TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 356, Number 9, Pages 3505–3515
S 0002-9947(03)03513-X
Article electronically published on November 18, 2003

AN EXTENDED URN MODEL WITH APPLICATION TO APPROXIMATION

FENGXIN CHEN

Abstract. Pólya's urn model from probability theory is extended to obtain a class of approximation operators for which the Weierstrass Approximation Theorem holds.

1. Introduction

The binomial distributions in probability theory generate the Bernstein approximations of a continuous function which converge to the original function uniformly. One of the remarkable characteristics of the Bernstein polynomials is the shape preserving property, that is, the Bernstein polynomials not only converge the function itself but they also approximate, in a general way, the shape of the function. This property is the key to their very successful applications in the field of computer aided geometric design.

In earlier work, a probabilistic urn model, known variously as Pólya's urn, Markov-Pólya urn, or Pólya-Eggenberger urn, was extended (see [1]–[4], [10] and [11]) to obtain a class of probability distributions that include the Bernstein polynomial basis, the uniform B-Splines, and the Stancu polynomial basis. A class of linear approximations that approximate continuous functions uniformly was constructed. These approximations include the well-known Bernstein approximations and Weierstrass approximations. These probability models provide fresh insight into the discovery and proof of many algebraic results on polynomial splines.

Nevertheless, for Goldman's approximation operators, the variation diminishing property, which implies the shape preserving property, has not yet been verified, although it is true for the case $n \leq 3$. It is also proved that, for the Polya-Eggerberger urn model, the Weierstrass Approximation Theorem does not hold for general approximation operators [4].

In this paper, we study an extended version of Pólya's urn model, where a ball is drawn for each trial from an urn, initially containing some white and black balls, and a different number of balls of the color chosen and a different number of balls of the opposite color are added to the urn after each distinct trial. We obtain a class of general distributions and approximation operators with $2n$ parameters. The model includes various models above as special cases, corresponding to different choices of parameters. One may select different parameters such that the corre-

Received by the editors March 1, 2003 and, in revised form, March 26, 2003.
2000 *Mathematics Subject Classification.* Primary 35L75, 35B40.

sponding approximation operator has the shape preserving property. In this paper, however, we will give a class of approximation operators for which the Weierstrass Approximation Theorem holds.

An extension of Pólya's urn model with $2n$ parameters is introduced in [5] and [6] to study interpolation and approximation of curves and surfaces, where only balls of the color chosen are allowed to be added to (or subtracted from) the urn. Our model allows us to add balls of both the same and opposite colors. It also gives the connection between probability urn models and parameters used in their model.

The paper is organized as follows. In section 2, we introduce the urn model and study its basic distribution properties. In section 3, we apply this urn model to introduce and select a class of approximation operators for which the Weierstrass Approximation Theorem holds.

2. Urn models and distributions

Consider an urn initially containing w white balls and b black balls. One ball at a time is drawn at random from the urn and its color is inspected. It is then returned to the urn and an additional c_{11} balls of the same color and c_{21} balls of the opposite color are added to the urn. Then another ball is drawn from the urn, its color is inspected, and c_{12} balls of the same color and c_{22} balls of the opposite color are added to the urn after returning the selected ball to the urn. Repeat this procedure. After the $(i-1)^{th}$ step is completed, a ball is drawn from the urn, its color is inspected, and c_{1i} balls of the same color and c_{2i} balls of the opposite color are added into the urn after returning the selected ball to the urn. Then proceed with the $(i+1)^{th}$ step. When c_{1i} $(i = 1, 2, \cdots)$ are all the same and c_{2i} $(i = 1, 2, \cdots)$ are all the same, this model is just the Friedman model considered in [4]. The case $c_{2i} = 0$ for all $i = 1, 2, \cdots$ is studied in [5] and [6].

We study the discrete distributions generated by the probabilities of selecting exactly k white balls in the first n trials. We first introduce some notation. Let $t = \frac{w}{w+b}$, which is the probability of selecting a white ball on the first trial, $a_{1i} = \frac{c_{1i}}{w+b}$, which is the ratio of the number of balls (of the same color) added to the urn after i^{th} trial to the initial number of balls, and $a_{2i} = \frac{c_{2i}}{w+b}$, which is the ratio of the number of balls (of the opposite color) added to the urn after i^{th} trial to the initial number of balls. We consider a_{1i}, a_{2i} $(i = 1, 2, \cdots)$ as parameters and consider the probabilities as functions of t. First, let $\theta_n = (a_{11}, \cdots, a_{1n}, a_{21}, \cdots, a_{2n})$. For $k = 0, 1, \cdots, n$, denote by

$D_k^n(t) = D_k^n(\theta_n, t) =$ the probability of selecting exactly k white balls in the first n trials;

$D_n(t) = D_n(\theta_n, t) =$ the probability distribution consisting of the functions $D_0^n(t), \cdots, D_n^n(t)$;

$s_k^n(t) = s_k^n(\theta_n, t) =$ the probability of selecting a white ball after selecting exactly k white balls in the first n trials;

$f_k^n(t) = f_k^n(\theta_n, t) =$ the probability of selecting a black ball after selecting exactly k white balls in the first n trials;

$S_n(t) = S_n(\theta_n, t) =$ the a priori probability of selecting a white ball on the nth trial;

$M_r^n = M_r^n(\theta_n, t) =$ the rth moment of the probability distribution $D_n(t) = \sum_{k=0}^{n} k^r D_k^n(t)$.

We easily deduce the following proposition from the definition (see [4]).

Proposition 2.1. *For all* $t \in [0,1]$,

(i) $D_k^n(t) \geq 0$ *and* $\sum_{k=0}^{n} D_k^n(t) = 1$;
(ii) $f_k^n(t) \geq 0$, $s_k^n(t) \geq 0$, *and* $f_k^n(t) + s_k^n(t) = 1$;
(iii) $D_k^n(t) = D_{n-k}^n(1-t)$;
(iv) $f_{n-k}^n(1-t) = s_k^n(t)$ *and* $s_{n-k}^n(1-t) = f_k^n(t)$.

In order to deduce formulas for the distributions, we need explicit formulas for $s_k^n(t)$ and $f_k^n(t)$. Let Γ_k denote the set $\{(l_1, l_2, \cdots, l_k) |\ l_i \in \{1, 2, \cdots, n\}$, for $i = 1, \cdots, k$, and $l_j < l_{j+1}$, for each $j \in \{1, 2, \cdots, k-1\}\}$. There are $\binom{n}{k}$ elements in Γ_k. Therefore, we have

Lemma 2.2.

$$\sum_{(l_1, l_2, \cdots, l_k) \in \Gamma_k} \sum_{j=1}^{k} c_{1l_j} = \binom{n-1}{k-1} \sum_{i=1}^{n} c_{1i},$$

$$\sum_{(\theta_1, \theta_2, \cdots, \theta_{n-k}) \in \Gamma_{n-k}} \sum_{j=1}^{n-k} c_{2\theta_j} = \binom{n-1}{n-k-1} \sum_{i=1}^{n} c_{2i}.$$

Proof. For each $i \in \{1, 2, \cdots, n\}$, there are exactly $\binom{n-1}{k-1}$ c_{1i}'s in the sums on the left side of the first formula. □

With these preparations, we have

Proposition 2.3.

$$s_k^n(t) = \frac{t + k\overline{a}_1^n + (n-k)\overline{a}_2^n}{1 + n(\overline{a}_1^n + \overline{a}_2^n)},$$

$$f_k^n(t) = \frac{1 - t + (n-k)\overline{a}_1^n + k\overline{a}_2^n}{1 + n(\overline{a}_1^n + \overline{a}_2^n)},$$

where $\overline{a}_1^n = \frac{1}{n}\sum_{1}^{n} a_{1i}$ *and* $\overline{a}_2^n = \frac{1}{n}\sum_{1}^{n} a_{2i}$.

Proof. By the procedure of trials, the event that we draw exactly k white balls in the first n trials has altogether $\binom{n}{k}$ different cases. So we imagine there are $\binom{n}{k}$ extended Friedman urns, each of which corresponds a different ordered set $\{l_1, l_2, \cdots, l_k\} \in \Gamma_k$. Then we make n trials of drawing and adding balls as described above so that the k white balls are drawn exactly at trials $l_1, l_2, \cdots, l_k$ for the corresponding urn. After those trials, we put the balls of $\binom{n}{k}$ urns together to form a larger urn. Then the probability of selecting a white ball in the first trial on this

new urn is just $s_k^n(t)$. Therefore,

$$\begin{aligned} s_k^n(t) &= \frac{\text{the number of white balls in the new urn}}{\text{the total number of balls in the new urn}} \\ &= \frac{\binom{n}{k}w + \sum\limits_{(l_1,l_2,\cdots,l_k)} \sum\limits_{j=1}^{k} c_{1l_j} + \sum\limits_{(l_1,l_2,\cdots,l_k)} \sum\limits_{j=1}^{n-k} c_{2\theta_j}}{\binom{n}{k}(w+b) + \binom{n}{k}\sum\limits_{i=1}^{n}(c_{1i}+c_{2i})} \\ &= \frac{\binom{n}{k}w + \binom{n-1}{k-1}\sum\limits_{i=1}^{n} c_{1i} + \binom{n-1}{n-k-1}\sum\limits_{i=1}^{n} c_{2i}}{\binom{n}{k}(w+b) + \binom{n}{k}\sum\limits_{i=1}^{n}(c_{1i}+c_{2i})}, \end{aligned} \tag{2.1}$$

where we have used Lemma 2.2. Dividing numerator and denominator by $\binom{n}{k}(w+b)$, we deduce the first conclusion of the proposition. The expression for $f_k^n(t)$ can be derived by a similar argument or simply by applying Proposition 2.1(ii). □

Remark 2.4. (i) If $c_{11} = c_{12} = \cdots = c_{1n} = c_1$, and $c_{21} = c_{22} = \cdots = c_{2n} = c_2$, these expressions for $f_k^n(t)$ and $s_k^n(t)$ are those of Proposition 2.3.1 in [4].

(ii) This proposition gives the relationship of parameters μ_k and ν_k used in [5] with the sequence of numbers of balls added into the urn.

For $D_0^n(t)$ and $D_n^n(t)$, similar to [4], we have

Proposition 2.5. (i) *$D_0^n(1) = 0$, $D_n^n(0) = 0$, for $n > 0$,*

(ii) $D_0^n(t) = \prod\limits_{k=0}^{n-1} f_0^k(t) = \prod\limits_{k=0}^{n-1} \frac{(1-t+k\overline{a}_1^k)}{[1+k(\overline{a}_1^k+\overline{a}_2^k)]}$,

(iii) $D_n^n(t) = \prod\limits_{k=0}^{n-1} s_k^k(t) = \prod\limits_{k=0}^{n-1} \frac{(t+k\overline{a}_1^k)}{[1+k(\overline{a}_1^k+\overline{a}_2^k)]}$.

Now we are ready to give the recursion formula for $D_k^n(t)$. By definition $D_k^{n+1}(t)$ is the probability of choosing exactly k white balls in the first $n+1$ trials. This equals the sum of the probabilities of two mutually exclusive events: (1) selecting exactly k white balls in the first n trials and then selecting a black ball on the $(n+1)^{th}$ trial; (2) selecting exactly $k-1$ white balls in the first n trials and then selecting a white ball on the $(n+1)^{th}$ trial. By the definitions of $D_k^n(t)$, $s_k^n(t)$ and $f_k^n(t)$, we have

Proposition 2.6. $D_k^{n+1}(t) = f_k^n(t)D_k^n(t) + s_{k-1}^n(t)D_{k-1}^n(t)$.

Now that we have explicit formulas for $s_k^n(t)$ and $f_k^n(t)$ and

$$D_0^1(t) = 1-t, \quad D_1^1(t) = t,$$

we can get $D_k^n(t)$ by recursion for any n and k. It is easy to see that $D_k^n(t)$ are polynomials of degree n in t with parameters $a_{11}, \cdots, a_{1n}, a_{21}, \cdots, a_{2n}$.

Now let us study the moments $M_r^n(t)$, $r = 0, 1, \cdots, n$. By definition, we know that $M_r^n(t)$, $r = 0, 1, \cdots, n$, are also $n+1$ polynomials of degree n in t. It is easy to get (see [4])

Proposition 2.7. (i) $M_0^n(t) = 1$;

(ii) $M_1^n(t) = \sum\limits_{k=1}^{n} S_k(t)$;

(iii) $S_{n+1}(t) = M_1^{n+1}(t) - M_1^n(t)$;

(iv) $S_{n+1}(t) = \sum_{k=0}^{n} s_k^n(t) D_k^n(t)$.

Proposition 2.8. $S_{n+1}(t) = \frac{t+n\overline{a}_2^n+(\overline{a}_1^n-\overline{a}_2^n)M_1^n(t)}{1+n(\overline{a}_1^n+\overline{a}_2^n)}$.

Proof. By Propositions 2.7 and 2.3, we have

$$\begin{aligned} S_{n+1}(t) &= \sum_{k=0}^{n} s_k^n(t) D_k^n(t) \\ &= \sum_{k=0}^{n} \frac{t + k\overline{a}_1^n + (n-k)\overline{a}_2^n}{1 + n(\overline{a}_1^n + \overline{a}_2^n)} D_k^n(t) \\ &= \left[\frac{t + n\overline{a}_2^n}{1 + n(\overline{a}_1^n + \overline{a}_2^n)}\right] \sum_{k=0}^{n} D_k^n(t) + \left[\frac{\overline{a}_1^n - \overline{a}_2^n}{1 + n(\overline{a}_1^n + \overline{a}_2^n)}\right] \sum_{k=0}^{n} k D_k^n(t) \\ &= \frac{t + n\overline{a}_2^n + (\overline{a}_1^n - \overline{a}_2^n) M_1^n(t)}{1 + n(\overline{a}_1^n + \overline{a}_2^n)}. \end{aligned} \tag{2.2}$$

□

Corollary 2.9. (i) $M_1^{n+1}(t) = \frac{t+n\overline{a}_2^n+[1+(n+1)\overline{a}_1^n+(n-1)\overline{a}_2^n]M_1^n(t)}{1+n(\overline{a}_1^n+\overline{a}_2^n)}$.

(ii) *There exist constants* p_n, q_n *such that*

$$M_1^n(t) = p_n t + q_n,$$

and (a) $p_n > 0$, (b) $q_n \geq 0$, (c) $p_n + 2q_n = n$.

(iii) *If* $\overline{a}_2^j = 0$, *for* $j = 1, 2, \cdots, n$, *then* (a) $S_n(t) = t$, (b) $M_1^n(t) = nt$.

Proof. The corollary can be proved by induction (see [4]). □

For the rth moment $M_r^n(t)$, we have the following recursion formula.

Proposition 2.10.

$$\begin{aligned} M_r^{n+1}(t) &= \left[\frac{1 + (n+r)\overline{a}_1^n + (n-r)\overline{a}_2^n}{1 + n(\overline{a}_1^n + \overline{a}_2^n)}\right] M_r^n(t) \\ &+ \sum_{i=1}^{r-1} \left[\frac{\binom{r}{i} t + \binom{r}{i-1}\overline{a}_1^n + [n\binom{r}{i} - \binom{r}{i-1}]\overline{a}_2^n}{1 + n(\overline{a}_1^n + \overline{a}_2^n)}\right] M_i^n(t) \\ &+ \frac{t + n\overline{a}_2^n}{1 + n(\overline{a}_1^n + \overline{a}_2^n)}. \end{aligned} \tag{2.3}$$

The proof is similar to that of Proposition 2.5.10 in [4], and is therefore omitted. This proposition implies that $M_r^n(t)$ are polynomials in t of degree less than or equal to n. In fact, we have the following result.

Corollary 2.11. *If* $0 \leq r \leq n$, *there exist* $r+1$ *constants* $p_r^{n,r}, p_r^{n,r-1}, \cdots, p_r^{n,0}$, *such that*

$$M_r^n(t) = p_r^{n,r} t^r + \cdots + p_r^{n,1} t + p_r^{n,0}$$

and

(i) $p_r^{n,r} > 0$, (ii) $p_r^{n,i} \geq 0$, $i = 0, 1, \cdots, r-1$, (iii) $p_n^{n,n} = \frac{n!}{\prod_{k=0}^{n-1}[1+k(\overline{a}_1^n+\overline{a}_2^n)]}$.

Proof. Since $M_0^1(t) = 1$, by Corollary 2.9, we know the result holds for $n = 1$. Assume the result is true for n. Then by the recursion formula of the moments, we have

$$p_r^{n+1,r} = \frac{[1 + (n+r)\overline{a}_1^n + (n-r)\overline{a}_2^n]p_r^{n,r} + rp_{r-1}^{n,r-1}}{1 + n(\overline{a}_1^n + \overline{a}_2^n)} > 0$$

and

$$p_r^{n+1,i} \geq 0, \ i = 0, 1, \cdots, r-1.$$

Also, by Proposition 2.10, we have

$$p_{n+1}^{n+1,n+1} = \frac{\binom{n+1}{n}}{1 + n(\overline{a}_1^n + \overline{a}_2^n)} p_n^{n,n} = \frac{(n+1)!}{\prod\limits_{k=0}^{n} [1 + k(\overline{a}_1^n + \overline{a}_2^n)]}.$$

The corollary follows from induction. □

Corollary 2.12. (i) *The $n+1$ moments $M_0^n(t), \cdots, M_n^n(t)$ are a basis for the degree n polynomials in t.*

(ii) *The $n+1$ distribution functions $D_0^n(t), \cdots, D_n^n(t)$ are a basis for the degree n polynomials in t.*

3. Approximation operators and uniform convergence

It is noted in [4] that the Weierstrass Approximation Theorem fails to hold for the distributions constructed by the Polya-Eggenberger urn model for general functions. Our model allows more freedom to choose parameters. We will introduce a class of approximation operators and prove that the Weierstrass Approximation Theorem holds for this class of approximation operators with an appropriate choice of parameters. In this section, we restrict ourselves to the case $\overline{a}_2^j = 0$, that is, $a_{2j} = 0$ for $j = 1, 2, \cdots$.

Suppose D_n is the distribution defined in section 2 and g is a continuous real function defined on some interval I. Assume $I \supset [0, 1]$. As in [4], define a linear operator $\mathcal{L}_n : C[I] \to C[0, 1]$ by

$$\mathcal{L}_n[g](t) = \sum_{k=0}^{n} g(\frac{k}{n}) D_k^n(t),$$

for $g \in C(I)$.

We have the following theorem:

Theorem 3.1 (Weierstrass Approximatiom Theorem)**.** *Suppose $\lim\limits_{j\to\infty} \overline{a}_1^j = 0$ and $\sum\limits_{j=1}^{\infty} \frac{1}{j\,(1+j\overline{a}_1^j)} = +\infty$. Then $\mathcal{L}_n[g]$ converges to g uniformly on $[0, 1]$ as $n \to +\infty$.*

Remark 3.2. (i) If $\sum\limits_{i=1}^{\infty} a_{1i} < \infty$, then, obviously, the conditions of Theorem 3.1 hold.

(ii) The conditions of Theorem 3.1 hold if $\sum\limits_{i=1}^{j} a_{1i} = O(\log^p j)$ for some $p \in (0, 1)$.

To prove Theorem 3.1, we need some lemmas. First we prove that the operator $\mathcal{L}_n$ is invariant for linear functions.

Lemma 3.3. *$\mathcal{L}_n[at + b](t) = at + b$, where a and b are constants.*

Proof. From the definition of $\mathcal{L}_n$, we have

$$\begin{aligned}\mathcal{L}_n[at+b](t) =& \sum_{k=0}^{n}(\frac{a}{n}k+b)D_k^n(t) = \frac{a}{n}\sum_{k=0}^{n}kD_k^n(t) + b\sum_{k=0}^{n}D_k^n(t)\\ =& \frac{a}{n}M_1^n(t) + b = at+b,\end{aligned}$$

where we have used Corollary 2.9. □

To prove the theorem, we need to show that approximations $\mathcal{L}_n[g]$ converges to g uniformly for $g(t) = t^2$. Therefore, we need the following explicit formula for the second moment of the distribution D_n:

Lemma 3.4.

$$M_2^n(t) = \frac{n(n-1)t^2 + n(1+nA_n)t}{1+B_n},$$

where

$$A_2 = B_2 = \overline{a}_1^1 = a_{11},$$

and

$$B_n = \frac{n[1+(n-1)\overline{a}_1^{n-1}]B_{n-1} - 2[B_{n-1} - \overline{a}_1^{n-1}]}{n[1+(n-1)\overline{a}_1^{n-1}] + 2[B_{n-1} - \overline{a}_1^{n-1}]},$$

$$A_n = \frac{(n-1)[1+(n+1)\overline{a}_1^{n-1}](1+B_n)[1+(n-1)A_{n-1}]}{n^2[1+(n-1)\overline{a}_1^{n-1}](1+B_{n-1})} + \frac{1+B_n-n}{n^2}$$

for $n \geq 3$.

Proof. We prove the lemma by induction on n. By Corollary 2.11, there exist constants $p_2^{n,2} > 0$, $p_2^{n,1} \geq 0$, and $p_2^{n,0} \geq 0$ such that

$$M_2^n(t) = p_2^{n,2}t^2 + p_2^{n,1}t + p_2^{n,0}.$$

From the proof of Corollary 2.11, we see that $p_2^{n,0} = 0$ when $\overline{a}_2^j = 0$ for $j = 1, 2, \cdots, n-1$. Therefore, we may assume that

$$M_2^n(t) = \frac{n(n-1)t^2 + n(1+nA_n)t}{1+B_n} \tag{3.1}$$

and go on to get the recursion formulas for A_n and B_n. By Proposition 2.10, Corollary 2.9 and (3.1), we have

$$\begin{aligned}M_2^n(t) =& H_nM_2^{n-1}(t) + \frac{2t+\overline{a}_1^{n-1}}{[1+(n-1)\overline{a}_1^{n-1}]}M_1^{n-1}(t) + \frac{t}{[1+(n-1)\overline{a}_1^{n-1}]}\\ =& \frac{H_n[(n-1)(n-2)t^2 + (n-1)(1+(n-1)A_{n-1})t]}{(1+B_{n-1})}\\ &+ \frac{2t+\overline{a}_1^{n-1}}{[1+(n-1)\overline{a}_1^{n-1}]}(n-1)t + \frac{t}{[1+(n-1)\overline{a}_1^{n-1}]}\\ =& \frac{\{(n-2)[1+(n+1)\overline{a}_1^{n-1}] + 2(1+B_{n-1})\}(n-1)}{[1+(n-1)\overline{a}_1^{n-1}](1+B_{n-1})}t^2 + K_nt,\end{aligned} \tag{3.2}$$

where $K_n := \frac{(n-1)[1+(n+1)\overline{a}_1^{n-1}][1+(n-1)A_{n-1}]+[1+(n-1)\overline{a}_1^{n-1}](1+B_{n-1})}{[1+(n-1)\overline{a}_1^{n-1}](1+B_{n-1})}$ and $H_n := \frac{[1+(n+1)\overline{a}_1^{n-1}]}{[1+(n-1)\overline{a}_1^{n-1}]}$.

Comparing (3.1) with (3.2), we get

$$\frac{n}{1+B_n} = \frac{(n-2)[1+(n+1)\overline{a}_1^{n-1}]+2(1+B_{n-1})}{[1+(n-1)\overline{a}_1^{n-1}](1+B_{n-1})} \tag{3.3}$$

and

$$\frac{n(1+nA_n)}{1+B_n} = K_n. \tag{3.4}$$

The formulae for A_n and B_n follow from (3.3) and (3.4). That completes the deduction of the recursion formulas.

Now let us compute A_2 and B_2. From (3.1), when $n=2$,

$$M_2^2(t) = \frac{2t^2+2(1+2A_2)t}{1+B_2}, \tag{3.5}$$

and, by (3.2),

$$\begin{aligned} M_2^2(t) &= \frac{1+3a_{11}}{1+a_{11}}M_2^1(t) + \frac{2t+a_{11}}{1+a_{11}}M_1^1(t) + \frac{t}{1+a_{11}} \\ &= \frac{1+3a_{11}}{1+a_{11}}t + \frac{2t+a_{11}}{1+a_{11}}t + \frac{t}{1+a_{11}} \\ &= \frac{2t^2+2(1+2a_{11})t}{1+a_{11}}. \end{aligned} \tag{3.6}$$

Comparing (3.5) with (3.6), we get $A_2 = a_{11}$, $B_2 = a_{11}$. □

Lemma 3.5. *If* $\lim_{n\to\infty} A_n = A$, *and* $\lim_{n\to\infty} B_n = B$, *then*

$$\lim_{n\to\infty} \mathcal{L}_n[t^2](t) = \frac{t^2+At}{1+B}. \tag{3.7}$$

Proof.

$$\begin{aligned} \mathcal{L}_n[t^2](t) &= \sum_k \left(\frac{k}{n}\right)^2 D_2^n(t) = \frac{1}{n^2}\sum_k k^2 D_k^n(t) \\ &= \frac{1}{n^2}M_2^n(t) = \frac{(n-1)t^2+(1+nA_n)t}{n(1+B_n)}. \end{aligned}$$

Letting $n\to\infty$ gives (3.7). □

Therefore, we need $A=0$ and $B=0$ in (3.7). The following lemma gives a set of sufficient conditions.

Lemma 3.6. *If*

$$\text{(i)} \quad \lim_{n\to\infty} \overline{a}_1^n = 0 \tag{3.8}$$

and

$$\text{(ii)} \quad \sum_{n=1}^{\infty} \frac{1}{n\,(1+n\,\overline{a}_1^n)} = +\infty, \tag{3.9}$$

then

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = 0.$$

Proof. From the expression for B_n, we have

$$\begin{aligned} B_n &= \frac{(n[1+(n-1)\overline{a}_1^{n-1}]-2)B_{n-1}+2\overline{a}_1^{n-1}}{n[1+(n-1)\overline{a}_1^{n-1}]+2(B_{n-1}-\overline{a}_1^{n-1})} \\ &= \xi_{n-1}B_{n-1}+\eta_{n-1}, \end{aligned} \tag{3.10}$$

where

$$\xi_{n-1} := \frac{n[1+(n-1)\overline{a}_1^{n-1}]-2}{n[1+(n-1)\overline{a}_1^{n-1}]+2(B_{n-1}-\overline{a}_1^{n-1})} \tag{3.11}$$

and

$$\eta_{n-1} := \frac{2\overline{a}_1^{n-1}}{n[1+(n-1)\overline{a}_1^{n-1}]+2(B_{n-1}-\overline{a}_1^{n-1})}. \tag{3.12}$$

Since $\lim_{n\to\infty}\overline{a}_1^n=0$, we know that $0<\xi_{n-1}<1$, for n sufficiently large. In fact

$$\xi_{n-1} \le \frac{n[1+(n-1)\overline{a}_1^{n-1}]-2}{n[1+(n-1)\overline{a}_1^{n-1}]-2\overline{a}_1^{n-1}} < 1. \tag{3.13}$$

Moreover,

$$\begin{aligned} \xi_{n-1} &\le 1-\frac{2(1-\overline{a}_1^{n-1})}{n[1+(n-1)\overline{a}_1^{n-1}]-2\overline{a}_1^{n-1}} \\ &\le 1-\frac{1}{n[1+(n-1)\overline{a}_1^{n-1}]}, \end{aligned} \tag{3.14}$$

provided that n is sufficiently large so that $\overline{a}_1^{n-1}<1/2$ for $n\ge n_0$. As for η_{n-1}, we have

$$\eta_{n-1} \le \frac{2\overline{a}_1^{n-1}}{n[1+(n-1)\overline{a}_1^{n-1}]-2\overline{a}_1^{n-1}} \le \frac{2}{n(n-1)}. \tag{3.15}$$

From (3.10), (3.14) and (3.15), for $k>0$ and n sufficiently large, we have

$$\begin{aligned} B_{n+k} =& \xi_{n+k-1}B_{n+k-1}+\eta_{n+k-1} \\ =& \xi_{n+k-1}\xi_{n+k-2}\cdots\xi_n B_n+\xi_{n+k-1}\cdots\xi_{n+1}\eta_n+\cdots \\ &+\xi_{n+k-1}\eta_{n+k-2}+\eta_{n+k-1} \\ \le& \prod_{j=n}^{n+k-1}\left(1-\frac{1}{j[1+(j-1)\overline{a}_1^j]}\right)B_n+\sum_{j=n}^{n+k-1}\frac{2}{j(j-1)}, \end{aligned} \tag{3.16}$$

since $0\le\xi_j\le 1$. From (3.16), we know that B_j is bounded. Since $\sum_{j=2}^{\infty}\frac{1}{j[1+(j-1)\overline{a}_1^j]}$ diverges to ∞, we know that $\prod_{j=2}^{\infty}\left(1-\frac{1}{j[1+(j-1)\overline{a}_1^j]}\right)$ converges to zero. Therefore $\prod_{j=n}^{n+k-1}\left(1-\frac{1}{j[1+(j-1)\overline{a}_1^j]}\right)$ converges to zero as $k\to\infty$. From (3.16), it follows that $\lim_{n\to\infty}B_n=0$.

Now let us deal with A_n. From the expression for A_n,

$$\begin{aligned} A_n =& \frac{(n-1)[1+(n+1)\overline{a}_1^{n-1}][1+(n-1)A_{n-1}][1+B_n]}{n^2[1+(n-1)\overline{a}_1^{n-1}][1+B_{n-1}]}+\frac{1+B_n-n}{n^2} \\ =& \alpha_{n-1}A_{n-1}+\gamma_{n-1}, \end{aligned}$$

where

$$\alpha_{n-1} := \frac{(n-1)^2(1+B_n)}{n^2(1+B_{n-1})}[1+\frac{2\overline{a}_1^{n-1}}{1+(n-1)\overline{a}_1^{n-1}}]$$

and

$$\begin{aligned}\gamma_{n-1} =& \frac{2(n-1)(1+B_n)\overline{a}_1^{n-1}}{n^2[1+(n-1)\overline{a}_1^{n-1}][1+B_{n-1}]} \\ &- \frac{2(B_{n-1}-\overline{a}_1^{n-1})}{n^2[1+(n-1)\overline{a}_1^{n-1}]+2n(B_{n-1}-\overline{a}_1^{n-1})} + \frac{B_{n-1}(1+B_n)}{n^2(1+B_{n-1})}.\end{aligned}$$

When n is sufficiently large, we have

$$0 < \alpha_{n-1} \leq 3(1-\frac{1}{n})^2$$

and

$$0 < \gamma_n < \frac{3}{n},$$

since $\lim_{n\to\infty} B_n = 0$. We apply a similar argument to that for the case of B_n to prove that $\lim_{n\to\infty} A_n = 0$, since $\sum_n \frac{1}{n}$ diverges. □

From Lemmas 3.5 and 3.6, it is easy to see that

Corollary 3.7. *If the conditions of Lemma 3.6 hold, then* $\lim_{n\to\infty} \mathcal{L}_n[t^2](t) = t^2$.

Proof of Theorem 3.1. By the approximation theorem of positive operators (see Theorem 2.2, p. 62 in [8]; or Theorem 3, pp. 9-11 in [7]), we only need to prove (1) $\mathcal{L}_n$ is a positive linear operator; (2) the sequence $\{\mathcal{L}_n[g]\}$ uniformly converges to $g(t)$ for $g(t) = t^k$, for $k = 0, 1, 2$. (1) is obvious and (2) is the consequence of Lemma 3.3 and Corollary 3.7.

Acknowledgement

The author thanks the referee for his helpful comments and suggestions.

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Department of Applied Mathematics, University of Texas at San Antonio, 6900 North Loop 1604 West, San Antonio, Texas 78249

E-mail address: feng@math.utsa.edu