

L^2 -METRICS, PROJECTIVE FLATNESS AND FAMILIES OF POLARIZED ABELIAN VARIETIES

WING-KEUNG TO AND LIN WENG

ABSTRACT. We compute the curvature of the L^2 -metric on the direct image of a family of Hermitian holomorphic vector bundles over a family of compact Kähler manifolds. As an application, we show that the L^2 -metric on the direct image of a family of ample line bundles over a family of abelian varieties and equipped with a family of canonical Hermitian metrics is always projectively flat. When the parameter space is a compact Kähler manifold, this leads to the poly-stability of the direct image with respect to any Kähler form on the parameter space.

§0. INTRODUCTION

It is well known that for a family $\pi : \mathcal{A} \rightarrow S$ of principally polarized abelian varieties, the theta bundles on the fibers of π form a line bundle Θ over \mathcal{A} . From the fact that the theta characters of level n , $n \in \mathbb{Z}^+$, satisfy a certain heat equation, one knows that the direct image vector bundles $\pi_*\Theta^{\otimes n}$ admit projectively flat connections such that the theta characters are parallel sections. Moreover, from the fact that the theta characters form an orthonormal basis with respect to a certain natural L^2 -pairing, one knows that the above projectively flat connections are indeed Hermitian connections (cf. [APW, §5]).

As a generalization of the above, one considers in conformal field theory a family of Riemann surfaces and an associated family $\pi : \mathcal{M} \rightarrow S$ of moduli spaces of stable vector bundles over the Riemann surfaces. As before, the so-called generalized theta bundles over the fibers of π form a line bundle Θ over \mathcal{M} . An important result in conformal field theory in [TUY], [H] and [APW] was to show that for $n \in \mathbb{Z}^+$, $\pi_*\Theta^{\otimes n}$ admits a projectively flat connection, whose construction depends on the fact that certain generalized heat equations, or KZ equations, hold for sections of generalized theta bundles.

In another direction, instead of just considering the theta bundles, Welters [W], following [M1], showed that for a family of polarized abelian varieties $\pi : \mathcal{A} \rightarrow S$ and any relatively ample line bundle \mathcal{L} over \mathcal{A} , $\pi_*\mathcal{L}$ always admits a projectively flat connection. This was established from hypercohomological considerations in [W], which led to certain generalized heat equations for sections of \mathcal{L} (cf. also [H]).

However the projectively flat connections constructed in [H] and [W] are not Hermitian connections in general. From the viewpoint of representation theory, a vector bundle of rank r over S admits a projectively flat connection (resp. projectively flat Hermitian connection) if and only if E arises from a representation of

Received by the editors November 14, 2002.

2000 *Mathematics Subject Classification*. Primary 14K99, 32G08, 32G13, 32Q20.

$\pi_1(S)$ into $\mathbb{P}GL(r, \mathbb{C})$ (resp. $\mathbb{P}U(r)$). Thus a natural and important question that arises is whether the direct image bundles in the above cases admit projectively flat Hermitian connections.

In this article, we study the problem from a different approach. We consider Hermitian metrics on direct image bundles arising from certain L^2 -pairings on the sections of the original vector bundles. Our first main result is to give a computation of the curvature tensor of such L^2 -metrics for a family of Hermitian holomorphic vector bundles over a family of compact Kähler manifolds (cf. Theorem 1 in §2). Our approach has been inspired by that of Schumacher [Sch] in his computation of the curvature of the Weil-Petersson metric, which was defined similarly in terms of certain L^2 -pairing. In particular, the technique of ‘horizontal’ lifting of vector fields in [Sch] also plays an important part in our calculations. As in the case of the Weil-Petersson metric, we also give an interpretation of our curvature formula in terms of Kodaira-Spencer theory (cf. Proposition 2.3.2).

An interesting application of Theorem 1 is Welters’ case of a relatively ample line bundle \mathcal{L} over a family of abelian varieties $\pi : \mathcal{A} \rightarrow S$. Our second main result is to show that the Hermitian connection of the L^2 -metric on $\pi_*\mathcal{L}$ associated to a family of ‘canonical’ Hermitian metrics on \mathcal{L} is indeed projectively flat (cf. Theorem 3 in §2). Thus our result generalizes the result mentioned above for theta bundles over families of principally polarized abelian varieties, and it improves Welters’ result [W] in terms of representation of $\pi_1(S)$.

We sketch the proof of Theorem 3 briefly as follows. First, we reduce Theorem 3 to the case when the smooth $(1, 1)$ -form on the total space of the family of abelian varieties (which is flat on each fiber) is given as the first Chern form of a smooth Hermitian metric on the relatively ample line bundle over the total space, so that Corollary 2 applies. Then an important step in simplifying the curvature tensor of the L^2 metric is to show that the values of the above-mentioned $(1, 1)$ -form evaluated at the horizontal lifting of tangent vectors from the base space are constant along each fiber. The latter is achieved by comparing the $(1, 1)$ -form with another $(1, 1)$ -form on the total space with the same values along the tangent spaces of the fibers but with zero eigenspaces along the tangent spaces of the leaves of certain natural smooth foliation on the total space. We remark that our original approach in proving Theorem 3 consists of studying the classifying map from the families of ample line bundles over abelian varieties to the associated universal Poincaré line bundles over families of abelian varieties parametrized by Siegel modular varieties. The present simpler and more direct approach is suggested by the referee.

This paper is organized as follows. In §1, we introduce some notations and list our main results. In §2, we compute the curvature of the L^2 -metric. In §3, we treat the case of a family of Hermitian holomorphic line bundles whose first Chern forms give the Kähler metrics on the manifolds. In §4, we make some useful observations on L^2 -metrics associated to families of polarized abelian varieties. The results in §4 are needed in §5, where we give the proof of Theorem 3 in full generality. Finally in the Appendix (§6), we also give a description of the canonical Hermitian metrics in the ‘local universal’ case of families of Poincaré line bundles over abelian varieties parametrized by Siegel modular varieties.

The authors express their thanks to Professors A. Fujiki and Ngaiming Mok for their discussions and enlightenment. The authors also thank the referee for his clarifications and suggestions, which led to substantial improvements of this article.

§1. NOTATION AND STATEMENT OF RESULTS

(1.1). Let $\pi : \mathcal{X} \rightarrow S$ be a surjective holomorphic map between complex manifolds \mathcal{X} and S such that $X_s := \pi^{-1}(s)$ is an n -dimensional compact complex manifold for each $s \in S$. Moreover, suppose that \mathcal{X} admits a smooth $(1,1)$ -form $\omega_{\mathcal{X}} \in \mathcal{A}^{1,1}(\mathcal{X})$ such that the restriction $\omega_s := \omega_{\mathcal{X}}|_{X_s}$ is a Kähler form on X_s for each $s \in S$. Then such $(\pi : \mathcal{X} \rightarrow S, \omega_{\mathcal{X}})$ is called a family of compact Kähler manifolds parametrized by S . With some slight abuse of notation, we will simply call $\omega_{\mathcal{X}} \in \mathcal{A}^{1,1}(\mathcal{X})$ a family of Kähler metrics for the family $\pi : \mathcal{X} \rightarrow S$. Observe that two different $(1,1)$ -forms on \mathcal{X} may lead to the same restrictions $\{\omega_s\}_{s \in S}$. Let E be a holomorphic vector bundle over \mathcal{X} , and let h be a smooth Hermitian metric on E . Then for each $s \in S$, the restriction $(\mathcal{E}_s \rightarrow X_s, h_s)$ forms a Hermitian holomorphic vector bundle over X_s , where $\mathcal{E}_s = \mathcal{E}|_{X_s}$ and $h_s = h|_{X_s}$. Thus (E, h) can be understood as a family of Hermitian holomorphic vector bundles over the family of complex manifolds $\pi : \mathcal{X} \rightarrow S$. By a result of Grauert [G], the direct image sheaf $\pi_*\mathcal{E}$ is coherent over S , and is thus locally free on the complement of a proper subvariety Z in S . Moreover, changing Z if necessary, one may assume that $\pi_*\mathcal{E}$ is given by

$$(\pi_*\mathcal{E})_s = H^0(X_s, \mathcal{E}_s) \quad \text{for } s \in S \setminus Z.$$

For simplicity, we will also denote by $\pi_*\mathcal{E}$ its underlying holomorphic vector bundle over $S \setminus Z$. It is easy to see that h and $\omega_{\mathcal{X}}$ induce a smooth Hermitian metric, known as the L^2 -metric, on $\pi_*\mathcal{E}$ over $S \setminus Z$ given by

$$(1.1.1) \quad H_{h, \omega_{\mathcal{X}}}(t, t') := \int_{X_s} \langle t, t' \rangle \frac{\omega_s^n}{n!} \quad \text{for } t, t' \in H^0(X_s, \mathcal{E}_s), \quad s \in S \setminus Z,$$

where $\langle t, t' \rangle$ denotes $h_s(t, t')$. We will also denote by $\langle \cdot, \cdot \rangle$ the inner product on \mathcal{E}_s -valued differential forms induced by h_s and ω_s .

Consider a family of compact Kähler manifolds $(\pi : \mathcal{X} \rightarrow S, \omega_{\mathcal{X}})$. The orthogonal complement of $\text{Ker}(\pi_* : T\mathcal{X} \rightarrow TS)$ in $T\mathcal{X}$ with respect to $\omega_{\mathcal{X}}$ defines a smooth ‘horizontal’ vector subbundle $T^H\mathcal{X} \subset T\mathcal{X}$. For $s \in S$ and a holomorphic tangent vector $u \in T_sS$, there exists a unique lifting of u to a smooth vector field $v_u \in \mathcal{A}^0(X_s, T\mathcal{X}|_{X_s})$ such that $\pi_*v_u(z) = u$ and $v_u(z) \in T^H\mathcal{X}$ for all $z \in X_s$. Such v_u is called the horizontal lifting of u (with respect to $\omega_{\mathcal{X}}$). Moreover one can easily check that

$$(1.1.2) \quad A_u := \bar{\partial}v_u \in \mathcal{A}^{0,1}(X_s, TX_s)$$

(cf. (2.1.2)), and its class $[A_u] \in H^{0,1}(X_s, TX_s)$ is the Kodaira-Spencer class of u (cf. e.g. [FS], [Sch] and (2.3)).

We will denote by $s = (s^i)_{1 \leq i \leq m}$, where $m = \dim_{\mathbb{C}}S$, the local holomorphic coordinates for S , and we will denote $z = (z^\alpha)_{1 \leq \alpha \leq n}$ the local holomorphic coordinates for the fibers X_s . Thus $(z, s) = (z^1, \dots, z^n, s^1, \dots, s^m)$ gives local holomorphic coordinates for X_s . For $s \in S$, a coordinate tangent vector $\partial/\partial s^i \in T_sS$, we simply denote its horizontal lifting by

$$(1.1.3) \quad v_i := v_{\partial/\partial s^i} \quad \text{and thus} \quad A_{\partial/\partial s^i} = \bar{\partial}v_i.$$

Denote the components of $A_{\partial/\partial s^i}$ by

$$(1.1.4) \quad A_{\partial/\partial s^i} =: A_{i\bar{\beta}}^\alpha \frac{\partial}{\partial z^\alpha} \otimes d\bar{z}^\beta.$$

Here we adopt the Einstein summation notation. Also for a Hermitian holomorphic vector bundle (\mathcal{E}, h) over $(\mathcal{X}, \omega_{\mathcal{X}})$, we denote the curvature tensor of (\mathcal{E}, h) by $\Omega \in \mathcal{A}^{1,1}(\mathcal{X}, \text{End}(\mathcal{E}))$. Also denote the smooth family of Kähler metrics associated to $\omega_{\mathcal{X}}$ by $g := \{g_s\}_{s \in S}$. For $s \in S$, we also denote by $G_{\mathcal{E}_s}$ the Green's operator on \mathcal{E}_s -valued differential forms on X_s associated to the $\bar{\partial}$ -Laplacian $\square := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ with respect to h_s and g_s . Thus the Laplacian for functions is given by $\square = -g^{\bar{\beta}\alpha} \partial^2 / \partial z^\alpha \partial \bar{z}^\beta$. We will also adopt the semi-colon notation to denote covariant derivatives, so that $t_{;\alpha} := \nabla_{\partial/\partial z^\alpha} t$ for $t \in H^0(X_s, \mathcal{E}_s)$, etc.

With the above notations, we state our first main result in this paper as follows:

Theorem 1. *Let $(\mathcal{E} \rightarrow \mathcal{X}, h)$ be a family of Hermitian holomorphic vector bundles of rank r over a family of n -dimensional compact Kähler manifolds $(\pi : \mathcal{X} \rightarrow S, \omega_{\mathcal{X}})$ and parametrized by an m -dimensional compact complex manifold S . Suppose that $\omega_{\mathcal{X}}$ is d -closed on \mathcal{X} . Let $Z \subset S$ be a proper analytic subvariety such that $\pi_*\mathcal{E}$ is locally free over $S \setminus Z$ with $(\pi_*\mathcal{E})_s = H^0(X_s, \mathcal{E}_s)$ for $s \in S \setminus Z$. Then in terms of local holomorphic coordinates $s = (s^i)_{1 \leq i \leq m}$ of S , $z = (z^\alpha)_{1 \leq \alpha \leq n}$ of X_s and a local holomorphic trivialization $\{t_a\}_{1 \leq a \leq p}$ of $\pi_*\mathcal{E}$, where $p := \text{rank}(\pi_*\mathcal{E})$, the curvature tensor $\Theta \in \mathcal{A}^{1,1}(S \setminus Z, \text{End}(\pi_*\mathcal{E}))$ of the associated L^2 -metric $H_{h, \omega_{\mathcal{X}}}$ on $\pi_*\mathcal{E}$ is given by*

$$(1.1.5) \quad \Theta_{\bar{a}b\bar{i}j}(s) = - \int_{X_s} \langle G_{\mathcal{E}_s}(\Omega_{v_i\bar{\alpha}}(t_a)dz^\alpha - A_{i\bar{\beta}}^\gamma t_{a;\gamma}d\bar{z}^\beta), \Omega_{v_j\bar{\alpha}}(t_b)dz^\alpha - A_{j\bar{\beta}}^\gamma t_{b;\gamma}d\bar{z}^\beta \rangle \frac{\omega_s^n}{n!} \\ + \int_{X_s} \langle \Omega_{v_i\bar{v}_j}(t_a), t_b \rangle \frac{\omega_s^n}{n!} + \int_{X_s} \square(g_{v_i\bar{v}_j})(t_a, t_b) \frac{\omega_s^n}{n!}$$

for $s \in S \setminus Z$. Here $\Theta_{\bar{a}b\bar{i}j} := \Theta(t_a, t_b; \partial/\partial s^i, \partial/\partial s^j)$ with $t_a, t_b \in H^0(X_s, \mathcal{E}_s)$.

Remark 1.1.1. (i) When h (and thus also $H_{h, \omega_{\mathcal{X}}}$) is multiplied by a smooth function $\lambda(s)$ on S , the second term of Θ in (1.1.5) is modified by $-\partial\bar{\partial} \log \lambda \cdot H_{h, \omega_{\mathcal{X}}}(t_a, t_b)$, while the first and the third terms remain unchanged.

(ii) We refer the reader to (2.3) for an interpretation of the first term of Θ in (1.1.5) in terms of Kodaira-Spencer representatives.

(iii) Theorem 1 in the special case of a family of Hermitian-Einstein vector bundles over a fixed compact Kähler manifold was treated earlier by the authors in [TW], where the curvature of the L^2 metric was further expressed in terms of harmonic representatives of the Kodaira-Spencer class of the tangent vectors of the base manifold.

(iv) For a family of compact Kähler-Einstein manifolds with constant non-zero scalar curvature, one can always glue the Kähler forms (associated to the Kähler-Einstein metrics) on the fibers appropriately to get a d -closed (1,1)-form on the total space (cf. e.g. [Sch]). In particular, Theorem 1 applies to such families of Kähler manifolds.

(1.2). Next we consider a family of Hermitian holomorphic line bundles $(\mathcal{L} \rightarrow \mathcal{X}, h)$ over a family of n -dimensional compact Kähler manifolds $(\pi : \mathcal{X} \rightarrow S, \omega_{\mathcal{X}})$ such that for some $k \in \mathbb{R}$,

$$(1.2.1) \quad c_1(\mathcal{L}, h) = \frac{k}{2\pi} \omega_{\mathcal{X}} \quad \text{on } \mathcal{X}.$$

Thus in particular, one has, for all $s \in S$,

$$(1.2.2) \quad c_1(L_s, h_s) = \frac{k}{2\pi} \omega_s \quad \text{on } X_s.$$

Then Theorem 1 gives rise to

Corollary 2. *Let $(\pi : \mathcal{X} \rightarrow S, \omega_{\mathcal{X}})$ be as in Theorem 1 and let $(\mathcal{L} \rightarrow \mathcal{X}, h)$ be a family of Hermitian holomorphic line bundles satisfying (1.2.1) for some $k \in \mathbb{R}$. Let $Z \subset S$ be a proper analytic subvariety such that $\pi_*\mathcal{L}$ is locally free over $S \setminus Z$ with $(\pi_*\mathcal{L})_s = H^0(X_s, L_s)$ for $s \in S \setminus Z$. Then the curvature tensor $\Theta \in \mathcal{A}^{1,1}(S \setminus Z, \text{End}(\pi_*\mathcal{L}))$ of the associated L^2 -metric $H_{h, \omega_{\mathcal{X}}}$ on $\pi_*\mathcal{L}$ is given by*

$$(1.2.3) \quad \begin{aligned} \Theta_{\bar{a}\bar{b}i\bar{j}}(s) = & - \int_{X_s} \langle G_{\mathcal{L}_s}(A_{i\bar{j}}^\gamma t_a, \gamma d\bar{z}^\beta), A_{j\bar{b}}^\gamma t_b, \gamma d\bar{z}^\beta \rangle \frac{\omega_s^n}{n!} \\ & + \int_{X_s} (kg_{v_i \bar{v}_j} + \square(g_{v_i \bar{v}_j})) \langle t_a, t_b \rangle \frac{\omega_s^n}{n!} \end{aligned}$$

for $s \in S \setminus Z$ and $t_a, t_b \in H^0(X_s, L_s)$.

(1.3). To facilitate ensuing discussion, we recall that a connection D on a smooth complex vector bundle F of rank r over a manifold M is said to be projectively flat if there exists a complex 2-form $\alpha \in \mathcal{A}^2(M)$ such that the curvature form $R = D^2 \in \mathcal{A}^2(M, \text{End}(F))$ satisfies

$$(1.3.1) \quad R = \alpha \cdot \text{Id}_F,$$

where Id_F is the identity endomorphism on F . It is well known that F admits a projectively flat connection if and only if F arises from a representation $\Phi : \pi_1(M) \rightarrow \mathbb{PGL}(r, \mathbb{C})$. Here $\pi_1(M)$ denotes the fundamental group of M , $\mathbb{PGL}(r, \mathbb{C}) := \text{GL}(r, \mathbb{C})/\mathbb{C}^* \cdot I_r$ denotes the projective general linear group, and I_r is the $r \times r$ identity matrix. Moreover, F admits a projectively flat Hermitian connection if and only if F arises from a representation $\Phi : \pi_1(M) \rightarrow \mathbb{PU}(r)$, where $\mathbb{PU}(r) := U(r)/U(1) \cdot I_r$ denotes the projective unitary group (cf. e.g. [Ko2, pp. 7 and 14] for the above background materials).

Let A_0 be an abelian variety, and let L_0 be a holomorphic line bundle over A_0 .

Definition 1.3.1. A Hermitian metric h_0 on L_0 is said to be canonical if $c_1(L_0, h_0)$ is invariant under translations of A_0 .

Remark 1.3.2. It is well known that a canonical Hermitian metric always exists on L_0 and it is unique up to a positive multiplicative constant. In fact, this follows from the standard fact that a de Rham cohomology class of a compact complex torus is represented uniquely by a translation-invariant form (see e.g. [GH, p. 302]).

Now we consider a (holomorphic) family of abelian varieties $\pi : \mathcal{A} \rightarrow S$ parametrized by a complex manifold S , i.e., each fiber $A_s := \pi^{-1}(s)$ is an n -dimensional abelian variety. Let $\mathcal{L} \rightarrow \mathcal{A}$ be a (holomorphic) family of ample line bundles over the family $\pi : \mathcal{A} \rightarrow S$, i.e., \mathcal{L} is a holomorphic line bundle over the total space \mathcal{A} such that $L_s := \mathcal{L}|_{A_s}$ is an ample line bundle over A_s for each $s \in S$. (Such an \mathcal{L} is usually called a relatively ample line bundle.) Then it follows from the Kodaira vanishing theorem that $H^i(A_s, L_s) = 0$ for each $i > 0$ and $s \in S$, and thus by a result of Grauert [G], $\pi_*\mathcal{L}$ is locally free over S with $(\pi_*\mathcal{L})_s = H^0(A_s, L_s)$ for

each $s \in S$. Moreover, one can always construct a Hermitian metric ρ on \mathcal{L} such that $\rho_s := \rho|_{L_s}$ is a canonical Hermitian metric on L_s for all $s \in S$ (this follows, for example, from the argument in [Siu2, p. 17] or (6.4) and using a partition of unity on S). With slight abuse of notation, such ρ will be called a smooth family of canonical Hermitian metrics on the family of line bundles $\mathcal{L} \rightarrow \mathcal{A}$. Also, it is obvious that such ρ is unique up to a smooth positive function on S . A (smooth) family of flat Kähler metrics $\omega_{\mathcal{A}}$ on the family $\pi : \mathcal{A} \rightarrow S$ is simply a smooth (1,1)-form on the total space \mathcal{A} such that $\omega_s := \omega_{\mathcal{A}}|_{A_s}$ is a flat Kähler metric on A_s for each $s \in S$. Our second main result is the following

Theorem 3. *Let $\pi : \mathcal{A} \rightarrow S$ be a family of abelian varieties parametrized by a complex manifold S , and let $\mathcal{L} \rightarrow \mathcal{A}$ be a family of ample line bundles parametrized by S .*

(i) *Then for any smooth family ρ of canonical Hermitian metrics on the family $\mathcal{L} \rightarrow \mathcal{A}$ and any smooth family of flat Kähler metrics $\omega_{\mathcal{A}}$ on the family $\pi : \mathcal{A} \rightarrow S$, the Hermitian connection of the associated L^2 -metric $H_{\rho, \omega_{\mathcal{A}}}$ on $\pi_*\mathcal{L}$ is projectively flat.*

(ii) *In particular, if the parameter space S is a compact Kähler manifold, then $\pi_*\mathcal{L}$ is poly-stable with respect to any Kähler form on S .*

Remark 1.3.3. (i) Welters [W] proved earlier that under the hypothesis of Theorem 3, $\pi_*\mathcal{L}$ admits a projectively flat connection, and hence $\pi_*\mathcal{L}$ arises from a representation of $\pi_1(S)$ into $\mathbb{PGL}(r, \mathbb{C})$, where $r := \text{rank}(\pi_*\mathcal{L})$. However, the projectively flat connection in [W] is not a Hermitian connection in general (cf. also [H]), and thus Welters' result does not lead to statement (i) or (ii) of Theorem 3. Theorem 3, in particular, implies that $\pi_*\mathcal{L}$ arises from a representation of $\pi_1(S)$ into $\mathbb{P}U(r)$.

(ii) Theorem 3 in the special case of the family of theta line bundles (or their powers) over a family of principally polarized abelian varieties is well known and follows from the heat equation and orthonormality of theta characters (cf. e.g. [APW, §5]).

(iii) We remark that, unlike Theorem 1, the (1,1)-form $\omega_{\mathcal{A}}$ in Theorem 3(i) need not be d -closed on \mathcal{A} .

§2. CURVATURE OF THE L^2 -METRIC

We are going to prove Theorem 1 in this section, and we will follow the notations in (1.1) throughout §2.

(2.1). Let $(\pi : \mathcal{X} \rightarrow S, \omega_{\mathcal{X}})$ be a family of compact Kähler manifolds parametrized by a complex manifold S , and write $X_s := \pi^{-1}(s)$ and $\omega_s := \omega_{\mathcal{X}}|_{X_s}$ for each $s \in S$. Let $m = \dim_{\mathbb{C}} S$ and $n = \dim_{\mathbb{C}} X_s$. We will use $s = (s^i)_{1 \leq i \leq m}$ to denote local holomorphic coordinates of S , which will be indexed by the letters i, j, k , etc. Also we will use $z = (z^\alpha)_{1 \leq \alpha \leq n}$ to denote local holomorphic coordinates of the fibers X_s , which will be indexed by the Greek letters α, β, γ , etc. Thus $(z, s) = (z^\alpha, s^i)_{1 \leq \alpha \leq n, 1 \leq i \leq m}$ gives local holomorphic coordinates for \mathcal{X} and we will use capital letters I, J, K to index coordinates on \mathcal{X} so that I can be i or α , etc. Also we write $\omega_{\mathcal{X}} = \sqrt{-1}g_{I\bar{J}}(z, s)dw^I \wedge d\bar{w}^{\bar{J}}$, where w can be z or s . For a coordinate tangent vector $\partial/\partial s^i \in T_s S, s \in S$, its horizontal lifting v_i (with respect to $\omega_{\mathcal{X}}$)

defined in (1.1) is given by

$$(2.1.1) \quad v_i = \frac{\partial}{\partial s^i} - g^{\bar{\beta}\alpha} g_{i\bar{\beta}} \frac{\partial}{\partial z^\alpha} \quad \text{at } (z, s) \in \mathcal{X}$$

(see [Sch, §1, equation (1.2)]). Here $g^{\bar{\beta}\alpha}$ denotes the components of the inverse of $g_{\alpha\bar{\beta}}$ (and not that of $g_{I\bar{J}}$, which may not be invertible). Then it is easy to see that the associated tensor $A_{\partial/\partial s^i}$ defined in (1.1.3) is given locally by

$$(2.1.2) \quad A_{i\bar{\beta}}^\alpha = -\frac{\partial}{\partial \bar{z}^\beta} (g^{\bar{\gamma}\alpha} g_{i\bar{\gamma}}).$$

We have the following simple lemma:

Lemma 2.1.1. (i) $[v_i, \frac{\partial}{\partial \bar{z}^\alpha}] = -A_{i\bar{\alpha}}^\beta \frac{\partial}{\partial z^\beta}$.
 (ii) For a smooth (n, n) -form η on \mathcal{X} ,

$$\frac{\partial}{\partial s^i} \int_{X_s} \eta = \int_{X_s} L_{v_i}(\eta)$$

and

$$\frac{\partial}{\partial \bar{s}^i} \int_{X_s} \eta = \int_{X_s} L_{\bar{v}_i}(\eta).$$

Moreover, if $\omega_{\mathcal{X}}$ is d -closed, then

- (iii) $[v_i, \bar{v}_j] = g^{\bar{\gamma}\alpha} \frac{\partial}{\partial \bar{z}^\gamma} (g_{v_i \bar{v}_j}) \frac{\partial}{\partial z^\alpha} - g^{\bar{\beta}\gamma} \frac{\partial}{\partial z^\gamma} (g_{v_i \bar{v}_j}) \frac{\partial}{\partial \bar{z}^\beta}$, and
- (iv) $L_{v_i}(\omega_s^n) = 0$.

Here $[,]$ denotes the Lie bracket of two vector fields, and L_{v_i} denotes the Lie derivative in the direction of v_i , etc.

Proof. (i) follows immediately from (2.1.1) and (2.1.2). (ii), (iii) and (iv) can be found in [Sch, Lemma 2.1], [Sch, Lemma 2.6] and [Sch, Lemma 2.2], respectively. We remark that among the conditions assumed in [Sch], the proofs of [Sch, Lemma 2.6] and [Sch, Lemma 2.2] work under the mere condition that $\omega_{\mathcal{X}}$ is d -closed. \square

(2.2). Now we give the proof of Theorem 1 as follows:

Proof of Theorem 1. Let $(\mathcal{E} \rightarrow \mathcal{X}, h)$ be a family of Hermitian holomorphic vector bundles of rank r over $(\pi : \mathcal{X} \rightarrow S, \omega_{\mathcal{X}})$ as in Theorem 1, where $\dim_{\mathbb{C}} S = m$ and $\dim_{\mathbb{C}} X_s = n$. Here and thereafter, X_s and ω_s are as in (2.1). For simplicity, we simply write $H := H_{h, \omega_{\mathcal{X}}}$. Choose local holomorphic coordinates $s = (s^1, \dots, s^m)$ for S so that $\pi_* \mathcal{E}$ is locally free at $s = 0$ and of rank p . Choose a holomorphic trivialization $\{t_1, t_2, \dots, t_p\}$ of $\pi_* \mathcal{E}$ over a coordinate neighborhood U of S containing 0 such that the L^2 -metric H on $\pi_* \mathcal{E}$ satisfies

$$(2.2.1) \quad \frac{\partial H_{a\bar{b}}}{\partial s^i} \Big|_{s=0} = 0 \quad \text{for } 1 \leq a, b \leq p, 1 \leq i \leq m.$$

Here $H_{a\bar{b}} := H(t_a, t_b)$. In the sequel, covariant derivatives will be with respect to the Hermitian connection on (\mathcal{E}, h) . For $1 \leq i \leq m$, recall from (2.1) the horizontal lifting v_i of $\partial/\partial s^i$ with respect to $\omega_{\mathcal{X}}$. With respect to the above trivialization, the

curvature tensor Θ of $(\pi_*\mathcal{E}, H)$ at $s = 0$ is given by

$$\begin{aligned}
 \Theta_{a\bar{b}i\bar{j}}(0) &= -\frac{\partial^2 H_{a\bar{b}}}{\partial s^i \partial \bar{s}^j} \Big|_{s=0} \\
 &= -\frac{\partial^2}{\partial s^i \partial \bar{s}^j} \int_{X_s} \langle t_a, t_b \rangle \frac{\omega_s^n}{n!} \Big|_{s=0} \\
 &= -\frac{\partial}{\partial \bar{s}^j} \int_{X_s} L_{v_i} (\langle t_a, t_b \rangle \frac{\omega_s^n}{n!}) \Big|_{s=0} \quad (\text{by Lemma 2.1.1(ii)}) \\
 (2.2.2) \quad &= -\frac{\partial}{\partial \bar{s}^j} \int_{X_s} L_{v_i} (\langle t_a, t_b \rangle) \frac{\omega_s^n}{n!} \Big|_{s=0} \quad (\text{by Lemma 2.1.1(iv)}) \\
 &= -\frac{\partial}{\partial \bar{s}^j} \int_{X_s} \langle \nabla_{v_i} t_a, t_b \rangle \frac{\omega_s^n}{n!} \Big|_{s=0} \quad (\text{by holomorphicity of } t_b) \\
 &= -\int_{X_s} L_{\bar{v}_j} (\langle \nabla_{v_i} t_a, t_b \rangle) \frac{\omega_s^n}{n!} \Big|_{s=0} \quad (\text{by Lemma 2.1.1(ii) and (iv) again}) \\
 &= -\int_{X_s} \langle \nabla_{\bar{v}_j} \nabla_{v_i} t_a, t_b \rangle \frac{\omega_s^n}{n!} \Big|_{s=0} - \int_{X_s} \langle \nabla_{v_i} t_a, \nabla_{v_j} t_b \rangle \frac{\omega_s^n}{n!} \Big|_{s=0} \\
 &=: I_1 + I_2.
 \end{aligned}$$

First we deal with the integral I_2 . Denote by $H_{\mathcal{E}_0}$ the harmonic projection operator on \mathcal{E}_0 with respect to the $\bar{\partial}$ -Laplacian. The Hodge decomposition theorem gives, at $s = 0$,

$$(2.2.3) \quad \nabla_{v_i} t_a = H_{\mathcal{E}_0}(\nabla_{v_i} t_a) + \square G_{\mathcal{E}_0}(\nabla_{v_i} t_a),$$

where $G_{\mathcal{E}_0}$ is as in (1.1). As in (2.2.2), it follows from (2.2.1) that we have

$$(2.2.4) \quad \int_{X_0} \langle \nabla_{v_i} t_a, t_b \rangle \frac{\omega_0^n}{n!} = 0 \quad \text{for } 1 \leq a, b \leq p, 1 \leq i \leq m.$$

Since $\{t_b\}_{1 \leq b \leq p}$ is a basis of $H^0(X_0, \mathcal{E}_0)$, and $H_{\mathcal{E}_0}(\nabla_{v_i} t_a) \in H^0(X_0, \mathcal{E}_0)$, it follows from (2.2.4) that

$$(2.2.5) \quad H_{\mathcal{E}_0}(\nabla_{v_i} t_a) = 0 \quad \text{for } 1 \leq a \leq p, 1 \leq i \leq m.$$

Combining (2.2.3) and (2.2.5), we have at $s = 0$,

$$\begin{aligned}
 \int_{X_0} \langle \nabla_{v_i} t_a, \nabla_{v_j} t_b \rangle \frac{\omega_0^n}{n!} &= \int_{X_0} \langle \square G_{\mathcal{E}_0}(\nabla_{v_i} t_a), \nabla_{v_j} t_b \rangle \frac{\omega_0^n}{n!} \\
 (2.2.6) \quad &= \int_{X_0} \langle G_{\mathcal{E}_0} \bar{\partial}(\nabla_{v_i} t_a), \bar{\partial}(\nabla_{v_j} t_b) \rangle \frac{\omega_0^n}{n!} \\
 &\quad (\text{since } [G_{\mathcal{E}_0}, \bar{\partial}] = 0, \text{ and } \bar{\partial}^*(\nabla_{v_i} t_a) = 0 \text{ trivially}).
 \end{aligned}$$

Taking $\bar{\partial}$ along the X_0 direction, we have, from the Ricci identity,

$$\begin{aligned}
 \bar{\partial}(\nabla_{v_i} t_a) &= (\nabla_{\bar{\alpha}} \nabla_{v_i} t_a) d\bar{z}^\alpha \\
 &= (\nabla_{v_i} \nabla_{\bar{\alpha}} t_a - \nabla_{[v_i, \partial/\partial \bar{z}^\alpha]} t_a - \Omega_{v_i \bar{\alpha}}(t_a)) d\bar{z}^\alpha \\
 (2.2.7) \quad &= (0 + A_{i\bar{\alpha}}^\gamma \nabla_\gamma t_a - \Omega_{v_i \bar{\alpha}}(t_a)) d\bar{z}^\alpha \\
 &\quad (\text{by holomorphicity of } t_a \text{ and Lemma 2.1.1(i)}) \\
 &= -\Omega_{v_i \bar{\alpha}}(t_a) d\bar{z}^\alpha + A_{i\bar{\beta}}^\gamma t_{a;\gamma} d\bar{z}^\beta
 \end{aligned}$$

using the semicolon notation. Combining (2.2.6) and (2.2.7), we have

$$(2.2.8) \quad I_2 = - \int_{X_0} \langle G_{\mathcal{E}_0}(-\Omega_{v_i\bar{\alpha}}(t_a)d\bar{z}^\alpha + A_{i\bar{\beta}}^\gamma t_{a;\gamma}d\bar{z}^\beta), -\Omega_{v_j\bar{\alpha}}(t_b)d\bar{z}^\alpha + A_{j\bar{\beta}}^\gamma t_{b;\gamma}d\bar{z}^\beta \rangle \frac{\omega_0^n}{n!}.$$

Next we consider the integral I_1 . From the Ricci identity, we have

$$(2.2.9) \quad \begin{aligned} \nabla_{\bar{v}_j} \nabla_{v_i} t_a &= \nabla_{v_i} \nabla_{\bar{v}_j} t_a - \nabla_{[v_i, \bar{v}_j]} t_a - \Omega_{v_i \bar{v}_j}(t_a) \\ &= \nabla_{v_i} \nabla_{\bar{v}_j} t_a - g^{\bar{\gamma}\alpha} \frac{\partial}{\partial \bar{z}^\gamma} (g_{v_i \bar{v}_j}) \nabla_\alpha t_a + g^{\bar{\beta}\gamma} \frac{\partial}{\partial z^\gamma} (g_{v_i \bar{v}_j}) \nabla_{\bar{\beta}} t_a - \Omega_{v_i \bar{v}_j}(t_a) \\ &\quad \text{(by Lemma 2.1.1(iii))} \\ &= -g^{\bar{\gamma}\alpha} \frac{\partial}{\partial \bar{z}^\gamma} (g_{v_i \bar{v}_j}) \nabla_\alpha t_a - \Omega_{v_i \bar{v}_j}(t_a) \\ &\quad \text{(by holomorphicity of } t_a \text{).} \end{aligned}$$

By (2.2.2) and (2.2.9), we have

$$(2.2.10) \quad \begin{aligned} I_1 &= \int_{X_0} \langle g^{\bar{\gamma}\alpha} \frac{\partial}{\partial \bar{z}^\gamma} (g_{v_i \bar{v}_j}) \nabla_\alpha t_a + \Omega_{v_i \bar{v}_j}(t_a), t_b \rangle \frac{\omega_0^n}{n!} \\ &= \int_{X_0} g^{\bar{\gamma}\alpha} \frac{\partial}{\partial \bar{z}^\gamma} (g_{v_i \bar{v}_j}) \overline{\frac{\partial}{\partial \bar{z}^\alpha} \langle t_a, t_b \rangle} \frac{\omega_0^n}{n!} + \int_{X_0} \langle \Omega_{v_i \bar{v}_j}(t_a), t_b \rangle \frac{\omega_0^n}{n!} \\ &\quad \text{(by holomorphicity of } t_b \text{)} \\ &= \int_{X_0} \langle \bar{\partial}(g_{v_i \bar{v}_j}), \bar{\partial}(\langle t_a, t_b \rangle) \rangle \frac{\omega_0^n}{n!} + \int_{X_0} \langle \Omega_{v_i \bar{v}_j}(t_a), t_b \rangle \frac{\omega_0^n}{n!} \\ &= \int_{X_0} \square(g_{v_i \bar{v}_j}) \cdot \langle t_a, t_b \rangle \frac{\omega_0^n}{n!} + \int_{X_0} \langle \Omega_{v_i \bar{v}_j}(t_a), t_b \rangle \frac{\omega_0^n}{n!}, \end{aligned}$$

where the last line follows from the definition $\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ and the adjoint property of $\bar{\partial}^*$. Then (1.1.5) follows immediately from (2.2.2), (2.2.8) and (2.2.10), and we have finished the proof of Theorem 1. \square

(2.3). Notation as in (1.1). In this subsection, we are going to interpret the terms $\Omega_{v_i\bar{\beta}}d\bar{z}^\beta \in \mathcal{A}^{0,1}(X_s, \text{End}(\mathcal{E}_s))$ and $A_{i\bar{\beta}}^\alpha \frac{\partial}{\partial z^\alpha} \otimes d\bar{z}^\beta \in \mathcal{A}^{0,1}(X_s, T_s)$ which appear in (1.1.5) in Theorem 1 in terms of Kodaira-Spencer theory.

Let $(p : \mathcal{E} \rightarrow \mathcal{X}, h)$ and $(\pi : \mathcal{X} \rightarrow S, \omega_{\mathcal{X}})$ be as in Theorem 1. Here p and p_s denote the projection maps. For each rank- r holomorphic vector bundle $\mathcal{E}_s := \mathcal{E}|_{X_s}$, where $X_s = p^{-1}(s)$, we denote by P_s the associated principal $\text{GL}(r, \mathbb{C})$ -bundle over X_s . By abuse of notation, we also let $p_s : P_s \rightarrow X_s$ denote the projection map. The structure group $\text{GL}(r, \mathbb{C})$ acts naturally on the tangent bundle TP_s of P_s , and we denote the quotient by $Q_s := TP_s/\text{GL}(r, \mathbb{C})$. Then Q_s is a holomorphic bundle over X_s (with holomorphic structure induced from TP_s) such that $\mathcal{O}_{X_s}(Q_s)$ is isomorphic to the sheaf of germs of $\text{GL}(r, \mathbb{C})$ -invariant tangent vector fields on P_s . Denote the associated quotient map by $\tau_s : TP_s \rightarrow Q_s$. It is easy to see that the differential $p_{s*} : TP_s \rightarrow TX_s$ factors through τ_s to a bundle epimorphism $q_s : Q_s \rightarrow TX_s$, i.e., one has $p_{s*} = q_s \circ \tau_s$. By [A, Theorem 1 and Proposition 9], there exists a short exact sequence of holomorphic vector bundles

$$(2.3.1) \quad 0 \rightarrow \text{End}(\mathcal{E}_s) \xrightarrow{i} Q_s \xrightarrow{q_s} TX_s \rightarrow 0$$

over X_s , which is usually known as the Atiyah sequence associated to \mathcal{E}_s . Similarly, for the vector bundle \mathcal{E} over \mathcal{X} , we have the associated principal $\text{GL}(r, \mathbb{C})$ -bundle

$p : P \rightarrow \mathcal{X}$, the quotient $Q := TP/\text{GL}(r, \mathbb{C})$, the quotient map $\tau : TP \rightarrow Q$ and $q : Q \rightarrow T\mathcal{X}$ with $p_* = q \circ \tau$, and the Atiyah sequence

$$(2.3.2) \quad 0 \rightarrow \text{End}(\mathcal{E}) \xrightarrow{i} Q \xrightarrow{q} T\mathcal{X} \rightarrow 0$$

over \mathcal{X} . Of course, the restriction of (2.3.2) to X_s simply yields (2.3.1) for each $s \in S$. The Hermitian connection of (\mathcal{E}, h) induces an associated principal connection on P which is a $gl(r, \mathbb{C})$ -valued $(1,0)$ -form $\tilde{\theta}$ on P invariant under $\text{GL}(r, \mathbb{C})$. By identifying the fibers of $\text{End}(\mathcal{E})$ with $gl(r, \mathbb{C})$, $\tilde{\theta}$ descends to a homomorphism $\theta : Q \rightarrow \text{End}(\mathcal{E})$. Moreover $\theta \circ i = \text{Id}_{\mathcal{E}}$, which leads to a smooth splitting of (2.3.1) (see [A] for the above discussion).

Also consider the following short exact sequence of vector bundles

$$(2.3.3) \quad 0 \rightarrow T_{\mathcal{X}|S} \rightarrow T\mathcal{X} \rightarrow \pi^*TS \rightarrow 0$$

over \mathcal{X} . Here $T_{\mathcal{X}|S}$ denotes the relative tangent bundle over \mathcal{X} associated to the family $\pi : \mathcal{X} \rightarrow S$. For each $s \in S$, we have the Kodaira-Spencer map $\rho_s : T_sS \rightarrow H^1(X_s, TX_s)$ associated to (2.3.3). Here we do not distinguish between a holomorphic vector bundle and its sheaf of germs of holomorphic sections. From (2.3.2) and (2.3.3), we may regard π^*TS as a quotient bundle of Q , and this leads to the following short exact sequence of vector bundles

$$(2.3.4) \quad 0 \rightarrow Q_{\mathcal{X}|S} \rightarrow Q \rightarrow \pi^*TS \rightarrow 0$$

over \mathcal{X} , where $Q_{\mathcal{X}|S} \subset Q$ is the associated relative vector bundle. For each $s \in S$, we similarly have the Kodaira-Spencer map $\hat{\rho}_s : T_sS \rightarrow H^1(X_s, Q_s)$ associated to (2.3.4) (cf. [FS, §4] for the case of line bundles). We also let $Z^1(X_s, TX_s)$ (resp. $Z^1(X_s, Q_s)$) denote the space of $\bar{\partial}$ -closed TX_s (resp. Q_s)-valued $(0,1)$ -forms on X_s . Denote by $s = (s^i)_{1 \leq i \leq m}$ local holomorphic coordinates for S . We recall from (1.1) the horizontal lifting v_i of $\partial/\partial s^i$ with respect to $\omega_{\mathcal{X}}$, and let $A_{\partial/\partial s^i} = \bar{\partial}v_i$ be as in (1.1.3). Obviously, $A_{\partial/\partial s^i} \in Z^1(X_s, TX_s)$. It is well known that $A_{\partial/\partial s^i}$ is a representative of the Kodaira-Spencer class of $\rho_s(\partial/\partial s^i)$, i.e. the Dolbeault cohomology class $[A_{\partial/\partial s^i}]$ of $A_{\partial/\partial s^i}$ satisfies $[A_{\partial/\partial s^i}] = \rho_s(\partial/\partial s^i)$ in $H^1(X_s, TX_s)$ via the Dolbeault isomorphism. Note however that $A_{\partial/\partial s^i}$, in general, may not be harmonic (see e.g. [Sch, §1]). Let $\partial/\partial s^i$ and v_i be as above. One can further lift v_i uniquely to a smooth vector field \tilde{v}_i on P such that

$$(2.3.5) \quad p_*(\tilde{v}_i) = v_i \quad \text{and} \quad \tilde{v}_i \in \text{Ker}(\tilde{\theta})$$

(cf. e.g. [KN, p. 65]). Since $\tilde{\theta}$ is $\text{GL}(r, \mathbb{C})$ -invariant, it is easy to see that \tilde{v}_i is also $\text{GL}(r, \mathbb{C})$ -invariant and thus \tilde{v}_i descends to a smooth section \hat{v}_i of Q . Define

$$(2.3.6) \quad \hat{A}_{\partial/\partial s^i} := \bar{\partial}\hat{v}_i$$

along each fiber X_s . Then $\hat{A}_{\partial/\partial s^i} \in Z^1(X_s, Q_s)$ for $s \in S$. Using the argument in [FS, Lemma 4.4] (which deals with the case of line bundles), one sees that $\hat{A}_{\partial/\partial s^i}$ is a $\bar{\partial}$ -closed representative of the Kodaira-Spencer class $\hat{\rho}_s(\partial/\partial s^i)$, i.e. the Dolbeault cohomology class $[\hat{A}_{\partial/\partial s^i}]$ of $\hat{A}_{\partial/\partial s^i}$ satisfies $[\hat{A}_{\partial/\partial s^i}] = \hat{\rho}_s(\partial/\partial s^i)$ in $H^1(X_s, Q_s)$ via the Dolbeault isomorphism. Moreover, it follows easily from (2.3.5) and the holomorphicity of p_{s*} and τ_s that

$$(2.3.7) \quad q_s(\hat{A}_{\partial/\partial s^i}) = A_{\partial/\partial s^i} \quad \text{for } s \in S$$

(cf. also [FS, Lemma 4.6]). Next we consider $\theta_s(\hat{A}_{\partial/\partial s^i}) \in \mathcal{A}^{0,1}(X_s, \text{End}(\mathcal{E}_s))$. Let $\Omega \in \mathcal{A}^{1,1}(\mathcal{X}, \text{End}(\mathcal{E}))$ be as in (1.1). We have

Lemma 2.3.1. For $s \in S$, $\theta_s(\hat{A}_{\partial/\partial s^i}) = \Omega_{v_i, \bar{\alpha}} d\bar{z}^\alpha \in \mathcal{A}^{0,1}(X_s, \text{End}(\mathcal{E}_s))$.

Proof. Let $\partial/\partial \bar{z}^\alpha$ be a local coordinate tangent vector field on \mathcal{X} of type (0,1). We lift $\partial/\partial \bar{z}^\alpha$ to a $\text{GL}(r, \mathbb{C})$ -invariant (0,1)-tangent vector field $\widetilde{\partial/\partial \bar{z}^\alpha}$ such that $p_*(\widetilde{\partial/\partial \bar{z}^\alpha}) = \partial/\partial \bar{z}^\alpha$. From the structure equation of $\tilde{\theta}$ (cf. e.g. [KN, pp. 77-78]), we have

$$\begin{aligned}
 d\tilde{\theta}(\tilde{v}_i, \widetilde{\partial/\partial \bar{z}^\alpha}) &= -\frac{1}{2}[\tilde{\theta}(\tilde{v}_i), \widetilde{\partial/\partial \bar{z}^\alpha}] + (p^*\Omega)(\tilde{v}_i, \widetilde{\partial/\partial \bar{z}^\alpha}) \\
 &= (p^*\Omega)(\tilde{v}_i, \widetilde{\partial/\partial \bar{z}^\alpha}) \quad (\text{by (2.3.5)}) \\
 (2.3.8) \qquad &= \Omega(p_*\tilde{v}_i, p_*\widetilde{\partial/\partial \bar{z}^\alpha}) \\
 &= \Omega(v_i, \partial/\partial \bar{z}^\alpha) \\
 &= \Omega_{v_i, \bar{\alpha}}.
 \end{aligned}$$

From consideration of type, we have $\tilde{\theta}(\tilde{v}_i) = 0$ and $\tilde{\theta}(\tilde{v}_i(\widetilde{\partial/\partial \bar{z}^\alpha})) = 0$. Together with (2.3.5), we have

$$\begin{aligned}
 d\tilde{\theta}(\tilde{v}_i, \widetilde{\partial/\partial \bar{z}^\alpha}) &= \tilde{v}_i(\tilde{\theta}(\widetilde{\partial/\partial \bar{z}^\alpha})) - \widetilde{\partial/\partial \bar{z}^\alpha}(\tilde{\theta}(\tilde{v}_i)) - \tilde{\theta}([\tilde{v}_i, \widetilde{\partial/\partial \bar{z}^\alpha}]) \\
 (2.3.9) \qquad &= \tilde{\theta}(\widetilde{\partial/\partial \bar{z}^\alpha}(\tilde{v}_i)) \\
 &= \theta(\partial/\partial \bar{z}^\alpha(\hat{v}_i)),
 \end{aligned}$$

where the last equality follows from the descendance of $\tilde{\theta}$, $\widetilde{\partial/\partial \bar{z}^\alpha}$, \tilde{v}_i to θ , $\partial/\partial \bar{z}^\alpha$, \hat{v}_i respectively (as a result of their $\text{GL}(r, \mathbb{C})$ -invariance). We finally have

$$\begin{aligned}
 \theta_s(\hat{A}_{\partial/\partial s^i}) &= \theta_s(\partial/\partial \bar{z}^\alpha(\hat{v}_i)d\bar{z}^\alpha) \quad (\text{by (2.3.6)}) \\
 &= \theta_s(\partial/\partial \bar{z}^\alpha(\hat{v}_i))d\bar{z}^\alpha \\
 &= \Omega_{v_i, \bar{\alpha}} d\bar{z}^\alpha \quad (\text{by (2.3.8) and (2.3.9)}).
 \end{aligned}$$

This finishes the proof of Lemma 2.3.1. □

For $k, l \geq 0$, we also denote the extension of the Hermitian connection ∇ of (\mathcal{E}_s, h_s) induced from the tensor product $\mathcal{A}^{k,l}(X_s, TX_s) = \mathcal{A}^{k,l}(X_s) \otimes_{\mathcal{A}^{0,0}(X_s)} \mathcal{A}^{0,0}(X_s, TX_s)$ by the same symbol $\nabla : \mathcal{A}^{k,l}(X_s, TX_s) \times \mathcal{A}^{0,0}(X_s, \mathcal{E}_s) \rightarrow \mathcal{A}^{k,l}(X_s, \mathcal{E}_s)$. We summarize our discussion in this subsection in the following

Proposition 2.3.2. In terms of the $\bar{\partial}$ -closed representative $\hat{A}_{\partial/\partial s^i}$ of the Kodaira-Spencer class $\hat{\rho}(\partial/\partial s^i)$ in $H^1(X_s, Q_s)$, the curvature formula (1.1.5) of Theorem 1 can be re-written as follows:

$$\begin{aligned}
 &\Theta_{a\bar{b}i\bar{j}}(s) \\
 &= -\int_{X_s} \langle G_{\mathcal{E}_s}(\theta_s(\hat{A}_{\partial/\partial s^i})(t_a) - \nabla_{q_s(\hat{A}_{\partial/\partial s^i})} t_a), \theta_s(\hat{A}_{\partial/\partial s^i})(t_b) - \nabla_{q_s(\hat{A}_{\partial/\partial s^j})} t_b \rangle \frac{\omega_s^n}{n!} \\
 &\quad + \int_{X_s} \langle \Omega_{v_i \bar{v}_j}(t_a, t_b) \rangle \frac{\omega_s^n}{n!} + \int_{X_s} \square(g_{v_i \bar{v}_j}) \langle t_a, t_b \rangle \frac{\omega_s^n}{n!}.
 \end{aligned}$$

Proof. Proposition 2.3.2 follows immediately from (1.1.5), (2.3.7) and Lemma 2.3.1. □

§3. PROOF OF COROLLARY 2

(3.1). Notation as in (1.1) and (1.2). We deduce Corollary 2 from Theorem 1 as follows:

Proof of Corollary 2. Let $(\mathcal{L} \rightarrow \mathcal{X}, h)$ and $(\pi : \mathcal{X} \rightarrow S, \omega_{\mathcal{X}})$ be as in Corollary 2, so that (1.2.1) is satisfied for some $k \in \mathbb{R}$. Denote by $s = (s^i)_{1 \leq i \leq m}$ and $z = (z^\alpha)_{1 \leq \alpha \leq n}$ the local holomorphic coordinates for S and the fibers X_s , respectively. Then (1.2.1) implies that $\omega_{\mathcal{X}}$ is d -closed, and thus Theorem 1 applies. By the construction in (1.1) (or (2.1.1)), the horizontal lifting v_i of a coordinate tangent vector field $\partial/\partial s^i$ in S necessarily satisfies $g_{v_i \bar{\alpha}} = 0$. Together with (1.2.2), we have

$$(3.1.1) \quad \Omega_{v_i \bar{v}_j} = kg_{v_i \bar{v}_j} \quad \text{and} \quad \Omega_{v_i \bar{\alpha}} = kg_{v_i \bar{\alpha}} = 0.$$

Then (1.2.3) follows immediately from (1.1.5) and (3.1.1), and we have finished the proof of Corollary 2. \square

§4. L^2 -METRICS AND FAMILIES OF ABELIAN VARIETIES

(4.1). In this subsection, we make some simple observations on the L^2 -metrics on direct image bundles associated to families of ample line bundles over families of abelian varieties. These will be needed in the proof of Theorem 3 in §5.

Let $\pi : \mathcal{A} \rightarrow S$ be a family of abelian varieties parametrized by a complex manifold S , and let $\mathcal{L} \rightarrow \mathcal{A}$ be a family of ample line bundles as in Theorem 3. Recall that for any smooth family of Hermitian metrics ρ on the family of holomorphic line bundles $\mathcal{L} \rightarrow \mathcal{A}$ and any smooth family of Kähler metrics $\omega_{\mathcal{A}}$ on the family $\pi : \mathcal{A} \rightarrow S$, one has an associated L^2 -metric $H_{\rho, \omega_{\mathcal{A}}}$ on $\pi_* \mathcal{L}$ as defined in (1.1.1). For a holomorphic line bundle F over S and a Hermitian metric h on F , it is easy to see that $(\mathcal{L} \otimes \pi^* F, \rho \otimes \pi^* h)$ forms a family of Hermitian holomorphic line bundles. Thus one also has the associated L^2 -metric $H_{\rho \otimes \pi^* h, \omega_{\mathcal{A}}}$ on $\pi_*(\mathcal{L} \otimes \pi^* F)$.

Proposition 4.1.1. *Let $\omega_{\mathcal{A}}$ and $\omega'_{\mathcal{A}}$ be two smooth families of flat Kähler metrics on the family $\pi : \mathcal{A} \rightarrow S$, and let ρ and ρ' be two smooth families of canonical Hermitian metrics on the family of ample line bundles $\mathcal{L} \rightarrow \mathcal{A}$ (cf. (1.3)). Also let (F, h) be a Hermitian holomorphic line bundle over S . Then the following statements hold:*

- (i) $(\pi_*(\mathcal{L} \otimes \pi^* F), H_{\rho \otimes \pi^* h, \omega_{\mathcal{A}}})$ is projectively flat if and only if $(\pi_* \mathcal{L}, H_{\rho, \omega_{\mathcal{A}}})$ is projectively flat.
- (ii) $(\pi_* \mathcal{L}, H_{\rho, \omega_{\mathcal{A}}})$ is projectively flat if and only if $(\pi_* \mathcal{L}, H_{\rho', \omega_{\mathcal{A}}})$ is projectively flat.
- (iii) $(\pi_* \mathcal{L}, H_{\rho, \omega_{\mathcal{A}}})$ is projectively flat if and only if $(\pi_* \mathcal{L}, H_{\rho, \omega'_{\mathcal{A}}})$ is projectively flat.

Proof. For each $s \in S$, one has the isomorphism

$$H^0(A_s, L_s) \otimes F_s \simeq H^0(A_s, L_s \otimes \pi^* F_s)$$

given by $t \otimes f \mapsto t \otimes \pi^* f$ for $t \in H^0(A_s, L_s)$ and $f \in F_s$. It is easy to verify that under the above correspondence, we have the following isometry of Hermitian vector bundles over S :

$$(4.1.1) \quad (\pi_*(\mathcal{L} \otimes \pi^* F), H_{\rho \otimes \pi^* h, \omega_{\mathcal{A}}}) \simeq (\pi_* \mathcal{L} \otimes F, H_{\rho, \omega_{\mathcal{A}}} \otimes h)$$

(the underlying vector bundle isomorphism is usually known as the projection formula). Denote the curvature tensor of $(\pi_*\mathcal{L}, H_{\rho, \omega_{\mathcal{A}}})$ and $(\pi_*(\mathcal{L} \otimes \pi^*F), H_{\rho \otimes \pi^*h, \omega_{\mathcal{A}}})$ by $\Theta^{H_{\rho, \omega_{\mathcal{A}}}}$ and $\Theta^{H_{\rho \otimes \pi^*h, \omega_{\mathcal{A}}}}$, respectively. Under the identification

$$\text{End}(\pi_*(\mathcal{L} \otimes \pi^*F)) \simeq \text{End}(\pi_*\mathcal{L})$$

induced by the isomorphism $\text{End}(F) \simeq \mathcal{O}_S$, it follows from (4.1.1) that

$$(4.1.2) \quad \Theta^{H_{\rho \otimes \pi^*h, \omega_{\mathcal{A}}}} = \Theta^{H_{\rho, \omega_{\mathcal{A}}}} + \frac{2\pi}{\sqrt{-1}}c_1(F, h) \otimes \text{Id}_{\pi_*\mathcal{L}}.$$

Then Proposition 4.1.1(i) follows from (4.1.2) and the definition in (1.3.1). To prove Proposition 4.1.1(ii), we observe that canonical Hermitian metrics on holomorphic line bundles over abelian varieties are unique up to a multiplicative constant. This implies that $\rho' = e^{\pi^*\lambda} \cdot \rho$ for some smooth function λ on S . Regarding e^λ as a Hermitian metric on the trivial line bundle \mathcal{O}_S , one sees that Proposition 4.1.1(i) (with $(F, h) \simeq (\mathcal{O}_S, e^\lambda)$) readily implies Proposition 4.1.1(ii). To prove Proposition 4.1.1(iii), we observe that the volume forms associated to flat Kähler metrics on an abelian variety are constant multiples of each other. This implies that $H_{\rho, \omega'_{\mathcal{A}}} = e^\mu \cdot H_{\rho, \omega_{\mathcal{A}}}$ for some smooth function μ on S . Then a simple calculation similar to (4.1.2) readily leads to Proposition 4.1.1(iii), and this finishes the proof of Proposition 4.1.1. \square

(4.2). In this subsection, we are going to give a simplification of the formula (1.2.3) in Corollary 2 under the hypothesis of Theorem 3 (cf. Proposition 4.2.4). This does not lead directly to Theorem 3 yet, and the conclusion of Theorem 3 will only be arrived in §5.

Throughout (4.2), we let $\pi : \mathcal{A} \rightarrow S, \mathcal{L} \rightarrow \mathcal{A}, \rho = \{\rho_s\}_{s \in S}$ and $\omega_{\mathcal{A}}$ be as in Theorem 3. Furthermore, we assume that

$$(4.2.1) \quad c_1(\mathcal{L}, \rho) = \frac{1}{2\pi}\omega_{\mathcal{A}} \quad \text{on } \mathcal{A},$$

i.e. (1.2.1) is satisfied with $k = 1$. As in (4.1), we have the associated L^2 -metric $H_{\rho, \omega_{\mathcal{A}}}$ on $\pi_*\mathcal{L}$. For $s \in S$ and a coordinate tangent vector $\partial/\partial s^i \in T_sS$, we let v_i be the horizontal lifting of $\partial/\partial s^i$ with respect to $\omega_{\mathcal{A}}$ as in (1.1), and $A_{\partial/\partial s^i} = A_{i\bar{\beta}}^\alpha \frac{\partial}{\partial z^\alpha} \otimes d\bar{z}^\beta \in A^{0,1}(A_s, TA_s)$ be the associated Kodaira-Spencer representative as in (1.1.3) and (1.1.4). Denote also by g the metric tensor associated to $\omega_{\mathcal{A}}$ as in §1. First we have

Lemma 4.2.1. *For $s \in S$ and $\partial/\partial s^i \in T_sS$,*

(i) *$A_{\partial/\partial s^i}$ is invariant under translations of A_s . In particular, $A_{\partial/\partial s^i}$ is parallel with respect to ω_s .*

(ii) *We have $A_{i\bar{\beta}}^\alpha g_{\alpha\bar{\gamma}} = A_{i\bar{\gamma}}^\alpha g_{\alpha\bar{\beta}}$ for $1 \leq \beta, \gamma \leq n$.*

Proof. Being a flat metric, each ω_s is Kähler-Einstein. Then by [Sch, Proposition 1.1], $A_{\partial/\partial s^i}$ is a harmonic tensor (with respect to ω_s). Then it follows from standard facts on compact complex tori that $A_{\partial/\partial s^i}$ is necessarily invariant under translations of A_s (cf. e.g. [GH, p. 302]), which, in turn, implies readily that $A_{\partial/\partial s^i}$ is parallel with respect to ω_s . This proves Lemma 4.2.1(i). Lemma 4.2.1(ii) follows from the Kähler-Einstein condition of ω_s , and it can also be found in [Sch, Proposition 1.1]. \square

For $s \in S$, $\partial/\partial s^i \in T_s(A_s)$ and $t \in H^0(A_s, L_s)$, we let $\nabla_{A_{\partial/\partial s^i}} t \in A^{0,1}(A_s, L_s)$ be as interpreted in Proposition 2.3.2 (cf. also 2.3.6). In terms of local holomorphic coordinates $(z^\alpha)_{1 \leq \alpha \leq n}$ on A_s , we have $\nabla_{A_{\partial/\partial s^i}} t = A_{i\bar{\beta}}^\alpha t_\alpha d\bar{z}^\beta$ (cf. (1.1.4)).

Lemma 4.2.2. *On A_s , we have $\square(\nabla_{A_{\partial/\partial s^i}} t) = 2\nabla_{A_{\partial/\partial s^i}} t$, i.e.,*

$$\square(A_{i\bar{\beta}}^\alpha t_\alpha d\bar{z}^\beta) = 2A_{i\bar{\beta}}^\alpha t_\alpha d\bar{z}^\beta.$$

Proof. By the construction of v_i , we have $g_{v_i\bar{\alpha}} (= g(v_i, \partial/\partial z^\alpha)) = 0$. Together with (4.2.1) and the identity (2.2.7) in the proof of Theorem 1, it follows that $A_{i\bar{\beta}}^\alpha t_\alpha d\bar{z}^\beta$ is $\bar{\partial}$ -exact, and thus $\bar{\partial}(A_{i\bar{\beta}}^\alpha t_\alpha d\bar{z}^\beta) = 0$. Then

$$\begin{aligned} & \square(A_{i\bar{\beta}}^\alpha t_\alpha d\bar{z}^\beta) \\ &= \bar{\partial}\bar{\partial}^*(A_{i\bar{\beta}}^\alpha t_\alpha d\bar{z}^\beta) \\ &= \bar{\partial}(-g^{\bar{\beta}\gamma}\nabla_\gamma(A_{i\bar{\beta}}^\alpha t_\alpha)) \\ &= \bar{\partial}(-g^{\bar{\beta}\gamma}A_{i\bar{\beta}}^\alpha t_{;\alpha\gamma}) \quad (\text{since } A_{i\bar{\beta};\gamma}^\alpha = 0 \text{ by Lemma 4.2.1(i)}) \\ &= -g^{\bar{\beta}\gamma}A_{i\bar{\beta}}^\alpha t_{;\alpha\gamma\bar{\delta}} d\bar{z}^\delta \quad (\text{since } g^{\bar{\beta}\gamma}_{;\bar{\delta}} = 0, A_{i\bar{\beta};\bar{\delta}}^\alpha = 0 \text{ by Lemma 4.2.1(i)}) \\ &= -g^{\bar{\beta}\gamma}A_{i\bar{\beta}}^\alpha (t_{;\alpha\bar{\delta}\gamma} - g_{\gamma\bar{\delta}} t_{;\alpha}) d\bar{z}^\delta \\ & \quad (\text{by Ricci identity, (4.2.1) and flatness of } \omega_s) \\ &= -g^{\bar{\beta}\gamma}A_{i\bar{\beta}}^\alpha (t_{;\bar{\delta}\alpha\gamma} - (g_{\alpha\bar{\delta}} t)_{;\gamma} - g_{\gamma\bar{\delta}} t_{;\alpha}) d\bar{z}^\delta \\ & \quad (\text{by Ricci identity and (4.2.1)}) \\ &= -g^{\bar{\beta}\gamma}A_{i\bar{\beta}}^\alpha (-g_{\alpha\bar{\delta}} t_{;\gamma} - g_{\gamma\bar{\delta}} t_{;\alpha}) d\bar{z}^\delta \\ & \quad (\text{since } t \text{ is holomorphic and } g_{\alpha\bar{\delta};\gamma} = 0) \\ &= g^{\bar{\beta}\gamma}A_{i\bar{\beta}}^\alpha g_{\alpha\bar{\delta}} t_{;\gamma} d\bar{z}^\delta + A_{i\bar{\delta}}^\alpha t_{;\alpha} d\bar{z}^\delta \\ &= g^{\bar{\beta}\gamma}A_{i\bar{\delta}}^\alpha g_{\alpha\bar{\beta}} t_{;\gamma} d\bar{z}^\delta + A_{i\bar{\delta}}^\alpha t_{;\alpha} d\bar{z}^\delta \quad (\text{by Lemma 4.2.1(ii)}) \\ &= 2A_{i\bar{\delta}}^\alpha t_{;\alpha} d\bar{z}^\delta. \end{aligned}$$

Thus we have finished the proof of Lemma 4.2.2. □

Next we fix a Euclidean coordinate system $(z^\alpha)_{1 \leq \alpha \leq n}$ on A_s so that each $\partial/\partial z^\alpha$ can be considered as a global translation-invariant vector field on A_s .

Lemma 4.2.3. *For $s \in S$, $t_a, t_b \in H^0(A_s, L_s)$ and $\partial/\partial z^\alpha, \partial/\partial z^\beta$ as above, there exists a constant $\Xi_{\alpha\bar{\beta}}$ depending only on $\partial/\partial z^\alpha$ and $\partial/\partial z^\beta$ such that*

$$\int_{A_s} \langle t_{a;\alpha}, t_{b;\beta} \rangle \frac{\omega_s^n}{n!} = \Xi_{\alpha\bar{\beta}} \cdot H_{\rho, \omega_A}(t_a, t_b).$$

Proof. Fix a Euclidean coordinate system $(z^\alpha)_{1 \leq \alpha \leq n}$ on A_s as above. Since ω_s is flat, we have

$$\omega_s^n = C \cdot \left(\frac{\sqrt{-1}}{2}\right)^n \cdot dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \quad \text{on } A_s,$$

where C is some positive constant. Observe that for $1 \leq \alpha \leq n$, $dz^\alpha, d\bar{z}^\alpha$ are global 1-forms on A_s . Then for $t_a, t_b \in H^0(A_s, L_s)$,

$$\Phi := \langle t_{a;\alpha}, t_b \rangle \cdot C \cdot \left(\frac{\sqrt{-1}}{2}\right)^n \cdot dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge \widehat{dz^\alpha} \wedge d\bar{z}^\beta \cdots \wedge dz^n \wedge d\bar{z}^n$$

is a global $(2n - 1)$ -form on A_s . Here $\widehat{\wedge dz^\beta}$ means that the factor $\wedge dz^\beta$ is omitted. It is easy to see that

$$\begin{aligned} d\Phi &= \frac{\partial}{\partial \bar{z}^\beta} (\langle t_{a;\alpha}, t_b \rangle) \cdot \omega_s^n \\ &= (\langle t_{a;\alpha\bar{\beta}}, t_b \rangle + \langle t_{a;\alpha}, t_{b;\beta} \rangle) \omega_s^n \\ &= (\langle t_{a;\bar{\beta}\alpha} - g_{\alpha\bar{\beta}} t_a, t_b \rangle + \langle t_{a;\alpha}, t_{b;\beta} \rangle) \omega_s^n \\ &\quad \text{(by Ricci identity and (4.2.1))} \\ &= (-g_{\alpha\bar{\beta}} \langle t_a, t_b \rangle + \langle t_{a;\alpha}, t_{b;\beta} \rangle) \omega_s^n \\ &\quad \text{(by holomorphicity of } t_a \text{).} \end{aligned}$$

Since ω_s is flat, $g_{\alpha\bar{\beta}}$ is a (globally) constant function on A_s . Then by Stokes' theorem, we have $\int_{A_s} d\Phi = 0$, and thus

$$\int_{A_s} \langle t_{a;\alpha}, t_{b;\beta} \rangle \frac{\omega_s^n}{n!} = g_{\alpha\bar{\beta}} \int_{A_s} \langle t_a, t_b \rangle \frac{\omega_s^n}{n!},$$

which leads to Lemma 4.2.3 with the constant $\Xi_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}}$. □

Now we summarize our discussion in (4.2) in the following

Proposition 4.2.4. *Let $\pi : \mathcal{A} \rightarrow S, \mathcal{L}, \rho, \omega_{\mathcal{A}}$ be as in Theorem 3. Suppose furthermore that (4.2.1) is satisfied. Then the curvature Θ of the L^2 -metric $H_{\rho, \omega_{\mathcal{A}}}$ on $\pi_* \mathcal{L}$ is given by*

$$\Theta_{ab\bar{i}\bar{j}}(s) = C_{i\bar{j}} \cdot H_{\rho, \omega_{\mathcal{A}}}(t_a, t_b) + \int_{A_s} (g_{v_i \bar{v}_j} + \square(g_{v_i \bar{v}_j})) \langle t_a, t_b \rangle \frac{\omega_s^n}{n!}$$

for $s \in S$ and $t_a, t_b \in H^0(A_s, L_s)$, where $C_{i\bar{j}}$ is some constant depending on s , $\partial/\partial s^i$ and $\partial/\partial s^j$.

Proof. Let $s \in S, \partial/\partial s^i, \partial/\partial s^j \in T_s S$ and $t_a, t_b \in H^0(A_s, L_s)$ be as above. As in Lemma 4.2.3, we fix a Euclidean coordinate system $(z^\alpha)_{1 \leq \alpha \leq n}$ on A_s . By Lemma 4.2.2, $\nabla_{A_{\partial/\partial s^i}} t_a = A_{i\bar{\alpha}}^\gamma t_{a;\gamma} d\bar{z}^\alpha$ is an eigenvector of the ∂ -Laplacian \square on A_s corresponding to the eigenvalue 2. Thus we have

$$(4.2.2) \quad G_{L_s}(A_{i\bar{\alpha}}^\gamma t_{a;\gamma} d\bar{z}^\alpha) = \frac{1}{2} A_{i\bar{\alpha}}^\gamma t_{a;\gamma} d\bar{z}^\alpha,$$

where G_{L_s} is the Green's operator on L_s . By Lemma 4.2.2(i) and as in Lemma 4.2.3, the functions $A_{i\bar{\beta}}^\alpha, g_{\alpha\bar{\beta}}$ and $g^{\bar{\beta}\alpha}, 1 \leq \alpha, \beta \leq n$, are all constant on the entirety of A_s . Then by (4.2.2), we have

$$\begin{aligned} &\int_{A_s} \langle G_{L_s}(A_{i\bar{\alpha}}^\gamma t_{a;\gamma} d\bar{z}^\alpha), A_{j\bar{\beta}}^\delta t_{b;\delta} d\bar{z}^\beta \rangle \frac{\omega_s^n}{n!} \\ &= \int_{A_s} \langle \frac{1}{2} \cdot A_{i\bar{\alpha}}^\gamma t_{a;\gamma} d\bar{z}^\alpha, A_{j\bar{\beta}}^\delta t_{b;\delta} d\bar{z}^\beta \rangle \frac{\omega_s^n}{n!} \\ (4.2.3) \quad &= \frac{1}{2} \cdot A_{i\bar{\alpha}}^\gamma A_{j\bar{\beta}}^\delta g^{\bar{\alpha}\beta} \int_{A_s} \langle t_{a;\gamma}, t_{b;\delta} \rangle \frac{\omega_s^n}{n!} \\ &= \frac{1}{2} \cdot A_{i\bar{\alpha}}^\gamma A_{j\bar{\beta}}^\delta g^{\bar{\alpha}\beta} \Xi_{\gamma\bar{\delta}} H_{\rho, \omega_{\mathcal{A}}}(t_a, t_b) \quad \text{(by Lemma 4.2.3)} \\ &= -C_{i\bar{j}} \cdot H_{\rho, \omega_{\mathcal{A}}}(t_a, t_b), \end{aligned}$$

where $C_{i\bar{j}} := -\frac{1}{2} \cdot A_{i\bar{\alpha}}^\gamma A_{j\bar{\beta}}^\delta g^{\bar{\alpha}\beta} \Xi_{\gamma\bar{\delta}}$ is easily seen to be a constant on the entirety of A_s and depending only on $\partial/\partial s^i$ and $\partial/\partial s^j$. Here $\Xi_{\gamma\bar{\delta}}$ is as in Lemma 4.2.3. Then Proposition 4.2.4 follows immediately by combining Corollary 2 (with $k = 1$) and (4.2.3). \square

§5. PROOF OF THEOREM 3

(5.1). In this subsection, we are going to deduce Theorem 3 from Corollary 2 under the additional assumption (4.2.1) as embarked in §4. Theorem 3 in full generality will be proved in (5.2). As remarked in §0, the approach taken in this section is largely suggested by the referee.

First we describe a natural smooth foliation on any holomorphic family of n -dimensional abelian varieties $\pi : \mathcal{A} \rightarrow S$ as follows: Recall that each fiber $A_s = \pi^{-1}(s)$ is of the form $\mathbb{C}^n/\Lambda(s)$ for some lattice $\Lambda(s)$ of \mathbb{C}^n . Moreover, over a small open subset U of S , we may fix a choice of $\Lambda(s)$ and a choice of generators $e_1(s), \dots, e_{2n}(s)$ of $\Lambda(s)$, $s \in U$, such that each $e_i(s)$ varies holomorphically on U , shrinking U if necessary. For simplicity, we simply denote the corresponding holomorphic family of generators over U by e_1, \dots, e_{2n} . It is easy to see that one has the following holomorphic isomorphism (as families over U) given by

$$(5.1.1) \quad \mathcal{A}|_U \cong (U \times \mathbb{C}^n)/\mathbb{Z}^{2n},$$

where \mathbb{Z}^{2n} is identified with the family of lattices generated by e_1, \dots, e_{2n} over U . Next we define a foliation $\mathcal{F}(U \times \mathbb{C}^{2n})$ on $U \times \mathbb{C}^{2n}$, whose leaves are of the form

$$(5.1.2) \quad \mathcal{F}_a = \{(s, a_1 e_1(s) + \dots + a_{2n} e_{2n}(s)) \in U \times \mathbb{C}^{2n} \mid s \in U\},$$

where $a = (a_1, \dots, a_{2n}) \in \mathbb{R}^{2n}$. In other words, a leaf of the foliation is given by a fixed real linear combination of e_1, \dots, e_{2n} over U . It is easy to see that the foliation $\mathcal{F}(U \times \mathbb{C}^{2n})$ is smooth and invariant under the action of \mathbb{Z}^{2n} in (5.1.1). Moreover it does not depend on the choice of the family of generators e_1, \dots, e_{2n} . Thus it descends to a smooth foliation on $\mathcal{A}|_U$, which we denote by $\mathcal{F}(\mathcal{A}|_U)$. Moreover, one can easily check that $\mathcal{F}(\mathcal{A}|_U)$ agrees with $\mathcal{F}(\mathcal{A}|_V)$ on $\mathcal{A}|_{U \cap V}$ for any open subsets $U, V \subset S$. It follows that one obtains a smooth foliation $\mathcal{F}(\mathcal{A})$ on \mathcal{A} such that $\mathcal{F}(\mathcal{A})|_{\mathcal{A}|_U} = \mathcal{F}(\mathcal{A}|_U)$ for any open subset $U \subset S$. Moreover, one easily sees that the leaves of $\mathcal{F}(\mathcal{A})$ are holomorphic, and $\mathcal{F}(\mathcal{A})$ is transversal and complementary to the holomorphic distribution $\text{Ker}(d\pi)$ of vertical tangent vectors of the family $\pi : \mathcal{A} \rightarrow S$. In particular, the restriction $d\pi|_{T_p(\mathcal{F}(\mathcal{A}))}$ gives rise to an isomorphism $T_p(\mathcal{F}(\mathcal{A})) \cong T_{\pi(p)}(S)$ at each $p \in \mathcal{A}$. Here $T_p(\mathcal{F}(\mathcal{A}))$ denotes the tangent space of the leaf of $\mathcal{F}(\mathcal{A})$ at p .

Lemma 5.1.1. *Let $\pi : \mathcal{A} \rightarrow S$ and $\mathcal{F}(\mathcal{A})$ be as above, and let $\omega_{\mathcal{A}}$ be a d -closed $(1, 1)$ -form on \mathcal{A} such that the restriction $\omega_s := \omega_{\mathcal{A}}|_{A_s}$ is a flat Kähler form on $A_s = \pi^{-1}(s)$ for each $s \in S$. Then there exists a unique smooth non-negative d -closed $(1, 1)$ -form ν on \mathcal{A} such that*

$$(5.1.3) \quad \nu|_{A_s} = \omega_s \quad \text{and} \quad \ker(\nu_p) = T_p(\mathcal{F}(\mathcal{A}))$$

for each $s \in S$ and $p \in \mathcal{A}$.

Proof. It is easy to see that the conditions in (5.1.3) define uniquely a non-negative $(1, 1)$ -form ν on \mathcal{A} . To see that ν is smooth and d -closed, we first fix a small connected open subset U of S and a reference point $o \in S$. Shrinking U if necessary, we

have an isomorphism (as families) of $\mathcal{A}|_U$ with a holomorphic family of generators e_1, \dots, e_{2n} of lattices in \mathbb{C}^n as given in (5.1.1). Then for each $s \in U$, we consider the *real* linear vector space isomorphism on \mathbb{C}^n sending $e_i(o)$ to $e_i(s)$, $i = 1, \dots, 2n$, which is easily seen to descend to a diffeomorphism $\kappa_s : A_o \rightarrow A_s$ which is an isomorphism with respect to the underlying *real* abelian Lie group structures on A_o and A_s . In particular, $\kappa_s^* \omega_s$ is a translation-invariant 2-form on A_o . Moreover, one easily checks that κ_s varies smoothly with s , and thus one obtains a diffeomorphic trivialization of the family $\pi|_U : \mathcal{A}|_U \rightarrow U$ given by $\kappa : A_o \times U \rightarrow \mathcal{A}|_U$ satisfying $\kappa|_{A_o \times \{s\}} = \kappa_s$ for all $s \in U$. Since $\omega_{\mathcal{A}}$ is d -closed, it follows from Stokes' theorem that $\kappa_s^* \omega_s$, $s \in U$, are all cohomologous to $\kappa_o^* \omega_o = \omega_o$ on A_o (noting that $\kappa_o = \text{Id}$ on A_o). Together with the translation-invariance of $\kappa_s^* \omega_s$, it follows that for all $s \in U$,

$$(5.1.4) \quad \kappa_s^* \omega_s = \omega_o \quad \text{on } A_o$$

(cf. Remark 1.3.2). Next one easily sees from the definition of κ and $\mathcal{F}(\mathcal{A})$ that $\kappa^* \mathcal{F}(\mathcal{A})$ is a foliation on $A_o \times U$ whose leaves are of the form $\{z\} \times U$ for some fixed $z \in A_o$. Together with (5.1.4) and the definition of ν , it follows readily that

$$(5.1.5) \quad \kappa^* \nu = p_1^* \omega_o \quad \text{on } A_o \times U,$$

where p_1 denotes the projection of $A_o \times U$ onto the first factor A_o . Since ω_o is both smooth and d -closed, it follows that $\kappa^* \nu$ is both smooth and d -closed, and thus ν is also smooth and d -closed on $\mathcal{A}|_U$. Upon varying U , one sees that ν is smooth and d -closed on \mathcal{A} . □

Proposition 5.1.2. *Let $\pi : \mathcal{A} \rightarrow S$ and $\omega_{\mathcal{A}}$ be as in Lemma 5.1.1. Then for $s \in S$ and $\partial/\partial s_i, \bar{\partial}/\partial s_j \in T_s S$,*

- (i) $[v_i, \bar{v}_j]$ is a translation-invariant vector field on A_s , where v_i, v_j denote the horizontal lifting of $\partial/\partial s_i, \bar{\partial}/\partial s_j$ with respect to $\omega_{\mathcal{A}}$ respectively; and
- (ii) $g_{v_i \bar{v}_j} := \omega_{\mathcal{A}}(v_i, \bar{v}_j)$ is a constant function on A_s .

Proof. As in (4.2), we will fix a Euclidean coordinate system $(z^\alpha)_{1 \leq \alpha \leq n}$ on A_s , so that each $\partial/\partial z^\alpha$ can be regarded as a global translation-invariant vector field on A_s . Let ν be the smooth d -closed (1,1)-form in Lemma 5.1.1, and let v'_i, v'_j be the horizontal lifting of $\partial/\partial s_i, \bar{\partial}/\partial s_j$ with respect to ν , respectively. Let

$$(5.1.6) \quad w_i = v_i - v'_i \quad \text{and} \quad w_j = v_j - v'_j.$$

It is easy to see that $w_i, w_j \in \ker(d\pi)$, and thus w_i and w_j restrict to smooth vector fields on each fiber A_s . Since both $\omega_{\mathcal{A}}|_{A_s}$ and $\nu|_{A_s}$ are the same flat Kähler metric ω_s on A_s , it follows that $A_{\partial/\partial s_i} := \bar{\partial} v_i$ and $A'_{\partial/\partial s_i} := \bar{\partial} v'_i$ are harmonic tensors with respect to ω_s (cf. Lemma 4.2.1). On the other hand, one recalls from (2.3) that both $A_{\partial/\partial s_i}$ and $A'_{\partial/\partial s_i}$ are representatives of the same Kodaira-Spencer class of $\rho_s(\partial/\partial s_i)$ via the Dolbeault isomorphism, where $\rho_s : T_s S \rightarrow H^1(A_s, T A_s)$ is the Kodaira-Spencer map associated to the family $\pi : \mathcal{A} \rightarrow S$. Therefore, one has $A_{\partial/\partial s_i} = A'_{\partial/\partial s_i}$, and thus

$$(5.1.7) \quad \bar{\partial} w_i = \bar{\partial} v_i - \bar{\partial} v'_i = A_{\partial/\partial s_i} - A'_{\partial/\partial s_i} = 0 \quad \text{on } A_s.$$

Hence w_i is a global holomorphic (and thus translation-invariant) vector field on A_s , and so is w_j . It follows that we may write

$$(5.1.8) \quad w_i = w_i^\alpha \frac{\partial}{\partial z^\alpha} \quad \text{and} \quad w_j = w_j^\alpha \frac{\partial}{\partial z^\alpha},$$

where each w_i^α or w_j^α is a constant function on A_s . Since v'_i is the horizontal lifting of $\partial/\partial s_i$ with respect to ν , it follows that $g'_{v'_i \bar{w}} = 0$ for all $p \in A_s$ and $w \in T_p A_s$. Here and henceforth, we denote $g'_{v'_i \bar{w}} = \nu(v'_i, \bar{w})$, etc. Together with the fact that ν is a semi-positive (1,1)-form on \mathcal{A} whose positive eigenspace at each $p \in A_s$ is given by $T_p A_s$, it follows readily that $v'_i(p) \in \ker(\nu_p)$ for each $p \in A_s$. Thus one has $g'_{v'_i \bar{v}'_j} \equiv 0$ on A_s . Together with the commutation relation in Lemma 2.1.1(iii), it follows that

$$(5.1.9) \quad [v'_i, \bar{v}'_j] \equiv 0 \quad \text{on } A_s.$$

(We remark that (5.1.9) can also be obtained by using the diffeomorphic trivialization of the family $\pi : \mathcal{A}|_U \rightarrow U$ over an open neighborhood U of s in S as given in the proof of Lemma 5.1.1.) From (5.1.8), one also easily checks that

$$(5.1.10) \quad [w_i, \bar{w}_j] \equiv 0 \quad \text{on } A_s.$$

Write $A'_{\partial/\partial s_i} = A'_{i\beta} \frac{\partial}{\partial z^\alpha} \otimes d\bar{z}^\beta$. Then by Lemma 4.2.1(i) (and as remarked above), each $A'_{i\beta}$ is a constant function on A_s . Thus, by (5.1.6), we have, on A_s ,

$$(5.1.11) \quad \begin{aligned} [v_i, \bar{v}_j] &= [v'_i + w_i, \bar{v}'_j + \bar{w}_j] \\ &= 0 + [v'_i, \bar{w}_j] + [w_i, \bar{v}'_j] + 0 \quad (\text{by (5.1.9) and (5.1.10)}) \\ &= -A'_{i\beta} \bar{w}_j^\beta \frac{\partial}{\partial z^\alpha} + \bar{A}'_{j\beta} w_i^\beta \frac{\partial}{\partial \bar{z}^\alpha} \quad (\text{by Lemma 2.1.1(i)}), \end{aligned}$$

which readily implies Proposition 5.1.2(i), since $A'_{i\beta}$, $A'_{j\beta}$, w_i^β and w_j^β are all constant functions on A_s . Next we proceed to prove Proposition 5.1.2(ii). Since $\omega_{\mathcal{A}}$ is d -closed, it follows from Lemma 2.1.1(iii) (and upon taking inner product with $\partial/\partial z^\alpha$) that for $1 \leq \alpha \leq n$,

$$(5.1.12) \quad g_{[v_i, \bar{v}_j] \bar{\alpha}} = \frac{\partial}{\partial \bar{z}^\alpha} (g_{v_i \bar{v}_j}) \quad \text{on } A_s.$$

From Proposition 5.1.2(i), one easily sees that $g_{[v_i, \bar{v}_j] \bar{\alpha}}$ (and thus also $\frac{\partial}{\partial \bar{z}^\alpha} (g_{v_i \bar{v}_j})$) is a constant function on A_s . Hence,

$$(5.1.13) \quad \square(g_{v_i \bar{v}_j}) = -g^{\bar{\alpha}\beta} \frac{\partial}{\partial z^\beta} \left(\frac{\partial}{\partial \bar{z}^\alpha} (g_{v_i \bar{v}_j}) \right) = 0 \quad \text{on } A_s.$$

Thus $g_{v_i \bar{v}_j}$ is a constant function on A_s , and we have finished the proof of Proposition 5.1.2(ii). \square

Now we summarize our discussion in (4.2) and (5.1) in the following

Proposition 5.1.3. *Let $\pi : \mathcal{A} \rightarrow S$, $\mathcal{L} \rightarrow \mathcal{A}$, $\rho, \omega_{\mathcal{A}}$ be as in Theorem 3. Suppose furthermore that (4.2.1) is satisfied. Then the L^2 -metric $H_{\rho, \omega_{\mathcal{A}}}$ on $\pi_* \mathcal{L}$ is projectively flat.*

Proof. For $s \in S$, $\partial/\partial s^i$ and $\partial/\partial s^j \in T_s S$, it follows from Proposition 5.1.2 that on A_s , $g_{v_i \bar{v}_j} = \kappa_{i\bar{j}}$ for some constant $\kappa_{i\bar{j}}$. In particular, $\square(g_{v_i \bar{v}_j}) = 0$ on A_s . Together with Proposition 4.2.4, we have, for $t_a, t_b \in H^0(A_s, L_s)$, the curvature tensor Θ of $H_{\rho, \omega_{\mathcal{A}}}$ on $\pi_* \mathcal{L}$ is given by

$$\begin{aligned} \Theta_{a\bar{i}\bar{j}}(s) &= C_{i\bar{j}} \cdot H_{\rho, \omega_{\mathcal{A}}}(t_a, t_b) + \int_{A_s} (\kappa_{i\bar{j}} + 0) \langle t_a, t_b \rangle \frac{\omega_s^n}{n!} \\ &= \mu_{i\bar{j}} \cdot H_{\rho, \omega_{\mathcal{A}}}(t_a, t_b), \end{aligned}$$

where $C_{i\bar{j}}$ is as in Proposition 4.2.4, and the constant $\mu_{i\bar{j}} := C_{i\bar{j}} + \kappa_{i\bar{j}}$ depends only on $\partial/\partial s^i$ and $\partial/\partial s^j$. Thus we have finished the proof of Proposition 5.1.3. \square

(5.2). Finally we are ready to complete the proof of Theorem 3.

Proof of Theorem 3. First we are going to prove Theorem 3(i). Let $\pi : \mathcal{A} \rightarrow S$, $\mathcal{L} \rightarrow \mathcal{A}$, ρ and $\omega_{\mathcal{A}}$ be as in Theorem 3(i). As indicated in (1.3), one can always construct a smooth family of canonical Hermitian metrics ρ' on the family $\mathcal{L} \rightarrow \mathcal{A}$. Then $\omega'_{\mathcal{A}} := c_1(\mathcal{L}, \rho')$ is a family of flat Kähler metrics on the family $\mathcal{A} \rightarrow S$ such that ρ' and $\omega'_{\mathcal{A}}$ satisfy (4.2.1). Thus by Proposition 5.1.3, the L^2 -metric $H_{\rho', \omega'_{\mathcal{A}}}$ on $\pi_*\mathcal{L}$ is projectively flat. Together with Proposition 4.1.1(ii) and (iii), it follows that $H_{\rho, \omega_{\mathcal{A}}}$ on $\pi_*\mathcal{L}$ is also projectively flat over S , and this finishes the proof of Theorem 3(i).

Finally, we deduce Theorem 3(ii) as follows. As mentioned above, one can always construct a smooth family of canonical Hermitian metrics ρ on the family $\mathcal{L} \rightarrow \mathcal{A}$ and a smooth family of flat Kähler metrics $\omega_{\mathcal{A}}$ on the family $\pi : \mathcal{A} \rightarrow S$. By Theorem 3(i), the Hermitian connection of the L^2 -metric $H_{\rho, \omega_{\mathcal{A}}}$ on $\pi_*\mathcal{L}$ is projectively flat, i.e., the curvature tensor Θ of $H_{\rho, \omega_{\mathcal{A}}}$ satisfies $\Theta = \alpha \cdot \text{Id}_{\pi_*\mathcal{L}}$ for some smooth 2-form α on S . Then for any Kähler form μ on S , we have $\Lambda\Theta = \lambda(s) \cdot \text{Id}_{\pi_*\mathcal{L}}$, where Λ denotes the contraction with respect to μ and $\lambda(s)$ is the smooth function given by $\lambda = \Lambda\alpha$. Since S is compact, it is well known that the above identity implies that $H_{\rho, \omega_{\mathcal{A}}}$ is conformally equivalent to a Hermitian-Einstein metric (with respect to μ) on $\pi_*\mathcal{L}$ (see e.g. [Siu2, p. 16]). Then by a result of Kobayashi [Ko1] and Lübke [Lü], this implies that $\pi_*\mathcal{L}$ is poly-stable with respect to μ . This finishes the proof of Theorem 3(ii), and thus we have completed the proof of Theorem 3. \square

§6. APPENDIX: SIEGEL MODULAR VARIETIES AND THEIR POINCARÉ LINE BUNDLES

(6.1). In this appendix, we are going to briefly discuss the Poincaré line bundles over families of abelian varieties parametrized by Siegel modular varieties, which may be regarded as the universal objects associated to the families of line bundles studied in Theorem 3. We will also explicitly describe their canonical metrics, and indicate briefly how canonical Hermitian metrics on families of ample line bundles over abelian varieties can be obtained via their classifying maps.

Denote the Siegel upper half plane by $\mathcal{H}_n := \{\tau \in \mathcal{M}_n(\mathbb{C}) : \tau = \tau^t, \text{Im}(\tau) > 0\}$, where $\mathcal{M}_n(\mathbb{C})$ is the set of $n \times n$ matrices with entries in \mathbb{C} . Fix a type of polarization $\delta := \text{diag}(d_1, d_2, \dots, d_n) \in \mathcal{M}_n(\mathbb{Z})$, where $d_1, d_2, \dots, d_n \in \mathbb{Z}^+$ (with $d_i | d_{i+1}$) are the elementary divisors. The symplectic group $\text{Sp}(2n, \mathbb{R})$ acts as a group of biholomorphisms on \mathcal{H}_n via the following action:

$$(6.1.1) \quad \tau \mapsto (A\tau + B)(C\tau + D)^{-1} =: \tau' \quad \text{for} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2n, \mathbb{R}).$$

Moreover, this action extends to an action of the semi-direct product $\text{Sp}(2n, \mathbb{R}) \triangleright \mathbb{R}^{2n}$ on $\mathcal{H}_n \times \mathbb{C}^n$ such that the action of $(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, [\alpha]) \in \text{Sp}(2n, \mathbb{R}) \triangleright \mathbb{R}^{2n}$ is given by

$$(6.1.2) \quad (\tau, z) \mapsto (\tau', [(C\tau + D)^t]^{-1}z + \tau'\alpha + \beta),$$

where τ' is as in (6.1.1). Let Γ be a discrete torsion-free subgroup of $\mathrm{Sp}(2n, \mathbb{R})$. Then $X_\Gamma := \mathcal{H}_n/\Gamma$ is smooth, and one gets an associated analytic family of δ -polarized abelian varieties given by

$$(6.1.3) \quad \pi : A_\Gamma := (\mathcal{H}_n \times \mathbb{C}^n)/\Gamma \triangleright \mathbb{Z}^{2n} \rightarrow X_\Gamma.$$

In general, it is well known that the set of δ -polarized abelian varieties is in one-to-one correspondence with $\mathcal{H}_n/\mathrm{Sp}(2n, \mathbb{Z}, \delta)$ for some arithmetic subgroup $\mathrm{Sp}(2n, \mathbb{Z}, \delta) \subset \mathrm{Sp}(2n, \mathbb{R})$, but $\mathrm{Sp}(2n, \mathbb{Z}, \delta)$ may not be torsion-free (cf. e.g. [MT, §1], [MFK], [SD, pp. 69-73] for the above background materials). By [Ku, p. 82], there exists the following Kähler form given by

$$(6.1.4) \quad \omega := \sqrt{-1} \partial \bar{\partial} [(\mathrm{Im} Z)^t (\mathrm{Im} \tau)^{-1} (\mathrm{Im} Z) + \log \det (\mathrm{Im} \tau)^{-1}] \quad \text{on } \mathcal{H}_n \times \mathbb{C}^n,$$

which can easily be seen to be invariant under the $\mathrm{Sp}(2n, \mathbb{R}) \triangleright \mathbb{R}^{2n}$ action in (6.1.2) (cf. e.g. [MT, §1]). Thus ω descends to a Kähler form on X_Γ , which we denote by ω_Γ . Write $A_s := \pi^{-1}(s)$ and $\omega_s := \omega_\Gamma|_{A_s}$ for $s \in X_\Gamma$ as in (1.3). Then it is easy to see that each ω_s is a flat Kähler metric on A_s .

Now suppose that \mathcal{L}_Γ is a holomorphic line bundle over A_Γ and ρ_Γ is a Hermitian metric on \mathcal{L}_Γ such that

$$(6.1.5) \quad c_1(\mathcal{L}_\Gamma, \rho_\Gamma) = \frac{\omega_\Gamma}{2\pi}.$$

Write $L_s := \mathcal{L}_\Gamma|_{A_s}$ and $\rho_s := \rho_\Gamma|_{L_s}$ for $s \in X_\Gamma$. Then it follows easily from (6.1.5) that each ρ_s is a canonical Hermitian metric on L_s .

Remark 6.1.1. It is proved in [Ku] (see also [Sa, pp. 203-208]) that when $\Gamma \subset \mathrm{Sp}(2n, \mathbb{R})$ is discrete and torsion-free, such an $(\mathcal{L}_\Gamma, \rho_\Gamma)$ over X_Γ and satisfying (6.1.5) always exists. Here X_Γ need not be compact. If X_Γ is indeed compact, then as concluded there, it follows that ω_Γ is a Hodge metric.

(6.2). Let $\pi : \mathcal{A} \rightarrow S$ be a family of abelian varieties parametrized by a complex manifold S . For each $s \in S$ and $A_s = \pi^{-1}(s)$, the set of isomorphism classes of holomorphic line bundles with zero first Chern class is isomorphic to the dual abelian variety \hat{A}_s of A_s . It is well known that $\{\hat{A}_s\}_{s \in S}$ also form an analytic family of abelian varieties parametrized by S , which we denote by $\hat{\mathcal{A}}$. Fix a type of polarization δ of \mathcal{A} , and let $\hat{A}_s(\delta)$ denote the set of isomorphism classes of holomorphic line bundles over A_s with first Chern class determined by δ . Then it is also well known that $\hat{A}_s(\delta)$ is isomorphic to \hat{A}_s (as an algebraic variety), and thus one also has a family $\hat{\pi} : \hat{\mathcal{A}}(\delta) \rightarrow S$ (with $\hat{\mathcal{A}}(\delta)|_{\hat{\pi}^{-1}(s)} = \hat{A}_s(\delta)$ for each $s \in S$), which is analytically isomorphic to $\hat{\mathcal{A}}$. Consider the fibered product

$$\mathcal{A} \times_S \hat{\mathcal{A}}(\delta) := \{(x, y) \in \mathcal{A} \times \hat{\mathcal{A}}(\delta) \mid \pi(x) = \hat{\pi}(y)\}.$$

It is easy to see that $\mathcal{A} \times_S \hat{\mathcal{A}}(\delta)$ is a smooth complex submanifold of $\mathcal{A} \times \hat{\mathcal{A}}(\delta)$. We will denote by $p^{(1)} : \mathcal{A} \times_S \hat{\mathcal{A}}(\delta) \rightarrow \mathcal{A}$ and $p^{(2)} : \mathcal{A} \times_S \hat{\mathcal{A}}(\delta) \rightarrow \hat{\mathcal{A}}(\delta)$ the projection maps onto the factors \mathcal{A} and $\hat{\mathcal{A}}(\delta)$, respectively. (Throughout this appendix, we will simply denote by $p^{(1)}$ and $p^{(2)}$ the projection maps of any fibered products onto the first and second factors, respectively, when no confusion arises.) The projection maps $\pi, \hat{\pi}$ induce the projection map $\bar{\pi} : \mathcal{A} \times_S \hat{\mathcal{A}}(\delta) \rightarrow S$ such that $\bar{\pi}^{-1}(s) = A_s \times \hat{A}_s(\delta)$ for $s \in S$. For each $s \in S$, one has the associated Poincaré line bundle $P_s(\delta)$ over $A_s \times \hat{A}_s(\delta)$ such that (i) for $w \in \hat{A}_s(\delta)$, $P_s(\delta)|_{A_s \times \{w\}}$ is

the line bundle over A_s represented by w ; and (ii) $P_s(\delta)|_{\{0\} \times \hat{A}_s(\delta)} = \mathcal{O}_{\hat{A}_s(\delta)}$. Here, condition (ii) ensures that $P_s(\delta)$ is uniquely defined.

Consider $p^{(2)} : \mathcal{A} \times_S \hat{\mathcal{A}}(\delta) \rightarrow \hat{\mathcal{A}}(\delta)$ as a family of abelian varieties parametrized by $\hat{\mathcal{A}}(\delta)$, and let $\varepsilon : \hat{\mathcal{A}}(\delta) \rightarrow \mathcal{A} \times_S \hat{\mathcal{A}}(\delta)$ denote the zero section. Then it is well known that one can uniquely glue the $P_s(\delta)$'s together to form a holomorphic line bundle $\mathcal{P}_S(\delta)$ over $\mathcal{A} \times_S \hat{\mathcal{A}}(\delta)$ such that (i) $\mathcal{P}_S(\delta)|_{\bar{\pi}^{-1}(s)} = P_s(\delta)$ for $s \in S$; and (ii) $\mathcal{P}_S(\delta)|_{\varepsilon(\hat{\mathcal{A}}(\delta))} \simeq \mathcal{O}_{\hat{\mathcal{A}}(\delta)}$. $\mathcal{P}_S(\delta)$ will be called the Poincaré line bundle over $\mathcal{A} \times_S \hat{\mathcal{A}}(\delta)$. Observe that we may consider $\mathcal{P}_S(\delta)$ as a family of ample line bundles parametrized by $\hat{\mathcal{A}}(\delta)$. We refer the reader to [MFK, Chapter 6] for the above background materials.

(6.3). Now let Γ be a discrete torsion-free subgroup of $\mathrm{Sp}(2n, \mathbb{R})$. Fix a type of polarization δ , and let $\pi : \mathcal{A}_\Gamma \rightarrow X_\Gamma$ be the family of δ -polarized abelian varieties as constructed in (6.1). Then it follows from (6.2) that we have an associated family $\hat{\pi} : \hat{\mathcal{A}}_\Gamma(\delta) \rightarrow X_\Gamma$ and an associated Poincaré line bundle, denoted by $\mathcal{P}_\Gamma(\delta)$, over the fibered product $\mathcal{A}_\Gamma \times_{X_\Gamma} \hat{\mathcal{A}}_\Gamma(\delta)$. Also recall from Remark 6.1.1 that there exists a Hermitian holomorphic line bundle $(\mathcal{L}_\Gamma, \rho_\Gamma)$ over \mathcal{A}_Γ such that (6.1.5) is satisfied. As for $\mathcal{P}_\Gamma(\delta)$, we denote by $\mathcal{P}_\Gamma(0)$ the ‘usual’ universal Poincaré line bundle over $\mathcal{A}_\Gamma \times_{X_\Gamma} \hat{\mathcal{A}}_\Gamma$ associated to line bundles over fibers of $\pi : \mathcal{A}_\Gamma \rightarrow X_\Gamma$ with zero first Chern class (cf. (6.2)). It is well known that the polarization δ induces a morphism $\lambda_\delta : \mathcal{A}_\Gamma \rightarrow \hat{\mathcal{A}}_\Gamma$ satisfying the following properties: (i) one has $\hat{\pi} \circ \lambda_\delta = \pi$, and for $s \in X_\Gamma$, $\lambda_\delta|_{A_s} : A_s \rightarrow \hat{A}_s$ is an isogeny; and (ii) if we denote by $\Phi_\delta : \mathcal{A}_\Gamma \times_{X_\Gamma} \mathcal{A}_\Gamma \rightarrow \mathcal{A}_\Gamma \times_{X_\Gamma} \hat{\mathcal{A}}_\Gamma$ the restriction of the product map $(\mathrm{Id}_{\mathcal{A}_\Gamma}, \lambda_\delta) : \mathcal{A}_\Gamma \times \mathcal{A}_\Gamma \rightarrow \mathcal{A}_\Gamma \times \hat{\mathcal{A}}_\Gamma$ to $\mathcal{A}_\Gamma \times_{X_\Gamma} \hat{\mathcal{A}}_\Gamma$, where $\mathrm{Id}_{\mathcal{A}_\Gamma}$ is the identity map on \mathcal{A}_Γ , then

$$(6.3.1) \quad \Phi_\delta^* \mathcal{P}_\Gamma(0) = \mu^* \mathcal{L}_\Gamma \otimes (p^{(2)})^* \mathcal{L}_\Gamma^{-1} \otimes (p^{(1)})^* \mathcal{L}_\Gamma^{-1} \quad \text{on } \mathcal{A}_\Gamma \times_{X_\Gamma} \mathcal{A}_\Gamma$$

(cf. [MFK, Chapter 6, §2]). Here $\mu : \mathcal{A}_\Gamma \times_{X_\Gamma} \mathcal{A}_\Gamma \rightarrow \mathcal{A}_\Gamma$ denotes the holomorphic map given by the group law on each fiber of the projection map $\bar{\pi} : \mathcal{A}_\Gamma \times_{X_\Gamma} \mathcal{A}_\Gamma \rightarrow X_\Gamma$, i.e., $\mu(z, w) = z + w$ for $z, w \in \bar{\pi}^{-1}(s) = A_s \times A_s$, $s \in X_\Gamma$. Since \mathcal{L}_Γ induces the polarization δ on \mathcal{A}_Γ , it follows easily from (6.3.1) that under the identification $\hat{\mathcal{A}}_\Gamma(\delta) \cong \hat{A}_\Gamma$ (cf. (6.2)), one has

$$(6.3.2) \quad \Phi_\delta^* \mathcal{P}_\Gamma(\delta) = \mu^* \mathcal{L}_\Gamma \otimes (p^{(2)})^* \mathcal{L}_\Gamma^{-1} \quad \text{on } \mathcal{A}_\Gamma \times_{X_\Gamma} \mathcal{A}_\Gamma.$$

Moreover, it is easy to see that $h_\delta := \mu^* \rho_\Gamma \otimes (p^{(2)})^* \rho_\Gamma^{-1}$ is a smooth Hermitian metric on $\Phi_\delta^* \mathcal{P}_\Gamma(\delta)$ such that $h_\delta|_{L_w}$ is a canonical Hermitian metric on $L_w := \Phi_\delta^* \mathcal{P}_\Gamma(\delta)|_{(p^{(2)})^{-1}(w)}$ for each $w \in \mathcal{A}_\Gamma$. From (6.1.5), one easily sees that

$$(6.3.3) \quad c_1(\Phi_\delta^* \mathcal{P}_\Gamma(\delta), h_\delta) = \mu^* \omega_\Gamma - (p^{(2)})^* \omega_\Gamma \quad \text{on } \mathcal{A}_\Gamma \times_{X_\Gamma} \mathcal{A}_\Gamma.$$

It is easy to see that the universal cover of $\mathcal{A}_\Gamma \times_{X_\Gamma} \mathcal{A}_\Gamma$ is given by

$$(6.3.4) \quad \mathcal{H}_n \times \mathbb{C}^n \times \mathbb{C}^n = \{(\tau, Z, W) \in \mathbb{C}^{n^2} \times \mathbb{C}^n \times \mathbb{C}^n \mid \tau \in \mathcal{H}_n\}.$$

Thus, the coordinates (τ, Z, W) may also be regarded as local holomorphic coordinates for $\mathcal{A}_\Gamma \times_{X_\Gamma} \mathcal{A}_\Gamma$. Using (6.1.4) and in terms of the coordinates (τ, Z, W) , one

easily sees from (6.3.3) that $c_1(\Phi_\delta^* \mathcal{P}_\Gamma(\delta), h_\delta)$ can be locally given as

(6.3.5)

$$c_1(\Phi_\delta^* \mathcal{P}_\Gamma(\delta), h_\delta) = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} [(\operatorname{Im} Z)^t (\operatorname{Im} \tau)^{-1} (\operatorname{Im} Z) + (\operatorname{Im} Z)^t (\operatorname{Im} \tau)^{-1} (\operatorname{Im} W) + (\operatorname{Im} W)^t (\operatorname{Im} \tau)^{-1} (\operatorname{Im} Z)].$$

(6.4). Let $\pi : \mathcal{A} \rightarrow S$ be a family of abelian varieties parametrized by a complex manifold S , and let $\mathcal{L} \rightarrow \mathcal{A}$ be a holomorphic line bundle over \mathcal{A} such that $L_s := \mathcal{L}|_{A_s}$ is an ample line bundle over $A_s := \pi^{-1}(s)$ for each $s \in S$. Then \mathcal{L} determines a fixed polarization type δ of \mathcal{A} . From [MFK, Chapter 7], one knows that for each $s_o \in S$ and $k \in \mathbb{Z}^+$, one can always introduce a level k structure on $\mathcal{A}|_V$ for some open neighborhood $V \subset S$ containing s_o . Moreover, for sufficiently large k and upon shrinking V if necessary, one has the base change diagram

$$(6.4.1) \quad \begin{array}{ccccc} \mathcal{A}|_V & \xrightarrow{\Xi} & \mathcal{A}_\Gamma \times_{X_\Gamma} \mathcal{A}_\Gamma & \xrightarrow{\Phi_\delta} & \mathcal{A}_\Gamma \times_{X_\Gamma} \hat{\mathcal{A}}_\Gamma(\delta) \\ \pi \downarrow & & \downarrow p^{(2)} & & \downarrow p^{(2)} \\ V & \xrightarrow{i} & \mathcal{A}_\Gamma & \xrightarrow{\lambda_\delta} & \hat{\mathcal{A}}_\Gamma(\delta) \end{array}$$

for some torsion-free arithmetic subgroup $\Gamma \subset \operatorname{Sp}(2n, \mathbb{Z}, \delta)$, and such that

$$(6.4.2) \quad \lambda_\delta \circ i(s) = [L_s] \in \hat{\mathcal{A}}(\delta) \quad \text{and} \quad L_s = (\Phi_\delta \circ \Xi)^* \mathcal{P}_\Gamma(\delta)|_{A_s}$$

for each $s \in V$. Here $[L_s]$ denotes the point in $\hat{\mathcal{A}}(\delta)$ corresponding to L_s , and λ_δ , Φ_δ , $\mathcal{P}_\Gamma(\delta)$ are as in (6.3). In other words, the map $i \circ \lambda_\delta$ is the classifying map associated to $\mathcal{L}|_V$, and the map i is a lifting of $i \circ \lambda_\delta$ to the finite cover \mathcal{A}_Γ of $\hat{\mathcal{A}}_\Gamma(\delta)$. Let h_δ be as in (6.3). Then one easily sees from (6.3.3) that $\Xi^* h_\delta$ is a smooth Hermitian metric on $\mathcal{L}|_V$ such that $\Xi^* h_\delta|_{L_s}$ is a canonical metric on L_s for each $s \in V$. Finally we remark that one can easily use the above construction and a partition of unity on S to construct a smooth Hermitian metric h on \mathcal{L} such that $h|_{L_s}$ is a canonical Hermitian metric on L_s for each $s \in S$.

REFERENCES

- [A] M.F. Atiyah, Complex analytic connections in fiber bundles, *Trans. Amer. Math. Soc.* **85**(1957), 181-207. MR **19**:172c
- [APW] S. Axelrod, S.D. Pietra and E. Witten, Geometric quantization of Chern-Simons gauge theory, *J. Diff. Geom.* **33**(1991), 787-902. MR **92i**:58064
- [Bo] A. Borel, *Introduction aux groupes arithmétiques*, Hermann, Paris, 1969. MR **39**:5577
- [D1] S.K. Donaldson, A new proof of a theorem of Narasimhan and Seshadri, *J. Diff. Geom.* **18** (1993), 269-277. MR **85a**:32036
- [D2] S.K. Donaldson, Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles, *Proc. London Math. Soc.* **50**(1985), 1-26. MR **86h**:58038
- [FS] A. Fujiki and G. Schumacher, The moduli space of compact extremal Kähler manifolds and generalized Petersson-Weil metrics, *Publ. RIMS Kyoto Univ.* **26**(1990), 101-183. MR **92e**:32012
- [G] H. Grauert, Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen, *Publ. Math. IHES* **5**(1960), 233-292. MR **22**:12544
- [GH] P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley and Sons, 1978. MR **80b**:14001
- [H] N.J. Hitchin, Flat connections and geometric quantization, *Commun. Math. Phys.* **131**(1990), 347-380. MR **91g**:32022
- [Ko1] S. Kobayashi, Curvature and stability of vector bundles, *Proc. Japan Acad. Ser. A. Math. Sci.* **58**(1982), 158-162. MR **83i**:53090

- [Ko2] S. Kobayashi, *Differential geometry of complex vector bundles*, Princeton, NJ: Iwanami Shoten and Princeton University Press, 1987.
- [KN] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol. 1, Interscience Publishers, New York, 1963. MR **27**:2945
- [Ku] M. Kuga, *Fiber varieties over a symmetric space whose fibers are abelian varieties*, I, II, Lect. Notes, Univ. of Chicago, 1963-64.
- [Lü] M. Lübke, Stability of Einstein-Hermitian vector bundles, *Manuscripta Math.* **42**(1983), 245-247. MR **85e**:53087
- [MFK] D. Mumford, J. Fogarty and F. Kirwan, *Geometric invariant theory*, 3rd Edition, Springer-Verlag, Berlin-Heidelberg-New York, 1994. MR **95m**:14012
- [Mok] N. Mok, Aspects of Kähler geometry on arithmetic varieties, *Proc. Symp. Pure Math.* **52**, part 2(1991), 335-396. MR **92j**:32027
- [M1] D. Mumford, On the equations defining abelian varieties I, *Invent. Math.* **1**(1966) 287-354; II, *Invent. Math.* **3**(1967) 75-135; III, *Invent. Math.* **3**(1967) 215-244.
- [M2] D. Mumford, Towards an enumerative geometry of the moduli space of curves, M. Artin and J. Tate (eds.), *Arithmetic and geometry*, Birkhäuser, Basel-Boston, 1983, pp. 271-328. MR **85j**:14046
- [MT] N. Mok and W.-K. To, Eigensections on Kuga families of abelian varieties and finiteness of their Mordell-Weil groups, *J. Reine Angew. Math.* **444**(1993), 29-78. MR **94m**:32045
- [NS] M.S. Narasimhan and C.S. Seshadri, Stable and unitary vector bundles on a compact Riemann surface, *Ann. Math.* **82**(1965), 540-564. MR **32**:1725
- [Sa] I. Satake, *Algebraic structures on symmetric domains*, Iwanami Shoten and Princeton Univ. Press, 1980. MR **82i**:32003
- [Sch] G. Schumacher, The curvature of the Petersson-Weil metric on the moduli space of Kähler-Einstein manifolds, in V. Ancona and A. Silva (eds.), *Complex Analysis and Geometry*, Plenum Press, New York, 1993, pp. 339-354. MR **94g**:32038
- [SD] H.P.F. Swinnerton-Dyer, *Analytic theory of abelian varieties*, Cambridge Univ. Press, Cambridge, 1974. MR **51**:3180
- [Siu1] Y.-T. Siu, Complex analyticity of harmonic maps, vanishing and Lefschetz theorems, *J. Diff. Geom.* **17**(1982), 55-138. MR **83j**:58039
- [Siu2] Y.-T. Siu, *Lectures on Hermitian-Einstein metrics for stable bundles and Kähler-Einstein metrics*, Birkhäuser, Basel-Boston, 1987. MR **89d**:32020
- [ST] G. Schumacher and M. Toma, On the Petersson-Weil metric for the moduli space of Hermitian-Einstein bundles and its curvature, *Math. Ann.* **293**(1992), 101-107. MR **93h**:32040
- [TUY] A. Tsuchiya, K. Ueno and Y. Yamada, Conformal field theory on universal family of stable curves with gauge symmetries, *Advanced Studies in Pure Math.* **19**(1989), 459-566. MR **92a**:81191
- [TW] W.-K. To and L. Weng, curvature of the L²-metric on the direct image of a family of Hermitian-Einstein vector bundles, *Amer. J. Math.* **120** (1998), 649-661. MR **99d**:32038
- [UY] K. Uhlenbeck and S.T. Yau, On the existence of Hermitian-Yang-Mills connections in stable vector bundles, *Comm. Pure Appl. Math.* **39**(1986), 257-293. MR **88i**:58154
- [W] G. Welters, Polarized abelian varieties and the heat equation, *Compositio Math.* **49**(1983), 173-194. MR **85f**:14045

DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, 2 SCIENCE DRIVE 2, SINGAPORE 117543

E-mail address: mattowk@nus.edu.sg

GRADUATE SCHOOL OF MATHEMATICS, KYUSHU UNIVERSITY, HAKOZAKI, HIGASHI-KU, FUKUOKA 812-8581, JAPAN

E-mail address: weng@math.kyushu-u.ac.jp