cap K(w) \neq \emptyset$.

Denote by $V_L$ the set $\{R(w)\}$, where the sets $R(u)$ form the tessellation of $L = L_P$. There is a unique graph $\overline{G}_L = (V_L, \overline{E}_L)$ associated with $L$ whose set of vertices is $V_L$, and there is an edge $(R, R') \in \overline{E}_L$ if and only if $R, R' \in V_L$ are neighbors.

**Algorithm II.** This algorithm finds a spanning tree $G_L = (V_L, E_L)$ of the graph $\overline{G}_L$. An edge $(R, R') \in E_L$ is labeled by $T \in \Gamma$ if $R = L \cap K[g]$ and $R' = L \cap K[Tg]$.

If the dimension of $L$ is one, then $\overline{G}_L = G_L$.

**Input.** A simplex $L \subset P$ with vertices at $A_i \in B$, where $A_i = a_i a_i^T$ for $i = 1, \ldots, s$, $A_{s+i} = \alpha_{s+i} + \alpha_{s+i}^T$ for $i = 1, \ldots, t$, $a_i, \beta_{s+i} \in \mathbb{R}^n$, and $\det(a_1, \ldots, a_s, \beta_{s+1}, \alpha_{s+1}, \ldots, \beta_{s+t}, \alpha_{s+t}) \neq 0$.

$R_0 = L \cap K(w) \neq \emptyset$.

**Output.** A tree $G_L = (V_L, E_L)$ where $V_L = \{R(u)\}$, where sets $R(u)$ form the tessellation of $L$.

Denote by $(V, E)$ a subtree of $G_L$ where $V \subset V_L$ and $E \subset E_L$ are current sets of vertices and edges of $G_L$ found.

Let $L$ be an ordered list of leaves of the subtree $(V, E)$ of $G_L$.

Let $R \in L$. Let $\{R_k, k = 1, 2, \ldots\}$ be the set of neighbors of $R$ such that $R_k \notin V$.

Denote by $N(R)$ an ordered set of faces $\psi_k = R \cap R_k$.

The root of $G_L$ is $R_0 = R(w)$.

$V = L = \{R_0\}$ and $E = \{\emptyset\}$.

**Step 1.** Let $\psi_k \in N(R_0)$. Find $U_k \in \Gamma_{\infty}$ such that $\psi_k \subset \phi_k[U_k]$ for some $\phi_k \in \mathcal{F}_D$. Let $L_k = L[U_k]$.

Then the face $\psi'_k = \psi_k[U_k^{-1}]$ of $R' = L_k \cap K(w)$ lies in $\phi_k$.

**Step 2.** Let $\psi'_k \subset \phi_k \subset L^+(S_k^{-1})$, where $S_k \in \mathcal{S}_D$, be as in step 1. Denote $T_k = S_k U_k$.

(Then $L_k = L_k[S_k^{-1}] = L[T_k^{-1}]$ cuts $K(w)$, and $L = L_k[T_k]$.)

Add $R_k = L \cap K[T_k]$ to $V$ and $L$, and add the edge $(R_0, R_k)$ labeled by $T_k$ to $E$.

**Step 3.** If $N(R_0) \neq \emptyset$, then go to step 1. Otherwise, remove $R_0$ from $L$.

(When $R_0$ is removed from $L$, $L$ consists of all the neighbors of $R_0$ and $V = L \cup R_0$.)

**Step 4.** Let $R \in L$. Let $(R_0, R_1, \ldots, R_{i-1}, R)$ be the path in $G_L$ from $R_0$ to $R$, whose edges $(R_{s-1}, R_s)$ are labeled by $T_s \in \Gamma$, $s = 1, \ldots, i$, so that $R_s = L \cap K[T_s \ldots T_1]$ and $R = L \cap K[g]$, where $g = T_i \ldots T_1$.

(Then $R[g^{-1}] = L[g^{-1}] \cap K(w) \neq \emptyset$. If $L[g^{-1}] = L$, then $R[g^{-1}] = R_0$ and $g \in \Gamma_L$.)

**Step 5.** Let $\psi_k \in N(R)$. Find $U_k \in \Gamma_{\infty}$ such that $\psi_k[g^{-1}] \subset \phi_k[U_k]$ for some $\phi_k \in \mathcal{F}_D$. Let $L_k = L[g^{-1}U_k^{-1}]$.

Then the face $\psi'_k = \psi_k[g^{-1}U_k^{-1}]$ of $R' = L_k \cap K(w)$ lies in $\phi_k$.

**Step 6.** Let $\psi'_k \subset \phi_k \subset L^+(S_k^{-1})$, where $S_k \in \mathcal{S}_D$, be as in step 6. Denote $T_k = S_k U_k$.

(Then $L_k = L_k[S_k^{-1}] = L[g^{-1}T_k^{-1}]$ cuts $K(w)$, and $L = L_k[T_k g]$.)

Add $R_k = L \cap K[T_k g]$ to $V$ and $L$, and add the edge $(R, R_k)$ labeled by $T_k$ to $E$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Step 7. If \( \mathcal{N}(R) \neq \emptyset \), then go to step 5. Otherwise, remove \( R \) from \( \mathcal{L} \) and go to step 4.

(When \( R \) is removed from \( \mathcal{L} \), all the neighbors of \( R \) have been added to \( \mathcal{V} \).)

Remarks. 1. By [2], \( R = L \cap K[g] = R(g^{-1}w) \), and the convergent associated with the set \( R \) is \( u = g^{-1}w \), where \( g = T_{i_1} \ldots T_{i_r} \). Thus, Algorithm II can be used to find all the convergents of \( L \).

2. Assume that the sets \( R_i \) form the tessellation of the axis \( L_P \) of an irreducible \( g \in \Gamma \). Then the number of non-congruent sets \( R_i \) is finite and the tessellation of \( L_P \) is periodic. Let \( R_1 \cup \ldots \cup R_p = D_L \) be a fundamental domain of \( \Gamma_L = \langle E_t, \ldots, E_{n+t-1} \rangle \) in \( L_P \). Let \( V_L' = \{u_1, \ldots, u_p\} \) be the set of all convergents of \( L_P \) associated with the vertices \( R_1, \ldots, R_p \in V_L \). Then the set of all convergents of \( L_P \) is \( \{gu : g \in \Gamma_L, u \in V_L'\} \). In Example 1 above, the set of convergents of \( L_P \) is \( \{E^n w, n \in \mathbb{Z}\} \). In Example 4 below, \( p = 4 \). In all other examples, \( p = 1 \), i.e., it will be shown in each of these cases that \( R(u) \) is a fundamental domain of \( \Gamma_L \).

3. If vertices \( R_i \) on level \( i \) of the tree \( G_L \) have been added to \( \mathcal{V} \) in step 6, then all the vertices of \( G_L \) on levels \( < i \) belong to \( \mathcal{V} \). If \( L \) is the axis of an irreducible \( g \in \Gamma \), then in a finite number of steps \( \mathcal{V} \) will contain a complete set of non-congruent sets \( R(u) \). Thus, Algorithm II finds a fundamental domain and a set of generators of the group \( \Gamma_L \) in a finite number of steps.

4. For the axis \( L_P \) defined by [4], it can be shown that \( \langle u, a_1 \rangle \to 0 \) as \( R(u) \to A_1 = a_1a_1^T \) (see [32], Lemma 9).

4. Units in totally real fields

Let \((1, \omega_2, \ldots, \omega_n)\) be a \( \mathbb{Z} \)-basis of the ring of integers \( \mathbb{Z}_F \) of a number field \( F \) of degree \( n \). Let \( a_1 = (1, \omega_2, \ldots, \omega_n)^T \). Let \( \gamma \in \mathbb{Z}_F \). Then \( \gamma \omega_i = \sum m_{ij} \omega_j \) or \( \gamma a_1 = M a_1 \), where \( \omega_1 = 1 \), \( m_{ij} \in \mathbb{Z} \) and \( M_\gamma = (m_{ij}) \) is a square matrix of order \( n \). As explained in Section 1, the map \( \gamma \mapsto M_\gamma \) is an isomorphism of the ring of integers \( \mathbb{Z}_F \) and the commutative ring of \( \mathbb{Z} \)-integral square matrices of order \( n \) with the common axis \( L_P \) defined by [1], where \((s, t)\) is the signature of \( F \). The norm of \( \gamma \) equals \( \det(M_\gamma) \), so that \( \gamma \) is a unit in \( \mathbb{Z}_F \) if and only if \( M_\gamma \in \text{GL}_n(\mathbb{Z}) \). The group of units in \( \mathbb{Z}_F \) is not isomorphic to \( \text{Stab}(L_P, \Gamma) \). On the other hand, \( \Gamma = \text{GL}_n(\mathbb{Z})/\{ \pm 1 \} \). On the other hand, the group \( \text{Stab}(L_P, \Gamma) \) is not necessarily commutative, since there may exist a non-trivial homomorphism \( \text{Gal}(F) \to \text{Stab}(L_P, \Gamma) \). However, the torsion-free subgroup \( \Gamma_L \) of \( \text{Stab}(L_P, \Gamma) \) is isomorphic to \( \mathbb{Z}_F^* / \mu_F \). Thus, the problem of finding a system of fundamental units of \( F \) is equivalent to the problem of finding a set of generators of \( \Gamma_L \). If \( s > 0 \), then the torsion group \( \mu_F = \{ \pm 1 \} \), and \( \Gamma_L \) is isomorphic to \( \mathbb{Z}_F^* / \{ \pm 1 \} \). Note that, by Lemma 5,

\[
\text{ht}(L_P) = \frac{1}{n} |d(F)|^{1/n},
\]

where \( d(F) = \det^2(P) \) is the discriminant of \( F \).

A point \( X = (x_{ij}) \in \mathcal{P} \) is said to be rational over the field \( K \) if all \( x_{ij} \in K \). A subset \( S \) of \( \mathcal{P} \) is rational over \( K \) if the set of rational points of \( S \) is dense in \( S \). It is clear that the summit \( q_m \) (see [1]) of \( L_P \) is rational over some real subfield \( F_L \) of the Galois closure of \( F \). Let \( \Gamma_L(\mathbb{Q}) \) be the stabilizer of \( L_P \) in \( \text{GL}_n(\mathbb{Q}) \). The \( \Gamma_L(\mathbb{Q}) \)-orbit of \( q_m \) is dense in \( L_P \). Hence \( L_P \) is rational over \( F_L \). Assume that \( \dim(L_P) = r \). Then any vertex of \( R = L_P \cap K(w) \) is the intersection of \( L_P \) with \( r \) rational (over \( \mathbb{Q} \)) planes \( L^+(g_i) \). Hence all the vertices of \( R \) are rational over \( F_L \). In this section,
$F$ is a totally real field, $\text{dim}(L_P) = n - 1$, and $F_L = \mathbb{Q}$. In Section 5, the signature of $F$ is $(2,1)$. Let $f(x) = x^n - c_{n-1}x^{n-1} - \ldots - c_1x - c_0$ be an irreducible polynomial with integral coefficients. Let $f(\delta) = 0$. Let $F = \mathbb{Q}(\delta)$. Assume that $\mathbb{Z}_F$ has the power basis $\{1, \delta, \ldots, \delta^{n-1}\}$. Let $a_1 = (1, \delta, \ldots, \delta^{n-1})^T$. The integral matrix $C$ such that $Ca_1 = \delta a_1$ is said to be the companion matrix of $f(x)$. Let $a_i = (1, \delta, \ldots, \delta^{i-1})^T$, $i = 1, 2, \ldots, n$, where $\delta_1 = \delta$. An equation of the axis $L_P$ of $C^T$ is

$$q = \sum \mu_i A_i, \quad A_i = (a_{km}) = a_i a_i^T, \quad \mu > 0, \quad \mu_1 + \ldots + \mu_n = 1,$$

where $a_{km} = \delta_i^{km} - \delta_{i-1}^{km}$. Thus a point $q = (q_{km})$ of $L_P$ can be identified with the vector $[p_0, p_1, p_2, \ldots, p_{2n-2}]$, where $q_{km} = p_{k+m-2}$ and

$$p_i = \mu_1 \delta_1^i + \ldots + \mu_n \delta_n^i, \quad i = 0, 1, \ldots, n - 1.$$

Since $\delta_j = c_{n-1} \delta_j^{n-1} + \ldots + c_1 \delta_j + c_0$,

$$p_i = c_{n-1} p_{i-1} + c_{n-2} p_{i-2} + \ldots + c_1 p_{i-n+1} + c_0 p_{i-n}, \quad i = n, n + 1, \ldots, 2n - 2.$$

Using $p_0, \ldots, p_{n-1}$ as parameters on $L_P$ instead of $\{\mu_i, i = 1, \ldots, n\}$, we obtain the following.

**Lemma 9.** Assume that the ring of integers $\mathbb{Z}_F$ of a totally real field $F$ has a power basis. A point $q = [p_0, p_1, p_2, \ldots, p_{2n-2}]$, defined as above, belongs to the plane spanned by $L_P$ if and only if (7) holds, in which case the point $q \in L_P$ is uniquely determined by its first row:

$$q = [p_0, p_1, \ldots, p_{n-1}].$$

In Example 1 above, $I = [1,0]$ and $I[E] = [1,1]$ in these coordinates. In the examples below, we shall use these coordinates for a point $q \in L_P$ whenever the ring of integers $\mathbb{Z}_F$ of a totally real field $F$ has a power basis.

**Example 2.** Here we consider the case of the simplest cubic fields (see [25]). These are the cyclic fields of discriminant $(t^2 + 3t + 9)^2$. The field $F = \mathbb{Q}(\epsilon_1)$ is generated by a root $\epsilon_1$ of $f(x) = x^3 - tx^2 - (t+3)x - 1$. Assume that $t^2 + 3t + 9$ is square-free. Then $\{1, \epsilon_1, \epsilon_1^2\}$ is a basis of $\mathbb{Z}_F$, units $\epsilon_1$ and $\epsilon_2 = 1/(1 + \epsilon_1)$ both are the roots of this polynomial, and $\mathbb{Z}_F/\langle \pm 1 \rangle = \langle \epsilon_1, \epsilon_2 \rangle$ (see [24]). For $t = -1, 0, 1$, the discriminants of $F$ are $7^2, 9^2$, and $13^2$ respectively. These fields were considered by Davenport [9], [10], [11] (see also [3]), and Swinnerton-Dyer [26].

Let $E^T$ be the companion matrix of $f(x)$ and let $E_1 = E + I$. Let $L_P$ be the axis of $E$. Then the torsion-free subgroup $\Gamma_L$ of the stabilizer of $L_P$ in $\Gamma$ is generated by $E$ and $E_1$. Let $E^T a_i = \epsilon_i a_i$ and $A_i = a_i a_i^T$, where $a_i = (1, \epsilon_i, \epsilon_i^2)$, $i = 1, 2, 3$. Then

$q(\mu_1, \mu_2, \mu_3) = \mu_1 A_1 + \mu_2 A_2 + \mu_3 A_3, \quad \mu > 0, \quad \mu_1 + \mu_2 + \mu_3 = 1$, is an equation of $L_P$.

Let $R = L_P \cap K(w)$. An edge of $R$ is the intersection of $L_P$ with some $L^+(g)$, $g \in \Gamma$, which contains a face of $K(w)$, and the vertices of $R$ are the points of intersection of $L_P$ with some faces of $K(w)$ of codimension two. Denote $E_2 = EE_1^{-1}$.

Let $L_1$ be the intersection of $L_P$, $L^+(E)$, and $L^+(E_2)$, and let $G_1$ be the intersection of $L_P$, $L^+(E)$, and $L^+(E_1)$. Since $\mathbb{Z}_F$ has a power basis, Lemma 9 is applicable. In our case, $R$ is the hexagon with vertices at $F_1 = [1,0,1], F_2 = F_1[E], F_3 = F_1[F_3], G_1 = [1,-1/2,1], G_2 = G_1[F_1], G_3 = G_1[E]$ with $\det(F_1) = 1$,

$\det(G_1) = (t^2 + 3t + 9)/8$. The sides of $R$ are identified as follows: $E : F_1 G_1 \rightarrow F_2 G_3$;
$E_1 : F_3 G_1 \rightarrow F_2 G_2$; $E_2 : F_1 G_2 \rightarrow F_3 G_3$. Since the forms $F_i$ are integral, they are extremal.

Let $t = 3u + k$, where $|k| \leq 1$. Denote

$$h = \begin{bmatrix} 1 & 0 & h_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}, \quad G_0 = \begin{bmatrix} 1 & -1/2 & 0 \\ -1/2 & 1 & k/2 \\ 0 & k/2 & g_{33} \end{bmatrix},$$

where $h_{13} = -1 - u$, $h_{23} = -2u$, and $g_{33} = (4 \det(G_1) + k^2)/3$. Then $G_0 = G_1[h]$ is Minkowski reduced ([8], pp. 396-397). Hence the points $G_i$ are extremal.

Hence $R = L_P \cap K(w)$ is a fundamental domain of $\langle E, E_1 \rangle$, and therefore we have given a new proof that $\mathbf{Z}_F^*/\{\pm 1\} = \langle \epsilon, \epsilon + 1 \rangle$.

Note that if $F_1 = q(\mu_1, \mu_2, \mu_3)$, then $F_2 = q(\mu_2, \mu_3, \mu_1)$ and $F_3 = q(\mu_3, \mu_1, \mu_2)$. The same relations hold for $G_1, G_2$, and $G_3$. Also, for the sum $q_m$ of $L_P$, we have

$$q_m = \frac{1}{3} \sum A_i = \frac{1}{3} \sum F_i = \frac{1}{3} \sum G_i.$$

**Example 3.** Let $t \geq 3$ be a positive integer. Let $f(x) = x^3 + (t-1)x^2 - tx - 1$. Let $f(\epsilon) = 0$. Assume that the discriminant of $f(x)$ is square-free. Then $\{1, \epsilon, \epsilon^2\}$ is a basis of $\mathbf{Z}_F$, where $F = \mathbf{Q}(\epsilon)$ is a totally real field. The field $F$ is non-Galois, and it is exceptional; that is, $\mathbf{Z}_F^*/\{\pm 1\} = \langle \epsilon, \epsilon - 1 \rangle$ (see e.g. [14], [21]). Since $\mathbf{Z}_F$ has a power basis, Lemma 9 is applicable.

Let $E^T$ be the companion matrix of $f(x)$, $E_1 = E - I$, and $E_2 = EE_1^{-1}$. Let $L_P$ be the axis of $E$. Let $F_1$ be the intersection of $L_P$, $L^+(E)$, and $L^+(E_1)$, and let $G_1$ be the intersection of $L_P$, $L^+(E)$, and $L^+(E_2)$. Let $c = t^2 + 3t + 1$. The region $R = K(w) \cap L_P$ is the hexagon with vertices at $F_1 = [1,1/2,1]$, $F_2 = [1,1/2,0]$, $F_3 = [1,1,0]$, $G_1 = [1,-(2t+4)/c, 1]$, $G_2 = [1,-(t+2)/c, 1]$, and $G_3 = [1,-(t+2)/c, 0]$. Let $\det(F_i) = (t^2+3t-9)/8$ and $\det(G_i) \sim 1$ as $t \to \infty$. Let $h$ be as in Example 2 with $h_{13} = -u$, $h_{23} = -2 + 2u$. Then $F_1[h]$ is Minkowski reduced. Hence the points $F_i$, $i = 1, 2, 3$, are extremal. Denote

$$h = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}, \quad G_0 = \begin{bmatrix} 1 & -(2t + 4)/c & (2t + 2)/c \\ -(2t + 4)/c & 1 & -2/c \\ (2t + 2)/c & -2/c & 1 \end{bmatrix}.$$

The point $G_0 = G_1[h]$ is Minkowski reduced ([8], pp. 396-397); hence the points $G_i$, $i = 0, 1, 2, 3$, are extremal. The sides of $R$ are identified as follows: $E : F_1 G_1 \rightarrow G_2 E_2$; $E_1 : F_1 G_3 \rightarrow F_3 G_2$; $E_2 : F_3 G_1 \rightarrow F_2 G_3$. Hence $R$ is a fundamental domain of $\langle E, E_1 \rangle$, and therefore we have given a new proof that $\mathbf{Z}_F^*/\{\pm 1\} = \langle \epsilon, \epsilon - 1 \rangle$.

Note that now neither $(\sum F_i)/3$ nor $(\sum G_i)/3$ is the summit of $L_P$.

**Example 4.** Let $f(x) = x^3 + (t + 2)x^2 + (2t - 1)x - 1$, where $t \in \mathbf{Z}$. Let $f(\epsilon) = 0$. Assume that the discriminant of $f(x)$ is square-free. Then $\{1, \epsilon, \epsilon^2\}$ is a basis of $\mathbf{Z}_F$, where $F = \mathbf{Q}(\epsilon)$ is a totally real field. The field $F$ is non-Galois, and $\mathbf{Z}_F^*/\{\pm 1\} = \langle \epsilon, \epsilon + 2 \rangle$ (see [21]). When $t = 3$, the discriminant of the field is 148. This case was considered by Swinnerton-Dyer [20] (see also [23]).

Let $E^T$ be the companion matrix of $f(x)$, $E_1 = E + 2I$, and $E_2 = E E_1$. Let $L_P$ be the axis of $E$. Since $\mathbf{Z}_F$ has a power basis, Lemma 9 is applicable.
Let \( d = 2t^2 - 5t + 1 \). Denote
\[
F_1 = [1, -1/2 + 1/(4t - 6), 1], \quad F_2 = F_1[3], \quad F_3 = F_1[2],\]
\[
G_1 = [1, -1, 2], \quad G_2 = G_1[2],
\]
\[
H_1 = [1, -3/2, 3], \quad H_2 = H_1[1],
\]
\[
K_1 = [1, (2 - t)/d, 1], \quad K_2 = K_1[1],
\]
\[
M_1 = [1, -1/(2t - 1), 1], \quad M_2 = M_1[E^{-1}], \quad M_3 = M_2[E_2]
\]
with \( \det(G_1) = 1, \det(H_1) = (3t + 1)/8, \det(M_1) = (6t + 1)(t - 1)^2/(2t - 1)^3 \), and \( \det(F_1), \det(K_1) \sim 3/4 \) as \( t \to \infty \).

Denote \( u_1 = (0 t 1)^T, u_2 = (1 1 0)^T, u_3 = (t 1 0)^T \) and \( h_1 = [u_1, e_1, e_2] \), \( h_2 = [u_2, e_3] \), \( i = 2, 3 \), where \([e_1, e_2, e_3]\) is the identity matrix. Then \( F_1 = L_P \cap L^+(E) \cap L^+(E_2), G_1 = L_P \cap L^+(E_2) \cap L^+(u_2), H_1 = L_P \cap L^+(E_1) \cap L^+(u_2), K_1 = L_P \cap L^+(E) \cap L^+(u_1) \) and \( M_1 = L_P \cap L^+(E) \cap L^+(E_1^{-1}) \).

The points \( G_1 \) and \( H_1 \) are extremal since they are integral. Let
\[
h(s) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & s \\
0 & 0 & 1
\end{bmatrix}.
\]

Then \( F_1[h(2)] \) and \( K_1[h(t)] \) are Minkowski reduced ([8], pp. 396-397). Hence \( F_i \) and \( K_i \) are also extremal. The points \( M_i, i = 1, 2, 3 \), are not extremal. But the points \( M'_i = (2t - 1)/(2t - 2)M_i[h_i] \) are extremal with \( \det(M'_i) = (6t + 1)/(8t - 8) \). In particular, \( M_1[h_1] \) is Minkowski reduced.

The fundamental domain of \( \Gamma_L = \langle E, E_1 \rangle \) is the hexagon \( D_L = R \cup R_1 \cup R_2 \cup R_3 \) with vertices at \( F_1, M_1, F_3, M_3, F_2 \) and \( M_2 \). The region \( R = L_P \cap K(w) \) is the 9-gon with vertices at \( F_1, K_1, H_2, F_3, G_2, K_2, F_2, H_1 \) and \( G_1 \). Regions \( R_1, R_2 \) and \( R_3 \) are triangles. The vertices of \( R_1 \) are \( M_1, K_1 \) and \( H_2 \), of \( R_2 \) are \( M_2, H_1 \) and \( G_1 \), and of \( R_3 \) are \( M_1, G_2 \) and \( K_2 \). The intersection of the plane \( L^+(u_1) \) with \( D_L \) is the interval \( K_1H_2 \) which is the common boundary of \( R \) and \( R_1 \). Similarly, \( L^+(u_2) \cap D_L = G_1H_1 = R \cap R_2 \) and \( L^+(u_3) \cap D_L = G_2K_2 = R \cap R_3 \). Note that \( R_i[h_i] = L_P[h_i] \cap K(w), i = 1, 2, 3 \).

The sides of \( D_L \) are identified as follows: \( E : F_1M_1 \to F_2M_3; E_1 : F_2M_2 \to F_3M_1; E_2 : F_1M_2 \to F_3M_3 \). Hence \( D_L \) is a fundamental domain of \( \langle E, E_1 \rangle \), and therefore we have given a new proof that \( \mathbb{Z}_F^\times / \{ \pm 1 \} = \langle \epsilon, \epsilon + 2 \rangle \).

The set of convergents of \( L_P \) is \( \{ gu, g \in \Gamma_L, u \in V'_L \} \), where \( V'_L = \{ w, u_1, u_2, u_3 \} \).

Remark. Six sides of \( R \) are identified as follows: \( E : F_1K_1 \to F_2K_2; E_1 : F_2H_1 \to F_3H_2; E_2 : F_1G_1 \to F_3G_2 \).

Example 5. Assume that \( t > 3 \) is a positive integer which is not divisible by 3. Then \( 4t^2 + 9 \) is a square-free integer. Let \( f(x) = x^4 + tx^3 - x^2 - tx + 1 = x(x^2 - 1)(x + t) + 1 \). Let \( f(\epsilon) = 0 \). The discriminant of \( f(x) \) is \( (4t^2 + 9)(t^2 - 4)^2 \). Let \( \eta = \epsilon - \epsilon^{-1} \). Since
\[
x^{-2}f(x) = (x - x^{-1})^2 + t(x - x^{-1}) + 1,
\]
it follows that \( \eta^2 + t\eta + 1 = 0 \), \( \eta = -t \pm \sqrt{t^2 - 4}/2 \), and \( K = \mathbb{Q}(\eta) \) is a quadratic subfield of the totally real quartic field \( F = \mathbb{Q}(\epsilon) \). We have \( \epsilon^2 - \eta \epsilon - 1 = 0 \), and the roots of \( f(x) \) are
\[
\epsilon_{i,i+2} = \frac{\eta_i}{2} \pm \sqrt{\frac{\eta_i^2}{4} + 1}, \quad \eta_i = -\frac{t}{2} \pm \sqrt{\frac{t^2}{4} - 1}, \quad i = 1, 2.
\]
Clearly, \( \epsilon_1 \epsilon_3 = \epsilon_2 \epsilon_4 = -1 \). Hence, \( F_1 = K(\sqrt{\eta_1^2 + 4}) \) and the discriminant of \( F_1 \) is

\[
D_F = D_{F/K}^2 N(D_{F/K}) = (t^2 - 4)^2 N(\eta_1^2 + 4) = (t^2 - 4)(4t^2 + 9)
\]

provided \( t^2 - 4 \) is the discriminant of \( K \), in which case \( \mathbb{Z}_K^* / \{ \pm 1 \} = \{ \eta \} \), where \( \mathbb{Z}_K^* \) is the unit group in the maximal order \( \mathbb{Z}_K \) of \( K \). Since \( N(D_{F/K}) = 4t^2 + 9 \) is square-free, \( \{ 1, \epsilon \} \) is a basis of \( \mathbb{Z}_F \) (see e.g. [7], p. 79). It follows that \( \{ 1, \epsilon, \epsilon^2, \epsilon^3 \} \) is a basis of \( \mathbb{Z}_F \). The regulator obtained from the units \( \epsilon_1, \epsilon_2 \), and \( \eta \) is of order \( \log^3 t \) as \( t \to \infty \). To show that the Galois group of \( \mathbb{Q}(\epsilon) \) is \( S_3 \), it is enough to show that \( F_1 \) is not a normal field. But, if it is, then \( \sqrt{\eta_1^2 + 4} \sqrt{\eta_2^2 + 4} = \sqrt{N(\eta_1^2 + 4)} = \sqrt{4t^2 + 9} \) belongs to \( F_1 \) and therefore \( D_F \) is divisible by \( (4t^2 + 9)^2 \), which is not the case since \( t > 3 \).

Let \( E^T \) be the companion matrix of \( f(x) \), \( E_1 = E + I \), \( E_2 = E - E^{-1} \), and \( E_3 = E - I = EE_2 E_3^{-1} \). Let \( L_P \) be the axis of \( E \). Since \( \mathbb{Z}_F \) has a power basis, Lemma 9 is applicable. Now we shall show that the following 28 points are the vertices in each of the 396-397), then

\[
s = 1/(2t^2 - t - 2). \quad \text{The points } K = \left[ 1, -s, 1, 2/t - t \right] \text{ and } L = \left[ 1, s, 1, s + 2/t - t \right]
\]

with \( \det(L) = \det(K) \sim 3/4 \) as \( t \to \infty \) are the intersections of \( L_P \) with \( L^+(E) \), \( L^+(E_1) \), \( L^+(E_2) \) and \( L^+(E_3) \) respectively. Denote \( K_1 = K[E], K_2 = K[E_1 E_2^{-1}], K_3 = K[E_2^{-1}], L_1 = L[E_1], L_2 = L[E_1 E_2^{-1}], L_3 = L[E_3 E_2^{-1}] \).

The point \( M = [1, 0, 1, -2/t] \) with \( \det(M) = (1 - 4/t^2)^2 \) is the intersection of \( L_P \) with \( L^+(E_1), L^+(E_2), \) and \( L^+(E_3) \). Denote \( M_1 = M[E], M_2 = M[E_2], M_3 = M[E E_2] \). Note that the summit \( q_m \) of the 3-flat \( L_P \) is \( q_m = \frac{1}{4}(M + M_1 + M_2 + M_3) \) with \( \det(q_m) = D_{F/K}^{-4} \).

To show that all the points enumerated above are extremal we shall use Minkowski reduction. (If for some \( h \in \Gamma \), then \( X[h] \) is Minkowski reduced ([8], pp. 396-397), then \( X \) is extremal). Below we shall indicate such an \( h \) for one of the vertices in each of the \( \Gamma \)-orbits of vertices of \( R \). The polytope \( R \) is bounded by four octagons lying in \( L^+(g^{\pm 1}), g = E \), \( E_2 \), eight pentagons lying in \( L^+(g^{\pm 1}), g = E_1, E_2, \) \( EE_1^{-1}, E_2^{-1} \), and four triangular faces lying in \( L^+(g^{\pm 1}), g = EE_2, EE_1^{-1} \). It has 28 vertices, 42 edges and 16 faces. The projections of the boundary of \( R \) into a plane which is ‘perpendicular’ to its octagonal faces are shown in Figure 1.

Let

\[
U_A = \begin{bmatrix}
1 & 1 & -t & -2 \\
0 & 1 & 0 & 1 \\
0 & 0 & t & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}, \quad U_B = \begin{bmatrix}
1 & 0 & -1 & 1 - t \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & t \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]
Figure 1.
Let $b = 1/2 - 2/t$, $c = t/2 + 1 - 4/t$, $d = -1/2 - 2/t$. Then

$$A^U = \begin{bmatrix} 1 & 1/2 & -2/t & -1/2 \\ 1/2 & 1 & 0 & -2/t \\ -2/t & 0 & 1 & -b \\ -1/2 & -2/t & -b & c \end{bmatrix}, \quad B^U = \begin{bmatrix} 1 & -1/2 & 0 & b \\ -1/2 & 1 & -2/t & -1/2 \\ 0 & -2/t & 1 & 1/2 \\ b & -1/2 & 1/2 & c \end{bmatrix},$$

where $A^U = A[U_A]$ and $B^U = B[U_B]$, are Minkowski reduced. Hence $A$, $A_i$ and $B$, $B_i$, $i = 1, 2, 3$, are extremal. Let

$$U_K = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & t+1 & -t \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad U_L = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & t & -t \\ 0 & 0 & 1 & t-1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $a = 2t^3 - t^2 - 2t$, $b = 3t^2 - t - 4$, $c = -t^3 + t^2 + t$, $d = t - b$. Then

$$aK^U = \begin{bmatrix} a & -t & -d & -2a/t \\ -t & a & b & t^2 \\ -d & b & a & c \\ -2a/t & t^2 & c & a \end{bmatrix}, \quad aL^U = \begin{bmatrix} a & t & t^2 & b \\ t & a & 2a/t & d \\ t^2 & 2a/t & a & c \\ b & d & c & a \end{bmatrix},$$

where $K^U = K[U_K]$ and $L^U = L[U_L]$, are Minkowski reduced. Hence $K$, $K_i$ and $L$, $L_i$, $i = 1, 2, 3$, are extremal. For

$$U = \begin{bmatrix} 1 & 0 & -1 & -t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

the matrix

$$tM[U] = \begin{bmatrix} t & 0 & 0 & -2 \\ 0 & t & -2 & 0 \\ 0 & -2 & t & 0 \\ -2 & 0 & 0 & t \end{bmatrix}$$

is Minkowski reduced. Thus, $M$, $M_i$, $i = 1, 2, 3$, are extremal.

Reduction of $C$ and $D$ is more difficult. Let $t = 22k + l$. Let

$$U_D = \begin{bmatrix} 1 & 0 & u_{14} \\ 0 & 1 & -t \\ 0 & 0 & t-1 \\ 0 & 0 & 1 \end{bmatrix},$$

where $u_{44} = -[t/11]$, $u_{34} = u_{44}(t - 1) + 1$, $u_{14} = -9k - a$, while

$$u_{24} = 44k^2 + (2l + 15)k + b,$$

where $a = [l/3] + 1$, $b = [(2l + 1)/3]$ if $1 \leq l \leq 10$, and

$$u_{24} = 44k^2 + (2l - 7)k + b,$$

where $a = -[l/3]$, $b = [(1 - l)/3]$ if $-11 \leq l \leq 0$. Then $D[U_D]$ is Minkowski reduced. For example, when $l = 0$,

$$D[U_D] = \begin{bmatrix} 1 & 1/2 & 1/4 + 1/t & 7/44 + 1/t \\ 1/2 & 1 & 1/4 - 1/t & 1/11 \\ 1/4 + 1/t & 1/4 - 1/t & 1 & 1/2 \\ 7/44 + 1/t & 1/11 & 1/2 & 7/44(t^2 - 4) \end{bmatrix}.$$
Let

\[
U_C = \begin{bmatrix}
1 & 0 & -1 & u_{14} \\
0 & 1 & t & u_{24} \\
0 & 0 & t + 1 & u_{34} \\
0 & 0 & 1 & u_{44}
\end{bmatrix},
\]

where \(u_{44} = -[t/11] \), \(u_{34} = u_{44}(t + 1) + 1 \), \(u_{14} = 9k + a \), while

\[
u_{24} = -44k^2 - (2l + 7)k - b,
\]

where \(a = [t/3] \), \(b = [(l + 1)/3] \) if \(0 \leq l \leq 11 \), and

\[
u_{24} = -44k^2 - (2l - 15)k - b,
\]

where \(a = [t/3] - 1 \), if \(-10 \leq l \leq -1 \), \(l \neq -5 \), and \(a = [l/3] - 2 \) if \(l = -5 \), and \(b = [(1 - 2l)/3] \) if \(-10 \leq l \leq -1 \). Then \(C[U_C] \) is Minkowski reduced. For example, when \(l = 0 \),

\[
C[U_C] = \begin{bmatrix}
1 & -1/2 & 1/t - 1/4 & -1/11 \\
-1/2 & 1 & 1/t + 1/4 & 7/44 + 1/t \\
1/t - 1/4 & 1/t + 1/4 & 1 & 1/2 \\
-1/11 & 7/44 + 1/t & 1/2 & 7/44(t^2 - 4)
\end{bmatrix}.
\]

Hence, \(C, C_i \) and \(D, D_i \), \(i = 1, 2, 3 \), are extremal.

Thus, \(R = L_P \cap K(w) \) is a fundamental domain of \(\Gamma_L = \langle E, E_1, E_2 \rangle \), and therefore \(Z_P^* / \{ \pm 1 \} = \langle \epsilon, \epsilon + 1, \epsilon - 1/\epsilon \rangle = \langle \epsilon, \epsilon + 1, \epsilon - 1 \rangle \).

**Example 6.** Let \(f(x) = x^4 + tx^3 - x^2 - tx - 1 = x(x^2 - 1)(x + 1) - 1 \). Let \(f(\epsilon) = 0 \). The field \(F = \mathbb{Q}(\epsilon) \) is totally real and \(\text{Gal}(F) = S_4 \). Assume that the discriminant of \(f(x) \) is square-free. Then \(Z_P \) has a power basis, and Lemma 9 is applicable.

Let \(E \) be the companion matrix of \(f(x) \), \(E_1 = E - I \), \(E_2 = E - E^{-1} \), and \(E_3 = E + I = EE_2E_1^{-1} \). Let \(L_P \) be the axis of \(E \). We shall show that the region \(R = L_P \cap K(w) \) is the same as in Example 5.

The point \(A = [1, -1/2, 1, -1/2 - t] \) with \(\det(A) \sim \frac{3}{8}t \) as \(t \to \infty \) is the intersection of \(L_P \) with \(L^+(E), L^+(E_2) \) and \(L^+(E_3) \). Denote \(A_1 = A[E], A_2 = A[E_2], A_3 = A[E_3] \). Let \(c = -1/2 + 2/t \),

\[
U_A = \begin{bmatrix}
1 & 0 & -1 & -t \\
0 & 1 & 0 & -2 \\
0 & 0 & 1 & t \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad A^U = \begin{bmatrix}
1 & -1/2 & -2/t & c \\
-1/2 & 1 & 0 & -c \\
-2/t & 0 & 1 & c + 2/t \\
c & -c & c + 2/t & t/2 - 6c
\end{bmatrix}.
\]

Then \(A[U_A] = A^U \) is Minkowski reduced. Hence \(A, A_1, A_2 \) and \(A_3 \) are extremal.

The points \(B_3 = [1, 1/2, 1, 1/2] \) with \(\det(B) = (6t - 19)/16 \) is the intersection of \(L_P \) with \(L^+(E - E_2^{-1}) \), \(L^+(E_2^{-1}) \) and \(L^+(E_3^{-1}) \). Denote \(B = B_3[E^{-1}E_2^{-1}], B_1 = B_3[E_3^{-1}], B_2 = B_3[E_2^{-1}] \). Since \(B_3 \) is an integral matrix, it is extremal.

The points \(C = [1, 1/2, 1, -t + 1/2 + 7/4 + 1/(2t - 4)] \) and \(D = [1, -1/2, 1, -t/2 - 7/4 + 1/(2t + 4)] \) with \(\det(C), \det(D) \sim \frac{2}{7}\pi^2 \) as \(t \to \infty \) are the intersections of \(L_P \) with \(L^+(E), L^+(E_1), L^+(E_1E_2^{-1}) \) and with \(L^+(E), L^+(E_3), L^+(EE_1^{-1}) \) respectively. Denote \(C_1 = C[E_1], C_2 = C[E_2E_1^{-1}], C_3 = C[E] \) and \(D_1 = D[E], D_2 = D[EE_1^{-1}], D_3 = D[E_3] \). As in Example 5, reduction of \(C \) and \(D \) is divided into particular cases modulo 22. We consider only one case of such reduction. Let \(t = 22k + 13 \).
Let

$$U_C = \begin{bmatrix}
1 & 0 & -1 & -u_{14} \\
0 & 1 & -t & -u_{24} \\
0 & 0 & t - 1 & u_{34} \\
0 & 0 & 1 & u_{44}
\end{bmatrix},$$

where $u_{44} = 2 + 2k$, $u_{34} = u_{44}(t - 1) + 1$, $u_{14} = 9k + 6$, $u_{24} = 44k^2 + 55k + 19$. Then

$$C[U_C] = \begin{bmatrix}
1 & 1/2 & -1/4 + c & 1/22 + c \\
1/2 & 1 & -1/4 - c & -1/4 - 1/22 \\
1/4 + c & 1/4 - c & 1 & c \\
1/22 + c & -1/4 - 1/22 & c & 7/44t^2 + 3/11t - 31/22 - c
\end{bmatrix},$$

where $c = 1/(22 + 44)$, is Minkowski reduced. Hence $C$ is extremal. Similarly, the other cases of reduction of $C$ and $D$ can be considered.

Denote by $K_3$ the intersection of $L_P$ with $L^+(E_3)$, $L^+(E_2)$ and $L^+(EE_2)$, and by $L_3$ the intersection of $L_P$ with $L^+(E)$, $L^+(EE_2^{-1})$ and $L^+(EE_1^{-1})$. Let $K = K_3(E_2)$, $K = K[E], K_2 = K[E_1E_2^{-1}]$ and $L = L_3(EE_1^{-1})$, $L_1 = L[E_1], L_2 = L[E_1E_2^{-1}]$. Let

$$U_K = \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & -t & 0 \\
0 & 0 & t - 1 & t \\
0 & 0 & 1 & 1
\end{bmatrix}, \quad U_L = \begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & t & t \\
0 & 0 & 1 & t + 1 \\
0 & 0 & 0 & 1
\end{bmatrix},$$

and $a_1 = t(t - 2)(2t + 3)$, $b_1 = (t - 4)/a_1$, $c_1 = (3t^2 - 4)/a_1$, $d_1 = (t^2 - 3t - 4)/a_2$, $a_2 = t(t + 2)(2t - 5)$, $b_2 = (t + 4)/a_2$, $c_2 = (3t^2 - 3t - 8)/a_2$, $d_2 = -(t^2 + 2t - 8)/a_2$. Then

$$K[U_K] = \begin{bmatrix}
1 & -2/t - b_1 & c_1 & 2/t - 2b_1 \\
-2/t - b_1 & 1 & d_1 & tb_1 \\
c_1 & d_1 & 1 & 1/2 - tb_1/2 \\
2/t - 2b_1 & tb_1 & 1/2 - tb_1/2 & 1
\end{bmatrix},$$

$$L_A[U_L] = \begin{bmatrix}
1 & -2/t - b_2 & -tb_2 & c_2 \\
-2/t - b_2 & 1 & 2b_2 & d_2 \\
-tb_2 & 2b_2 & 1 & 1/2 - tb_2/2 \\
c_2 & d_2 & 1/2 - tb_2/2 & 1
\end{bmatrix},$$

are Minkowski reduced. Hence $K$ and $L$ are extremal. Note that $\det(K) \sim 3/4$, $\det(L) \sim 3/4$ as $t \to \infty$.

The point $M = [1, 2/t, 1, 2/t]$ with $\det(M) = (1 - 4/t^2)^2 - 4/t^2$ is the intersection of $L_p$ with $L^+(E)$, $L^+(E_2)$ and $L^+(EE_2)$. Denote $M_1 = M[E], M_2 = M[E_2], M_3 = M[EE_2]$. Let

$$U_M = \begin{bmatrix}
1 & 0 & -1 & -t \\
0 & 1 & 0 & -2 \\
0 & 0 & 1 & t \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad M_U = \begin{bmatrix}
1 & 2/t & 0 & -2/t \\
2/t & 1 & 0 & 0 \\
0 & 0 & 1 & 2/t \\
-2/t & 0 & 2/t & 1
\end{bmatrix},$$

Since $M[U_M] = M_U$ is Minkowski reduced, the points $M$, $M_1$, $M_2$, $M_3$ are extremal.

Thus, the polytope $R = L_P \cap K(w)$ is the same as in Example 5 (see Figure 1). It is a fundamental domain of $\Gamma_L = \langle E, E_1, E_2 \rangle$. Hence $\mathbb{Z}_F^\oplus/\{\pm 1\} = \langle \epsilon, \epsilon + 1, \epsilon - 1/\epsilon \rangle = \langle \epsilon, \epsilon + 1, \epsilon - 1 \rangle$. 
5. Units in fields with signature (2,1)

Let \( n = 4 \). Let \( g \in \Gamma \). Assume that the characteristic polynomial of \( g \) is irreducible with signature (2,1). Let \( \epsilon_1, \epsilon_2 \) be real and \( \epsilon_3, \epsilon_4 \) non-real complex eigenvalues of \( g \). Let \( g_{a_1} = \epsilon \alpha_1 \). Assume that the field \( F = \mathbb{Q}(\epsilon) \) is dihedral and \( K \) is its real quadratic subfield. Let \( \sigma \) be the non-trivial automorphism of \( F/K \). Then \( \sigma(\epsilon_1) = \epsilon_{i+1} \), \( i = 1, 3 \). Hence \( \sigma(A_1) = A_2 \) and \( \sigma(A_3) = A_3 \), where \( A_1, A_2, \) and \( A_3 \) are vertices of \( L_P \), the axis of \( g \). Thus, the entries of \( A_1 + A_2 \) and \( A_3 \) lie in \( K \), and \( q_m = (A_1 + A_2 + 2A_3)/3 \) is rational over \( K \). Hence \( L_P \) and the vertices of \( R = L_P \cap K(w) \) are rational over \( K \). It follows that \( F_L = K \) in this case. In Examples 7 and 9 below, the signature of the field \( F = (2,1) \), \( F \) has a real quadratic subfield \( K \), and \( F_L = K \). In Example 8, the degree of \( F_L \) is six.

**Example 7.** Let \( f(x) = x^4 + tx^3 + x^2 + tx + 1 = x(x^2 + 1)(x + t) + 1 \). Let \( f(\epsilon) = 0 \), where \( \epsilon \in \mathbb{R} \). The discriminant of \( f(x) \) is \(-(4t^2 - 9)(t^2 + 4)^2\). Let \( \eta = \epsilon + \epsilon^{-1} \). Since \[ x^{-2}f(x) = (x + x^{-1})^2 + t(x + x^{-1}) - 1, \] it follows that \( \eta^2 + t\eta - 1 = 0 \), \( \eta = (-t \pm \sqrt{t^2 + 4})/2 \), and \( K = \mathbb{Q}(\sqrt{d}) \), \( d = t^2 + 4 \), is a quadratic subfield of the dihedral quartic field \( F = \mathbb{Q}(\epsilon) \) with signature (2,1). We have \( \psi(\epsilon) = \epsilon^2 - \eta + 1 = 0 \), and the roots of \( f(x) \) are \[ \epsilon_{1,2} = \frac{1}{2}(\eta \pm \sqrt{\eta^2 - 4}) \text{, } \eta = \frac{1}{2}(-t \pm \sqrt{t^2 + 4}) \text{, } i = 1, 2. \] Hence \( \epsilon_1\epsilon_3 = \epsilon_2\epsilon_4 = -1 \). If \( N(\eta^2 - 4) = 4t^2 - 9 \), the norm of the discriminant of \( \psi(x) \), is square-free, then \( \{1, \epsilon\} \) is a basis of \( \mathbb{Z}_F/K \). Hence \( \{1, \epsilon, \eta, \eta \} \) or \( \{1, \epsilon, \epsilon^2, \epsilon^3\} \) is a basis of \( \mathbb{Z}_F \) provided \( \eta \) is a fundamental unit of \( K \).

Let \( E^T \) be the companion matrix of \( f(x) \). Then \( E^T a_i = \epsilon_i a_i \), where \( a_i = (1, \epsilon_i, \epsilon_i^2, \epsilon_i^3) \). Denote \( E_1 = E + E^{-1} \). Let \( M \) be the intersection of \( L_P, L^+(E) \), and \( L^+(E_1) \). Then \[ M[x] = \mu_1(x, a_1)^2 + \mu_3(x, a_3)^2 + \mu_2((x, a_2))^2, \] where \( \mu_{1,3} = (1 + (t \pm (2t\sqrt{d} + 2t^2 - 12)^{1/2})/\sqrt{d})/2, \mu_2 = (1 - t/\sqrt{d})/2, \) and \( \det(M) = (4t^2 - 9)(t^2 + 2 - t\sqrt{d})/2 \).

Let \( b = -2t(\eta\sqrt{d})^{-1} - 3(2\sqrt{d})^{-1}, c = -2t(\eta\sqrt{d})^{-1} + 3(2\sqrt{d})^{-1}, e = 5(4\eta\sqrt{d})^{-1} \) and \[ h = \begin{bmatrix} 1 & 0 & 1 & t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ } M_0 = \begin{bmatrix} 1 & b & e & 0 \\ b & 1 & 0 & -e \\ e & 0 & 1 & c \\ 0 & -e & c & 1 \end{bmatrix}. \] Then \( M_0 = M[h] \) is Minkowski reduced (see e.g. [8], p. 397). Hence the points \( M, M[E], M[E_1], \) and \( M[EE_1] \) are extremal, and they are the vertices of the ‘square’ \( R = L_P \cap K(w) \), which is a fundamental domain of \((E, E_1)\). Thus \( \mathbb{Z}_F^2/\{\pm 1\} = \{\epsilon, \eta\} \).

Note that \( q_m = (M + M[E] + M[E_1] + M[EE_1])/4 \) is the summit of \( L_P \).

**Example 8 (cf. [20]).** Let \( t \geq 4 \) and \( f_1(x) = f(x) = x^4 + tx^3 + x^2 + tx - 1 = x(x^2 + 1)(x + t) - 1 \). Note that \( f_{-1}(-x) = f_1(x) \). Let \( \epsilon \) and \( \epsilon_1 \) be the real and \( \epsilon_3, \epsilon_4 \) non-real complex roots of \( f(x) = 0 \). The signature of \( F = \mathbb{Q}(\epsilon) \) is \((2,1)\), and \( \text{Gal}(F) = S_4 \). Assume that the discriminant of \( F \) is square-free. Then \( \{1, \epsilon, \epsilon^2, \epsilon^3\} \) is a basis of the maximal order \( \mathbb{Z}_F \) of \( F \).
Let $E^T$ be the companion matrix of $f(x)$. Let $E_1 = E^2(E + tI)$. Let $L_P$ be the axis of $E$. It can be identified with the set

$$F(\nu_1, \nu_2, \nu_3) = \nu_1 a^T a + \nu_2 a_1^T a_1 + \nu_3 (a_2^T a_2 R + a_2^T a_2 t),$$

where $\nu_k > 0, \nu_1 + \nu_2 + \nu_3 = 1$. Here $a = (1, \epsilon, \epsilon^2, \epsilon^3), a_i = (1, \epsilon, \epsilon^2, \epsilon^3), i = 1, 2$, are the eigenvectors of $E$ corresponding to its eigenvalues $\epsilon, \epsilon_1, \epsilon_2$ respectively. Let $A$ and $B$ be the intersections of $L_P$ with $L^+(E), L^+(E_1)$ and with $L^+(E^{-1}), L^+(E_1^{-1})$ respectively.

Let $\alpha = |\epsilon_2|^2 = -1/(\epsilon_1)$. Then $\alpha \to 1$ as $t \to \infty$. Over $F_L = Q(\alpha)$ we have $f(x) = f_1(x)f_2(x)$, where

$$f_1(x) = x^2 + t \frac{\alpha + 1}{\alpha^2 + 1} x - \frac{1}{\alpha},$$

$$f_2(x) = x^2 + t \frac{\alpha^2 - \alpha}{\alpha^2 + 1} x + \alpha,$$

so that $f_i(\epsilon_i) = 0, i = 1, 2$. The degree of $F_L$ is 6, since $g(\alpha) = 0$, where $g(x) = x^6 - x^5 + (t^2 + 1)x^4 - 2x^3 - (t^2 + 1)x - x - 1$.

Let $d_1$ be the discriminant of $f_1(x)$. Let $\nu_1, 2 = \mu_1 \pm \mu_2 \sqrt{d_1}$, $\mu_3 = \nu_3$, $a^T a = M_1 + M_2 \sqrt{d_1}$, $a_1^T a_1 = M_1 - M_2 \sqrt{d_1}$. Then $L_P$ can be identified with the set $\mu_1 M_1 + \mu_2 d_1 M_2 + \mu_3 M_3$, where the entries of $M_k, k = 1, 2, 3$, belong to $F_L$. Thus, the summit $q_m$ of $L_P$, $L_P$ itself and all the vertices of $R = L_P \cap K(w)$ are rational over $F_L$.

Let $\Delta_A$ be the determinant of the system of three linear equations

$$2\mu_1 + \mu_3 = 1,$$

$$(\mu_1 M_1 + \mu_2 d_1 M_2 + \mu_3 M_3)[wE] = 1,$$

$$(\mu_1 M_1 + \mu_2 d_1 M_2 + \mu_3 M_3)[wE_1] = 1,$$

with respect to $\mu_1, \mu_2, \mu_3$. Then

$$D_A = \Delta_A (\alpha^2 + 1)/4t$$

$$= \alpha^5 + 2\alpha^4 + (2t^2 - 1)\alpha^3 + (t^2 - 2)(\alpha^2 - \alpha) - 2.$$

Let $\Delta_B$ be the determinant of the system of three linear equations where $E$ and $E_1$ are replaced by $E$ and $EE_1^{-1}$. Then

$$D_B = \Delta_B (\alpha^2 + 1)/4t$$

$$= - (\alpha^3 + (t^2 + 5)\alpha + t^2 + 2).$$

Let

$$U_A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & t \\ 0 & 0 & t & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad U_B = \begin{bmatrix} 1 & 0 & 1 & t \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix},$$
and let $A_U = A[U_A] = (a_{ij})$ and $B_U = B[U_B] = (b_{ij})$. Then $a_{11} = a_{22} = a_{33} = 1$,

$$
2tD_Aa_{12} = (1 - \alpha)(1 + \alpha^2 - \alpha^3)t^2 - 4\alpha(2 - \alpha + 2\alpha^2 - \alpha^3),
2tD_Aa_{13} = (\alpha - \alpha^3)t^4 + (1 + 5\alpha - 7\alpha^2 + 4\alpha^3 + 2\alpha^4 - \alpha^5)t^2 - 6 - 14\alpha^2 + 6\alpha^3 - 8\alpha^4 + 6\alpha^5,
2D_Aa_{14} = (1 - \alpha)(1 + \alpha - 4\alpha^2)t^2 - 6 - 3\alpha - 13\alpha^2 + 2\alpha^3 - 7\alpha^4 + 5\alpha^5.
$$

$$
2D_Aa_{23} = (1 - \alpha)(\alpha + 6\alpha^2 + \alpha^3)t^2 + 11 - 3\alpha + 18\alpha^2 - 9\alpha^3 + 7\alpha^4 - 6\alpha^5,
2tD_Aa_{24} = (1 + 2\alpha - 2\alpha^2 + 7\alpha^3 + \alpha^4 - \alpha^5)t^2 - 6 + 8\alpha - 18\alpha^2 + 14\alpha^3 - 12\alpha^4 + 6\alpha^5,
2tD_Aa_{34} = 4(\alpha^2 - \alpha^3)t^4 + (9 - 9\alpha + 23\alpha^2 - 14\alpha^3 + 5\alpha^4 - 4\alpha^5)t^2 + 12 - 8\alpha + 32\alpha^2 - 20\alpha^3 + 20\alpha^4 - 12\alpha^5.
$$

$D_Aa_{44} = (2\alpha + 3\alpha^2 - 3\alpha^3)t^2 + 8 + 12\alpha^2 + \alpha^3 + 2\alpha^4 - 3\alpha^5,$

and $b_{11} = b_{22} = b_{33} = 1$,

$$
2tD_Bb_{12} = 4(\alpha^3 - \alpha)t^4 + (-3 - 9\alpha - 12\alpha^2 + 10\alpha^3 - 4\alpha^4 + 4\alpha^5)t^2 - 4 + 2\alpha - 14\alpha^2 + 8\alpha^3 - 10\alpha^4 + 6\alpha^5,
2D_Bb_{13} = (\alpha^2 - 1)(2 - 3\alpha)t^2 - 4 - 11\alpha + 11\alpha^2 - 4\alpha^3 + 5\alpha^4 - 3\alpha^5.
$$

$$
2tD_Bb_{14} = (\alpha^2 - 1)(\alpha + 2\alpha^2 - \alpha^3 + \alpha^4 + 5\alpha^5)t^2 + 4\alpha(1 - \alpha + \alpha^2 - \alpha^3),
2tD_Bb_{23} = 2(\alpha^3 - \alpha)t^4 + (5 + 4\alpha - 7\alpha^2 - 2\alpha^3 - 2\alpha^4 + 2\alpha^5)t^2 + 8 + 4\alpha + 8\alpha^2 - 4\alpha^5,
2D_Bb_{24} = 2(\alpha^3 - \alpha)t^4 + (1 - 8\alpha - 7\alpha^2 + 6\alpha^3 - 2\alpha^4 + 2\alpha^5)t^2 - 3 - 13\alpha - 8\alpha^2 + 5\alpha^3 - 7\alpha^4 + 4\alpha^5,
2tD_Bb_{34} = 2(\alpha - \alpha^3)t^4 + (7 + 7\alpha - 4\alpha^2 - 10\alpha^3 + 2\alpha^4 - 2\alpha^5)t^2 + 10 + 14\alpha + 8\alpha^2 + 6\alpha^3 - 2\alpha^4 - 8\alpha^5.
$$

$D_Bb_{44} = (2 - 3\alpha - 5\alpha^2)t^2 - 4 - 5\alpha - 4\alpha^2 + 7\alpha^3 - 6\alpha^4.$

Let $c = (\alpha^2 + 1)^2$. The identity

$$
t^2(\alpha^3 - \alpha) = c - \left(\frac{c}{\alpha t}\right)^2 + \frac{c^3}{\alpha^2 t^2(\alpha^2 t^2 + c)}
$$

implies $a_{ij} \to 0$ and $b_{ij} \to 0$ as $t \to \infty$ for $i \neq j$. It follows that $\det(A) \sim 1$ and $\det(B) \sim 3$ as $t \to \infty$. Thus, $A_U$ and $B_U$ are Minkowski reduced. Hence the points $A, A[E], A[E_1], B, B[E^{-1}], \text{ and } B[E_1^{-1}]$ are extremal, and they are the vertices of the hexagon $R = L_F \cap K(w)$. Thus, $R$ is a fundamental domain of $\langle E, E_1 \rangle$, and $\mathbb{Z}_F^*/\{\pm 1\} = \langle \epsilon, \epsilon + t \rangle$.

Example 9. Let

$$
f(x) = x^4 + s tx^3 + (t - \alpha s^2)x^2 + s(t^2 + 2\alpha)x - \alpha = x(x^2 + t)(x + st) - \alpha(s x - 1)^2,
$$

where $t \geq s$ are positive integers and $\alpha = \pm 1$. Let $d = t^2 + 4\alpha$. It can be easily verified that

$$
f(x) = (x^2 + s\eta x - \alpha/\eta)(x^2 - \alpha sx/\eta + \eta),
$$
where \( \eta = \frac{1}{2}(t + \sqrt{d}) \) satisfies the equation \( \eta^2 - t\eta - \alpha = 0 \). The roots of \( f(x) = 0 \) are

\[
\epsilon_{i,i+2} = \frac{1}{2}(-s\eta_i \pm \sqrt{s^2\eta_i^2 + 4\alpha/\eta_i}), \quad \eta_i = \frac{1}{2}(t \pm \sqrt{d}), \quad i = 1, 2,
\]

where \( \eta_1 = \eta \) and real \( \epsilon_1 \) and \( \epsilon_3 \) are the roots of \( x^2 + s\eta x - \alpha/\eta = 0 \). The discriminant of \( f(x) \) is \( \Delta(f) = -d^2(4s^2t^3 + 12\alpha s^2t - 8\eta^3 + 16\alpha)(1 + 2\eta^2) \). Let \( \epsilon = \epsilon_1 \). Since \( \eta = (\epsilon^2 + t)/(1 - s\epsilon) \), \( K = \mathbb{Q}(\eta) \) is a quadratic subfield of the dihedral quartic field \( F = \mathbb{Q}(\epsilon) \) with signature \((2,1)\). Assume that \( \eta \) is a fundamental unit of \( K \). Then \( \{1, \eta\} \) is a basis of \( \mathbb{Z}_K \). Let \( c = ts^2 + 1 \). Denote \( p(x) = (s + (at - s^2)x + \alpha x^3)/c \). Then \( \epsilon \eta = p(\epsilon) \). Assume that \( 4s^2t^3 + 12\alpha s^2t - 8\eta^3 + 16\alpha \) is square-free. Then \( \{1, \epsilon, \epsilon^2, p(\epsilon)\} \) is a basis of \( \mathbb{Z}_F \), and the discriminant of \( F \) is \( D_F = \Delta(f)/(ts^2 + 1)^2 \).

Let \( a_i = (1, \epsilon, \epsilon^2, p(\epsilon)) \). Let \( E_0 \) be the companion matrix of \( f \). Denote

\[
\tau = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
s/c & (at - s^2)/c & 0 & \alpha/c \\
\end{bmatrix}.
\]

Let \( E^T = \tau E_0 \tau^{-1}, E_1 = p(E), E_2 = E_1 E^{-1} \). Then \( E^T a_i = \epsilon_i a_i \), and the axis \( L_P \) of \( E \) has equation \( q(x) = \mu_1(x, a_1)^2 + \mu_2(x, a_3)^2 + \mu_3(x, a_2)^2, \mu_1 > 0, \mu_1 + \mu_3 + \mu_2 = 1 \).

First let \( \alpha = 1 \). Let \( A \) and \( B \) be the intersections of \( L_P \) with \( L^+(E), L^+(E_1) \) and with \( L^+(E), L^+(E_2^{-1}) \) respectively. Then \( \det(A) = \det(B) \). Let \( a_{ii} = b_{ii} = 1, i = 1, 2, 3, a_{44} = b_{44} = \sqrt{d} - 1 + \eta^{-2}, a_{23} = b_{13} = 0, a_{14} = b_{24} = \eta^{-1}, \)

\[
a_{24} = -b_{14} = \left( -\frac{s}{2\sqrt{d}} + \frac{t}{s\eta\sqrt{d}} \right) (1 - \eta^{-1}),
\]

\[
a_{34} = -b_{34} = \left( \frac{s}{2\eta\sqrt{d}} + \frac{t}{s\eta\sqrt{d}} \right) (1 - \eta^{-1}),
\]

\[
a_{13} = -b_{23} = \frac{1}{2\eta} - \frac{2}{s\eta\sqrt{d}} - \frac{s}{2\eta^2\sqrt{d}},
\]

\[
a_{12} = -b_{12} = \frac{1}{2s\eta} - \frac{2}{s\eta^2\sqrt{d}} + \frac{s}{2\eta\sqrt{d}}.
\]

Let

\[
U_A = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & s \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad U_B = \begin{bmatrix}
1 & 0 & t & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & s & 1
\end{bmatrix}.
\]

Then \( A_U = (a_{ij}) = A[U_A] \) and \( B_U = (b_{ij}) = B[E_2^{-1}][U_B] \) are Minkowski reduced. Hence the points \( A, A[E], A[E_1], B, B[E], B[E_2^{-1}] \) are extremal, and they are the vertices of the hexagon \( R = L_P \cap K(w) \).

Now let \( \alpha = -1 \). Let \( A \) and \( B \) be the intersections of \( L_P \) with \( L^+(E_1^{-1}), L^+(E_2) \) and with \( L^+(E_1), L^+(E_2) \) respectively. Then \( \det(A) = \det(B) \). Note that if \( A = q(\gamma_1, \gamma_2, \gamma_3), \) then \( B = q(\gamma_3, \gamma_2, \gamma_1) \). Here \( \gamma_2 = \eta \) and \( \sigma(\gamma_1) = \gamma_3 \), where \( \sigma \in \text{Gal}(F/K) \).
Let $a_{ii} = b_{ii} = 1$, $i = 1, 2, 3$, $a_{44} = b_{44} = \sqrt{d} - 1 - 2\eta^{-1} + \eta^{-2} - 4/(t\eta)$, $a_{14} = -b_{24} = -(1 - 2/t)/\eta$, $a_{23} = b_{13} = -2/t$.

Then $A = (a_{ij}) = A[E^{-1}]U^t_P$ and $B_U = (b_{ij}) = B[U]$ are Minkowski reduced. Hence the points $A$, $A[E^{-1}]$, $A[E_2]$, $B$, $B[E_1]$, $B[E_2]$ are extremal, and they are the vertices of the hexagon $R = L_P \cap K(w)$.

Thus, for any $\alpha$, $R$ is a fundamental domain of $(E, E_2)$, and $\mathbb{Z}_F^2/\{\pm 1\} = \langle \epsilon, \eta \rangle$.

References


Department of Mathematics, The Cooper Union, 51 Astor Place, New York, New York 10003

E-mail address: vulakh@cooper.edu