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# $L^2$-METRICS, PROJECTIVE FLATNESS AND FAMILIES OF POLARIZED ABELIAN VARIETIES

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Abstract. We compute the curvature of the $L^2$-metric on the direct image of a family of Hermitian holomorphic vector bundles over a family of compact Kähler manifolds. As an application, we show that the $L^2$-metric on the direct image of a family of ample line bundles over a family of abelian varieties and equipped with a family of canonical Hermitian metrics is always projectively flat. When the parameter space is a compact Kähler manifold, this leads to the poly-stability of the direct image with respect to any Kähler form on the parameter space.

## §0. Introduction

It is well known that for a family $\pi : \mathcal{A} \to S$ of principally polarized abelian varieties, the theta bundles on the fibers of $\pi$ form a line bundle $\Theta$ over $\mathcal{A}$. From the fact that the theta characters of level $n$, $n \in \mathbb{Z}^+$, satisfy a certain heat equation, one knows that the direct image vector bundles $\pi_*\Theta^{\otimes n}$ admit projectively flat connections such that the theta characters are parallel sections. Moreover, from the fact that the theta characters form an orthonormal basis with respect to a certain natural $L^2$-pairing, one knows that the above projectively flat connections are indeed Hermitian connections (cf. [APW, §5]).

As a generalization of the above, one considers in conformal field theory a family of Riemann surfaces and an associated family $\pi : \mathcal{M} \to S$ of moduli spaces of stable vector bundles over the Riemann surfaces. As before, the so-called generalized theta bundles over the fibers of $\pi$ form a line bundle $\Theta$ over $\mathcal{M}$. An important result in conformal field theory in [TUY], [H] and [APW] was to show that for $n \in \mathbb{Z}^+$, $\pi_*\Theta^{\otimes n}$ admits a projectively flat connection, whose construction depends on the fact that certain generalized heat equations, or KZ equations, hold for sections of generalized theta bundles.

In another direction, instead of just considering the theta bundles, Welters [W], following [M1], showed that for a family of polarized abelian varieties $\pi : \mathcal{A} \to S$ and any relatively ample line bundle $\mathcal{L}$ over $\mathcal{A}$, $\pi_*\mathcal{L}$ always admits a projectively flat connection. This was established from hypercohomological considerations in [W], which led to certain generalized heat equations for sections of $\mathcal{L}$ (cf. also [H]).

However the projectively flat connections constructed in [H] and [W] are not Hermitian connections in general. From the viewpoint of representation theory, a vector bundle of rank $r$ over $S$ admits a projectively flat connection (resp. projectively flat Hermitian connection) if and only if $E$ arises from a representation of

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$\pi_1(S)$ into $\mathbb{P}GL(r,\mathbb{C})$ (resp. $\mathbb{P}U(r)$). Thus a natural and important question that arises is whether the direct image bundles in the above cases admit projectively flat Hermitian connections.

In this article, we study the problem from a different approach. We consider Hermitian metrics on direct image bundles arising from certain $L^2$-pairings on the sections of the original vector bundles. Our first main result is to give a computation of the curvature tensor of such $L^2$-metrics for a family of Hermitian holomorphic vector bundles over a family of compact Kähler manifolds (cf. Theorem 1 in §2). Our approach has been inspired by that of Schumacher [Sch] in his computation of the curvature of the Weil-Petersson metric, which was defined similarly in terms of certain $L^2$-pairing. In particular, the technique of 'horizontal' lifting of vector fields in [Sch] also plays an important part in our calculations. As in the case of the Weil-Petersson metric, we also give an interpretation of our curvature formula in terms of Kodaira-Spencer theory (cf. Proposition 2.3.2).

An interesting application of Theorem 1 is Welters' case of a relatively ample line bundle $\mathcal{L}$ over a family of abelian varieties $\pi : \mathcal{A} \to S$. Our second main result is to show that the Hermitian connection of the $L^2$-metric on $\pi_*\mathcal{L}$ associated to a family of 'canonical' Hermitian metrics on $\mathcal{L}$ is indeed projectively flat (cf. Theorem 3 in §2). Thus our result generalizes the result mentioned above for theta bundles over families of principally polarized abelian varieties, and it improves Welters' result [W] in terms of representation of $\pi_1(S)$.

We sketch the proof of Theorem 3 briefly as follows. First, we reduce Theorem 3 to the case when the smooth $(1,1)$-form on the total space of the family of abelian varieties (which is flat on each fiber) is given as the first Chern form of a smooth Hermitian metric on the relatively ample line bundle over the total space, so that Corollary 2 applies. Then an important step in simplifying the curvature tensor of the $L^2$ metric is to show that the values of the above-mentioned (1,1)-form evaluated at the horizontal lifting of tangent vectors from the base space are constant along each fiber. The latter is achieved by comparing the (1,1)-form with another (1,1)-form on the total space with the same values along the tangent spaces of the fibers but with zero eigenspaces along the tangent spaces of the leaves of certain natural smooth foliation on the total space. We remark that our original approach in proving Theorem 3 consists of studying the classifying map from the families of ample line bundles over abelian varieties to the associated universal Poincaré line bundles over families of abelian varieties parametrized by Siegel modular varieties. The present simpler and more direct approach is suggested by the referee.

This paper is organized as follows. In §1, we introduce some notations and list our main results. In §2, we compute the curvature of the $L^2$-metric. In §3, we treat the case of a family of Hermitian holomorphic line bundles whose first Chern forms give the Kähler metrics on the manifolds. In §4, we make some useful observations on $L^2$-metrics associated to families of polarized abelian varieties. The results in §4 are needed in §5, where we give the proof of Theorem 3 in full generality. Finally in the Appendix (§6), we also give a description of the canonical Hermitian metrics in the 'local universal' case of families of Poincaré line bundles over abelian varieties parametrized by Siegel modular varieties.

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## §1. Notation and statement of results

**(1.1).** Let $\pi : \mathcal{X} \to S$ be a surjective holomorphic map between complex manifolds $\mathcal{X}$ and $S$ such that $X_s := \pi^{-1}(s)$ is an $n$-dimensional compact complex manifold for each $s \in S$. Moreover, suppose that $\mathcal{X}$ admits a smooth (1,1)-form $\omega_{\mathcal{X}} \in \mathcal{A}^{1,1}(\mathcal{X})$ such that the restriction $\omega_s := \omega_{\mathcal{X}}|_{X_s}$ is a Kähler form on $X_s$ for each $s \in S$. Then such $(\pi : \mathcal{X} \to S, \omega_{\mathcal{X}})$ is called a family of compact Kähler manifolds parametrized by $S$. With some slight abuse of notation, we will simply call $\omega_{\mathcal{X}} \in \mathcal{A}^{1,1}(\mathcal{X})$ a family of Kähler metrics for the family $\pi : \mathcal{X} \to S$. Observe that two different $(1,1)$-forms on $\mathcal{X}$ may lead to the same restrictions $\{\omega_s\}_{s\in S}$. Let $E$ be a holomorphic vector bundle over $\mathcal{X}$, and let $h$ be a smooth Hermitian metric on $E$. Then for each $s \in S$, the restriction $(\mathcal{E}_s \to X_s, h_s)$ forms a Hermitian holomorphic vector bundle over $X_s$, where $\mathcal{E}_s = \mathcal{E}|_{X_s}$ and $h_s = h|_{X_s}$. Thus $(E, h)$ can be understood as a family of Hermitian holomorphic vector bundles over the family of complex manifolds $\pi : \mathcal{X} \to S$. By a result of Grauert [G], the direct image sheaf $\pi_*\mathcal{E}$ is coherent over $S$, and is thus locally free on the complement of a proper subvariety $Z$ in $S$. Moreover, changing $Z$ if necessary, one may assume that $\pi_*\mathcal{E}$ is given by

$$(\pi_*\mathcal{E})_s = H^0(X_s, \mathcal{E}_s) \quad \text{for } s \in S\backslash Z.$$

For simplicity, we will also denote by $\pi_*\mathcal{E}$ its underlying holomorphic vector bundle over $S\backslash Z$. It is easy to see that $h$ and $\omega_{\mathcal{X}}$ induce a smooth Hermitian metric, known as the $L^2$-metric, on $\pi_*\mathcal{E}$ over $S\backslash Z$ given by

$$H_{h,\omega_{\mathcal{X}}}(t, t') := \int_{X_s} \langle t, t'\rangle \frac{\omega_s^n}{n!} \quad \text{for } t, t' \in H^0(X_s, \mathcal{E}_s),\ s \in S\backslash Z, \tag{1.1.1}$$

where $\langle t, t'\rangle$ denotes $h_s(t, t')$. We will also denote by $\langle\, ,\, \rangle$ the inner product on $\mathcal{E}_s$-valued differential forms induced by $h_s$ and $\omega_s$.

Consider a family of compact Kähler manifolds $(\pi : \mathcal{X} \to S,\ \omega_{\mathcal{X}})$. The orthogonal complement of $\mathrm{Ker}(\pi_* : T\mathcal{X} \to TS)$ in $T\mathcal{X}$ with respect to $\omega_{\mathcal{X}}$ defines a smooth 'horizontal' vector subbundle $T^H\mathcal{X} \subset T\mathcal{X}$. For $s \in S$ and a holomorphic tangent vector $u \in T_sS$, there exists a unique lifting of $u$ to a smooth vector field $v_u \in \mathcal{A}^0(X_s, T\mathcal{X}|_{X_s})$ such that $\pi_*v_u(z) = u$ and $v_u(z) \in T^H\mathcal{X}$ for all $z \in X_s$. Such $v_u$ is called the horizontal lifting of $u$ (with respect to $\omega_{\mathcal{X}}$). Moreover one can easily check that

$$A_u := \bar\partial v_u \in \mathcal{A}^{0,1}(X_s, TX_s) \tag{1.1.2}$$

(cf. (2.1.2)), and its class $[A_u] \in H^{0,1}(X_s, TX_s)$ is the Kodaira-Spencer class of $u$ (cf. e.g. [FS], [Sch] and (2.3)).

We will denote by $s = (s^i)_{1\le i\le m}$, where $m = \dim_{\mathbb{C}} S$, the local holomorphic coordinates for $S$, and we will denote $z = (z^\alpha)_{1\le\alpha\le n}$ the local holomorphic coordinates for the fibers $X_s$. Thus $(z, s) = (z^1, \ldots, z^n, s^1, \ldots, s^m)$ gives local holomorphic coordinates for $X_s$. For $s \in S$, a coordinate tangent vector $\partial/\partial s^i \in T_sS$, we simply denote its horizontal lifting by

$$v_i := v_{\partial/\partial s^i} \quad \text{and thus} \quad A_{\partial/\partial s^i} = \bar\partial v_i. \tag{1.1.3}$$

Denote the components of $A_{\partial/\partial s^i}$ by

$$A_{\partial/\partial s^i} =: A^\alpha_{i\bar\beta} \frac{\partial}{\partial z^\alpha} \otimes d\bar z^\beta. \tag{1.1.4}$$

Here we adopt the Einstein summation notation. Also for a Hermitian holomorphic vector bundle $(\mathcal{E}, h)$ over $(\mathcal{X}, \omega_{\mathcal{X}})$, we denote the curvature tensor of $(\mathcal{E}, h)$ by $\Omega \in \mathcal{A}^{1,1}(\mathcal{X}, \mathrm{End}(\mathcal{E}))$. Also denote the smooth family of Kähler metrics associated to $\omega_{\mathcal{X}}$ by $g := \{g_s\}_{s\in S}$. For $s \in S$, we also denote by $G_{\mathcal{E}_s}$ the Green's operator on $\mathcal{E}_s$-valued differential forms on $X_s$ associated to the $\bar\partial$-Laplacian $\Box := \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ with respect to $h_s$ and $g_s$. Thus the Laplacian for functions is given by $\Box = -g^{\bar\beta\alpha}\partial^2/\partial z^\alpha \partial \bar z^\beta$. We will also adopt the semi-colon notation to denote covariant derivatives, so that $t_{;\alpha} := \nabla_{\partial/\partial z^\alpha} t$ for $t \in H^0(X_s, \mathcal{E}_s)$, etc.

With the above notations, we state our first main result in this paper as follows:

**Theorem 1.** *Let $(\mathcal{E} \to \mathcal{X},\, h)$ be a family of Hermitian holomorphic vector bundles of rank $r$ over a family of $n$-dimensional compact Kähler manifolds $(\pi : \mathcal{X} \to S,\, \omega_{\mathcal{X}})$ and parametrized by an $m$-dimensional compact complex manifold $S$. Suppose that $\omega_{\mathcal{X}}$ is $d$-closed on $\mathcal{X}$. Let $Z \subset S$ be a proper analytic subvariety such that $\pi_*\mathcal{E}$ is locally free over $S\backslash Z$ with $(\pi_*\mathcal{E})_s = H^0(X_s, \mathcal{E}_s)$ for $s \in S\backslash Z$. Then in terms of local holomorphic coordinates $s = (s^i)_{1\le i\le m}$ of $S$, $z = (z^\alpha)_{1\le\alpha\le n}$ of $X_s$ and a local holomorphic trivialization $\{t_a\}_{1\le a\le p}$ of $\pi_*\mathcal{E}$, where $p := rank(\pi_*\mathcal{E})$, the curvature tensor $\Theta \in \mathcal{A}^{1,1}(S\backslash Z, End(\pi_*\mathcal{E}))$ of the associated $L^2$-metric $H_{h,\omega_{\mathcal{X}}}$ on $\pi_*\mathcal{E}$ is given by*

(1.1.5)
$$\begin{aligned}\Theta_{a\bar b i\bar j}(s) = &-\int_{X_s} \langle G_{\mathcal{E}_s}(\Omega_{v_i\bar\alpha}(t_a)d\bar z^\alpha - A^\gamma_{i\bar\beta} t_{a;\gamma} d\bar z^\beta), \Omega_{v_j\bar\alpha}(t_b)d\bar z^\alpha - A^\gamma_{j\bar\beta} t_{b;\gamma} d\bar z^\beta\rangle \frac{\omega_s^n}{n!}\\ &+\int_{X_s} \langle \Omega_{v_i\bar v_j}(t_a), t_b\rangle \frac{\omega_s^n}{n!} + \int_{X_s} \Box(g_{v_i\bar v_j})\langle t_a, t_b\rangle \frac{\omega_s^n}{n!}\end{aligned}$$

*for $s \in S\backslash Z$. Here $\Theta_{a\bar b i\bar j} := \Theta(t_a, t_b; \partial/\partial s^i, \partial/\partial s^j)$ with $t_a, t_b \in H^0(X_s, \mathcal{E}_s)$.*

*Remark* 1.1.1. (i) When $h$ (and thus also $H_{h,\omega_{\mathcal{X}}}$) is multiplied by a smooth function $\lambda(s)$ on $S$, the second term of $\Theta$ in (1.1.5) is modified by $-\partial\bar\partial\log\lambda \cdot H_{h,\omega_{\mathcal{X}}}(t_a, t_b)$, while the first and the third terms remain unchanged.

(ii) We refer the reader to (2.3) for an interpretation of the first term of $\Theta$ in (1.1.5) in terms of Kodaira-Spencer representatives.

(iii) Theorem 1 in the special case of a family of Hermitian-Einstein vector bundles over a fixed compact Kähler manifold was treated earlier by the authors in [TW], where the curvature of the $L^2$ metric was further expressed in terms of harmonic representatives of the Kodaira-Spencer class of the tangent vectors of the base manifold.

(iv) For a family of compact Kähler-Einstein manifolds with constant non-zero scalar curvature, one can always glue the Kähler forms (associated to the Kähler-Einstein metrics) on the fibers appropriately to get a $d$-closed (1,1)-form on the total space (cf. e.g. [Sch]). In particular, Theorem 1 applies to such families of Kähler manifolds.

**(1.2).** Next we consider a family of Hermitian holomorphic line bundles $(\mathcal{L} \to \mathcal{X}, h)$ over a family of $n$-dimensional compact Kähler manifolds $(\pi : \mathcal{X} \to S,\, \omega_{\mathcal{X}})$ such that for some $k \in \mathbb{R}$,

$$c_1(\mathcal{L}, h) = \frac{k}{2\pi}\omega_{\mathcal{X}} \quad \text{on } \mathcal{X}. \tag{1.2.1}$$

Thus in particular, one has, for all $s \in S$,

$$c_1(L_s, h_s) = \frac{k}{2\pi}\omega_s \quad \text{on } X_s. \tag{1.2.2}$$

Then Theorem 1 gives rise to

**Corollary 2.** *Let $(\pi : \mathcal{X} \to S, \omega_{\mathcal{X}})$ be as in Theorem 1 and let $(\mathcal{L} \to \mathcal{X}, h)$ be a family of Hermitian holomorphic line bundles satisfying (1.2.1) for some $k \in \mathbb{R}$. Let $Z \subset S$ be a proper analytic subvariety such that $\pi_*\mathcal{L}$ is locally free over $S\backslash Z$ with $(\pi_*\mathcal{L})_s = H^0(X_s, L_s)$ for $s \in S\backslash Z$. Then the curvature tensor $\Theta \in \mathcal{A}^{1,1}(S\backslash Z, End(\pi_*\mathcal{L}))$ of the associated $L^2$-metric $H_{h,\omega_{\mathcal{X}}}$ on $\pi_*\mathcal{L}$ is given by*

$$\begin{aligned}\Theta_{a\bar{b}i\bar{j}}(s) = &-\int_{X_s} \langle G_{\mathcal{L}_s}(A^{\gamma}_{i\bar{\beta}} t_{a;\gamma} d\bar{z}^{\beta}), A^{\gamma}_{j\bar{\beta}} t_{b;\gamma} d\bar{z}^{\beta}\rangle \frac{\omega_s^n}{n!} \\ &+ \int_{X_s} \big(k g_{v_i\bar{v}_j} + \Box(g_{v_i\bar{v}_j})\big)\langle t_a, t_b\rangle \frac{\omega_s^n}{n!}\end{aligned} \tag{1.2.3}$$

*for $s \in S\backslash Z$ and $t_a$, $t_b \in H^0(X_s, L_s)$.*

**(1.3).** To facilitate ensuing discussion, we recall that a connection $D$ on a smooth complex vector bundle $F$ of rank $r$ over a manifold $M$ is said to be projectively flat if there exists a complex 2-form $\alpha \in \mathcal{A}^2(M)$ such that the curvature form $R = D^2 \in \mathcal{A}^2(M, \mathrm{End}(F))$ satisfies

$$R = \alpha \cdot \mathrm{Id}_F, \tag{1.3.1}$$

where $\mathrm{Id}_F$ is the identity endomorphism on $F$. It is well known that $F$ admits a projectively flat connection if and only if $F$ arises from a representation $\Phi : \pi_1(M) \to \mathbb{P}\mathrm{GL}(r, \mathbb{C})$. Here $\pi_1(M)$ denotes the fundamental group of $M$, $\mathbb{P}\mathrm{GL}(r, \mathbb{C}) := \mathrm{GL}(r, \mathbb{C})/\mathbb{C}^* \cdot I_r$ denotes the projective general linear group, and $I_r$ is the $r \times r$ identity matrix. Moreover, $F$ admits a projectively flat *Hermitian* connection if and only if $F$ arises from a representation $\Phi : \pi_1(M) \to \mathbb{P}U(r)$, where $\mathbb{P}U(r) := U(r)/U(1) \cdot I_r$ denotes the projective unitary group (cf. e.g. [Ko2, pp. 7 and 14] for the above background materials).

Let $A_0$ be an abelian variety, and let $L_0$ be a holomorphic line bundle over $A_0$.

**Definition 1.3.1.** A Hermitian metric $h_0$ on $L_0$ is said to be canonical if $c_1(L_0, h_0)$ is invariant under translations of $A_0$.

*Remark* 1.3.2. It is well known that a canonical Hermitian metric always exists on $L_0$ and it is unique up to a positive multiplicative constant. In fact, this follows from the standard fact that a de Rham cohomology class of a compact complex torus is represented uniquely by a translation-invariant form (see e.g. [GH, p. 302]).

Now we consider a (holomorphic) family of abelian varieties $\pi : \mathcal{A} \to S$ parametrized by a complex manifold $S$, i.e., each fiber $A_s := \pi^{-1}(s)$ is an $n$-dimensional abelian variety. Let $\mathcal{L} \to \mathcal{A}$ be a (holomorphic) family of ample line bundles over the family $\pi : \mathcal{A} \to S$, i.e., $\mathcal{L}$ is a holomorphic line bundle over the total space $\mathcal{A}$ such that $L_s := \mathcal{L}\big|_{A_s}$ is an ample line bundle over $A_s$ for each $s \in S$. (Such an $\mathcal{L}$ is usually called a relatively ample line bundle.) Then it follows from the Kodaira vanishing theorem that $H^i(A_s, L_s) = 0$ for each $i > 0$ and $s \in S$, and thus by a result of Grauert [G], $\pi_*\mathcal{L}$ is locally free over $S$ with $(\pi_*\mathcal{L})_s = H^0(A_s, L_s)$ for each $s \in S$. Moreover, one can always construct a Hermitian metric $\rho$ on $\mathcal{L}$ such that $\rho_s := \rho|_{L_s}$ is a canonical Hermitian metric on $L_s$ for all $s \in S$ (this follows, for example, from the argument in [Siu2, p. 17] or (6.4) and using a partition of unity on $S$). With slight abuse of notation, such $\rho$ will be called a smooth family of canonical Hermitian metrics on the family of line bundles $\mathcal{L} \to \mathcal{A}$. Also, it is obvious that such $\rho$ is unique up to a smooth positive function on $S$. A (smooth) family of flat Kähler metrics $\omega_{\mathcal{A}}$ on the family $\pi : \mathcal{A} \to S$ is simply a smooth $(1,1)$-form on the total space $\mathcal{A}$ such that $\omega_s := \omega_{\mathcal{A}}\big|_{A_s}$ is a flat Kähler metric on $A_s$ for each $s \in S$. Our second main result is the following

**Theorem 3.** *Let $\pi : \mathcal{A} \to S$ be a family of abelian varieties parametrized by a complex manifold $S$, and let $\mathcal{L} \to \mathcal{A}$ be a family of ample line bundles parametrized by $S$.*

(i) *Then for any smooth family $\rho$ of canonical Hermitian metrics on the family $\mathcal{L} \to \mathcal{A}$ and any smooth family of flat Kähler metrics $\omega_{\mathcal{A}}$ on the family $\pi : \mathcal{A} \to S$, the Hermitian connection of the associated $L^2$-metric $H_{\rho,\omega_{\mathcal{A}}}$ on $\pi_*\mathcal{L}$ is projectively flat.*

(ii) *In particular, if the parameter space $S$ is a compact Kähler manifold, then $\pi_*\mathcal{L}$ is poly-stable with respect to any Kähler form on $S$.*

*Remark* 1.3.3. (i) Welters [W] proved earlier that under the hypothesis of Theorem 3, $\pi_*\mathcal{L}$ admits a projectively flat connection, and hence $\pi_*\mathcal{L}$ arises from a representation of $\pi_1(S)$ into $\mathbb{P}\mathrm{GL}(r,\mathbb{C})$, where $r := \mathrm{rank}(\pi_*\mathcal{L})$. However, the projectively flat connection in [W] is not a Hermitian connection in general (cf. also [H]), and thus Welters' result does not lead to statement (i) or (ii) of Theorem 3. Theorem 3, in particular, implies that $\pi_*\mathcal{L}$ arises from a representation of $\pi_1(S)$ into $\mathbb{P}U(r)$.

(ii) Theorem 3 in the special case of the family of theta line bundles (or their powers) over a family of principally polarized abelian varieties is well known and follows from the heat equation and orthonormality of theta characters (cf. e.g. [APW, §5]).

(iii) We remark that, unlike Theorem 1, the (1,1)-form $\omega_{\mathcal{A}}$ in Theorem 3(i) need not be $d$-closed on $\mathcal{A}$.

## §2. Curvature of the $L^2$-metric

We are going to prove Theorem 1 in this section, and we will follow the notations in (1.1) throughout §2.

**(2.1).** Let $(\pi : \mathcal{X} \to S, \omega_{\mathcal{X}})$ be a family of compact Kähler manifolds parametrized by a complex manifold $S$, and write $X_s := \pi^{-1}(s)$ and $\omega_s := \omega_{\mathcal{X}}\big|_{X_s}$ for each $s \in S$. Let $m = \dim_{\mathbb{C}} S$ and $n = \dim_{\mathbb{C}} X_s$. We will use $s = (s^i)_{1\le i\le m}$ to denote local holomorphic coordinates of $S$, which will be indexed by the letters $i, j, k$, etc. Also we will use $z = (z^\alpha)_{1\le\alpha\le n}$ to denote local holomorphic coordinates of the fibers $X_s$, which will be indexed by the Greek letters $\alpha, \beta, \gamma$, etc. Thus $(z, s) = (z^\alpha, s^i)_{1\le\alpha\le n, 1\le i\le m}$ gives local holomorphic coordinates for $\mathcal{X}$ and we will use capital letters $I, J, K$ to index coordinates on $\mathcal{X}$ so that $I$ can be $i$ or $\alpha$, etc. Also we write $\omega_{\mathcal{X}} = \sqrt{-1}g_{I\bar{J}}(z,s)dw^I \wedge d\bar{w}^J$, where $w$ can be $z$ or $s$. For a coordinate tangent vector $\partial/\partial s^i \in T_sS, s \in S$, its horizontal lifting $v_i$ (with respect to $\omega_{\mathcal{X}}$)

defined in (1.1) is given by

$$v_i = \frac{\partial}{\partial s^i} - g^{\bar\beta\alpha} g_{i\bar\beta} \frac{\partial}{\partial z^\alpha} \quad \text{at } (z,s) \in \mathcal{X} \tag{2.1.1}$$

(see [Sch, §1, equation (1.2)]). Here $g^{\bar\beta\alpha}$ denotes the components of the inverse of $g_{\alpha\bar\beta}$ (and not that of $g_{I\bar J}$, which may not be invertible). Then it is easy to see that the associated tensor $A_{\partial/\partial s^i}$ defined in (1.1.3) is given locally by

$$A^\alpha_{i\bar\beta} = -\frac{\partial}{\partial \bar z^\beta}(g^{\bar\gamma\alpha} g_{i\bar\gamma}). \tag{2.1.2}$$

We have the following simple lemma:

**Lemma 2.1.1.** (i) $[v_i, \frac{\partial}{\partial \bar z^\alpha}] = -A^\beta_{i\bar\alpha}\frac{\partial}{\partial \bar z^\beta}$.

(ii) *For a smooth* $(n,n)$*-form* $\eta$ *on* $\mathcal{X}$*,*

$$\frac{\partial}{\partial s^i}\int_{X_s} \eta = \int_{X_s} L_{v_i}(\eta)$$

*and*

$$\frac{\partial}{\partial \bar s^i}\int_{X_s} \eta = \int_{X_s} L_{\bar v_i}(\eta).$$

*Moreover, if* $\omega_{\mathcal{X}}$ *is d-closed, then*

(iii) $[v_i, \bar v_j] = g^{\bar\gamma\alpha}\frac{\partial}{\partial \bar z^\gamma}(g_{v_i\bar v_j})\frac{\partial}{\partial z^\alpha} - g^{\bar\beta\gamma}\frac{\partial}{\partial z^\gamma}(g_{v_i\bar v_j})\frac{\partial}{\partial \bar z^\beta}$*, and*

(iv) $L_{v_i}(\omega_s^n) = 0$.

*Here* $[\,,\,]$ *denotes the Lie bracket of two vector fields, and* $L_{v_i}$ *denotes the Lie derivative in the direction of* $v_i$*, etc.*

*Proof.* (i) follows immediately from (2.1.1) and (2.1.2). (ii), (iii) and (iv) can be found in [Sch, Lemma 2.1], [Sch, Lemma 2.6] and [Sch, Lemma 2.2], respectively. We remark that among the conditions assumed in [Sch], the proofs of [Sch, Lemma 2.6] and [Sch, Lemma 2.2] work under the mere condition that $\omega_{\mathcal{X}}$ is $d$-closed. □

**(2.2).** Now we give the proof of Theorem 1 as follows:

*Proof of Theorem* 1. Let $(\mathcal{E} \to \mathcal{X}, h)$ be a family of Hermitian holomorphic vector bundles of rank $r$ over $(\pi : \mathcal{X} \to S, \omega_{\mathcal{X}})$ as in Theorem 1, where $\dim_{\mathbb{C}} S = m$ and $\dim_{\mathbb{C}} X_s = n$. Here and thereafter, $X_s$ and $\omega_s$ are as in (2.1). For simplicity, we simply write $H := H_{h,\omega_{\mathcal{X}}}$. Choose local holomorphic coordinates $s = (s^1, \ldots, s^m)$ for $S$ so that $\pi_*\mathcal{E}$ is locally free at $s = 0$ and of rank $p$. Choose a holomorphic trivialization $\{t_1, t_2, \ldots, t_p\}$ of $\pi_*\mathcal{E}$ over a coordinate neighborhood $U$ of $S$ containing 0 such that the $L^2$-metric $H$ on $\pi_*\mathcal{E}$ satisfies

$$\frac{\partial H_{a\bar b}}{\partial s^i}\Big|_{s=0} = 0 \quad \text{for } 1 \le a, b \le p,\ 1 \le i \le m. \tag{2.2.1}$$

Here $H_{a\bar b} := H(t_a, t_b)$. In the sequel, covariant derivatives will be with respect to the Hermitian connection on $(\mathcal{E}, h)$. For $1 \le i \le m$, recall from (2.1) the horizontal lifting $v_i$ of $\partial/\partial s^i$ with respect to $\omega_{\mathcal{X}}$. With respect to the above trivialization, the curvature tensor $\Theta$ of $(\pi_*\mathcal{E}, H)$ at $s=0$ is given by

$$
\begin{aligned}
\Theta_{a\bar{b}i\bar{j}}(0) &= -\frac{\partial^2 H_{a\bar{b}}}{\partial s^i \partial \bar{s}^j}\Big|_{s=0} \\
&= -\frac{\partial^2}{\partial s^i \partial \bar{s}^j}\int_{X_s} \langle t_a, t_b\rangle \frac{\omega_s^n}{n!}\Big|_{s=0} \\
&= -\frac{\partial}{\partial \bar{s}^j}\int_{X_s} L_{v_i}\big(\langle t_a, t_b\rangle \frac{\omega_s^n}{n!}\big)\Big|_{s=0} \quad \text{(by Lemma 2.1.1(ii))} \\
&= -\frac{\partial}{\partial \bar{s}^j}\int_{X_s} L_{v_i}\big(\langle t_a, t_b\rangle\big) \frac{\omega_s^n}{n!}\Big|_{s=0} \quad \text{(by Lemma 2.1.1(iv))} \\
&= -\frac{\partial}{\partial \bar{s}^j}\int_{X_s} \langle \nabla_{v_i} t_a, t_b\rangle \frac{\omega_s^n}{n!}\Big|_{s=0} \quad \text{(by holomorphicity of } t_b) \\
&= -\int_{X_s} L_{\bar{v}_j}\big(\langle \nabla_{v_i} t_a, t_b\rangle\big) \frac{\omega_s^n}{n!}\Big|_{s=0} \quad \text{(by Lemma 2.1.1(ii) and (iv) again)} \\
&= -\int_{X_s} \langle \nabla_{\bar{v}_j}\nabla_{v_i} t_a, t_b\rangle \frac{\omega_s^n}{n!}\Big|_{s=0} - \int_{X_s} \langle \nabla_{v_i} t_a, \nabla_{v_j} t_b\rangle \frac{\omega_s^n}{n!}\Big|_{s=0} \\
&=: I_1 + I_2.
\end{aligned} \tag{2.2.2}
$$

First we deal with the integral $I_2$. Denote by $H_{\mathcal{E}_0}$ the harmonic projection operator on $\mathcal{E}_0$ with respect to the $\bar{\partial}$-Laplacian. The Hodge decomposition theorem gives, at $s=0$,

$$
\nabla_{v_i} t_a = H_{\mathcal{E}_0}(\nabla_{v_i} t_a) + \Box\, G_{\mathcal{E}_0}(\nabla_{v_i} t_a), \tag{2.2.3}
$$

where $G_{\mathcal{E}_0}$ is as in (1.1). As in (2.2.2), it follows from (2.2.1) that we have

$$
\int_{X_0} \langle \nabla_{v_i} t_a, t_b\rangle \frac{\omega_0^n}{n!} = 0 \quad \text{for } 1 \le a, b \le p,\ 1 \le i \le m. \tag{2.2.4}
$$

Since $\{t_b\}_{1\le b\le p}$ is a basis of $H^0(X_0, \mathcal{E}_0)$, and $H_{\mathcal{E}_0}(\nabla_{v_i} t_a) \in H^0(X_0, \mathcal{E}_0)$, it follows from (2.2.4) that

$$
H_{\mathcal{E}_0}(\nabla_{v_i} t_a) = 0 \quad \text{for } 1 \le a \le p,\ 1 \le i \le m. \tag{2.2.5}
$$

Combining (2.2.3) and (2.2.5), we have at $s=0$,

$$
\begin{aligned}
\int_{X_0} \langle \nabla_{v_i} t_a, \nabla_{v_j} t_b\rangle \frac{\omega_0^n}{n!} &= \int_{X_0} \langle \Box G_{\mathcal{E}_0}(\nabla_{v_i} t_a), \nabla_{v_j} t_b\rangle \frac{\omega_0^n}{n!} \\
&= \int_{X_0} \langle G_{\mathcal{E}_0}\bar{\partial}(\nabla_{v_i} t_a), \bar{\partial}(\nabla_{v_j} t_b)\rangle \frac{\omega_0^n}{n!} \\
&\qquad \text{(since } [G_{\mathcal{E}_0}, \bar{\partial}] = 0, \text{ and } \bar{\partial}^*(\nabla_{v_i} t_a) = 0 \text{ trivially).}
\end{aligned} \tag{2.2.6}
$$

Taking $\bar{\partial}$ along the $X_0$ direction, we have, from the Ricci identity,

$$
\begin{aligned}
\bar{\partial}(\nabla_{v_i} t_a) &= (\nabla_{\bar{\alpha}}\nabla_{v_i} t_a) d\bar{z}^\alpha \\
&= \big(\nabla_{v_i}\nabla_{\bar{\alpha}} t_a - \nabla_{[v_i, \partial/\partial \bar{z}^\alpha]} t_a - \Omega_{v_i\bar{\alpha}}(t_a)\big) d\bar{z}^\alpha \\
&= \big(0 + A^\gamma_{i\bar{\alpha}}\nabla_\gamma t_a - \Omega_{v_i\bar{\alpha}}(t_a)\big) d\bar{z}^\alpha \\
&\qquad \text{(by holomorphicity of } t_a \text{ and Lemma 2.1.1(i))} \\
&= -\Omega_{v_i\bar{\alpha}}(t_a) d\bar{z}^\alpha + A^\gamma_{i\bar{\beta}} t_{a;\gamma} d\bar{z}^\beta
\end{aligned} \tag{2.2.7}
$$

using the semicolon notation. Combining (2.2.6) and (2.2.7), we have

(2.2.8)
$$I_2 = -\int_{X_0} \langle G_{\mathcal{E}_0}(-\Omega_{v_i\bar{\alpha}}(t_a)d\bar{z}^\alpha + A^\gamma_{i\bar{\beta}}t_{a;\gamma}d\bar{z}^\beta), -\Omega_{v_j\bar{\alpha}}(t_b)d\bar{z}^\alpha + A^\gamma_{j\bar{\beta}}t_{b;\gamma}d\bar{z}^\beta\rangle \frac{\omega_0^n}{n!}.$$

Next we consider the integral $I_1$. From the Ricci identity, we have

$$\begin{aligned}
\nabla_{\bar{v}_j}\nabla_{v_i}t_a =& \nabla_{v_i}\nabla_{\bar{v}_j}t_a - \nabla_{[v_i,\bar{v}_j]}t_a - \Omega_{v_i\bar{v}_j}(t_a)\\
=& \nabla_{v_i}\nabla_{\bar{v}_j}t_a - g^{\bar{\gamma}\alpha}\frac{\partial}{\partial\bar{z}^\gamma}(g_{v_i\bar{v}_j})\nabla_\alpha t_a + g^{\bar{\beta}\gamma}\frac{\partial}{\partial z^\gamma}(g_{v_i\bar{v}_j})\nabla_{\bar{\beta}}t_a - \Omega_{v_i\bar{v}_j}(t_a)\\
&\quad \text{(by Lemma 2.1.1(iii))}\\
=& -g^{\bar{\gamma}\alpha}\frac{\partial}{\partial\bar{z}^\gamma}(g_{v_i\bar{v}_j})\nabla_\alpha t_a - \Omega_{v_i\bar{v}_j}(t_a)\\
&\quad \text{(by holomorphicity of } t_a).
\end{aligned} \tag{2.2.9}$$

By (2.2.2) and (2.2.9), we have

$$\begin{aligned}
I_1 =& \int_{X_0} \langle g^{\bar{\gamma}\alpha}\frac{\partial}{\partial\bar{z}^\gamma}(g_{v_i\bar{v}_j})\nabla_\alpha t_a + \Omega_{v_i\bar{v}_j}(t_a), t_b\rangle\frac{\omega_0^n}{n!}\\
=& \int_{X_0} g^{\bar{\gamma}\alpha}\frac{\partial}{\partial\bar{z}^\gamma}(g_{v_i\bar{v}_j})\overline{\frac{\partial}{\partial\bar{z}^\alpha}\overline{\langle t_a, t_b\rangle}}\,\frac{\omega_0^n}{n!} + \int_{X_0}\langle\Omega_{v_i\bar{v}_j}(t_a), t_b\rangle\frac{\omega_0^n}{n!}\\
&\quad \text{(by holomorphicity of } t_b)\\
=& \int_{X_0}\langle\bar{\partial}(g_{v_i\bar{v}_j}), \bar{\partial}(\langle t_a, t_b\rangle)\rangle\,\frac{\omega_0^n}{n!} + \int_{X_0}\langle\Omega_{v_i\bar{v}_j}(t_a), t_b\rangle\frac{\omega_0^n}{n!}\\
=& \int_{X_0}\square(g_{v_i\bar{v}_j})\cdot\langle t_a, t_b\rangle\frac{\omega_0^n}{n!} + \int_{X_0}\langle\Omega_{v_i\bar{v}_j}(t_a), t_b\rangle\frac{\omega_0^n}{n!},
\end{aligned} \tag{2.2.10}$$

where the last line follows from the definition $\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ and the adjoint property of $\bar{\partial}^*$. Then (1.1.5) follows immediately from (2.2.2), (2.2.8) and (2.2.10), and we have finished the proof of Theorem 1. □

**(2.3).** Notation as in (1.1). In this subsection, we are going to interpret the terms $\Omega_{v_i\bar{\beta}}d\bar{z}^\beta \in \mathcal{A}^{0,1}(X_s, \mathrm{End}(\mathcal{E}_s))$ and $A^\alpha_{i\bar{\beta}}\frac{\partial}{\partial z^\alpha}\otimes d\bar{z}^\beta \in \mathcal{A}^{0,1}(X_s, T_s)$ which appear in (1.1.5) in Theorem 1 in terms of Kodaira-Spencer theory.

Let $(p : \mathcal{E} \to \mathcal{X}, h)$ and $(\pi : \mathcal{X} \to S, \omega_{\mathcal{X}})$ be as in Theorem 1. Here $p$ and $p_s$ denote the projection maps. For each rank-$r$ holomorphic vector bundle $\mathcal{E}_s := \mathcal{E}\big|_{X_s}$, where $X_s = p^{-1}(s)$, we denote by $P_s$ the associated principal $\mathrm{GL}(r,\mathbb{C})$-bundle over $X_s$. By abuse of notation, we also let $p_s : P_s \to X_s$ denote the projection map. The structure group $\mathrm{GL}(r,\mathbb{C})$ acts naturally on the tangent bundle $TP_s$ of $P_s$, and we denote the quotient by $Q_s := TP_s/\mathrm{GL}(r,\mathbb{C})$. Then $Q_s$ is a holomorphic bundle over $X_s$ (with holomorphic structure induced from $TP_s$) such that $\mathcal{O}_{X_s}(Q_s)$ is isomorphic to the sheaf of germs of $\mathrm{GL}(r,\mathbb{C})$-invariant tangent vector fields on $P_s$. Denote the associated quotient map by $\tau_s : TP_s \to Q_s$. It is easy to see that the differential $p_{s*} : TP_s \to TX_s$ factors through $\tau_s$ to a bundle epimorphism $q_s : Q_s \to TX_s$, i.e., one has $p_{s*} = q_s \circ \tau_s$. By [A, Theorem 1 and Proposition 9], there exists a short exact sequence of holomorphic vector bundles

$$0 \to \mathrm{End}(\mathcal{E}_s) \xrightarrow{i} Q_s \xrightarrow{q_s} TX_s \to 0 \tag{2.3.1}$$

over $X_s$, which is usually known as the Atiyah sequence associated to $\mathcal{E}_s$. Similarly, for the vector bundle $\mathcal{E}$ over $\mathcal{X}$, we have the associated principal $\mathrm{GL}(r,\mathbb{C})$-bundle

$p : P \to \mathcal{X}$, the quotient $Q := TP/\mathrm{GL}(r,\mathbb{C})$, the quotient map $\tau : TP \to Q$ and $q : Q \to T\mathcal{X}$ with $p_* = q \circ \tau$, and the Atiyah sequence

$$0 \to \mathrm{End}(\mathcal{E}) \xrightarrow{i} Q \xrightarrow{q} T\mathcal{X} \to 0 \tag{2.3.2}$$

over $\mathcal{X}$. Of course, the restriction of (2.3.2) to $X_s$ simply yields (2.3.1) for each $s \in S$. The Hermitian connection of $(\mathcal{E}, h)$ induces an associated principal connection on $P$ which is a $gl(r,\mathbb{C})$-valued (1,0)-form $\tilde{\theta}$ on $P$ invariant under $\mathrm{GL}(r,\mathbb{C})$. By identifying the fibers of $\mathrm{End}(\mathcal{E})$ with $gl(r,\mathbb{C})$, $\tilde{\theta}$ descends to a homomorphism $\theta : Q \to \mathrm{End}(\mathcal{E})$. Moreover $\theta \circ i = \mathrm{Id}_{\mathcal{E}}$, which leads to a smooth splitting of (2.3.1) (see [A] for the above discussion).

Also consider the following short exact sequence of vector bundles

$$0 \to T_{\mathcal{X}|S} \to T\mathcal{X} \to \pi^* TS \to 0 \tag{2.3.3}$$

over $\mathcal{X}$. Here $T_{\mathcal{X}|S}$ denotes the relative tangent bundle over $\mathcal{X}$ associated to the family $\pi : \mathcal{X} \to S$. For each $s \in S$, we have the Kodaira-Spencer map $\rho_s : T_sS \to H^1(X_s, TX_s)$ associated to (2.3.3). Here we do not distinguish between a holomorphic vector bundle and its sheaf of germs of holomorphic sections. From (2.3.2) and (2.3.3), we may regard $\pi^*TS$ as a quotient bundle of $Q$, and this leads to the following short exact sequence of vector bundles

$$0 \to Q_{\mathcal{X}|S} \to Q \to \pi^* TS \to 0 \tag{2.3.4}$$

over $\mathcal{X}$, where $Q_{\mathcal{X}|S} \subset Q$ is the associated relative vector bundle. For each $s \in S$, we similarly have the Kodaira-Spencer map $\hat{\rho}_s : T_sS \to H^1(X_s, Q_s)$ associated to (2.3.4) (cf. [FS, §4] for the case of line bundles). We also let $Z^1(X_s, TX_s)$ (resp. $Z^1(X_s, Q_s)$) denote the space of $\bar{\partial}-$closed $TX_s$ (resp. $Q_s$)-valued (0,1)-forms on $X_s$. Denote by $s = (s^i)_{1\le i\le m}$ local holomorphic coordinates for $S$. We recall from (1.1) the horizontal lifting $v_i$ of $\partial/\partial s^i$ with respect to $\omega_{\mathcal{X}}$, and let $A_{\partial/\partial s^i} = \bar{\partial} v_i$ be as in (1.1.3). Obviously, $A_{\partial/\partial s^i} \in Z^1(X_s, TX_s)$. It is well known that $A_{\partial/\partial s^i}$ is a representative of the Kodaira-Spencer class of $\rho_s(\partial/\partial s^i)$, i.e. the Dolbeault cohomology class $[A_{\partial/\partial s^i}]$ of $A_{\partial/\partial s^i}$ satisfies $[A_{\partial/\partial s^i}] = \rho_s(\partial/\partial s^i)$ in $H^1(X_s, TX_s)$ via the Dolbeault isomorphism. Note however that $A_{\partial/\partial s^i}$, in general, may not be harmonic (see e.g. [Sch, §1]). Let $\partial/\partial s^i$ and $v_i$ be as above. One can further lift $v_i$ uniquely to a smooth vector field $\tilde{v}_i$ on $P$ such that

$$p_*(\tilde{v}_i) = v_i \quad \text{and} \quad \tilde{v}_i \in \mathrm{Ker}(\tilde{\theta}) \tag{2.3.5}$$

(cf. e.g. [KN, p. 65]). Since $\tilde{\theta}$ is $\mathrm{GL}(r,\mathbb{C})$-invariant, it is easy to see that $\tilde{v}_i$ is also $\mathrm{GL}(r,\mathbb{C})$-invariant and thus $\tilde{v}_i$ descends to a smooth section $\hat{v}_i$ of $Q$. Define

$$\hat{A}_{\partial/\partial s^i} := \bar{\partial} \hat{v}_i \tag{2.3.6}$$

along each fiber $X_s$. Then $\hat{A}_{\partial/\partial s^i} \in Z^1(X_s, Q_s)$ for $s \in S$. Using the argument in [FS, Lemma 4.4] (which deals with the case of line bundles), one sees that $\hat{A}_{\partial/\partial s^i}$ is a $\bar{\partial}$-closed representative of the Kodaira-Spencer class $\hat{\rho}_s(\partial/\partial s^i)$, i.e. the Dolbeault cohomology class $[\hat{A}_{\partial/\partial s^i}]$ of $\hat{A}_{\partial/\partial s^i}$ satisfies $[\hat{A}_{\partial/\partial s^i}] = \hat{\rho}_s(\partial/\partial s^i)$ in $H^1(X_s, Q_s)$ via the Dolbeault isomorphism. Moreover, it follows easily from (2.3.5) and the holomorphicity of $p_{s*}$ and $\tau_s$ that

$$q_s(\hat{A}_{\partial/\partial s^i}) = A_{\partial/\partial s^i} \quad \text{for } s \in S \tag{2.3.7}$$

(cf. also [FS, Lemma 4.6]). Next we consider $\theta_s(\hat{A}_{\partial/\partial s^i}) \in \mathcal{A}^{0,1}(X_s, \mathrm{End}(\mathcal{E}_s))$. Let $\Omega \in \mathcal{A}^{1,1}(\mathcal{X}, \mathrm{End}(\mathcal{E}))$ be as in (1.1). We have

**Lemma 2.3.1.** *For $s \in S$, $\theta_s(\hat{A}_{\partial/\partial s^i}) = \Omega_{v_i\bar{\alpha}} d\bar{z}^\alpha \in \mathcal{A}^{0,1}(X_s, End(\mathcal{E}_s))$.*

*Proof.* Let $\partial/\partial\bar{z}^\alpha$ be a local coordinate tangent vector field on $\mathcal{X}$ of type (0,1). We lift $\partial/\partial\bar{z}^\alpha$ to a $\mathrm{GL}(r,\mathbb{C})$-invariant (0,1)-tangent vector field $\widetilde{\partial/\partial\bar{z}^\alpha}$ such that $p_*(\widetilde{\partial/\partial\bar{z}^\alpha}) = \partial/\partial\bar{z}^\alpha$. From the structure equation of $\tilde{\theta}$ (cf. e.g. [KN, pp. 77-78]), we have

$$
\begin{aligned}
d\tilde{\theta}(\tilde{v}_i, \widetilde{\partial/\partial\bar{z}^\alpha}) =& -\frac{1}{2}[\tilde{\theta}(\tilde{v}_i), \widetilde{\partial/\partial\bar{z}^\alpha}] + (p^*\Omega)(\tilde{v}_i, \widetilde{\partial/\partial\bar{z}^\alpha}) \\
=&(p^*\Omega)(\tilde{v}_i, \widetilde{\partial/\partial\bar{z}^\alpha}) \quad \text{(by (2.3.5))} \\
=&\Omega(p_*\tilde{v}_i, p_*\widetilde{\partial/\partial\bar{z}^\alpha}) \\
=&\Omega(v_i, \partial/\partial\bar{z}^\alpha) \\
=&\Omega_{v_i\bar{\alpha}}.
\end{aligned}
\tag{2.3.8}
$$

From consideration of type, we have $\tilde{\theta}(\tilde{v}_i) = 0$ and $\tilde{\theta}(\tilde{v}_i(\widetilde{\partial/\partial\bar{z}^\alpha})) = 0$. Together with (2.3.5), we have

$$
\begin{aligned}
d\tilde{\theta}(\tilde{v}_i, \widetilde{\partial/\partial\bar{z}^\alpha}) =&\tilde{v}_i(\tilde{\theta}(\widetilde{\partial/\partial\bar{z}^\alpha})) - \widetilde{\partial/\partial\bar{z}^\alpha}(\tilde{\theta}(\tilde{v}_i)) - \tilde{\theta}([\tilde{v}_i, \widetilde{\partial/\partial\bar{z}^\alpha}]) \\
=&\tilde{\theta}(\widetilde{\partial/\partial\bar{z}^\alpha}(\tilde{v}_i)) \\
=&\theta(\partial/\partial\bar{z}^\alpha(\hat{v}_i)),
\end{aligned}
\tag{2.3.9}
$$

where the last equality follows from the descendence of $\tilde{\theta}, \widetilde{\partial/\partial\bar{z}^\alpha}, \tilde{v}_i$ to $\theta, \partial/\partial\bar{z}^\alpha, \hat{v}_i$ respectively (as a result of their $\mathrm{GL}(r,\mathbb{C})$-invariance). We finally have

$$
\begin{aligned}
\theta_s(\hat{A}_{\partial/\partial s^i}) =&\theta_s(\partial/\partial\bar{z}^\alpha(\hat{v}_i)d\bar{z}^\alpha) \quad \text{(by (2.3.6))} \\
=&\theta_s(\partial/\partial\bar{z}^\alpha(\hat{v}_i))d\bar{z}^\alpha \\
=&\Omega_{v_i\bar{\alpha}}d\bar{z}^\alpha \quad \text{(by (2.3.8) and (2.3.9))}.
\end{aligned}
$$

This finishes the proof of Lemma 2.3.1. □

For $k, l \geq 0$, we also denote the extension of the Hermitian connection $\nabla$ of $(\mathcal{E}_s, h_s)$ induced from the tensor product $\mathcal{A}^{k,l}(X_s, TX_s) = \mathcal{A}^{k,l}(X_s) \otimes_{\mathcal{A}^{0,0}(X_s)} \mathcal{A}^{0,0}(X_s, TX_s)$ by the same symbol $\nabla : \mathcal{A}^{k,l}(X_s, TX_s) \times \mathcal{A}^{0,0}(X_s, \mathcal{E}_s) \to \mathcal{A}^{k,l}(X_s, \mathcal{E}_s)$. We summarize our discussion in this subsection in the following

**Proposition 2.3.2.** *In terms of the $\bar{\partial}$-closed representative $\hat{A}_{\partial/\partial s^i}$ of the Kodaira-Spencer class $\hat{\rho}(\partial/\partial s^i)$ in $H^1(X_s, Q_s)$, the curvature formula* (1.1.5) *of Theorem* 1 *can be re-written as follows:*

$$
\begin{aligned}
&\Theta_{a\bar{b}i\bar{j}}(s) \\
=& -\int_{X_s} \langle G_{\mathcal{E}_s}\big(\theta_s(\hat{A}_{\partial/\partial s^i})(t_a) - \nabla_{q_s(\hat{A}_{\partial/\partial s^i})}t_a\big), \theta_s(\hat{A}_{\partial/\partial s^j})(t_b) - \nabla_{q_s(\hat{A}_{\partial/\partial s^j})}t_b\rangle \frac{\omega_s^n}{n!} \\
&+ \int_{X_s} \langle \Omega_{v_i\bar{v}_j}(t_a), t_b\rangle \frac{\omega_s^n}{n!} + \int_{X_s} \Box(g_{v_i\bar{v}_j})\langle t_a, t_b\rangle \frac{\omega_s^n}{n!}.
\end{aligned}
$$

*Proof.* Proposition 2.3.2 follows immediately from (1.1.5), (2.3.7) and Lemma 2.3.1. □

## §3. Proof of Corollary 2

**(3.1).** Notation as in (1.1) and (1.2). We deduce Corollary 2 from Theorem 1 as follows:

*Proof of Corollary* 2. Let $(\mathcal{L} \to \mathcal{X}, h)$ and $(\pi : \mathcal{X} \to S, \omega_{\mathcal{X}})$ be as in Corollary 2, so that (1.2.1) is satisfied for some $k \in \mathbb{R}$. Denote by $s = (s^i)_{1 \le i \le m}$ and $z = (z^\alpha)_{1 \le \alpha \le n}$ the local holomorphic coordinates for $S$ and the fibers $X_s$, respectively. Then (1.2.1) implies that $\omega_{\mathcal{X}}$ is $d$-closed, and thus Theorem 1 applies. By the construction in (1.1) (or (2.1.1)), the horizontal lifting $v_i$ of a coordinate tangent vector field $\partial/\partial s^i$ in $S$ necessarily satisfies $g_{v_i \bar{\alpha}} = 0$. Together with (1.2.2), we have

$$\Omega_{v_i \bar{v}_j} = k g_{v_i \bar{v}_j} \quad \text{and} \quad \Omega_{v_i \bar{\alpha}} = k g_{v_i \bar{\alpha}} = 0. \tag{3.1.1}$$

Then (1.2.3) follows immediately from (1.1.5) and (3.1.1), and we have finished the proof of Corollary 2. □

## §4. $L^2$-metrics and families of abelian varieties

**(4.1).** In this subsection, we make some simple observations on the $L^2$-metrics on direct image bundles associated to families of ample line bundles over families of abelian varieties. These will be needed in the proof of Theorem 3 in §5.

Let $\pi : \mathcal{A} \to S$ be a family of abelian varieties parametrized by a complex manifold $S$, and let $\mathcal{L} \to \mathcal{A}$ be a family of ample line bundles as in Theorem 3. Recall that for any smooth family of Hermitian metrics $\rho$ on the family of holomorphic line bundles $\mathcal{L} \to \mathcal{A}$ and any smooth family of Kähler metrics $\omega_{\mathcal{A}}$ on the family $\pi : \mathcal{A} \to S$, one has an associated $L^2$-metric $H_{\rho,\omega_{\mathcal{A}}}$ on $\pi_*\mathcal{L}$ as defined in (1.1.1). For a holomorphic line bundle $F$ over $S$ and a Hermitian metric $h$ on $F$, it is easy to see that $(\mathcal{L} \otimes \pi^* F, \rho \otimes \pi^* h)$ forms a family of Hermitian holomorphic line bundles. Thus one also has the associated $L^2$-metric $H_{\rho\otimes\pi^* h,\omega_{\mathcal{A}}}$ on $\pi_*(\mathcal{L} \otimes \pi^* F)$.

**Proposition 4.1.1.** *Let $\omega_{\mathcal{A}}$ and $\omega'_{\mathcal{A}}$ be two smooth families of flat Kähler metrics on the family $\pi : \mathcal{A} \to S$, and let $\rho$ and $\rho'$ be two smooth families of canonical Hermitian metrics on the family of ample line bundles $\mathcal{L} \to \mathcal{A}$ (cf. (1.3)). Also let $(F, h)$ be a Hermitian holomorphic line bundle over $S$. Then the following statements hold:*

(i) *$(\pi_*(\mathcal{L} \otimes \pi^* F), H_{\rho\otimes\pi^* h,\omega_{\mathcal{A}}})$ is projectively flat if and only if $(\pi_*\mathcal{L}, H_{\rho,\omega_{\mathcal{A}}})$ is projectively flat.*

(ii) *$(\pi_*\mathcal{L}, H_{\rho,\omega_{\mathcal{A}}})$ is projectively flat if and only if $(\pi_*\mathcal{L}, H_{\rho',\omega_{\mathcal{A}}})$ is projectively flat.*

(iii) *$(\pi_*\mathcal{L}, H_{\rho,\omega_{\mathcal{A}}})$ is projectively flat if and only if $(\pi_*\mathcal{L}, H_{\rho,\omega'_{\mathcal{A}}})$ is projectively flat.*

*Proof.* For each $s \in S$, one has the isomorphism

$$H^0(A_s, L_s) \otimes F_s \simeq H^0(A_s, L_s \otimes \pi^* F_s)$$

given by $t \otimes f \mapsto t \otimes \pi^* f$ for $t \in H^0(A_s, L_s)$ and $f \in F_s$. It is easy to verify that under the above correspondence, we have the following isometry of Hermitian vector bundles over $S$:

$$\big(\pi_*(\mathcal{L} \otimes \pi^* F), H_{\rho\otimes\pi^* h,\omega_{\mathcal{A}}}\big) \simeq \big(\pi_*\mathcal{L} \otimes F, H_{\rho,\omega_{\mathcal{A}}} \otimes h\big) \tag{4.1.1}$$

(the underlying vector bundle isomorphism is usually known as the projection formula). Denote the curvature tensor of $(\pi_*\mathcal{L}, H_{\rho,\omega_{\mathcal{A}}})$ and $\big(\pi_*(\mathcal{L}\otimes\pi^*F), H_{\rho\otimes\pi^*h,\omega_{\mathcal{A}}}\big)$ by $\Theta^{H_{\rho,\omega_{\mathcal{A}}}}$ and $\Theta^{H_{\rho\otimes\pi^*h,\omega_{\mathcal{A}}}}$, respectively. Under the identification

$$\mathrm{End}(\pi_*(\mathcal{L}\otimes\pi^*F)) \simeq \mathrm{End}(\pi_*\mathcal{L})$$

induced by the isomorphism $\mathrm{End}(F)\simeq\mathcal{O}_S$, it follows from (4.1.1) that

$$\Theta^{H_{\rho\otimes\pi^*h,\omega_{\mathcal{A}}}} = \Theta^{H_{\rho,\omega_{\mathcal{A}}}} + \frac{2\pi}{\sqrt{-1}}c_1(F,h)\otimes \mathrm{Id}_{\pi_*\mathcal{L}}. \tag{4.1.2}$$

Then Proposition 4.1.1(i) follows from (4.1.2) and the definition in (1.3.1). To prove Proposition 4.1.1(ii), we observe that canonical Hermitian metrics on holomorphic line bundles over abelian varieties are unique up to a multiplicative constant. This implies that $\rho' = e^{\pi^*\lambda}\cdot\rho$ for some smooth function $\lambda$ on $S$. Regarding $e^{\lambda}$ as a Hermitian metric on the trivial line bundle $\mathcal{O}_S$, one sees that Proposition 4.1.1(i) (with $(F,h)\simeq(\mathcal{O}_S,e^{\lambda})$) readily implies Proposition 4.1.1(ii). To prove Proposition 4.1.1(iii), we observe that the volume forms associated to flat Kähler metrics on an abelian variety are constant multiples of each other. This implies that $H_{\rho,\omega'_{\mathcal{A}}} = e^{\mu}\cdot H_{\rho,\omega_{\mathcal{A}}}$ for some smooth function $\mu$ on $S$. Then a simple calculation similar to (4.1.2) readily leads to Proposition 4.1.1(iii), and this finishes the proof of Proposition 4.1.1. □

**(4.2).** In this subsection, we are going to give a simplification of the formula (1.2.3) in Corollary 2 under the hypothesis of Theorem 3 (cf. Proposition 4.2.4). This does not lead directly to Theorem 3 yet, and the conclusion of Theorem 3 will only be arrived in §5.

Throughout (4.2), we let $\pi:\mathcal{A}\to S$, $\mathcal{L}\to\mathcal{A}$, $\rho=\{\rho_s\}_{s\in S}$ and $\omega_{\mathcal{A}}$ be as in Theorem 3. Furthermore, we assume that

$$c_1(\mathcal{L},\rho) = \frac{1}{2\pi}\omega_{\mathcal{A}} \quad \text{on } \mathcal{A}, \tag{4.2.1}$$

i.e. (1.2.1) is satisfied with $k=1$. As in (4.1), we have the associated $L^2$-metric $H_{\rho,\omega_{\mathcal{A}}}$ on $\pi_*\mathcal{L}$. For $s\in S$ and a coordinate tangent vector $\partial/\partial s^i\in T_sS$, we let $v_i$ be the horizontal lifting of $\partial/\partial s^i$ with respect to $\omega_{\mathcal{A}}$ as in (1.1), and $A_{\partial/\partial s^i} = A^{\alpha}_{i\bar\beta}\frac{\partial}{\partial z^\alpha}\otimes d\bar z^\beta \in A^{0,1}(A_s, TA_s)$ be the associated Kodaira-Spencer representative as in (1.1.3) and (1.1.4). Denote also by $g$ the metric tensor associated to $\omega_{\mathcal{A}}$ as in §1. First we have

**Lemma 4.2.1.** *For $s\in S$ and $\partial/\partial s^i\in T_sS$,*

(i) *$A_{\partial/\partial s^i}$ is invariant under translations of $A_s$. In particular, $A_{\partial/\partial s^i}$ is parallel with respect to $\omega_s$.*

(ii) *We have $A^{\alpha}_{i\bar\beta}\,g_{\alpha\bar\gamma} = A^{\alpha}_{i\bar\gamma}\,g_{\alpha\bar\beta}$ for $1\le\beta,\gamma\le n$.*

*Proof.* Being a flat metric, each $\omega_s$ is Kähler-Einstein. Then by [Sch, Proposition 1.1], $A_{\partial/\partial s^i}$ is a harmonic tensor (with respect to $\omega_s$). Then it follows from standard facts on compact complex tori that $A_{\partial/\partial s^i}$ is necessarily invariant under translations of $A_s$ (cf. e.g. [GH, p. 302]), which, in turn, implies readily that $A_{\partial/\partial s^i}$ is parallel with respect to $\omega_s$. This proves Lemma 4.2.1(i). Lemma 4.2.1(ii) follows from the Kähler-Einstein condition of $\omega_s$, and it can also be found in [Sch, Proposition 1.1]. □

For $s \in S$, $\partial/\partial s^i \in T_s(A_s)$ and $t \in H^0(A_s, L_s)$, we let $\nabla_{A_{\partial/\partial s^i}} t \in A^{0,1}(A_s, L_s)$ be as interpreted in Proposition 2.3.2 (cf. also 2.3.6). In terms of local holomorphic coordinates $(z^\alpha)_{1\leq\alpha\leq n}$ on $A_s$, we have $\nabla_{A_{\partial/\partial s^i}} t = A^\alpha_{i\bar\beta} t_{;\alpha} d\bar z^\beta$ (cf. (1.1.4)).

**Lemma 4.2.2.** *On $A_s$, we have $\Box(\nabla_{A_{\partial/\partial s^i}} t) = 2\nabla_{A_{\partial/\partial s^i}} t$, i.e.,*

$$\Box(A^\alpha_{i\bar\beta} t_{;\alpha} d\bar z^\beta) = 2\, A^\alpha_{i\bar\beta} t_{;\alpha} d\bar z^\beta.$$

*Proof.* By the construction of $v_i$, we have $g_{v_i\bar\alpha} (= g(v_i, \partial/\partial z^\alpha)) = 0$. Together with (4.2.1) and the identity (2.2.7) in the proof of Theorem 1, it follows that $A^\alpha_{i\bar\beta} t_{;\alpha} d\bar z^\beta$ is $\bar\partial$-exact, and thus $\bar\partial(A^\alpha_{i\bar\beta} t_{;\alpha} d\bar z^\beta) = 0$. Then

$$\begin{aligned}
&\Box(A^\alpha_{i\bar\beta} t_{;\alpha} d\bar z^\beta)\\
=&\bar\partial\bar\partial^*(A^\alpha_{i\bar\beta} t_{;\alpha} d\bar z^\beta)\\
=&\bar\partial\big(-g^{\bar\beta\gamma}\nabla_\gamma(A^\alpha_{i\bar\beta} t_{;\alpha})\big)\\
=&\bar\partial(-g^{\bar\beta\gamma} A^\alpha_{i\bar\beta} t_{;\alpha\gamma}) \quad (\text{since } A^\alpha_{i\bar\beta;\gamma} = 0 \text{ by Lemma 4.2.1(i)})\\
=&-g^{\bar\beta\gamma} A^\alpha_{i\bar\beta} t_{;\alpha\gamma\bar\delta} d\bar z^\delta \quad (\text{since } g^{\bar\beta\gamma}{}_{;\bar\delta} = 0,\ A^\alpha_{i\bar\beta;\bar\delta} = 0 \text{ by Lemma 4.2.1(i)})\\
=&-g^{\bar\beta\gamma} A^\alpha_{i\bar\beta}(t_{;\alpha\bar\delta\gamma} - g_{\gamma\bar\delta} t_{;\alpha}) d\bar z^\delta\\
&\qquad (\text{by Ricci identity, (4.2.1) and flatness of } \omega_s)\\
=&-g^{\bar\beta\gamma} A^\alpha_{i\bar\beta}(t_{;\bar\delta\alpha\gamma} - (g_{\alpha\bar\delta} t)_{;\gamma} - g_{\gamma\bar\delta} t_{;\alpha}) d\bar z^\delta\\
&\qquad (\text{by Ricci identity and (4.2.1)})\\
=&-g^{\bar\beta\gamma} A^\alpha_{i\bar\beta}(-g_{\alpha\bar\delta} t_{;\gamma} - g_{\gamma\bar\delta} t_{;\alpha}) d\bar z^\delta\\
&\qquad (\text{since } t \text{ is holomorphic and } g_{\alpha\bar\delta;\gamma} = 0)\\
=&g^{\bar\beta\gamma} A^\alpha_{i\bar\beta} g_{\alpha\bar\delta} t_{;\gamma} d\bar z^\delta + A^\alpha_{i\bar\delta} t_{;\alpha} d\bar z^\delta\\
=&g^{\bar\beta\gamma} A^\alpha_{i\bar\delta} g_{\alpha\bar\beta} t_{;\gamma} d\bar z^\delta + A^\alpha_{i\bar\delta} t_{;\alpha} d\bar z^\delta \quad (\text{by Lemma 4.2.1(ii)})\\
=&2A^\alpha_{i\bar\delta} t_{;\alpha} d\bar z^\delta.
\end{aligned}$$

Thus we have finished the proof of Lemma 4.2.2. □

Next we fix a Euclidean coordinate system $(z^\alpha)_{1\leq\alpha\leq n}$ on $A_s$ so that each $\partial/\partial z^\alpha$ can be considered as a global translation-invariant vector field on $A_s$.

**Lemma 4.2.3.** *For $s \in S$, $t_a$, $t_b \in H^0(A_s, L_s)$ and $\partial/\partial z^\alpha$, $\partial/\partial z^\beta$ as above, there exists a constant $\Xi_{\alpha\bar\beta}$ depending only on $\partial/\partial z^\alpha$ and $\partial/\partial z^\beta$ such that*

$$\int_{A_s} \langle t_{a;\alpha}, t_{b;\beta}\rangle \frac{\omega_s^n}{n!} = \Xi_{\alpha\bar\beta} \cdot H_{\rho,\omega_{\mathcal A}}(t_a, t_b).$$

*Proof.* Fix a Euclidean coordinate system $(z^\alpha)_{1\leq\alpha\leq n}$ on $A_s$ as above. Since $\omega_s$ is flat, we have

$$\omega_s^n = C \cdot \big(\frac{\sqrt{-1}}{2}\big)^n \cdot dz^1 \wedge d\bar z^1 \wedge \cdots \wedge dz^n \wedge d\bar z^n \quad \text{on} \quad A_s,$$

where $C$ is some positive constant. Observe that for $1 \leq \alpha \leq n$, $dz^\alpha, d\bar z^\alpha$ are global 1-forms on $A_s$. Then for $t_a$, $t_b \in H^0(A_s, L_s)$,

$$\Phi := \langle t_{a;\alpha}, t_b\rangle \cdot C \cdot \big(\frac{\sqrt{-1}}{2}\big)^n \cdot dz^1 \wedge d\bar z^1 \wedge \cdots \widehat{\wedge dz^\beta} \wedge d\bar z^\beta \cdots \wedge dz^n \wedge d\bar z^n$$

is a global $(2n-1)$-form on $A_s$. Here $\widehat{\wedge dz^\beta}$ means that the factor $\wedge dz^\beta$ is omitted. It is easy to see that

$$\begin{aligned}
d\Phi =&\frac{\partial}{\partial \bar{z}^\beta}\big(\langle t_{a;\alpha}, t_b\rangle\big)\cdot \omega_s^n\\
=&\big(\langle t_{a;\alpha\bar{\beta}}, t_b\rangle + \langle t_{a;\alpha}, t_{b;\beta}\rangle\big)\omega_s^n\\
=&\big(\langle t_{a;\bar{\beta}\alpha} - g_{\alpha\bar{\beta}}t_a, t_b\rangle + \langle t_{a;\alpha}, t_{b;\beta}\rangle\big)\omega_s^n\\
&\quad\text{(by Ricci identity and (4.2.1))}\\
=&\big(-g_{\alpha\bar{\beta}}\langle t_a, t_b\rangle + \langle t_{a;\alpha}, t_{b;\beta}\rangle\big)\omega_s^n\\
&\quad\text{(by holomorphicity of } t_a).
\end{aligned}$$

Since $\omega_s$ is flat, $g_{\alpha\bar{\beta}}$ is a (globally) constant function on $A_s$. Then by Stokes' theorem, we have $\int_{A_s} d\Phi = 0$, and thus

$$\int_{A_s}\langle t_{a;\alpha}, t_{b;\beta}\rangle\frac{\omega_s^n}{n!} = g_{\alpha\bar{\beta}}\int_{A_s}\langle t_a, t_b\rangle\frac{\omega_s^n}{n!},$$

which leads to Lemma 4.2.3 with the constant $\Xi_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}}$. □

Now we summarize our discussion in (4.2) in the following

**Proposition 4.2.4.** *Let $\pi : \mathcal{A} \to S, \mathcal{L}, \rho, \omega_{\mathcal{A}}$ be as in Theorem 3. Suppose furthermore that (4.2.1) is satisfied. Then the curvature $\Theta$ of the $L^2$-metric $H_{\rho,\omega_{\mathcal{A}}}$ on $\pi_*\mathcal{L}$ is given by*

$$\Theta_{a\bar{b}i\bar{j}}(s) = C_{i\bar{j}}\cdot H_{\rho,\omega_{\mathcal{A}}}(t_a, t_b) + \int_{A_s}\big(g_{v_i\bar{v}_j} + \Box(g_{v_i\bar{v}_j})\big)\langle t_a, t_b\rangle\frac{\omega_s^n}{n!}$$

*for $s \in S$ and $t_a$, $t_b \in H^0(A_s, L_s)$, where $C_{i\bar{j}}$ is some constant depending on $s$, $\partial/\partial s^i$ and $\partial/\partial s^j$.*

*Proof.* Let $s \in S$, $\partial/\partial s^i$, $\partial/\partial s^j \in T_sS$ and $t_a$, $t_b \in H^0(A_s, L_s)$ be as above. As in Lemma 4.2.3, we fix a Euclidean coordinate system $(z^\alpha)_{1\le\alpha\le n}$ on $A_s$. By Lemma 4.2.2, $\nabla_{A_{\partial/\partial s^i}} t_a = A^\gamma_{i\bar{\alpha}} t_{a;\gamma} d\bar{z}^\alpha$ is an eigenvector of the $\bar{\partial}$-Laplacian $\Box$ on $A_s$ corresponding to the eigenvalue 2. Thus we have

$$G_{L_s}(A^\gamma_{i\bar{\alpha}} t_{a;\gamma} d\bar{z}^\alpha) = \frac{1}{2}A^\gamma_{i\bar{\alpha}} t_{a;\gamma} d\bar{z}^\alpha, \tag{4.2.2}$$

where $G_{L_s}$ is the Green's operator on $L_s$. By Lemma 4.2.2(i) and as in Lemma 4.2.3, the functions $A^\alpha_{i\bar{\beta}}$, $g_{\alpha\bar{\beta}}$ and $g^{\bar{\beta}\alpha}$, $1 \le \alpha, \beta \le n$, are all constant on the entirety of $A_s$. Then by (4.2.2), we have

$$\begin{aligned}
&\int_{A_s}\langle G_{L_s}(A^\gamma_{i\bar{\alpha}} t_{a;\gamma} d\bar{z}^\alpha), A^\delta_{j\bar{\beta}} t_{b;\delta} d\bar{z}^\beta\rangle\frac{\omega_s^n}{n!}\\
=&\int_{A_s}\langle\frac{1}{2}\cdot A^\gamma_{i\bar{\alpha}} t_{a;\gamma} d\bar{z}^\alpha, A^\delta_{j\bar{\beta}} t_{b;\delta} d\bar{z}^\beta\rangle\frac{\omega_s^n}{n!}\\
=&\frac{1}{2}\cdot A^\gamma_{i\bar{\alpha}} A^\delta_{j\bar{\beta}} g^{\bar{\alpha}\beta}\int_{A_s}\langle t_{a;\gamma}, t_{b;\delta}\rangle\frac{\omega_s^n}{n!}\\
=&\frac{1}{2}\cdot A^\gamma_{i\bar{\alpha}} A^\delta_{j\bar{\beta}} g^{\bar{\alpha}\beta}\Xi_{\gamma\bar{\delta}} H_{\rho,\omega_{\mathcal{A}}}(t_a, t_b) \quad \text{(by Lemma 4.2.3)}\\
=&-C_{i\bar{j}}\cdot H_{\rho,\omega_{\mathcal{A}}}(t_a, t_b),
\end{aligned}\tag{4.2.3}$$

where $C_{i\bar{j}} := -\frac{1}{2} \cdot A^{\gamma}_{i\bar{\alpha}} A^{\delta}_{j\bar{\beta}} g^{\bar{\alpha}\beta} \Xi_{\gamma\bar{\delta}}$ is easily seen to be a constant on the entirety of $A_s$ and depending only on $\partial/\partial s^i$ and $\partial/\partial s^j$. Here $\Xi_{\gamma\bar{\delta}}$ is as in Lemma 4.2.3. Then Proposition 4.2.4 follows immediately by combining Corollary 2 (with $k = 1$) and (4.2.3). $\square$

## §5. Proof of Theorem 3

**(5.1).** In this subsection, we are going to deduce Theorem 3 from Corollary 2 under the additional assumption (4.2.1) as embarked in §4. Theorem 3 in full generality will be proved in (5.2). As remarked in §0, the approach taken in this section is largely suggested by the referee.

First we describe a natural smooth foliation on any holomorphic family of $n$-dimensional abelian varieties $\pi : \mathcal{A} \to S$ as follows: Recall that each fiber $A_s = \pi^{-1}(s)$ is of the form $\mathbb{C}^n/\Lambda(s)$ for some lattice $\Lambda(s)$ of $\mathbb{C}^n$. Moreover, over a small open subset $U$ of $S$, we may fix a choice of $\Lambda(s)$ and a choice of generators $e_1(s), \cdots, e_{2n}(s)$ of $\Lambda(s)$, $s \in U$, such that each $e_i(s)$ varies holomorphically on $U$, shrinking $U$ if necessary. For simplicity, we simply denote the corresponding holomorphic family of generators over $U$ by $e_1, \cdots, e_{2n}$. It is easy to see that one has the following holomorphic isomorphism (as families over $U$) given by

$$\mathcal{A}\big|_U \cong (U \times \mathbb{C}^n)/\mathbb{Z}^{2n}, \tag{5.1.1}$$

where $\mathbb{Z}^{2n}$ is identified with the family of lattices generated by $e_1, \cdots, e_{2n}$ over $U$. Next we define a foliation $\mathcal{F}(U \times \mathbb{C}^{2n})$ on $U \times \mathbb{C}^{2n}$, whose leaves are of the form

$$\mathcal{F}_a = \{(s, a_1 e_1(s) + \cdots + a_{2n} e_{2n}(s)) \in U \times \mathbb{C}^{2n} \,\big|\, s \in U\}, \tag{5.1.2}$$

where $a = (a_1, \cdots, a_{2n}) \in \mathbb{R}^{2n}$. In other words, a leaf of the foliation is given by a fixed real linear combination of $e_1, \cdots, e_{2n}$ over $U$. It is easy to see that the foliation $\mathcal{F}(U \times \mathbb{C}^{2n})$ is smooth and invariant under the action of $\mathbb{Z}^{2n}$ in (5.1.1). Moreover it does not depend on the choice of the family of generators $e_1, \cdots, e_{2n}$. Thus it descends to a smooth foliation on $\mathcal{A}\big|_U$, which we denote by $\mathcal{F}(\mathcal{A}\big|_U)$. Moreover, one can easily check that $\mathcal{F}(\mathcal{A}\big|_U)$ agrees with $\mathcal{F}(\mathcal{A}\big|_V)$ on $\mathcal{A}\big|_{U\cap V}$ for any open subsets $U, V \subset S$. It follows that one obtains a smooth foliation $\mathcal{F}(\mathcal{A})$ on $\mathcal{A}$ such that $\mathcal{F}(\mathcal{A})\big|_{\mathcal{A}|_U} = F(\mathcal{A}|_U)$ for any open subset $U \subset S$. Moreover, one easily sees that the leaves of $\mathcal{F}(\mathcal{A})$ are holomorphic, and $\mathcal{F}(\mathcal{A})$ is transversal and complementary to the holomorphic distribution $\mathrm{Ker}(d\pi)$ of vertical tangent vectors of the family $\pi : \mathcal{A} \to S$. In particular, the restriction $d\pi\big|_{T_p(\mathcal{F}(\mathcal{A}))}$ gives rise to an isomorphism $T_p(\mathcal{F}(\mathcal{A})) \cong T_{\pi(p)}(S)$ at each $p \in \mathcal{A}$. Here $T_p(\mathcal{F}(\mathcal{A}))$ denotes the tangent space of the leaf of $\mathcal{F}(\mathcal{A})$ at $p$.

**Lemma 5.1.1.** *Let $\pi : \mathcal{A} \to S$ and $\mathcal{F}(\mathcal{A})$ be as above, and let $\omega_{\mathcal{A}}$ be a $d$-closed $(1,1)$-form on $\mathcal{A}$ such that the restriction $\omega_s := \omega_{\mathcal{A}}\big|_{A_s}$ is a flat Kähler form on $A_s = \pi^{-1}(s)$ for each $s \in S$. Then there exists a unique smooth non-negative $d$-closed $(1,1)$-form $\nu$ on $\mathcal{A}$ such that*

$$\nu\big|_{A_s} = \omega_s \quad \textit{and} \quad ker(\nu_p) = T_p(\mathcal{F}(\mathcal{A})) \tag{5.1.3}$$

*for each $s \in S$ and $p \in \mathcal{A}$.*

*Proof.* It is easy to see that the conditions in (5.1.3) define uniquely a non-negative (1,1)-form $\nu$ on $\mathcal{A}$. To see that $\nu$ is smooth and $d$-closed, we first fix a small connected open subset $U$ of $S$ and a reference point $o \in S$. Shrinking $U$ if necessary, we have an isomorphism (as families) of $\mathcal{A}\big|_U$ with a holomorphic family of generators $e_1, \cdots, e_{2n}$ of lattices in $\mathbb{C}^n$ as given in (5.1.1). Then for each $s \in U$, we consider the *real* linear vector space isomorphism on $\mathbb{C}^n$ sending $e_i(o)$ to $e_i(s)$, $i = 1, \cdots, 2n$, which is easily seen to descend to a diffeomorphism $\kappa_s : A_o \to A_s$ which is an isomorphism with respect to the underlying *real* abelian Lie group structures on $A_o$ and $A_s$. In particular, $\kappa_s^*\omega_s$ is a translation-invariant 2-form on $A_o$. Moreover, one easily checks that $\kappa_s$ varies smoothly with $s$, and thus one obtains a diffeomorphic trivialization of the family $\pi\big|_U : \mathcal{A}\big|_U \to U$ given by $\kappa : A_o \times U \to \mathcal{A}\big|_U$ satisfying $\kappa\big|_{A_o\times\{s\}} = \kappa_s$ for all $s \in U$. Since $\omega_{\mathcal{A}}$ is $d$-closed, it follows from Stokes' theorem that $\kappa_s^*\omega_s$, $s \in U$, are all cohomologous to $\kappa_o^*\omega_o = \omega_o$ on $A_o$ (noting that $\kappa_o = \mathrm{Id}$ on $A_o$). Together with the translation-invariance of $\kappa_s^*\omega_s$, it follows that for all $s \in U$,

$$\kappa_s^*\omega_s = \omega_o \quad \text{on } A_o \tag{5.1.4}$$

(cf. Remark 1.3.2). Next one easily sees from the definition of $\kappa$ and $\mathcal{F}(\mathcal{A})$ that $\kappa^*\mathcal{F}(\mathcal{A})$ is a foliation on $A_o \times U$ whose leaves are of the form $\{z\} \times U$ for some fixed $z \in A_o$. Together with (5.1.4) and the definition of $\nu$, it follows readily that

$$\kappa^*\nu = p_1^*\omega_o \quad \text{on } A_o \times U, \tag{5.1.5}$$

where $p_1$ denotes the projection of $A_o \times U$ onto the first factor $A_o$. Since $\omega_o$ is both smooth and $d$-closed, it follows that $\kappa^*\nu$ is both smooth and $d$-closed, and thus $\nu$ is also smooth and $d$-closed on $\mathcal{A}\big|_U$. Upon varying $U$, one sees that $\nu$ is smooth and $d$-closed on $\mathcal{A}$. □

**Proposition 5.1.2.** *Let $\pi : \mathcal{A} \to S$ and $\omega_{\mathcal{A}}$ be as in Lemma* 5.1.1. *Then for $s \in S$ and $\partial/\partial s_i$, $\partial/\partial s_j \in T_sS$,*

(i) $[v_i, \overline{v_j}]$ *is a translation-invariant vector field on $A_s$, where $v_i$, $v_j$ denote the horizontal lifting of $\partial/\partial s_i$, $\partial/\partial s_j$ with respect to $\omega_{\mathcal{A}}$ respectively; and*

(ii) $g_{v_i\overline{v_j}} := \omega_{\mathcal{A}}(v_i, \overline{v_j})$ *is a constant function on $A_s$.*

*Proof.* As in (4.2), we will fix a Euclidean coordinate system $(z^\alpha)_{1\le i\le n}$ on $A_s$, so that each $\partial/\partial z^\alpha$ can be regarded as a global translation-invariant vector field on $A_s$. Let $\nu$ be the smooth $d$-closed (1,1)-form in Lemma 5.1.1, and let $v_i'$, $v_j'$ be the horizontal lifting of $\partial/\partial s_i$, $\partial/\partial s_j$ with respect to $\nu$, respectively. Let

$$w_i = v_i - v_i' \quad \text{and} \quad w_j = v_j - v_j'. \tag{5.1.6}$$

It is easy to see that $w_i$, $w_j \in \ker(d\pi)$, and thus $w_i$ and $w_j$ restrict to smooth vector fields on each fiber $A_s$. Since both $\omega_{\mathcal{A}}\big|_{A_s}$ and $\nu\big|_{A_s}$ are the same flat Kähler metric $\omega_s$ on $A_s$, it follows that $A_{\partial/\partial s_i} := \bar\partial v_i$ and $A'_{\partial/\partial s_i} := \bar\partial v_i'$ are harmonic tensors with respect to $\omega_s$ (cf. Lemma 4.2.1). On the other hand, one recalls from (2.3) that both $A_{\partial/\partial s_i}$ and $A'_{\partial/\partial s_i}$ are representatives of the same Kodaira-Spencer class of $\rho_s(\partial/\partial s_i)$ via the Dolbeault isomorphism, where $\rho_s : T_sS \to H^1(A_s, TA_s)$ is the Kodaira-Spencer map associated to the family $\pi : \mathcal{A} \to S$. Therefore, one has $A_{\partial/\partial s_i} = A'_{\partial/\partial s_i}$, and thus

$$\bar\partial w_i = \bar\partial v_i - \bar\partial v_i' = A_{\partial/\partial s_i} - A'_{\partial/\partial s_i} = 0 \quad \text{on } A_s. \tag{5.1.7}$$

Hence $w_i$ is a global holomorphic (and thus translation-invariant) vector field on $A_s$, and so is $w_j$. It follows that we may write

$$w_i = w_i^\alpha \frac{\partial}{\partial z^\alpha} \quad \text{and} \quad w_j = w_j^\alpha \frac{\partial}{\partial z^\alpha}, \tag{5.1.8}$$

where each $w_i^\alpha$ or $w_j^\alpha$ is a constant function on $A_s$. Since $v_i'$ is the horizontal lifting of $\partial/\partial s_i$ with respect to $\nu$, it follows that $g'_{v_i'\overline{w}}=0$ for all $p\in A_s$ and $w\in T_pA_s$. Here and henceforth, we denote $g'_{v_i'\overline{w}}=\nu(v_i',\overline{w})$, etc. Together with the fact that $\nu$ is a semi-positive (1,1)-form on $\mathcal{A}$ whose positive eigenspace at each $p\in A_s$ is given by $T_pA_s$, it follows readily that $v_i'(p)\in\ker(\nu_p)$ for each $p\in A_s$. Thus one has $g'_{v_i'\overline{v_j'}}\equiv 0$ on $A_s$. Together with the commutation relation in Lemma 2.1.1(iii), it follows that

$$[v_i',\overline{v_j'}]\equiv 0 \quad \text{on } A_s. \tag{5.1.9}$$

(We remark that (5.1.9) can also be obtained by using the diffeomorphic trivialization of the family $\pi:\mathcal{A}\big|_U\to U$ over an open neighborhood $U$ of $s$ in $S$ as given in the proof of Lemma 5.1.1.) From (5.1.8), one also easily checks that

$$[w_i,\overline{w_j}]\equiv 0 \quad \text{on } A_s. \tag{5.1.10}$$

Write $A'_{\partial/\partial s_i}=A'^{\alpha}_{i\bar\beta}\frac{\partial}{\partial z^\alpha}\otimes d\bar z^\beta$. Then by Lemma 4.2.1(i) (and as remarked above), each $A'^{\alpha}_{i\bar\beta}$ is a constant function on $A_s$. Thus, by (5.1.6), we have, on $A_s$,

$$\begin{aligned}
[v_i,\overline{v_j}]&=[v_i'+w_i,\overline{v_j'}+\overline{w_j}]\\
&=0+[v_i',\overline{w_j}]+[w_i,\overline{v_j'}]+0 \quad \text{(by (5.1.9) and (5.1.10))}\\
&=-A'^{\alpha}_{i\bar\beta}\overline{w_j^\beta}\frac{\partial}{\partial z^\alpha}+\overline{A'^{\alpha}_{j\bar\beta}}w_i^\beta\frac{\partial}{\partial \overline{z}^\alpha} \quad \text{(by Lemma 2.1.1(i))},
\end{aligned} \tag{5.1.11}$$

which readily implies Proposition 5.1.2(i), since $A'^{\alpha}_{i\bar\beta}$, $A'^{\alpha}_{j\bar\beta}$, $w_i^\beta$ and $w_j^\beta$ are all constant functions on $A_s$. Next we proceed to prove Proposition 5.1.2(ii). Since $\omega_{\mathcal{A}}$ is $d$-closed, it follows from Lemma 2.1.1(iii) (and upon taking inner product with $\partial/\partial z^\alpha$) that for $1\le\alpha\le n$,

$$g_{[v_i,\overline{v_j}]\overline{\alpha}}=\frac{\partial}{\partial\bar z^\alpha}\big(g_{v_i\overline{v_j}}\big) \quad \text{on } A_s. \tag{5.1.12}$$

From Proposition 5.1.2(i), one easily sees that $g_{[v_i,\overline{v_j}]\overline{\alpha}}$ (and thus also $\frac{\partial}{\partial\bar z^\alpha}(g_{v_i\overline{v_j}})$) is a constant function on $A_s$. Hence,

$$\Box(g_{v_i\overline{v_j}})=-g^{\overline{\alpha}\beta}\frac{\partial}{\partial z^\beta}\big(\frac{\partial}{\partial\bar z^\alpha}(g_{v_i\overline{v_j}})\big)=0 \quad \text{on } A_s. \tag{5.1.13}$$

Thus $g_{v_i\overline{v_j}}$ is a constant function on $A_s$, and we have finished the proof of Proposition 5.1.2(ii). $\Box$

Now we summarize our discussion in (4.2) and (5.1) in the following

**Proposition 5.1.3.** *Let $\pi:\mathcal{A}\to S$, $\mathcal{L}\to\mathcal{A}$, $\rho$, $\omega_{\mathcal{A}}$ be as in Theorem* 3. *Suppose furthermore that* (4.2.1) *is satisfied. Then the $L^2$-metric $H_{\rho,\omega_{\mathcal{A}}}$ on $\pi_*\mathcal{L}$ is projectively flat.*

*Proof.* For $s\in S$, $\partial/\partial s^i$ and $\partial/\partial s^j\in T_sS$, it follows from Proposition 5.1.2 that on $A_s$, $g_{v_i\bar v_j}=\kappa_{i\bar j}$ for some constant $\kappa_{i\bar j}$. In particular, $\Box(g_{v_i\bar v_j})=0$ on $A_s$. Together with Proposition 4.2.4, we have, for $t_a$, $t_b\in H^0(A_s,L_s)$, the curvature tensor $\Theta$ of $H_{\rho,\omega_{\mathcal{A}}}$ on $\pi_*\mathcal{L}$ is given by

$$\begin{aligned}
\Theta_{a\bar b i\bar j}(s)&=C_{i\bar j}\cdot H_{\rho,\omega_{\mathcal{A}}}(t_a,t_b)+\int_{A_s}(\kappa_{i\bar j}+0)\langle t_a,t_b\rangle\frac{\omega_s^n}{n!}\\
&=\mu_{i\bar j}\cdot H_{\rho,\omega_{\mathcal{A}}}(t_a,t_b),
\end{aligned}$$

where $C_{i\bar{j}}$ is as in Proposition 4.2.4, and the constant $\mu_{i\bar{j}} := C_{i\bar{j}} + \kappa_{i\bar{j}}$ depends only on $\partial/\partial s^i$ and $\partial/\partial s^j$. Thus we have finished the proof of Proposition 5.1.3. $\square$

**(5.2).** Finally we are ready to complete the proof of Theorem 3.

*Proof of Theorem 3.* First we are going to prove Theorem 3(i). Let $\pi : \mathcal{A} \to S$, $\mathcal{L} \to \mathcal{A}$, $\rho$ and $\omega_{\mathcal{A}}$ be as in Theorem 3(i). As indicated in (1.3), one can always construct a smooth family of canonical Hermitian metrics $\rho'$ on the family $\mathcal{L} \to \mathcal{A}$. Then $\omega'_{\mathcal{A}} := c_1(\mathcal{L}, \rho')$ is a family of flat Kähler metrics on the family $\mathcal{A} \to S$ such that $\rho'$ and $\omega'_{\mathcal{A}}$ satisfy (4.2.1). Thus by Proposition 5.1.3, the $L^2$-metric $H_{\rho',\omega'_{\mathcal{A}}}$ on $\pi_*\mathcal{L}$ is projectively flat. Together with Proposition 4.1.1(ii) and (iii), it follows that $H_{\rho,\omega_{\mathcal{A}}}$ on $\pi_*\mathcal{L}$ is also projectively flat over $S$, and this finishes the proof of Theorem 3(i).

Finally, we deduce Theorem 3(ii) as follows. As mentioned above, one can always construct a smooth family of canonical Hermitian metrics $\rho$ on the family $\mathcal{L} \to \mathcal{A}$ and a smooth family of flat Kähler metrics $\omega_{\mathcal{A}}$ on the family $\pi : \mathcal{A} \to S$. By Theorem 3(i), the Hermitian connection of the $L^2$-metric $H_{\rho,\omega_{\mathcal{A}}}$ on $\pi_*\mathcal{L}$ is projectively flat, i.e., the curvature tensor $\Theta$ of $H_{\rho,\omega_{\mathcal{A}}}$ satisfies $\Theta = \alpha \cdot \mathrm{Id}_{\pi_*\mathcal{L}}$ for some smooth 2-form $\alpha$ on $S$. Then for any Kähler form $\mu$ on $S$, we have $\Lambda\Theta = \lambda(s) \cdot \mathrm{Id}_{\pi_*\mathcal{L}}$, where $\Lambda$ denotes the contraction with respect to $\mu$ and $\lambda(s)$ is the smooth function given by $\lambda = \Lambda\alpha$. Since $S$ is compact, it is well known that the above identity implies that $H_{\rho,\omega_{\mathcal{A}}}$ is conformally equivalent to a Hermitian-Einstein metric (with respect to $\mu$) on $\pi_*\mathcal{L}$ (see e.g. [Siu2, p. 16]). Then by a result of Kobayashi [Ko1] and Lübke [Lü], this implies that $\pi_*\mathcal{L}$ is poly-stable with respect to $\mu$. This finishes the proof of Theorem 3(ii), and thus we have completed the proof of Theorem 3. $\square$

## §6. Appendix: Siegel modular varieties and their Poincaré line bundles

**(6.1).** In this appendix, we are going to briefly discuss the Poincaré line bundles over families of abelian varieties parametrized by Siegel modular varieties, which may be regarded as the universal objects associated to the families of line bundles studied in Theorem 3. We will also explicitly describe their canonical metrics, and indicate briefly how canonical Hermitian metrics on families of ample line bundles over abelian varieties can be obtained via their classifying maps.

Denote the Siegel upper half plane by $\mathcal{H}_n := \{\tau \in \mathcal{M}_n(\mathbb{C}) : \tau = \tau^t, \mathrm{Im}(\tau) > 0\}$, where $\mathcal{M}_n(\mathbb{C})$ is the set of $n \times n$ matrices with entries in $\mathbb{C}$. Fix a type of polarization $\delta := \mathrm{diag}(d_1, d_2, \ldots, d_n) \in \mathcal{M}_n(\mathbb{Z})$, where $d_1, d_2, \ldots, d_n \in \mathbb{Z}^+$ (with $d_i | d_{i+1}$) are the elementary divisors. The symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ acts as a group of biholomorphisms on $\mathcal{H}_n$ via the following action:

$$\tau \mapsto (A\tau + B)(C\tau + D)^{-1} =: \tau' \quad \text{for} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{R}). \tag{6.1.1}$$

Moreover, this action extends to an action of the semi-direct product $\mathrm{Sp}(2n,\mathbb{R}) \triangleright \mathbb{R}^{2n}$ on $\mathcal{H}_n \times \mathbb{C}^n$ such that the action of $([\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}], [\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}]) \in \mathrm{Sp}(2n, \mathbb{R}) \triangleright \mathbb{R}^{2n}$ is given by

$$(\tau, z) \mapsto (\tau', [(C\tau + D)^t]^{-1} z + \tau'\alpha + \beta), \tag{6.1.2}$$

where $\tau'$ is as in (6.1.1). Let $\Gamma$ be a discrete torsion-free subgroup of $\mathrm{Sp}(2n,\mathbb{R})$. Then $X_\Gamma := \mathcal{H}_n/\Gamma$ is smooth, and one gets an associated analytic family of $\delta$-polarized abelian varieties given by

$$\pi : A_\Gamma := (\mathcal{H}_n \times \mathbb{C}^n)/\Gamma \triangleright \mathbb{Z}^{2n} \to X_\Gamma. \tag{6.1.3}$$

In general, it is well known that the set of $\delta$-polarized abelian varieties is in one-to-one correspondence with $\mathcal{H}_n/\mathrm{Sp}(2n,\mathbb{Z},\delta)$ for some arithmetic subgroup $\mathrm{Sp}(2n,\mathbb{Z},\delta) \subset \mathrm{Sp}(2n,\mathbb{R})$, but $\mathrm{Sp}(2n,\mathbb{Z},\delta)$ may not be torsion-free (cf. e.g. [MT, §1], [MFK], [SD, pp. 69-73] for the above background materials). By [Ku, p. 82], there exists the following Kähler form given by

$$\omega := \sqrt{-1}\partial\bar\partial[(\operatorname{Im} Z)^t(\operatorname{Im}\tau)^{-1}(\operatorname{Im} Z) + \log\det(\operatorname{Im}\tau)^{-1}] \quad \text{on } \mathcal{H}_n \times \mathbb{C}^n, \tag{6.1.4}$$

which can easily be seen to be invariant under the $\mathrm{Sp}(2n,\mathbb{R}) \triangleright \mathbb{R}^{2n}$ action in (6.1.2) (cf. e.g. [MT, §1]). Thus $\omega$ descends to a Kähler form on $X_\Gamma$, which we denote by $\omega_\Gamma$. Write $A_s := \pi^{-1}(s)$ and $\omega_s := \omega_\Gamma|_{A_s}$ for $s \in X_\Gamma$ as in (1.3). Then it is easy to see that each $\omega_s$ is a flat Kähler metric on $A_s$.

Now suppose that $\mathcal{L}_\Gamma$ is a holomorphic line bundle over $\mathcal{A}_\Gamma$ and $\rho_\Gamma$ is a Hermitian metric on $\mathcal{L}_\Gamma$ such that

$$c_1(\mathcal{L}_\Gamma, \rho_\Gamma) = \frac{\omega_\Gamma}{2\pi}. \tag{6.1.5}$$

Write $L_s := \mathcal{L}_\Gamma|_{A_s}$ and $\rho_s := \rho_\Gamma|_{L_s}$ for $s \in X_\Gamma$. Then it follows easily from (6.1.5) that each $\rho_s$ is a canonical Hermitian metric on $L_s$.

*Remark* 6.1.1. It is proved in [Ku] (see also [Sa, pp. 203-208]) that when $\Gamma \subset \mathrm{Sp}(2n,\mathbb{R})$ is discrete and torsion-free, such an $(\mathcal{L}_\Gamma, \rho_\Gamma)$ over $X_\Gamma$ and satisfying (6.1.5) always exists. Here $X_\Gamma$ need not be compact. If $X_\Gamma$ is indeed compact, then as concluded there, it follows that $\omega_\Gamma$ is a Hodge metric.

**(6.2).** Let $\pi : \mathcal{A} \to S$ be a family of abelian varieties parametrized by a complex manifold $S$. For each $s \in S$ and $A_s = \pi^{-1}(s)$, the set of isomorphism classes of holomorphic line bundles with zero first Chern class is isomorphic to the dual abelian variety $\hat{A}_s$ of $A_s$. It is well known that $\{\hat{A}_s\}_{s\in S}$ also form an analytic family of abelian varieties parametrized by $S$, which we denote by $\hat{\mathcal{A}}$. Fix a type of polarization $\delta$ of $\mathcal{A}$, and let $\hat{A}_s(\delta)$ denote the set of isomorphism classes of holomorphic line bundles over $A_s$ with first Chern class determined by $\delta$. Then it is also well known that $\hat{A}_s(\delta)$ is isomorphic to $\hat{A}_s$ (as an algebraic variety), and thus one also has a family $\hat\pi : \hat{\mathcal{A}}(\delta) \to S$ (with $\hat{\mathcal{A}}(\delta)\big|_{\hat\pi(s)} = \hat{A}_s(\delta)$ for each $s \in S$), which is analytically isomorphic to $\hat{\mathcal{A}}$. Consider the fibered product

$$\mathcal{A} \times_S \hat{\mathcal{A}}(\delta) := \{(x,y) \in \mathcal{A} \times \hat{\mathcal{A}}(\delta) \,\big|\, \pi(x) = \hat\pi(y)\}.$$

It is easy to see that $\mathcal{A} \times_S \hat{\mathcal{A}}(\delta)$ is a smooth complex submanifold of $\mathcal{A} \times \hat{\mathcal{A}}(\delta)$. We will denote by $p^{(1)} : \mathcal{A} \times_S \hat{\mathcal{A}}(\delta) \to \mathcal{A}$ and $p^{(2)} : \mathcal{A} \times_S \hat{\mathcal{A}}(\delta) \to \hat{\mathcal{A}}(\delta)$ the projection maps onto the factors $\mathcal{A}$ and $\hat{\mathcal{A}}(\delta)$, respectively. (Throughout this appendix, we will simply denote by $p^{(1)}$ and $p^{(2)}$ the projection maps of any fibered products onto the first and second factors, respectively, when no confusion arises.) The projection maps $\pi$, $\hat\pi$ induce the projection map $\bar\pi : \mathcal{A} \times_S \hat{\mathcal{A}}(\delta) \to S$ such that $\bar\pi^{-1}(s) = A_s \times \hat{A}_s(\delta)$ for $s \in S$. For each $s \in S$, one has the associated Poincaré line bundle $P_s(\delta)$ over $A_s \times \hat{A}_s(\delta)$ such that (i) for $w \in \hat{A}_s(\delta)$, $P_s(\delta)|_{A_s\times\{w\}}$ is the line bundle over $A_s$ represented by $w$; and (ii) $P_s(\delta)\big|_{\{0\}\times\hat{A}_s(\delta)} = \mathcal{O}_{\hat{A}_s(\delta)}$. Here, condition (ii) ensures that $P_s(\delta)$ is uniquely defined.

Consider $p^{(2)} : \mathcal{A}\times_S \hat{\mathcal{A}}(\delta) \to \hat{\mathcal{A}}(\delta)$ as a family of abelian varieties parametrized by $\hat{\mathcal{A}}(\delta)$, and let $\varepsilon : \hat{\mathcal{A}}(\delta) \to \mathcal{A}\times_S \hat{\mathcal{A}}(\delta)$ denote the zero section. Then it is well known that one can uniquely glue the $P_s(\delta)$'s together to form a holomorphic line bundle $\mathcal{P}_S(\delta)$ over $\mathcal{A}\times_S\hat{\mathcal{A}}(\delta)$ such that (i) $\mathcal{P}_S(\delta)\big|_{\bar{\pi}^{-1}(s)} = P_s(\delta)$ for $s\in S$; and (ii) $\mathcal{P}_S(\delta)\big|_{\varepsilon(\hat{\mathcal{A}}(\delta))} \simeq \mathcal{O}_{\hat{\mathcal{A}}(\delta)}$. $\mathcal{P}_S(\delta)$ will be called the Poincaré line bundle over $\mathcal{A}\times_S\hat{\mathcal{A}}(\delta)$. Observe that we may consider $\mathcal{P}_S(\delta)$ as a family of ample line bundles parametrized by $\hat{\mathcal{A}}(\delta)$. We refer the reader to [MFK, Chapter 6] for the above background materials.

**(6.3).** Now let $\Gamma$ be a discrete torsion-free subgroup of $\mathrm{Sp}(2n,\mathbb{R})$. Fix a type of polarization $\delta$, and let $\pi : \mathcal{A}_\Gamma \to X_\Gamma$ be the family of $\delta$-polarized abelian varieties as constructed in (6.1). Then it follows from (6.2) that we have an associated family $\hat{\pi} : \hat{\mathcal{A}}_\Gamma(\delta) \to X_\Gamma$ and an associated Poincaré line bundle, denoted by $\mathcal{P}_\Gamma(\delta)$, over the fibered product $\mathcal{A}_\Gamma \times_{X_\Gamma} \hat{\mathcal{A}}_\Gamma(\delta)$. Also recall from Remark 6.1.1 that there exists a Hermitian holomorphic line bundle $(\mathcal{L}_\Gamma,\rho_\Gamma)$ over $\mathcal{A}_\Gamma$ such that (6.1.5) is satisfied. As for $\mathcal{P}_\Gamma(\delta)$, we denote by $\mathcal{P}_\Gamma(0)$ the 'usual' universal Poincaré line bundle over $\mathcal{A}_\Gamma\times_{X_\Gamma}\hat{\mathcal{A}}_\Gamma$ associated to line bundles over fibers of $\pi : \mathcal{A}_\Gamma \to X_\Gamma$ with zero first Chern class (cf. (6.2)). It is well known that the polarization $\delta$ induces a morphism $\lambda_\delta : \mathcal{A}_\Gamma \to \hat{\mathcal{A}}_\Gamma$ satisfying the following properties: (i) one has $\hat{\pi}\circ\lambda_\delta = \pi$, and for $s\in X_\Gamma$, $\lambda_\delta\big|_{A_s} : A_s \to \hat{A}_s$ is an isogeny; and (ii) if we denote by $\Phi_\delta : \mathcal{A}_\Gamma\times_{X_\Gamma}\mathcal{A}_\Gamma \to \mathcal{A}_\Gamma\times_{X_\Gamma}\hat{\mathcal{A}}_\Gamma$ the restriction of the product map $(\mathrm{Id}_{\mathcal{A}_\Gamma},\lambda_\delta) : \mathcal{A}_\Gamma\times\mathcal{A}_\Gamma \to \mathcal{A}_\Gamma\times\hat{\mathcal{A}}_\Gamma$ to $\mathcal{A}_\Gamma\times_{X_\Gamma}\hat{\mathcal{A}}_\Gamma$, where $\mathrm{Id}_{\mathcal{A}_\Gamma}$ is the identity map on $\mathcal{A}_\Gamma$, then

$$(6.3.1)\qquad \Phi_\delta^*\mathcal{P}_\Gamma(0) = \mu^*\mathcal{L}_\Gamma \otimes (p^{(2)})^*\mathcal{L}_\Gamma^{-1}\otimes (p^{(1)})^*\mathcal{L}_\Gamma^{-1} \quad \text{on } \mathcal{A}_\Gamma\times_{X_\Gamma}\mathcal{A}_\Gamma$$

(cf. [MFK, Chapter 6, §2]). Here $\mu : \mathcal{A}_\Gamma\times_{X_\Gamma}\mathcal{A}_\Gamma \to \mathcal{A}_\Gamma$ denotes the holomorphic map given by the group law on each fiber of the projection map $\bar{\pi} : \mathcal{A}_\Gamma\times_{X_\Gamma}\mathcal{A}_\Gamma \to X_\Gamma$, i.e., $\mu(z,w) = z+w$ for $z,w\in\bar{\pi}^{-1}(s) = A_s\times A_s,\ s\in X_\Gamma$. Since $\mathcal{L}_\Gamma$ induces the polarization $\delta$ on $\mathcal{A}_\Gamma$, it follows easily from (6.3.1) that under the identification $\hat{\mathcal{A}}_\Gamma(\delta)\cong\hat{A}_\Gamma$ (cf. (6.2)), one has

$$(6.3.2)\qquad \Phi_\delta^*\mathcal{P}_\Gamma(\delta) = \mu^*\mathcal{L}_\Gamma\otimes(p^{(2)})^*\mathcal{L}_\Gamma^{-1} \quad \text{on } \mathcal{A}_\Gamma\times_{X_\Gamma}\mathcal{A}_\Gamma.$$

Moreover, it is easy to see that $h_\delta := \mu^*\rho_\Gamma\otimes(p^{(2)})^*\rho_\Gamma{}^{-1}$ is a smooth Hermitian metric on $\Phi_\delta^*\mathcal{P}_\Gamma(\delta)$ such that $h_\delta\big|_{L_w}$ is a canonical Hermitian metric on $L_w := \Phi_\delta^*\mathcal{P}_\Gamma(\delta)\big|_{(p^{(2)})^{-1}(w)}$ for each $w\in\mathcal{A}_\Gamma$. From (6.1.5), one easily sees that

$$(6.3.3)\qquad c_1(\Phi_\delta^*\mathcal{P}_\Gamma(\delta),h_\delta) = \mu^*\omega_\Gamma - (p^{(2)})^*\omega_\Gamma \quad \text{on } \mathcal{A}_\Gamma\times_{X_\Gamma}\mathcal{A}_\Gamma.$$

It is easy to see that the universal cover of $\mathcal{A}_\Gamma\times_{X_\Gamma}\mathcal{A}_\Gamma$ is given by

$$(6.3.4)\qquad \mathcal{H}_n\times\mathbb{C}^n\times\mathbb{C}^n = \{(\tau,Z,W)\in\mathbb{C}^{n^2}\times\mathbb{C}^n\times\mathbb{C}^n \,\big|\, \tau\in\mathcal{H}_n\}.$$

Thus, the coordinates $(\tau,Z,W)$ may also be regarded as local holomorphic coordinates for $\mathcal{A}_\Gamma\times_{X_\Gamma}\mathcal{A}_\Gamma$. Using (6.1.4) and in terms of the coordinates $(\tau,Z,W)$, one easily sees from (6.3.3) that $c_1(\Phi_\delta^*\mathcal{P}_\Gamma(\delta), h_\delta)$ can be locally given as

(6.3.5)

$$c_1(\Phi_\delta^*\mathcal{P}_\Gamma(\delta), h_\delta) = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\big[(\operatorname{Im} Z)^t(\operatorname{Im}\tau)^{-1}(\operatorname{Im} Z) + (\operatorname{Im} Z)^t(\operatorname{Im}\tau)^{-1}(\operatorname{Im} W) + (\operatorname{Im} W)^t(\operatorname{Im}\tau)^{-1}(\operatorname{Im} Z)\big].$$

**(6.4).** Let $\pi : \mathcal{A} \to S$ be a family of abelian varieties parametrized by a complex manifold $S$, and let $\mathcal{L} \to \mathcal{A}$ be a holomorphic line bundle over $\mathcal{A}$ such that $L_s := \mathcal{L}\big|_{A_s}$ is an ample line bundle over $A_s := \pi^{-1}(s)$ for each $s \in S$. Then $\mathcal{L}$ determines a fixed polarization type $\delta$ of $\mathcal{A}$. From [MFK, Chapter 7], one knows that for each $s_o \in S$ and $k \in \mathbb{Z}^+$, one can always introduce a level $k$ structure on $\mathcal{A}\big|_V$ for some open neighborhood $V \subset S$ containing $s_o$. Moreover, for sufficiently large $k$ and upon shrinking $V$ if necessary, one has the base change diagram

$$\begin{array}{ccccc} \mathcal{A}\big|_V & \xrightarrow{\Xi} & \mathcal{A}_\Gamma \times_{X_\Gamma} \mathcal{A}_\Gamma & \xrightarrow{\Phi_\delta} & \mathcal{A}_\Gamma \times_{X_\Gamma} \hat{\mathcal{A}}_\Gamma(\delta) \\ \pi\downarrow & & \downarrow p^{(2)} & & \downarrow p^{(2)} \\ V & \xrightarrow{i} & \mathcal{A}_\Gamma & \xrightarrow{\lambda_\delta} & \hat{\mathcal{A}}_\Gamma(\delta) \end{array} \tag{6.4.1}$$

for some torsion-free arithmetic subgroup $\Gamma \subset \operatorname{Sp}(2n, \mathbb{Z}, \delta)$, and such that

$$\lambda_\delta \circ i(s) = [L_s] \in \hat{\mathcal{A}}(\delta) \quad \text{and} \quad L_s = (\Phi_\delta \circ \Xi)^*\mathcal{P}_\Gamma(\delta)\big|_{A_s} \tag{6.4.2}$$

for each $s \in V$. Here $[L_s]$ denotes the point in $\hat{\mathcal{A}}(\delta)$ corresponding to $L_s$, and $\lambda_\delta$, $\Phi_\delta$, $\mathcal{P}_\Gamma(\delta)$ are as in (6.3). In other words, the map $i \circ \lambda_\delta$ is the classifying map associated to $\mathcal{L}\big|_V$, and the map $i$ is a lifting of $i \circ \lambda_\delta$ to the finite cover $\mathcal{A}_\Gamma$ of $\hat{\mathcal{A}}_\Gamma(\delta)$. Let $h_\delta$ be as in (6.3). Then one easily sees from (6.3.3) that $\Xi^* h_\delta$ is a smooth Hermitian metric on $\mathcal{L}\big|_V$ such that $\Xi^* h_\delta\big|_{L_s}$ is a canonical metric on $L_s$ for each $s \in V$. Finally we remark that one can easily use the above construction and a partition of unity on $S$ to construct a smooth Hermitian metric $h$ on $\mathcal{L}$ such that $h\big|_{L_s}$ is a canonical Hermitian metric on $L_s$ for each $s \in S$.

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