A CLASS OF PROCESSES ON THE PATH SPACE OVER A COMPACT RIEMANNIAN MANIFOLD WITH UNBOUNDED DIFFUSION

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Abstract. A class of diffusion processes on the path space over a compact Riemannian manifold is constructed. The diffusion of such a process is governed by an unbounded operator. A representation of the associated generator is derived and the existence of a certain local second moment is shown.

1. Introduction and basic notation

Infinite dimensional diffusion processes have been studied from several points of view. For example, S. Kusuoka [10], introduced diffusion type Dirichlet forms on Banach spaces. The existence of associated processes is then obtained by using regularity arguments. On the other hand, M. Röckner and T.S. Zhang [16] and A. Eberle [6] used finite dimensional approximation methods to treat infinite dimensional diffusion processes. In these papers, the diffusion is governed by bounded operators.

In contrast, we show the existence of a class of processes with unbounded diffusion operators. For this, we use methods and results of modern Dirichlet form theory (N. Bouleau and F. Hirsch [3], B.K. Driver and M. Röckner [5], M. Fukushima, Y. Oshima, and M. Takeda [8], Z.M. Ma and M. Röckner [13]). The basic structure of a diffusion form we deal with is

$$\mathcal{E}(F, F) := \int \langle DF, ADF \rangle_{H} d\nu, \quad F \in D(\mathcal{E}),$$

where $\mathbb{H}$ is the Cameron-Martin space, $D$ denotes the corresponding gradient operator, and $\nu$ is the Wiener measure on the space $P_{m_0}(M)$ of all Brownian paths $\gamma$ on the compact Riemannian manifold $M$ with $\gamma(0) = m_0 \in M$. In our setting, the diffusion operator $A : L^2(P_{m_0}(M)) \to \mathbb{H}, \nu \supseteq D(A) \to L^2(P_{m_0}(M)) \to \mathbb{H}, \nu)$ is unbounded. Let us, however, mention that there are authors speaking in quite different situations of unbounded diffusion coefficients, namely when omitting the operator $A$ and replacing the measure $d\nu$ with $Cd\nu$ where $C$ is a possibly unbounded density function (see, e.g., S. Aida [1]).

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We verify closability (Section 2) as well as quasi-regularity which implies the existence of an associated right process \( M = (\Omega, \mathcal{F}, (X_t)_{t \geq 0}, (P_x)_{x \in \mathcal{P}_{m_0}(M)}) \) (Section 3). Furthermore, we provide a representation of the associated generator \( A \) (Section 4). In particular, the fact that a certain subspace of the space of the cylindrical functions over \( \mathcal{P}_{m_0}(M) \) is a subset of the domain of \( A \) is used to determine the following local second moment,

\[
\lim_{t \to 0} \frac{1}{t} \int \sum_{v=1}^{N} \|x^v(\gamma(\cdot)) - x^v(\tau(\cdot))\|_{L^2([0,1],ds)}^2 \, P_\tau(X_t \in d\gamma) = \sum_{v=1}^{N} \int_{s \in [0,1]} \Gamma(x^v(\tau(s)), x^v(\tau(s))) \, ds,
\]

weakly in \( L^1(\mathcal{P}_{m_0}(M), \nu) \) (Section 5). Here, \( M \) is considered as isometrically embedded in some \( \mathbb{R}^N \) and \( x^v(p) \), \( v \in \{1, \ldots, N\} \), denote the standard coordinates of \( p \in M \) embedded in \( \mathbb{R}^N \); finally, \( \Gamma \) is the carré du champ operator corresponding to \((\mathcal{E}, D(\mathcal{E}))\).

Let \( M \) be a compact connected Riemannian manifold of dimension \( d \) without boundary, isometrically embedded in some \( \mathbb{R}^N \). Let \( T_m \) denote the tangent space to \( M \) at \( m(\in M) \) and let \( \langle \ldots \rangle_T \) denote the inner product on \( T_m \). We fix a covariant derivative \( \nabla \) compatible with the underlying Riemannian metric and assume that \( \nabla \) is torsion skew symmetric, which means that, if \( T \) is the torsion tensor of \( \nabla \), then \( \langle T(\xi, \eta), \eta \rangle \equiv 0 \) for all vector fields \( \xi \) and \( \eta \) on \( M \). This convention guarantees compatibility with the works of B.K. Driver \cite{4}, B.K. Driver and M. Röckner \cite{5}, and E.P. Hsu \cite{9}.

Let \( O(M) \) denote the orthonormal frame bundle with respect to \( M \). Furthermore, we denote the canonical projection \( O(M) \to M \) by \( \pi \) and the canonical horizontal vector fields by \( H_1, \ldots, H_d \). Let \( X \) be the space of all Brownian trajectories on \([0, 1] \) with values in \( \mathbb{R}^d \). For fixed \( m_0 \in M \), we introduce the path space \( \mathcal{P}_{m_0}(M) \) by

\[
\mathcal{P}_{m_0}(M) := \{ \gamma \in C([0, 1] \to M) : \gamma(0) = m_0 \}
\]

and equip it with the topology of uniform convergence. Let \( \mu \) denote the Wiener measure on \( X \) and let \( r_0 \in O(M) \) such that \( \pi(r_0) = m_0 \). According to J. Eells and D. Elworthy \cite{7} and P. Malliavin \cite{12}, the solution \( r_x \) to the Stratonovich SDE

\[
\begin{align*}
\partial_x r_x(t) &= \sum_{i=1}^{d} H_i(r_x(t)) \partial x_i(t), \quad t \in [0, 1], \\
r_x(0) &= r_0,
\end{align*}
\]

\( x = (x_1, \ldots, x_d) \in X \), defines \( (\mu \text{-a.e.}) \) a mapping \( I : X \to \mathcal{P}_{m_0}(M) \) by \( I(x)(t) := \pi(r_x(t)), x \in X, t \in [0, 1] \). Considering, simultaneously, \( x \) as a \( d \)-dimensional standard Brownian motion, \( I(x) \) becomes a Brownian motion on \( M \) whose law on \( \mathcal{P}_{m_0}(M) \) (the Wiener measure on \( \mathcal{P}_{m_0}(M) \)) is denoted by \( \nu \).

Finally, as discussed in \cite{9}, Section 4, there is an inverse map \( L \) of \( I \) in the sense that \( L \circ I = \text{identity} \mu \text{-a.e. on } X \) and \( I \circ L = \text{identity} \nu \text{-a.e. on } \mathcal{P}_{m_0}(M) \). Note that, for \( x \in X \) and \( \gamma \in \mathcal{P}_{m_0}(M) \) with \( \gamma = I(x) \) and \( x = L(\gamma) \), the path \( r_x \) in \( O(M) \) is well defined and that, for \( a \in \mathbb{R}^d \) and \( 0 \leq s, t \leq 1 \),

\[
(r_x(s)a, r_x(s)a)_{T_{\gamma}(t)} = (r_x(t)a, r_x(t)a)_{T_{\gamma}(t)} = |a|^2_{\mathbb{R}^d}.
\]
The parallel transport from \( T_{s(t)} \) to \( T_{\gamma(t)} \) along \( \gamma \in \mathbf{P}_{m_0}(M) \) is \( \nu \)-a.e. defined as follows. For \( x \in X \) and \( \gamma \in \mathbf{P}_{m_0}(M) \) with \( \gamma = I(x) \) and \( x = L(\gamma) \), set

\[
T_{s,t} := r_x(t)r_{x}^{-1}(s), \quad 0 \leq s, t \leq 1.
\]

Introduce the abbreviation \( L^p(\nu) \) for \( L^p(\mathbf{P}_{m_0}(M), \nu) \), \( 1 \leq p \leq \infty \), and define the set of all cylindrical functions on \( \mathbf{P}_{m_0}(M) \),

\[
Z := \{ F(\gamma) = f(\gamma(s_1); \ldots; \gamma(s_k)), \gamma \in \mathbf{P}_{m_0}(M) : 0 < s_1 < \ldots < s_k = 1, f \in C^\infty(M^k), k \in \mathbb{N} \}.
\]

As the system of the Haar functions on \( [0, 1] \), \( Y \) is also dense in \( L^2(\nu) \) (see [5]). Define \( (e_j)_{j=1, \ldots, d} \) be a standard basis in \( \mathbb{R}^d \) and let

\[
H_1(t) = 1, \quad t \in [0, 1],
\]

\[
H_{2m+k}(t) = \begin{cases} 
2^{2k} & \text{if } t \in \left( \frac{k-1}{2n+1}, \frac{2k+1}{2n+1} \right), \\
-2^{2k} & \text{if } t \in \left( \frac{2k+1}{2n+1}, \frac{k}{2m} \right), \\
0 & \text{otherwise,}
\end{cases} \quad k = 1, \ldots, 2^m, m = 0, 1, \ldots,
\]

denote the system of the Haar functions on \( [0, 1] \). Furthermore, define

\[
g_{d(r-1)+j} := H_r \cdot e_j, \quad r \in \mathbb{N}, \quad j \in \{1, \ldots, d\}.
\]

As the system of the Haar functions \( (H_n)_{n \in \mathbb{N}} \) is complete in \( L^2([0, 1] \to \mathbb{R}, ds) \), the system \( (g_n)_{n \in \mathbb{N}} \) is complete in \( L^2([0, 1] \to \mathbb{R}^d, ds) \). Therefore, \( (S_n)_{n \in \mathbb{N}} \), defined by

\[
S_n(s) := \int_0^s g_n(u) \, du, \quad s \in [0, 1], \quad n \in \mathbb{N},
\]

is complete in the Cameron-Martin space \( \mathbb{H} \), the space of all \( \mathbb{R}^d \)-valued absolutely continuous functions \( h \) on \( [0, 1] \) with \( h(0) = 0 \) endowed with the norm

\[
|h|_\mathbb{H} := \left( \int_0^1 \left| h'(s) \right|^2 ds \right)^{\frac{1}{2}}.
\]

2. Definition of the form and closability

For \( F \in Y \) and \( \nu \)-a.e. \( \gamma \in \mathbf{P}_{m_0}(M) \), define

\[
(2.1) \quad D_x F(\gamma) := \sum_{i=1}^k \chi_{[0,s_i]}(s) \{ T_{s_i} \{ \nabla_s f \} (\gamma) \}, \quad s \in [0, 1],
\]

where \( (\nabla_s f)(\gamma) \equiv (\nabla_s f)(\gamma(s_1); \ldots; \gamma(s_k)) \in T_{\gamma(s)} \) denotes the gradient of the function \( f \) relative to the \( i \)-th variable while holding the remaining variables fixed.
Here, $f$ and the $s_1, \ldots, s_k$ are as in the definition of $Z$. Furthermore, for $F \in Z$ and

$$D_s F(\gamma) := \int_0^s r_{L(\gamma)}^{-1}(s') D_{\nu'} F(\gamma) \, ds'$$

$$= \sum_{i=1}^k \mathcal{A} \left( \mathcal{H}_s f \right) = \sum_{i=1}^k s \wedge s_i \cdot r_{L(\gamma)}^{-1}(s_i)(\nabla_s f)(\gamma), \quad s \in [0, 1], \ \gamma \in \mathbf{P}_{m_0}(M),$$

we have $DF \in \mathbb{H} \, \nu$-a.e. See also (2.1).

Any $F \in Y$ has the representation $F(\gamma) = f(\gamma(s_1); \ldots; \gamma(s_k))$ where $s_1 = \frac{1}{2m}, \ldots, s_k = \frac{1}{2m}$, for some $k \in \mathbb{N}$, $n' \in \mathbb{N}$, $l_1, \ldots, l_k \in \{1, \ldots, 2^{m'}\}$, and $f \in C^\infty(M^k)$. As $H_{2^{m'}+1}(t) = 0$ on $[0, 1] \setminus \left[ \frac{1}{2m}, \frac{1}{2m'} \right]$, $L_{(i-1)/2m} H_{2^{m'+1}}(t) dt = 0$, and either $(\frac{1}{2m}, \frac{1}{2m'}) \subseteq [0, s_i]$, $(\frac{1}{2m}, \frac{1}{2m'}) \subseteq (s_i, 1]$ if $m \geq n'$, $l \in \{1, \ldots, 2^{m'}\}$, and $i \in \{1, \ldots, k\}$, from (2.2), we obtain the following lemma which is crucial for the technical procedure.

**Lemma 2.1.** Let $F \in Y$. There exists $n_0 \in \mathbb{N}$ such that, for $\nu$-a.e. $\gamma \in \mathbf{P}_{m_0}(M)$,

$$\langle S_n, DF(\gamma) \rangle_\mathbb{H} = 0, \quad n > n_0.$$

Let us define the diffusion operator we are dealing with in this paper. Choose an increasing sequence $(\lambda_i)_{i \in \mathbb{N}}$ of positive real numbers and define the operator

$$D(A) := \left\{ \Phi \in L^2(\mathbf{P}_{m_0}(M) \to \mathbb{H}, \nu) : \int \sum_{i=1}^\infty \lambda_i^2 \langle S_i, \Phi \rangle_\mathbb{H}^2 \, d\nu < \infty \right\},$$

$$\mathcal{A} \Phi(\gamma) := \sum_{i=1}^\infty \lambda_i \langle S_i, \Phi(\gamma) \rangle_\mathbb{H} S_i, \ \gamma \in \mathbf{P}_{m_0}(M), \quad \Phi \in D(A),$$

mapping $L^2(\mathbf{P}_{m_0}(M) \to \mathbb{H}, \nu) \supseteq D(A) \to L^2(\mathbf{P}_{m_0}(M) \to \mathbb{H}, \nu)$.

For $F \in Y$, we have $\int \langle S_i, DF(\gamma) \rangle_\mathbb{H}^2 \, d\nu < \infty$, $i \in \mathbb{N}$. By Lemma 2.1, for $F \in Y$, there is $n_0 \in \mathbb{N}$ such that, for all $i > n_0$, it holds that $\int \langle S_i, DF(\gamma) \rangle_\mathbb{H}^2 \, d\nu = 0$. Therefore, we obtain $\{DF : F \in Y\} \subseteq D(A)$. Furthermore, for all $F \in Y$, we get

$$\int \langle DF, ADF(\gamma) \rangle_\mathbb{H} \, d\nu = \int \left\langle DF, \sum_{i=1}^\infty \lambda_i \langle S_i, DF(\gamma) \rangle_\mathbb{H} S_i \right\rangle_\mathbb{H} \, d\nu$$

$$= \sum_{i=1}^\infty \lambda_i \int \langle S_i, DF(\gamma) \rangle_\mathbb{H}^2 \, d\nu$$

$$< \infty.$$

Consequently, the nonnegative symmetric bilinear form

$$E(F, F) := \int \langle DF, ADF(\gamma) \rangle_\mathbb{H} \, d\nu$$

$$= \int \left| A^{1/2} DF(\gamma) \right|_\mathbb{H}^2 \, d\nu, \quad F \in Y,$$

is well defined.

**Remarks.** (1) It is known from [5], Lemma 3, or [9], Proposition 5.3, that the operator $D : Z \to L^2(\mathbf{P}_{m_0}(M) \to \mathbb{H}, \nu)$ is closable on $L^2(\nu)$. Let $(\mathbf{D}, \mathbf{D}(\mathbf{D}))$ denote...
Theorem 2.2. The bilinear form $\mathcal{E}(Y, Y)$ is closable on $L^2(\nu)$.

Proof. Suppose $F_n \in Y$, $n \in \mathbb{N}$, with $F_n \to 0$ in $L^2(\nu)$ and $\mathcal{E}(F_n - F_m, F_n - F_m) \to 0$. In particular, (2.4) implies

$$A^{1/2}DF_n \to \Psi \quad \text{in} \quad L^2(P_{m_0}(M) \to \mathbb{H}, \nu)$$

for some $\Psi \in L^2(P_{m_0}(M) \to \mathbb{H}, \nu)$. Define

$$JF := \sum_{i=1}^{\infty} \lambda_i^{-1/2} (S_i, F)_{\mathbb{H}} S_i, \quad F \in L^2(P_{m_0}(M) \to \mathbb{H}, \nu).$$

Since $J$ is a bounded operator on $L^2(P_{m_0}(M) \to \mathbb{H}, \nu)$, we verify

$$DF_n = JA^{1/2}DF_n \to J\Psi \quad \text{in} \quad L^2(P_{m_0}(M) \to \mathbb{H}, \nu)$$

from (2.5). As $(D, Z)$ is closable on $L^2(\nu)$, we obtain $J\Psi = 0$. It follows from (2.6) and $\lambda_i > 0$, $i \in \mathbb{N}$, that $\Psi = 0$. Thus, relation (2.5) leads to $A^{1/2}DF_n \to 0$ in $L^2(P_{m_0}(M) \to \mathbb{H}, \nu)$ which implies $\mathcal{E}(F_n, F_n) = \int |A^{1/2}DF_n|_{\mathbb{H}}^2 d\nu \to 0$. 

Let $(\mathcal{E}, D(\mathcal{E}))$ denote the closure of $(\mathcal{E}, Y)$ on $L^2(\nu)$.

Remark. (3) Let $F \in D(\mathcal{E})$ and let $F_n \in Y$, $n \in \mathbb{N}$, be a sequence converging to $F$ in $E^1 = (|f|^2_{L^2(\nu)} + \mathcal{E}(\cdot, \cdot))^{1/2}$-norm. Since $\lambda_i > 0$, $i \in \mathbb{N}$, is an increasing sequence of positive real numbers and $(\mathcal{E}, Y) = (\mathcal{E}^O_1, Y)$ if $\lambda_i = 1$, $i \in \mathbb{N}$, from (2.3) it follows that $F_n, n \in \mathbb{N}$, is a Cauchy sequence in $(\mathcal{E}^O_1)^{1/2} = (|f|^2_{L^2(\nu)} + \mathcal{E}^O(\cdot, \cdot))^{1/2}$-norm. Therefore, $F_n \to F$ in $(\mathcal{E}^O_1)^{1/2}$-norm. Thus, we have $D(\mathcal{E}) \subseteq D(\mathcal{E}^O_1) = D(D)$.

Since, by self-adjointness, $A^{1/2}$ is a closed operator, it holds that $\{DF : F \in D(\mathcal{E})\} \subseteq D(A^{1/2})$ and relations (2.3) and (2.4) yield

$$\mathcal{E}(F, F) = \sum_{i=1}^{\infty} \int (S_i, DF)^2_{\mathbb{H}} d\nu
= \int |A^{1/2}DF|_{\mathbb{H}}^2 d\nu, \quad F \in D(\mathcal{E}).$$
3. QUASI-REGULARITY AND ASSOCIATED PROCESS

Let $h \in \mathbb{H}$, $t \in \mathbb{R}$, $s \in [0,1]$, and $\gamma \in \operatorname{P}_{m_0}(M)$ and let $\sigma$ denote the solution to the geometric flow equation

\[
\begin{cases}
\sigma^h(t,s)(\gamma) = T_{s-t}^{\sigma^h(t,s)}(\gamma) r_0 h(s), \\
\sigma^h(0,s)(\gamma) = \gamma(s).
\end{cases}
\]

Note that \(\sigma\) stands for differentiation with respect to $t$. In particular, we have $\sigma^h(\cdot, s)(\gamma) \in C^1(\mathbb{R} \to M)$, $\sigma^h(t)(\gamma) \equiv \sigma^h(t, \cdot)(\gamma) \in \operatorname{P}_{m_0}(M)$. For $h \in \mathbb{H}$ and $\nu$-a.e. $\gamma \in \operatorname{P}_{m_0}(M)$, there exists a unique solution (see [4] and [9]). For $F \in Y$ given as in (1.1), the directional derivative along the direction $h \in \mathbb{H}$ satisfies

\[
\partial_h F := \lim_{t \to 0} \frac{F(\sigma^h(t)) - F}{t}
\]

\[
= \sum_{i=1}^{k} \langle \nabla_s f, \sigma^h(0, s_i) \rangle T_{(s_i)}
\]

\[
= \sum_{i=1}^{k} \langle \nabla_s f, T_{s_i} r_0 h(s_i) \rangle T_{(s_i)}
\]

\[
= \sum_{i=1}^{k} \langle T_{0-s_i} \nabla_s f, r_0 h(s_i) \rangle T_{m_0}
\]

\[
= \langle DF; h \rangle_{\mathbb{H}} \quad \nu\text{-a.e.}
\]

(3.1)

See also [2].

Remark. (4) For every $h \in \mathbb{H}$, the operator $\partial_h : Z \to L^2(\nu)$ is closable on $L^2(\nu)$. Let $\partial_h, D(\partial_h)$ denote the corresponding closure. It holds that $D(\mathcal{E}) \subseteq D(D) \subseteq D(\partial_h)$, $h \in \mathbb{H}$, and $\partial_h F = \langle DF; h \rangle_{\mathbb{H}}$, $F \in D(D)$; cf. [9], Theorem 5.2 and Proposition 5.3. Therefore,

\[
\mathcal{E}(F,F) = \sum_{i=1}^{\infty} \lambda_i \int (\partial_{s_i} F)^2 \, d\nu, \quad F \in D(\mathcal{E}).
\]

Proposition 3.1. The form $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form on $L^2(\nu)$.

Proof. We have

\[
\mathcal{E}(F,F) = \sum_{i=1}^{\infty} \lambda_i \int (\partial_{s_i} F)^2 \, d\nu
\]

(3.2)

\[
= \sum_{i=1}^{\infty} \lambda_i \int \left( \frac{d}{dt} \right)_0 F(\sigma^{S_i}(t)) \, d\nu, \quad F \in Y.
\]

It follows directly from [13], Proposition I, 4.10, and the chain rule that $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form. \(\square\)

An important tool for the subsequent technical procedure will be the following assertion; cf. [13], Chapter IV, Lemma 4.1. Note that, for $u, v \in D(\mathcal{E})$, we have $u \lor v \in D(\mathcal{E}) \subseteq D(D) \subseteq D(\partial S_i)$, $i \in \mathbb{N}$ (see Remarks (3) and (4)).
\textbf{Lemma 3.2.} Let \( u, v \in D(\mathcal{E}) \). For all \( i \in \mathbb{N} \), we have
\[
\partial S_i (u \vee v) = \chi_{(u>v)} \partial S_i u + \chi_{(u<v)} \partial S_i v + \frac{1}{2} \chi_{(u=v)} (\partial S_i u + \partial S_i v) \quad \nu\text{-a.e.}
\]

\textit{Proof.} Having representation (3.2) of \((\mathcal{E}, Y)\) in mind, the proof can be obtained from that of [13], Chapter IV, Lemma 4.1 by replacing therein \( \frac{\partial}{\partial x} \) with \( \partial S_i \) and \( \mathcal{F} \mathcal{C}^\infty \) with \( Y \).

\textbf{Proposition 3.3.} Suppose
\[
\lambda_0 \leq cn^{1-\varepsilon}, \quad n \in \mathbb{N}, \quad \text{for some } c > 0 \text{ and } \varepsilon \in (0, 1).
\]

Then the Dirichlet form \((\mathcal{E}, D(\mathcal{E}))\) is quasi-regular.

\textit{Proof.} In steps 1-3 below, we show that there is an \( \mathcal{E} \)-nest consisting of compact sets.

\textbf{Step 1.} For \( r \in \mathbb{N}, l \in \{0, \ldots, 2^{r-1} - 1\} \), and \( k = 2^{r-1} + l \), set \( s_k := (2l + 1)^2 - r \).

Let \( x^v(p) \) denote the standard coordinates of \( p \in M \) embedded in \( \mathbb{R}^N \), \( v \in \{1, \ldots, N\} \). Fix \( \tau \in \mathbf{P}_{m_0}(M) \), \( k = 2^{r-1} + l \), and \( v \in \{1, \ldots, N\} \). Consider the functions \( f_{v,k,\tau}(p) := x^v(p) - x^v(\tau(s_k)) \) and \( v \in \mathbf{P}_{m_0}(M) \).

Obviously, \( f_{v,k,\tau} \) belongs to \( Y \). Furthermore, let either \( i = j \) or \( i = d(2^m + u - 1) + j \) for some \( m \in \{0, 1, \ldots\}, u \in \{1, \ldots, 2^m\} \) and \( j \in \{1, \ldots, d\} \).

We have
\[
|\langle S_i, Df_{v,k,\tau}(\gamma) \rangle|_H = |\left\langle T_{0-s_k}^\gamma \nabla_{s_k} x^v(\gamma(s_k)), r_0 S_i(s_k) \right\rangle|_{T_{m_0}}^H
\leq \|T_{0-s_k}^\gamma \nabla_{s_k} x^v(\gamma(s_k))\|_{T_{m_0}} \|r_0 S_i(s_k)\|_{T_{m_0}}
= \|\nabla_{s_k} x^v(\gamma(s_k))\|_{T_{\gamma(s_k)}} \|S_i(s_k)\|_{L^2}
\leq \|\nabla_{s_k} x^v(\gamma(s_k))\|_{T_{\gamma(s_k)}} \|S_i(s_k)\|_H^2
\]

for \( \nu\text{-a.e. } \gamma \in \mathbf{P}_{m_0}(M) \). As mentioned in [3], proof of Proposition 5, \( \|\nabla x^v(p)\|_{T_{\gamma}} \) is bounded by some constant \( K \) since \( Y \) is compact. Furthermore, the definitons of \( s_k \) and \( S_i \) yield \( |\langle S_i(s_k) \rangle| \leq 1 \) if \( i = j \) for some \( j \in \{1, \ldots, d\} \). Moreover, \( |\langle S_i(s_k) \rangle| \leq 2^{-m/2+1} \) if \( i = d(2^m + u - 1) + j \) for some \( m \in \{0, 1, \ldots\}, u \in \{1, \ldots, 2^m\} \), \( j \in \{1, \ldots, d\} \) and \( m < r \) as well as \( s_k = (2l + 1)^2 - r \in (\frac{u-1}{2^m}, \frac{u+1}{2^m}) \).

Otherwise, we have \( S_i(s_k) = 0 \). Therefore, (3.5) implies
\[
\sum_{i=1}^{\infty} \lambda_i \langle \partial S_i F_{v,k,\tau}(\gamma) \rangle^2 = \sum_{i=1}^{\infty} \lambda_i \langle S_i, Df_{v,k,\tau}(\gamma) \rangle^2_H
= \sum_{j=1}^{d} \lambda_j \langle S_j, Df_{v,k,\tau}(\gamma) \rangle^2_H
+ \sum_{m=0}^{\infty} \sum_{u=1}^{2^m} \sum_{j=1}^{d} \lambda_{d(2^{m+u-1} + j)} \langle S_d(2^{m+u-1} + j), Df_{v,k,\tau}(\gamma) \rangle^2_H
\leq K^2 d \lambda_d + K^2 d \sum_{m=0}^{\infty} \lambda_{d2^{m+1}} 2^{-m+2}
\]
for $\nu$-a.e. $\gamma \in P_{m_0}(M)$. Finally, from (3.3), we obtain

\begin{equation}
\sum_{i=1}^{\infty} \lambda_i (\partial S_i F_{v,k,\tau}(\gamma))^2 \leq K^2 c d^2 - \tau \frac{2^{1+\tau}}{2^{1+\tau}} = C_1 \tag{3.7}
\end{equation}

for $\nu$-a.e. $\gamma \in P_{m_0}(M)$ and

\begin{equation}
\mathcal{E}(F_{v,k,\tau}, F_{v,k,\tau}) \leq C_1, \tag{3.8}
\end{equation}

where the right-hand side is independent of $k$ (resp. $s_k$), $v \in \{1, \ldots, N\}$, and $\tau$.

**Step 2.** We apply a method introduced in [5] and [15]. Set

\[ G_{n,\tau} := \sup_{k \in [1^{\ldots n}]} |F_{v,k,\tau}|, \quad n \in \mathbb{N}. \]

It follows now from (3.7), relation

\[ \mathcal{E}(G_{n,\tau}, G_{n,\tau}) = \int \sum_{i=1}^{\infty} \lambda_i (\partial S_i G_{n,\tau}(\gamma))^2 \, d\nu, \]

and Lemma 3.2 that

\[ \mathcal{E}(G_{n,\tau}, G_{n,\tau}) \leq C_1, \quad n \in \mathbb{N}. \]

Since $M$ is compact, there exists $C_2 > 0$, such that

\[ |x^v(p)| \leq \frac{1}{2} \sqrt{C_2}, \quad p \in M, \quad v \in \{1, \ldots, N\}. \]

From (3.4) and the definition of $G_{n,\tau}$ it follows that $\|G_{n,\tau}\|_{L^2(\nu)} \leq C_2$ and, thus,

\begin{equation}
\mathcal{E}_1(G_{n,\tau}, G_{n,\tau}) \leq C_1 + C_2 =: C_3, \quad n \in \mathbb{N}. \tag{3.9}
\end{equation}

**Step 3.** In this step, we proceed as in [5] and [15]. In particular, we apply the Banach-Saks property of the Hilbert space $(D(\mathcal{E}), \ell^1(1/2))$, which states that every bounded sequence in $(D(\mathcal{E}), \ell^1(1/2))$ has a subsequence whose Cesaro means converge strongly (see, for example, [14]). Accordingly, relation (3.9) and the fact that the sequence $(G_{n,\tau})_{n \in \mathbb{N}}$ satisfies $G_{n,\tau} \leq G_{n+1,\tau}$, $n \in \mathbb{N}$, imply that the function

\[ H_{\tau}(\gamma) := \sup_{v \in [1^{\ldots N}]} |x^v(\gamma(s)) - x^v(\tau(s))|, \quad \gamma \in P_{m_0}(M), \]

belongs to $D(\mathcal{E})$ and that

\[ \mathcal{E}_1(H_{\tau}, H_{\tau}) \leq C_3. \]

Let $\{\tau_k : k \in \mathbb{N}\}$ be a dense set in $P_{m_0}(M)$. Set

\[ K_n := \inf_{1 \leq k \leq n} H_{\tau_k}, \quad n \in \mathbb{N}. \]

We have $K_n \in D(\mathcal{E})$. Again, recalling the Banach-Saks property of $(D(\mathcal{E}), \ell^1(1/2))$, the last relation implies that $\mathcal{E}_1(K_n, K_n) \rightarrow 0$. According to [13], Chapter III, Proposition 3.5, there exists a subsequence $K_{n_k}$, $k \in \mathbb{N}$, and an $\mathcal{E}$-nest $F_m$, $m \in \mathbb{N}$, such that $K_{n_k}$ converges uniformly to zero (as $k \rightarrow \infty$) on each $F_m$. Consult also [3], Section I.8. As in [3], proof of Proposition 5, it follows now from the definition of $K_n$, $n \in \mathbb{N}$, that each $F_m$ is totally bounded. Thus, $F_m$, $m \in \mathbb{N}$, form an $\mathcal{E}$-nest consisting of compact sets.

**Step 4.** For fixed $\tau \in P_{m_0}(M)$, the system of functions $F_{v,k,\tau}$, $v \in \{1, \ldots, N\}$, $k \in \mathbb{N}$, introduced in (3.4) separates the points in $P_{m_0}(M)$.
Together with Theorem 2.2 and the result of Step 3, quasi-regularity follows now from its definition (see [13], Chapter IV, Definition 3.1). □

**Proposition 3.4.** The form \((\mathcal{E}, D(\mathcal{E}))\) is local.

**Proof.** We follow [3], proof of Proposition 5 (ii) and [13], Example V.1.12. Let \(F, G \in D(\mathcal{E}) \cap L^\infty(\nu)\) with \(\text{supp}[F] \cap \text{supp}[G] = \emptyset\). According to [13], Propositions I.4.17 (i) and V.1.2 (ii), we have to verify \(\mathcal{E}(F, G) = 0\). Since \(\mathcal{E}(F, G) = \int (A^{1/2}DF, A^{1/2}DG)_\mathcal{H} \, d\nu\) (cf. (2.7)), it is sufficient to show that

\[
(3.10) \quad DF = 0 \quad \nu\text{-a.e. on } P_{m_0}(M) \setminus \text{supp}[F].
\]

From \(D(\mathcal{E}) \subseteq D(\mathcal{E}^{OU})\) and [3], equation (11), we obtain

\[
(3.11) \quad D(U \cdot V) = U \cdot D(V) + V \cdot D(U), \quad U, V \in D(\mathcal{E}) \cap L^\infty(\nu).
\]

See also [13], Example V.1.12. Furthermore, from [13], Proposition V.1.7, we get the existence of \(V \in D(\mathcal{E}) \cap L^\infty(\nu)\) with \(0 \leq V \leq \chi_{P_{m_0}(M) \setminus \text{supp}[F]}\) and \(V > 0\) \(\nu\text{-a.e. on } P_{m_0}(M) \setminus \text{supp}[F]\); here \(\chi\) denotes the indicator function. Now, relation (3.11) implies

\[
0 = F \cdot D(V) + V \cdot D(F) \quad \nu\text{-a.e.}
\]

This yields (3.10). □

As a consequence of Propositions 3.3 and 3.4 we get with [13], Theorems IV.3.5 and V.1.11:

**Theorem 3.5.** There exists a diffusion process \(M\) associated with \((\mathcal{E}, D(\mathcal{E}))\).

4. Generator

We start with a technical lemma.

**Lemma 4.1.** Let \(F \in Y\) and \(n \in \mathbb{N}\). Then the derivatives \(\frac{d}{dt} \bigg|_0 \partial S_n F(\sigma^S_n(-t))\), \(\frac{d}{dt} \bigg|_0 \frac{dv \circ \sigma^S_n(-t)}{dv}\), and \(\frac{d}{dt} \bigg|_0 \left\{ \partial S_n F(\sigma^S_n(-t)) \frac{dv \circ \sigma^S_n(-t)}{dv} \right\}\) exist in \(L^2(\nu)\) and we have

\[
(4.1) \quad \frac{d}{dt} \bigg|_0 \left\{ \partial S_n F(\sigma^S_n(-t)) \frac{dv \circ \sigma^S_n(-t)}{dv} \right\} = -\partial S_n \partial S_n F + \frac{d}{dt} \bigg|_0 \frac{dv \circ \sigma^S_n(-t)}{dv} \cdot \partial S_n F \quad \nu\text{-a.e.}
\]

**Proof.** Step 1. Introduce

\[
\varphi_t(\gamma) := \frac{dv \circ \sigma^S_n(-t)}{dv}(\gamma), \quad \gamma \in P_{m_0}(M), \quad t \in \mathbb{R}.
\]

The existence of \(\frac{d}{dt} \bigg|_0 \varphi_t\) in \(L^2(\nu)\) is shown in [4], Theorem 8.5 and in the proof of Theorem 9.1. Note that the result for \(h \in C^1([0,1] \rightarrow \mathbb{R}^d)\) presented in [4] can be extended to general \(h \in \mathbb{H}\) by [13], Theorems 3.5 and 4.1 and the proof of Theorem 5.1.

Step 2. Let \(n \in \mathbb{N}\) and let \(F \in Y\) be given as in (1.1). According to (3.3), we have, for \(\nu\text{-a.e. } \gamma \in P_{m_0}(M),\)

\[
\psi_t(\gamma) := \partial S_n F(\sigma^S_n(-t)(\gamma))
\]

\[
= \sum_{i=1}^k \left\langle T_{0-s_i}^{\sigma^S_n(\cdot-s_i)}(\nabla s_i f)(\sigma^S_n(\cdot-s_i)(\gamma)), r_0 S_n(s_i) \right\rangle_{T_{m_0}}, \quad t \in \mathbb{R}.
\]
From \(|S_n(s)|_{R^d} \leq 1, s \in [0,1]\), it follows that, for \(\nu\)-a.e. \(\gamma \in P_{m_0}(M)\),
\[
|\psi_t(\gamma)| \leq \sum_{i=1}^{k} \left\| T_{\sigma_i}^{S_n(\gamma)}(\nabla_s f)(\sigma_i) \right\|_{T_{m_0}} \| r_0 S_n(s_i) \|_{T_{m_0}}
\leq \sum_{i=1}^{k} \left\| (\nabla_s f)(\sigma_i) \right\|_{T_{\sigma_i}^{S_n(\gamma)}} , \quad t \in \mathbb{R}.
\]

Since \(f \in C^\infty(M^k)\), and \(M\) is compact, there exists \(C_4 > 0\) such that
\[
(4.2) \quad |\psi_t(\gamma)| \leq C_4 , \quad \nu\text{-a.e. } \gamma \in P_{m_0}(M) , \quad t \in \mathbb{R}.
\]
Furthermore, in virtue of [9], Theorem 4.1 (iii),
\[
(4.3) \quad \psi_t \xrightarrow{t \to 0} \psi_0 \quad \nu\text{-a.e.}
\]

**Step 3.** The aim of this step is to verify the existence of \(\frac{d}{d\nu}|0| \psi_t \in L^2(\nu)\). To this end, fix \(i \in \{1, \ldots, k\}\). Since \((\nabla_s f)^v\) is then a smooth function on \(M^k\), from [9], Section 5, and [4], Lemma 9.1, it follows that
\[
(4.4) \quad \frac{d}{d\nu}|0| \left( (\nabla_s f)(\sigma_i) \right)^v \text{ exists in } L^2(\nu) , \quad v \in \{1, \ldots, N\}.
\]
Furthermore, there is a \(C_5 > 0\) such that
\[
(4.5) \quad |(\nabla_s f)^v| \leq C_5 , \quad v \in \{1, \ldots, N\}.
\]
By [4], Corollary 4.2 and inequalities (i) as well as (ii) of Lemma 4.1 of the same reference, for all \(v \in \{1, \ldots, N\}\),
\[
(4.6) \quad \frac{d}{d\nu}|0| \left( T_{\sigma_i}^{S_n(\gamma)}(r_0 S_n(s_i))^v \right) = \frac{d}{d\nu}|0| \left( r_{\xi i}^{S_n(\gamma)}(r_0 S_n(s_i))^v \right) \text{ exists in } L^2(\nu)
\]
if \(n = j \in \{1, \ldots, d\}\). Even though in [4] the geometric flow is generated by a \(C^1\)-function, for \(n = d(2^m + l - 1) + j\) with \(m \in \{0, 1, \ldots, l \in \{1, \ldots, 2^m\}, j \in \{1, \ldots, d\}\), we may obtain \(\frac{d}{d\nu}|0|\) from the above reference by decomposing \(S_n = S_n^1 + S_n^2 + S_n^3\) where
\[
S_n^1 = 2^m \chi_{[l, l+1]}(s) (s - \frac{l-1}{2^m}) , \\
S_n^2 = -2^{m+1} \chi_{[l, l+1]}(s) (s - \frac{2l-1}{2^{m+1}}) , \\
S_n^3 = 2^{m} \chi_{[l, l+1]}(s) (s - \frac{l}{2^m}).
\]
Moreover, by isometric embedding of \(M\) into \(\mathbb{R}^N\) we verify
\[
(4.7) \quad |(T_{\sigma_i}^{S_n(\gamma)}(r_0 S_n(s_i))^v|_{\mathbb{R}^N} \leq |T_{\sigma_i}^{S_n(\gamma)}(r_0 S_n(s_i))|_{\mathbb{R}^N} = |T_{\sigma_i}^{S_n(\gamma)}(r_0 S_n(s_i))|_{T_{\gamma_i}} = |S_n(s_i)|_{R^d} \leq 1, \quad v \in \{1, \ldots, N\}.
\]
Introduce
\[
\alpha_i^v := \left( (\nabla_s f)(\sigma_i) (\sigma_i) \right)^v , \quad \beta_i^v := \left( T_{\sigma_i}^{S_n(\gamma)}(r_0 S_n(s_i))^v , \right.
\]
\[
\beta_i^v := \left( T_{\sigma_i}^{S_n(\gamma)}(r_0 S_n(s_i))^v , \right.
\]

$t \in \mathbb{R}$, $v \in \{1, \ldots, N\}$. The existence of
\[
\frac{d}{dt} \left|_0 \sum_{v=1}^N \left( (\nabla_s f)(\sigma^{S_n}(-t)(\gamma)) \right)^v \left( T_{s_{i,-0}} \sigma^{S_n}(-t)(\gamma) \right)^v_{0} S_n(s_i) \right)^v = \frac{d}{dt} \left|_0 \sum_{v=1}^N \alpha^v_i \beta^v_i \right|
\]
in $L^2(\nu)$ follows from (4.4)-(4.7), $\alpha^v_i \rightarrow \alpha^v_0$ $\nu$-a.e., and the inequality
\[
\left| \sum_{v=1}^N \alpha^v_i \beta^v_i - \sum_{v=1}^N \alpha^v_0 \beta^v_0 \right| \leq \sum_{v=1}^N \left| \left( \frac{\alpha^v_i - \alpha^v_0}{t} \right) \beta^v_0 \right| L^2(\nu) + C_5 \sum_{v=1}^N \left| \frac{\beta^v_i - \beta^v_0}{t} - \beta^v_0 \right| L^2(\nu)
\]
\[
+ \sum_{v=1}^N \left| (\alpha^v_i - \alpha^v_0) \beta^v_0 \right| L^2(\nu)
\]
ote that, by (4.5), $((\beta^v_i - \beta^v_0) ^2 \in L^1(\nu) \nu \in \{1, \ldots, N\}$. Finally, by isometry
\[
\frac{d}{dt} \left|_0 \psi(\gamma) \right| = \sum_{i=1}^k \frac{d}{dt} \left|_0 \left( (\nabla_s f)(\sigma^{S_n}(-t)(\gamma)) \right)^v \left( T_{s_{i,-0}} \sigma^{S_n}(-t)(\gamma) \right)^v_{0} S_n(s_i) \right)^v
\]
extists $L^2(\nu)$.

Step 4. Having in mind that $\varphi_0 = 1$ and that $((\psi_t - \psi_0)\varphi_0)^2 \in L^1(\nu)$ is dominated by $4C^2_0(\varphi_0)^2 \in L^1(\nu)$, the existence of $\frac{d}{dt} \left|_0 \varphi_t \right|$ and $\frac{d}{dt} \left|_0 \psi_t \right|$ in $L^2(\nu)$ (cf. Steps 1 and 3), relations (4.2), (4.3), and
\]
\[
(4.8)
\]
\[
\left| \frac{\varphi_t \psi_t - \varphi_0 \psi_0}{t} - \varphi_0 \psi_0 - \varphi_0 \psi_0 \right| \leq C_4 \left| \frac{\varphi_t - \varphi_0}{t} - \varphi_0 \right| L^2(\nu) + \left| \frac{\psi_t - \psi_0}{t} - \psi_0 \right| L^2(\nu) + \left| (\varphi_t - \varphi_0) \varphi_t \right| L^1(\nu)
\]
\[
imply the existence of $\frac{d}{dt} \left|_0 \left\{ \frac{\partial S_n}{\partial \nu} \left( \sigma^{S_n}(-t) \right) \frac{\partial \sigma^{S_n}(-t)}{\partial \nu} \right\} \right| \in L^2(\nu)$.

Step 5. As shown, for example in [11], 4.1, it holds that $\partial S_n \eta = \frac{d}{dt} \left|_0 \eta(\sigma^{S_n}(-t)) \right.$, whenever $\eta \in D(D) \subseteq D(\partial S_n)$ and the limit $\lim_{t \rightarrow 0} \eta(\sigma^{S_n}(-t)) - \eta \in L^2(\nu)$. Since $F \in Y$ belongs to $D(D^i)$ (see [4], Section 6), we have $\partial S_n F = (D_1 F)^i \in D(D)$ if $n = j \in \{1, \ldots, d\}$ and we have $\partial S_n F = 2^{m/2}(-D_1 F + 2D \frac{1}{1} F - D_1 F) \in D(D)$ if $n = j \in \{1, \ldots, d\}$ and we have $\partial S_n F = 2^{m/2}(-D_1 F + \frac{1}{1} F - D_1 F) \in D(D)$ if $n = j \in \{1, \ldots, d\}$. Therefore, relation (4.1) is a consequence of Step 4, in particular, of (4.8).

Let $(A, D(A))$ denote the generator of $(\mathcal{E}, D(\mathcal{E}))$. Fix a version $\hat{H}$ of the map $\gamma \rightarrow T_{s_{i,-0}}$. Using B.K. Driver’s geometrical notation (see [4], Definition 6.2 and
Theorem 9.1), we introduce
\begin{equation}
  z_n(\gamma) = \frac{1}{2} \int_0^1 \left( \text{Ric}_{\overset{\cdot}{\gamma}}(S_n) + \overset{\cdot}{\Theta}_{\overset{\cdot}{\gamma}}(S_n) \right) \cdot dx + \int_0^1 S_n' \cdot dx, \quad n \in \mathbb{N},
\end{equation}
where \( \gamma = I(x) \in P_{m_0}(M) \) for \( \mu \)-a.e. \( x \in X \).

**Theorem 4.2.** Let \( F \in Y \). We have \( F \in D(A) \) and
\begin{equation}
  AF = \sum_{n=1}^{\infty} \lambda_n \left\{ \partial S_n \partial S_n F + z_n \partial S_n F \right\} \text{ \( \nu \)-a.e.}
\end{equation}

**Proof.** Let \( G \in Y \). From (3.2), we obtain
\begin{equation}
  \mathcal{E}(F, G) = \int \sum_{n=1}^{\infty} \lambda_n \left. \frac{d}{dt} \right|_0 G(\sigma^{S_n}(t)) \partial S_n F \, d\nu.
\end{equation}
Taking into consideration that \( \left. \frac{d}{dt} \right|_0 G(\sigma^{S_n}(t)) \) exists in \( L^2(\nu) \), that \( \partial S_n F = \langle S_n, DF \rangle \in L^2(\nu) \), and that the sum in (4.9) is, by Lemma 2.1 actually a finite one, we obtain
\begin{equation}
  \mathcal{E}(F, G) = \left. \frac{d}{dt} \right|_0 \int \sum_{n=1}^{\infty} \lambda_n G(\sigma^{S_n}(t)) \partial S_n F \, d\nu.
\end{equation}
Under the substitution \( \gamma \to \sigma^{S_n}(-t)(\gamma) \), we get
\begin{align}
  \mathcal{E}(F, G) &= \left. \frac{d}{dt} \right|_0 \int G \cdot \sum_{n=1}^{\infty} \lambda_n \partial S_n F(\sigma^{S_n}(-t)) \, d\nu \circ \sigma^{S_n}(-t) \\
  &= \left. \frac{d}{dt} \right|_0 \int G \cdot \sum_{n=1}^{\infty} \lambda_n \partial S_n F(\sigma^{S_n}(-t)) \frac{d\nu \circ \sigma^{S_n}(-t)}{dt} \, d\nu.
\end{align}
From Lemma 4.1 it can be concluded that
\begin{equation}
  \mathcal{E}(F, G) = \int G \cdot \sum_{n=1}^{\infty} \lambda_n \left. \frac{d}{dt} \right|_0 \left\{ \partial S_n F(\sigma^{S_n}(-t)) \frac{d\nu \circ \sigma^{S_n}(-t)}{dt} \right\} \, d\nu;
\end{equation}
again, take into consideration that the above sum is actually a finite one. Let \( G^* \) be an arbitrary function belonging to \( D(\mathcal{E}) \). As \( Y \) is by Theorem 2.2 dense in \( D(\mathcal{E}) \) with respect to \( \mathcal{E}_{1/2} \)-norm, we find a sequence \( G_m \in Y, m \in \mathbb{N} \), with \( G_m \xrightarrow{m \to \infty} G^* \) in \( \mathcal{E}_{1/2} \)-norm. In particular, \( G_m \xrightarrow{m \to \infty} G^* \) in \( L^2(\nu) \). On account of \( |\mathcal{E}(G_m - G^*, F) + (G_m - G^*, F)_{L^2(\nu)}| = |\mathcal{E}_1(G_m - G^*, F)| \xrightarrow{m \to \infty} 0 \), we verify \( \mathcal{E}(G_m, F) \xrightarrow{m \to \infty} \mathcal{E}(G^*, F) \). Now, relation (4.10) and Lemma 4.1 yield
\begin{equation}
  \mathcal{E}(F, G^*) = \int G^* \cdot \sum_{n=1}^{\infty} \lambda_n \left. \frac{d}{dt} \right|_0 \left\{ \partial S_n F(\sigma^{S_n}(-t)) \frac{d\nu \circ \sigma^{S_n}(-t)}{dt} \right\} \, d\nu.
\end{equation}
Therefore, we have \( F \in D(A) \) and
\begin{equation}
  AF = -\sum_{n=1}^{\infty} \lambda_n \left. \frac{d}{dt} \right|_0 \left\{ \partial S_n F(\sigma^{S_n}(-t)) \frac{d\nu \circ \sigma^{S_n}(-t)}{dt} \right\} \text{ \( \nu \)-a.e.}
From Lemma 4.1, it can be deduced that

\[ AF = \sum_{n=1}^{\infty} \lambda_n \left\{ \partial_{S_n} \partial_{S_n} F - \left. \frac{d}{dt} \right|_{t=0} \frac{d\nu \circ \sigma^S_n(t)}{d\nu} \right\} \quad \nu\text{-a.e.} \]  

According to [4], Lemma 9.2, or [9], proof of Theorem 5.1, it holds that

\[ \frac{d}{dt} \left. \frac{d\nu \circ \sigma^S_n(t)}{d\nu} \right|_{t=0} = -z_n \quad \nu\text{-a.e.} \]

From (4.11), it follows that

\[ AF = \sum_{n=1}^{\infty} \lambda_n \left\{ \partial_{S_n} \partial_{S_n} F + z_n \partial_{S_n} F \right\} \quad \nu\text{-a.e.} \]

Remark. (5) Keeping in mind that, for \( F \in Y \), the sum \( \sum_{n=1}^{\infty} \lambda_n z_n \partial_{S_n} F \) is actually a finite sum (cf. Lemma 2.1), we can verify the following identity:

\[ \sum_{n=1}^{\infty} \lambda_n z_n \partial_{S_n} F(\gamma) \]

\[ = \sum_{n=1}^{\infty} \lambda_n \left\{ \frac{1}{2} \int_0^1 \left( \text{Ric}_{\gamma} \langle S_n \rangle + \Theta_{\gamma} \langle S_n \rangle \right) \cdot dx + \int_0^1 S'_n \cdot dx \right\} \partial_{S_n} F(\gamma) \]

\[ = \frac{1}{2} \int_0^1 \left( \text{Ric}_{\gamma} \langle \sum_{n} \lambda_n \partial_{S_n} F S_n \rangle + \Theta_{\gamma} \langle \sum_{n} \lambda_n \partial_{S_n} F S_n \rangle \right) \cdot dx \]

\[ + \int_0^1 \left( \sum_{n} \lambda_n \partial_{S_n} F S'_n \right) \cdot dx \]

\[ = \frac{1}{2} \int_0^1 \left( \text{Ric}_{\gamma} \langle (\text{AD}F)(s)(\gamma) \rangle + \Theta_{\gamma} \langle (\text{AD}F)(s)(\gamma) \rangle \right) \cdot dx_s \]

\[ + \int_0^1 (\text{AD}F)'(s)(\gamma) \cdot dx_s \]

\( \nu\text{-a.e.} \), where \( \gamma = I(x) \in P_{m_0}(M) \) for \( \mu\text{-a.e.} \ x \in X \).

5. LOCAL SECOND MOMENT

We begin this section with a general lemma. For a moment, we introduce a new setting more general than the situation in Sections 1-4. Let \( E \) be a Hausdorff topological space and let \( \mathcal{B}(E) \) denote its Borel \( \sigma \)-algebra. Suppose, furthermore, that \( \mathcal{B}(E) = \sigma(C(E)) \) where \( C(E) \) denotes the set of all continuous functions on \( E \). Let \( \nu \) be a probability measure on \( (E, \mathcal{B}) \) and \( (E, D(\mathcal{E})) \) a quasi-regular Dirichlet form on \( L^2(E, \nu) \). Let \( (A, D(A)) \) denote the generator of \( (\mathcal{E}, D(\mathcal{E})) \) and assume that there exists a subspace \( \mathbb{G} \subseteq D(A) \), dense in \( (D(\mathcal{E}), \mathcal{E}^{1/2}) \), such that \( g \in \mathbb{G} \) implies that \( g^2 \in D(A) \). Then, according to [34], Proposition I.4.1.3 and Corollary I.4.2.3, there exists a unique carré du champ operator \( \Gamma : D(\mathcal{E}) \times D(\mathcal{E}) \to L^1(E, \nu) \). In particular, for all \( f, g \in D(\mathcal{E}) \cap L^\infty(E, \nu) \), it holds that

\[ \int g\Gamma(f, f) \, d\nu = -\mathcal{E}(g, f^2) + 2\mathcal{E}(fg, f) \]
Let $M = (\Omega, \mathcal{F}, (X_t)_{t \geq 0}, (P_x)_{x \in E})$ denote the associated right process, set $1(x) = 1$, $x \in E$, and define

$$T_t f(x) := \int f(y) P_x(X_t \in dy), \quad t \geq 0, \ x \in E, \ f \in L^2(E, \nu).$$

**Proposition 5.1.** Suppose $1 \in \mathbb{G}$, $T_t 1 = 1$ for all $t \geq 0$, and $f g \in D(\mathcal{E}) \cap L^\infty(E, \nu)$ if $f \in D(\mathcal{E}) \cap L^\infty(E, \nu)$ and $g \in \mathbb{G}$. Then we have

$$\lim_{t \to 0} \frac{1}{t} \int (f(y) - f(\cdot))^2 P(X_t \in dy) = \Gamma(f, f) \text{ weakly in } L^1(E, \nu), \ f \in D(\mathcal{E}) \cap L^\infty(E, \nu).$$

**Proof.** Step 1. Let $Id$ denote the identity in $L^2(E, \nu)$ and let $f \in D(\mathcal{E}) \cap L^\infty(E, \nu)$ and $g \in \mathbb{G}$. Since $g \in D(A) \subseteq L^2(E, \nu)$, $f^2 \in L^\infty(E, \nu) \subseteq L^2(E, \nu)$, and $T_t - Id$ is a selfadjoint operator in $L^2(E, \nu)$, $t > 0$, we can conclude

$$-\mathcal{E}(g, f^2) = (Ag, f^2)_{L^2(E, \nu)} = \left( \lim_{t \to 0} \frac{1}{t} (T_t - Id)g, f^2 \right)_{L^2(E, \nu)} = \lim_{t \to 0} \left( \frac{1}{t} (T_t - Id)g, f^2 \right)_{L^2(E, \nu)} = \lim_{t \to 0} \left( \frac{g}{t} (T_t - Id)f^2 \right)_{L^2(E, \nu)}.$$

(5.2)

Let $(E_\lambda)_{\lambda \geq 0}$ denote the (right continuous) resolution of the identity with respect to $-A$, i.e.,

$$-Af = \int_{[0, \infty)} \lambda dE_\lambda f, \ f \in D(A),$$

recall that $A$ is a nonpositive definite selfadjoint operator in $L^2(E, \nu)$. The closed form $(\mathcal{E}, D(\mathcal{E}))$ has, therefore, a representation

$$\mathcal{E}(f, f) = \int_{[0, \infty)} \lambda d\|E_\lambda f\|_{L^2(E, \nu)}^2, \ f \in D(\mathcal{E}).$$

Taking into consideration $-\lambda \leq \frac{e^{-\lambda t} - 1}{t} \leq 0$ if $\lambda \geq 0$ and $t > 0$, we obtain

$$\mathcal{E}(fg, f) = \int_{[0, \infty)} \lambda d(E_\lambda(fg), E_\lambda f)_{L^2(E, \nu)} = -\lim_{t \to 0} \int_{[0, \infty)} \frac{e^{-\lambda t} - 1}{t} d(E_\lambda(fg), E_\lambda f)_{L^2(E, \nu)} = -\lim_{t \to 0} \left( \int_{[0, \infty)} dE_\lambda(fg), \int_{[0, \infty)} \frac{e^{-\lambda t} - 1}{t} dE_\lambda f \right)_{L^2(E, \nu)} = -\lim_{t \to 0} \left( \frac{fg}{t} \left( T_t f - f \right) \right)_{L^2(E, \nu)}.$$

(5.3)
Combining $T_1^1 = 1$, $t \geq 0$, (5.1), (5.2), and (5.3), we verify
\[
\int g\Gamma(f, f) \, d\nu = -E(g, f^2) + 2E(fg, f)
\]
\[
= \lim_{t \to 0} \int g \cdot \left( \frac{T_1 f^2 - f^2}{t} - 2f \frac{T_1 f - f}{t} \right) \, d\nu
\]
(5.4)
\[
= \lim_{t \to 0} \int g(x) \cdot \frac{1}{t} \int (f(y) - f(x))^2 P_r(X_t \in dy) \, d\nu(dx).
\]

Step 2. Again, let $f \in D(E) \cap L^\infty(E, \nu)$. For $g = 1$, relation (5.4) reduces to
\[
2E(f, f) = \lim_{t \to 0} \frac{1}{t} \int (f(y) - f(x))^2 P(X_t \in dy) \quad \text{in } L^1(E, \nu).
\]

Setting
\[
\phi_t := \begin{cases} 
\Gamma(f, f) & \text{if } t = 0, \\
\frac{1}{t} \int (f(y) - f(x))^2 P(X_t \in dy) & \text{if } t > 0
\end{cases}
\]
from $\phi_t = \frac{T_1 f^2 - f^2}{t} - 2f \frac{T_1 f - f}{t}$, $t > 0$, and (5.5) it follows that $\|\phi_t\|_{L^1(E, \nu)}$ is continuous on $[0, \infty)$ and $\lim_{t \to 0} \|\phi_t\|_{L^1(E, \nu)} = 0$. Thus, the family $(\phi_t)_{t \geq 0}$ is uniformly bounded in $L^1(E, \nu)$. Now the statement of the lemma is a consequence of relation (5.4). 

In the remainder of this section, we follow the setting of Sections 1-4. In particular, let $M = (\Omega, F, (X_t)_{t \geq 0}, (P_t)_{\gamma \in P(\mu(M))})$ denote the right process associated with $(E, D(E))$; cf. Theorem 3.3. Furthermore, recall that $x^v(p)$, $v \in \{1, \ldots, N\}$, denote the standard coordinates of $p \in M$ embedded in $\mathbb{R}^N$. For fixed $s \in [0, 1]$ and $v \in \{1, \ldots, N\}$, introduce the function $x^v_s$ by $x^v_s(\gamma) := x^v(s(\gamma))$, $\gamma \in P(\mu(M))$.

**Lemma 5.2.** Suppose the validity of relation (3.3), i.e.,
\[
\lambda_i \leq c_i^{1-\varepsilon}, \quad i \in \mathbb{N}, \text{ for some } c > 0 \text{ and } \varepsilon \in (0, 1).
\]
Then $x^v_s \in D(E) \cap L^\infty(\nu)$ and there exists $C_0 > 0$ such that
\[
E(x^v_s, x^v_s) \leq C_0
\]
(5.6)
for all $s \in [0, 1]$ and $v \in \{1, \ldots, N\}$.

**Proof.** Let $v \in \{1, \ldots, N\}$, $s \in [0, 1]$, and $s_n \in \{\frac{m}{2^n} : t \in \{1, \ldots, 2^m\}, m \in \mathbb{N}\}$, $n \in \mathbb{N}$, be a sequence with $s_n \to s$. Then
\[
x^v_{s_n} \to x^v_s \quad \nu\text{-a.e.}
\]
(5.7)
Furthermore, as $M$ is compact, there is a constant $C_7 > 0$ such that
\[
\|x^v_{s_n}\|_{L^2(\nu)} \leq C_7,
\]
(5.8)
independent of $n \in \mathbb{N}$. As in (3.5)-(3.8), it follows from (3.3) that
\[
E(x^v_{s_n}, x^v_{s_n}) \leq C_1,
\]
(5.9)
independent of $n \in \mathbb{N}$, where $C_1$ is the constant introduced in (3.7). Now, the above mentioned Banach-Saks property of the Hilbert space $(D(E), E^{1/2})$ and (5.7)-(5.9) imply $x^v_s \in D(E) \cap L^\infty(\nu)$ and from the closedness of $(E, D(E))$ on $L^2(\nu)$, relation (5.6) with $C_6 := C_1 + C_7$ can be derived. 

\[\square\]
Set $G := Y$. According to Theorems 2.2 and 3.2 $G$ is dense in $(D(E), E^{1/2})$ and we have $G \subseteq D(A)$. Obviously, $g \in G$ implies $g^2 \in G$. Thus, there exists a carré du champ operator $\Gamma$ and we have (5.11). In order to formulate the following theorem, we notice that, for $h \in L^\infty(\nu)$,
\[
\int |h| \cdot \Gamma(x^v_s, x^v_s) \, d\nu \leq 2\|h\|_{L^\infty(\nu)} \cdot \mathcal{E}(x^v_s, x^v_s) \\
\leq 2C_0\|h\|_{L^\infty(\nu)},
\]
independent of $s \in [0,1]$ and $v \in \{1, \ldots, N\}$ (cf. (5.6)), which implies
\[
\text{(5.10)} \quad \int_{\gamma \in \mathbb{P}_{m_0}} |h| \cdot \sum_{v=1}^{N} \int_{[0,1]} \Gamma(x^v_s(\gamma), x^v_s(\gamma)) \, d\nu(\gamma) \leq 2NC_0\|h\|_{L^\infty(\nu)}.
\]

**Theorem 5.3.** Suppose that relation (3.3) is valid, i.e.,
\[
\lambda_i \leq c_i l^{1-\varepsilon}, \quad i \in \mathbb{N}, \quad \text{for some} \ c > 0 \ \text{and} \ \varepsilon \in (0,1).
\]
For all $h \in L^\infty(\nu)$, we have
\[
\lim_{t \to 0} \int h(\tau) \cdot \frac{1}{t} \int_{[0,1]} \sum_{v=1}^{N} \|x^v_s(\gamma) - x^v_s(\tau)\|_E^2 \, d\tau \gamma(X_t \in d\gamma) \, \nu(d\tau) = \int h(\tau) \cdot \frac{1}{t} \int_{[0,1]} \Gamma(x^v_s, x^v_s) \, d\nu(\gamma).
\]

**Proof.** In order to apply Proposition 5.1, we note that $1 \in G, P_\tau(X_t \in \mathbb{P}_{m_0}(M)) = 1, t \geq 0, \tau \in \mathbb{P}_{m_0}(M)$, and that because of $G = Y \subseteq D(E) \cap L^\infty(\nu)$, we have $fg \in D(E) \cap L^\infty(\nu)$ if $f \in D(E) \cap L^\infty(\nu)$ and $g \in G$.

By virtue of Lemma 5.2 and Proposition 5.1 for all $s \in [0,1], v \in \{1, \ldots, N\},$ and $h \in L^\infty(\nu)$, it holds that
\[
\lim_{t \to 0} \int h(\tau) \cdot \frac{1}{t} \int (x^v_s(\gamma) - x^v_s) P(X_t \in d\gamma) \, d\nu = \int h(\gamma) (x^v_s, x^v_s) \, d\nu.
\]
Since \(\frac{1}{t} \int (x^v_s(\gamma) - x^v_s) P(X_t \in d\gamma)\) is bounded for $t > 0$ (cf. proof of Proposition 5.1), it can be concluded from dominated convergence that, for all $v \in \{1, \ldots, N\},$
\[
\lim_{t \to 0} \int_{[0,1]} \int h(\tau) \cdot \frac{1}{t} \int (x^v_s(\gamma) - x^v_s) P(X_t \in d\gamma) \, d\nu(\gamma) \, d\tau = \int_{[0,1]} h(\gamma, x^v_s, x^v_s) \, d\nu(\gamma) = h \in L^\infty(\nu).
\]
Relation (5.11) is now a direct consequence of (5.10) and Fubini’s theorem. \(\square\)

**Remark.** (6) Condition (3.3) in Proposition 5.2, Lemma 5.2 and Theorem 5.3 can be weakened. Recalling (3.3) it turns out that it is sufficient to require
\[
\sum_{m=1}^{\infty} \lambda_{d2^m} 2^{-m} < \infty
\]
instead of (3.3).
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REFERENCES


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