TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 356, Number 10, Pages 3939–3950
S 0002-9947(04)03426-9
Article electronically published on January 16, 2004

IDENTITIES OF GRADED ALGEBRAS AND CODIMENSION GROWTH

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ABSTRACT. Let $A = \oplus_{g\in G} A_g$ be a G-graded associative algebra over a field of characteristic zero. In this paper we develop a conjecture that relates the exponent of the growth of polynomial identities of the identity component A_e to that of the whole of A, in the case where the support of the grading is finite. We prove the conjecture in several natural cases, one of them being the case where A is finite dimensional and A_e has polynomial growth.

1. Introduction

Let A be an associative algebra over a field F and let G be a finite group. We say that A is G-*graded* if there exists a decomposition of A into the direct sum of subspaces $A = \oplus_{g\in G} A_g$ such that $A_g A_h \subset A_{gh}$. If e is the identity element of G, then the subspace A_e is called the *identity* (or the *neutral*) component of A with respect to the given G-grading. Clearly, A_e is a subalgebra of A. We study the relationship between the polynomial identities of A and A_e.

In the case of an infinite group G we can easily construct an example of a G-graded algebra A without non-trivial identities such that A_e is a PI-algebra. For instance, any free associative algebra $A = F\langle X\rangle$ admits a natural $\mathbb{Z}$-grading by degrees of its elements in X. With respect to this grading the identity component A_0 is zero and satisfies all possible polynomial identities. Obviously, A itself has no non-trivial identities. The situation is quite different in the case of finite groups. It is well known that $A = \oplus_{g\in G} A_g$ is a PI-algebra if and only if A_e satisfies a non-trivial polynomial identity provided that $|G| < \infty$ [1]. Moreover, even if G is infinite but the number of elements $g \in G$ such that $A_g \neq 0$ is finite, any non-trivial identity of A_e implies a non-trivial identity on whole of A (see [3]).

There are a number of results concerning the relationship between the identities of A and A_e. For instance, if A_e is nilpotent and G finite, then it is not hard to show that A is also nilpotent. If A_e satisfies a standard identity of degree m, then A satisfies a power of the standard identity of degree dm where $d = |G|$ [1]. If A_e satisfies an identity of degree d, then A has a non-trivial identity of degree bounded by a function of d and $|G|$ (see [2], [3]).

Another approach to the connection between the polynomial identities of A and A_e is based on the notion of the codimension growth. Let $F\langle X\rangle$ be a free associative algebra over F with a countable set of generators $X = \{x_1, x_2, \dots\}$. We denote by

Received by the editors March 6, 2002 and, in revised form, May 29, 2003.
2000 *Mathematics Subject Classification.* Primary 16R10, 16W50.
The first author was partially supported by MUN Dean of Science Research Grant #38647.
The second author was partially supported by RFBR, grants 99-01-00233 and 00-15-96128.

P_n the subspace in $F\langle X\rangle$ consisting of all multilinear polynomials in the variables $x_1,\dots,x_n$. For an arbitrary algebra A we denote by $\mathrm{Id}(A)$ the T-ideal in $F\langle X\rangle$ consisting of all identities of A. In the case $\operatorname{char} F = 0$ the ideal $\mathrm{Id}(A)$ is uniquely defined by its intersections $\mathrm{Id}(A)\cap P_n$, $n = 1,2,\dots$. The integer

$$c_n(A) = \dim \frac{P_n}{P_n \cap \mathrm{Id}(A)}$$

is called the *nth codimension* of A. It is known ([4]) that for any PI-algebra A the sequence $c_n(A)$ is exponentially bounded. One of the ways of proving that any identity on A_e implies an identity on A is to show that $c_n(A) < n!$ for n sufficiently large (see [2]).

Recently it was shown [5], [6] that asymptotically $c_n(A) \simeq d^n$, where d is a positive integer depending on A. This integer is called the *exponent* of A, $d = \mathrm{EXP}(A)$. Clearly, $\mathrm{EXP}(B) \le \mathrm{EXP}(A)$ for any subalgebra B of A. In particular, if $A = \oplus_{g\in G} A_g$ is a G-graded algebra, then the identity component A_e is a subalgebra of A and $\mathrm{EXP}(A_e) \le \mathrm{EXP}(A)$. On the other hand, in all known examples the exponent of A can be bounded by a function depending on $\mathrm{EXP}(A_e)$ and the order of G. For example, if $A = E = E_0 \oplus E_1$ is an infinite-dimensional Grassmann algebra with the canonical $\mathbb{Z}_2$-grading, then E_0 is a commutative subalgebra and $\mathrm{EXP}(E_0) = 1$. It is also known that $\mathrm{EXP}(E) = 2$ and in this case one has $\mathrm{EXP}(E) \le |G|\,\mathrm{EXP}(E_0)$. Another example is given by a full matrix algebra $A = M_n(F)$ of order n graded by the cyclic group $G = \mathbb{Z}_n$,

$$A = A_0 \oplus \cdots \oplus A_{n-1},$$

where

$$A_k = \mathrm{Span}\{E_{ij} | j - i \equiv k (\mathrm{mod}\ n)\}.$$

In this case A_0 is a commutative algebra of all diagonal matrices and $\mathrm{EXP}(A_0) = 1$. The exponent of A is also known, $\mathrm{EXP}(M_n(F)) = \dim M_n(F) = n^2$, and we have $\mathrm{EXP}(A) = |G|^2\,\mathrm{EXP}(A_0)$.

Our conjecture is as follows.

Conjecture. *For any PI-algebra $A = \oplus_{g\in G} A_g$ graded by a finite group G, the following inequality holds:*

$$\mathrm{EXP}(A) \le |G|^2\,\mathrm{EXP}(A_e).$$

Our latter example shows that this bound cannot be lowered.

In the present paper we deal with abelian gradings and we prove that this conjecture is true in some particular cases. First, we consider the case of a finite-dimensional algebra A over a field of characteristic zero graded by a finite abelian group G. Since all exponents of growth are integral and there are no algebras whose growth is intermediate between polynomial and exponential, the inequality $\mathrm{EXP}(A_e) \le 1$ means that $c_n(A_e)$ is bounded by a polynomial function Cn^q for some constants C, q.

The main result of our Section 4 is as follows.

Theorem 1.1. *Let $A = \oplus_{g\in G} A_g$ be a finite-dimensional G-graded algebra over a field of characteristic zero and G a finite abelian group. If $c_n(A_e)$ is polynomially bounded, then* $\mathrm{EXP}(A) \le |G|^2$.

All necessary notions and definitions will be recalled in the next Section 2. Here we only reformulate the above result in the following way: if $A = \oplus_{g\in G} A_g$ and A_e has polynomially bounded codimension growth then

$$c_n(A) \le C n^a q^n$$

for some constants C, a where $q = |G|^2$.

In Section 5 we consider the case of a semisimple algebra A graded by a finite abelian group G. We prove that $\mathrm{EXP}(A) \le |G|^2 \,\mathrm{EXP}(A_e)$ provided that A is finite-dimensional semisimple (Theorem 5.1). As a consequence of this result, we prove that our conjecture is true for arbitrary finitely generated semisimple PI-algebras (Corollary 5.1).

2. Codimension growth and gradings of finite-dimensional algebras

Let A be a P.I. algebra over F. It was recently proved [5], [6] that there exists a limit

$$\mathrm{EXP}(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$$

which is called the *exponent* of A. Moreover, $\mathrm{EXP}(A)$ is a positive integer for any non-nilpotent A. If $\dim A < \infty$ and F is algebraically closed, then $\mathrm{EXP}(A)$ can be computed in the following way (see [5]). Let $A = B + J$ be a decomposition of A into the sum of a semisimple subalgebra A and the Jacobson radical J. We decompose B into the sum of simple ideals, $B = B_1 \oplus \cdots \oplus B_k$, and assume that F is algebraically closed. Then $\mathrm{EXP}(A)$ is equal to the maximal value of sum of dimensions

$$\dim B_{i_1} + \cdots + \dim B_{i_m},$$

where $B_{i_1}, \dots, B_{i_m}$ are distinct and satisfy the condition

$$B_{i_1} J B_{i_2} J \dots J B_{i_m} \ne 0.$$

This result immediately implies the following.

Lemma 2.1. *Let A be a finite-dimensional non-nilpotent algebra over an algebraically closed field F, $\mathrm{char}\, F = 0$, and $\mathrm{EXP}(A) = d$. Then for any decomposition $A = B + J$, where J is the Jacobson radical, $B = B_1 \oplus \cdots \oplus B_k$ is semisimple, and $B_1, \dots, B_k$ are simple, there exists $m \le k$ such that after a renumbering of B_i's one has $B_1 J B_2 J \dots J B_m \ne 0$ and*

$$d = \dim B_1 + \cdots + \dim B_m. \quad \square$$

Now we would like to recall the structure of finite-dimensional graded algebras and relations between abelian gradings and actions by automorphisms in the case of algebraically closed fields of characteristic 0. Let G be a finite abelian group and $\widehat{G}$ its dual group, i.e. the group of irreducible G-characters. Consider a G-graded algebra $A = \oplus_{g\in G} A_g$. Then $\widehat{G}$ acts on A by automorphisms

$$\chi : a_g \to \chi(g) a_g, \quad \chi \in \widehat{G}, \quad a_g \in A_g.$$

Moreover, a subspace $V \subseteq A$ is *graded* with respect to the G-grading (i.e. $V = \oplus_{g\in G} V \cap A_g$) if and only if $\chi(V) \subseteq V$ for any $\chi \in \widehat{G}$. This duality provides us with useful information about abelian gradings on finite-dimensional algebras.

Recall that an algebra is called *G-graded simple* if it has no non-trivial graded ideals.

Lemma 2.2. *Let $A = \oplus_{g \in G} A_g$ be a finite-dimensional algebra over an algebraically closed field of characteristic zero graded by a finite abelian group G. Then the Jacobson radical $J = J(A)$ is graded with respect to the G-grading and there exists a semisimple subalgebra C graded with respect to the G-grading such that $A = C + J$. Moreover, C can be decomposed as the direct sum $C = C_1 \oplus \ldots \oplus C_p$ of graded two-sided ideals and any C_j is a G-graded simple algebra.*

Proof. Consider the $\widehat{G}$-action on A defined above. Since J is stable under all automorphisms, one has $\widehat{G}(J) = J$, and J is graded with respect to the G-grading. In [8] it was proved that there exists a maximal semisimple subalgebra $C \subseteq A$ such that $\widehat{G}(C) = C$. Hence C is graded with respect to the G-grading and $A = C + J$. Now C is the direct sum of simple ideals, $C = B_1 \oplus \ldots \oplus B_k$, as a non-graded algebra, and $\widehat{G}$ acts on the set $\{B_1, \ldots, B_k\}$. Clearly, any

$$T_i = \sum_{\varphi \in \widehat{G}} \varphi(B_i)$$

is a two-sided ideal of C. On the other hand T_i is a minimal ideal of C stable under the $\widehat{G}$-action, that is, T_i is G-graded simple. Obviously, C is the direct sum of some of $T_i, 1 \le i \le k$. □

If G is an arbitrary group and $A = \oplus_{g \in G} A_g$ a G-graded algebra, then for any normal subgroup $H \subseteq G$ one can define a G/H-grading on A by setting

$$A_{gH} = \oplus_{t \in gH} A_t$$

for any left coset gH. In the case of abelian groups the G-graded simple algebras of finite dimensions have been described in [9].

Lemma 2.3 ([9], Theorem 7)**.** *Let G be a finite abelian group and $C = \oplus_{g \in G} C_g$ a finite-dimensional G-graded simple algebra over an algebraically closed field F,* char F *$= 0$. Let B be a simple (non-graded) two-sided ideal of C and $\Lambda = \{\lambda \in \widehat{G} | \lambda(B) = B\}$. Then B is a G/H-graded subalgebra in C, where $H = \Lambda^{\perp} = \{g \in G | \lambda(g) = 1$ for all $\lambda \in \Lambda\}$.* □

Since F is algebraically closed, the subalgebra B of the previous lemma is a matrix algebra over F. All abelian gradings on matrix algebras have also been described in [9]. Let G be an abelian group and S, T two subgroups in G. First, we consider an S-graded algebra $A = \oplus_{s \in S} A_s$ and a T-graded algebra $B = \oplus_{t \in T} B_t$. Then the tensor product $C = A \otimes B$ acquires a natural G-grading if we set

$$C = \oplus_{g \in G} C_g, \quad C_g = \oplus_{st = g} A_s \otimes B_t.$$

Now let $R = M_k(F)$ be the $k \times k$-matrix algebra and G an arbitrary group. Consider any k-tuple $(g_1, \ldots, g_k) \in G^k$ (i.e. any element of the direct kth power of G). A G-grading $R = \oplus_{g \in G} R_g$ is called *elementary* if all matrix units E_{ij} are homogeneous and $E_{ij} \in R_{g_i^{-1} g_j}, 1 \le i, j \le k$.

We call a G-grading on an algebra A *"fine"* if $\dim A_g \le 1$ for any $g \in G$. A subset $\operatorname{Supp} A = \{g \in G | A_g \ne 0\}$ is called the *support* of the grading. All "fine" gradings have been described in [9]. We will use some of their properties here.

Lemma 2.4 ([9], Theorem 5, Lemma 4)**.** *Let $M_n(F) = R = \oplus_{g \in G} R_g$ be a "fine" grading on the matrix algebra over an algebraically closed field F,* char $F = 0$*, and G an abelian group. Then $H = \operatorname{Supp} R$ is a subgroup of G of order n^2 and any non-zero homogeneous element of R is invertible.*

Any grading on a matrix algebra can be constructed from elementary and "fine" gradings.

Lemma 2.5 ([9], Theorem 6)**.** *Let $M_k(F) = R = \oplus_{g\in G} R_g$ be a G-grading on the $k \times k$-matrix algebra over an algebraically closed field F, $\operatorname{char} F = 0$, and G an abelian group. Then there exist a decomposition $k = pq$, a subgroup H of order q^2 and a tuple $(g_1, \dots, g_p) \in G^p$ such that M_k is isomorphic as a G-graded algebra to $M_p(F) \otimes M_q(F)$, where $M_p(F)$ has an elementary G-grading defined by $(g_1, \dots, g_p)$ and $M_q(F)$ has a "fine" H-grading.*

In the proof of Theorem 6 in [9] it was shown that $R = BC$, where $B \simeq M_p(F)$ and $C \simeq M_q(F)$ are commuting G-graded subalgebras of R with elementary and "fine" gradings, respectively. Moreover, $R_e = B_e$, where e is the identity of G. Set $T = \operatorname{Supp} B$, $H = \operatorname{Supp} C$ and suppose that $T \cap H$ contains a non-identity element g. Then $C_{g^{-1}} \neq 0$ by Lemma 2.4 and there exist $b \in B_g$, $c \in C_{g^{-1}}$ with $0 \neq bc \in R_e \setminus B$. This contradicts to the equality $R_e = B_e$ and we have proved the following.

Lemma 2.6. *In Lemma 2.5 we can also claim that*

$$\operatorname{Supp} M_p(F) \cap \operatorname{Supp} M_q(F) = \{e\}.$$

3. Graded simple algebras with polynomial codimension growth of the identity component

First we reduce our problem to the case of algebraically closed fields. Let $A = \oplus_{g\in G} A_g$ be a G-graded algebra over F, $\operatorname{char} F = 0$, and A_e has polynomially bounded codimension growth. If $\bar{F} \supset F$ is any extension of F and $\bar{A} = A \otimes_F \bar{F}$, then $\bar{A}$ is also a G-graded algebra, $\bar{A}_g = A_g \otimes \bar{F}$. It is not hard to see ([5], Remark 1) that the nth codimension $c_n(A)$ over F coincides with the nth codimension $c_n(\bar{A})$ over $\bar{F}$. In particular, the identity component $\bar{A}_e$ of an $\bar{F}$-algebra $\bar{A}$ has a polynomially bounded codimension growth. It follows that we need to prove our theorem only for algebraically closed fields.

Now let C be a G-graded simple algebra and B a non-graded minimal ideal as in Lemma 2.3.

Lemma 3.1. *Let C, B be as in Lemma 2.3. If C_e is the identity component of C with respect to the G-grading and $B_{\bar{e}}$ is the identity component of B with respect to the G/H-grading, then $B_{\bar{e}} \simeq C_e$.*

Proof. Consider a decomposition of $\widehat{G}$ into the disjoint union of cosets

$$\widehat{G} = \chi_1\Lambda \cup \ldots \cup \chi_k\Lambda$$

and set $B_i = \chi_i(B), i = 1, \dots, k$. Then all $B_1, \dots, B_k$ are simple and $C = B_1 \oplus \cdots \oplus B_k$. Set, for convenience, $\chi_1 = 1$ and $B_1 = B$. Recall that $\Lambda = \{\varphi \in \widehat{G} | \varphi(B) = B\}$. Now if $e, \bar{e}$ are the respective identity elements of $G, G/H$, then

$$C_e = \{c \in C | \varphi(c) = c,\ \forall \varphi \in \widehat{G}\},$$
$$B_{\bar{e}} = \{b \in B | \lambda(b) = b,\ \forall \lambda \in \Lambda\}.$$

Consider some $b \in B_{\bar{e}}$ and set $\widetilde{b} = \chi_1(b) + \cdots + \chi_k(b)$. Then $\lambda(\chi_i(b)) = \chi_i(\lambda(b)) = \chi_i(b)$ since $b \in B_{\bar{e}}$ and $\widehat{G}$ is commutative. Hence $\chi_i(\widetilde{b}) = \widetilde{b}$ for any $i = 1, \dots, k$ since

$\{\chi_1, \dots, \chi_k\}$ is a transversal for $\widehat{G}$ over Λ. It follows that $\varphi(\widetilde{b}) = \widetilde{b}$ for any $\varphi \in \widehat{G}$ and $\widetilde{b} \in C_e$.

Clearly, $b \mapsto \widetilde{b}$ is the injective homomorphism $B_{\bar{e}} \to C_e$. On the other hand, let $c \in C_e$, $c = b_1 + \cdots + b_k$, $b_i \in B_i$, $i = 1, \dots, k$. Then $\lambda(b_1) = b_1$ for any $\lambda \in \Lambda$ since $\lambda(c) = c$ and $\lambda(B_1) = B_1$, that is, $b_1 \in B_{\bar{e}}$. From the relations $\chi_i(B_1) = B_i, \chi_i(c) = c$ it follows that $\chi_i(b_1) = b_i, i = 1, \dots, k$, and $c = \widetilde{b}_1$. Thus the mapping $b \mapsto \widetilde{b}$ is an isomorphism of $B_{\bar{e}}$ onto C_e. □

The proof of the next statement is obvious for any non-trivial group H.

Lemma 3.2. *Let H be a finite group with the identity element e, $|H| = q^2$. Then for any $p \geq 1$ one can choose some elements $h_i^{(j)} \in H$, $i = 1, \dots, q-1$, $j = 1, \dots, p$, such that for any fixed $j \in \{1, \dots, p\}$ all products*

$$e, h_1^{(j)}, h_1^{(j)} h_2^{(j)}, \dots, h_1^{(j)} h_2^{(j)} \cdots h_{q-1}^{(j)}$$

are distinct and the total product

$$\prod_{\substack{1 \leq i \leq q-1 \\ 1 \leq j \leq p}} h_i^{(j)}$$

is not equal to the identity in H. □

Now we consider a G-graded algebra $R = \oplus_{g \in G} R_g$ and a right R-module W. We say that W is a G-graded R-module if $W = \oplus_{g \in G} W_g$ and $W_g R_h \subseteq W_{gh}$ for any $g, h \in G$. Any element $a \in W_g$ is called *homogeneous* and by definition its weight $\mathrm{wt}(a)$ is equal to g.

First we consider the case of matrix algebras with a polynomial codimension growth of the identity component, and their G-graded modules.

Lemma 3.3. *Let $k = pq$ and $R = M_k(F) = M_p \otimes M_q$ be a G-graded matrix algebra, where G is an abelian group, M_p has an elementary G-grading, M_q has a "fine" H-grading and H is a subgroup of G of order q^2. Suppose that the identity component R_e has a polynomially bounded codimension growth and W is a G-graded right R-module such that $y = yE_{ii} \neq 0$ for some homogeneous $y \in W$ and some $E_{ii} \in M_p$. Then there exist homogeneous $b_1, \dots, b_k \in R$ such that $\mathrm{wt}(y), \mathrm{wt}(yb_1), \dots, \mathrm{wt}(yb_1 \cdots b_{k-1})$ are distinct elements of G and $yb_1 \cdots b_k = y$.*

Proof. Let us prove first that $H \cap \mathrm{Supp}\, M_p = \{e\}$. Assume that $h \in H \cap \mathrm{Supp}\, M_p$ and $h \neq e$. By the definition of elementary gradings there exists a matrix unit E_{st} with $s \neq t$ and $\mathrm{wt}(E_{st}) = h$. Since $H = \mathrm{Supp}\, M_q$ is a subgroup, one can find $a \in M_q$ with $\mathrm{wt}(a) = h^{-1}$. Hence $E_{ss} \otimes 1, E_{st} \otimes a, E_{tt} \otimes 1 \in R_e$ and form the subalgebra C isomorphic to the algebra of upper triangular 2×2 matrices. Therefore $2 = \mathrm{EXP}(C) \leq \mathrm{EXP}\, R_e$, a contradiction.

By our assumption $W \ni y = yE_{ii} \neq 0$. Using Lemma 3.2 we will construct a sequence $b_1, \dots, b_k$. First we choose $b_{jq+2}, \dots, b_{(j+1)q}$ for $j = 0, \dots, p-2$ in the following way. By Lemma 3.2 one can find $h_i^{(j)} \in H$ for $j = 0, \dots, p-2$, $i = 1, \dots, q-1$ such that

$$e, h_1^{(j)}, h_1^{(j)} h_2^{(j)}, \dots, h_1^{(j)} \cdots h_{q-1}^{(j)} \tag{1}$$

are distinct and

$$h = \prod_{\substack{1 \leq i \leq q-1 \\ 1 \leq j \leq p-1}} h_i^{(j)} \neq e \tag{2}$$

in H. Now we take $0 \neq c_i^{(j)} \in M_q$ with $\mathrm{wt}(c_i^{(j)}) = h_i^{(j)}$ and define

$$b_{jq+2} = c_1^{(j)}, \dots, b_{(j+1)q} = c_{q-1}^{(j)} \in M_q, j = 0, \dots, p-2.$$

Clearly, $\mathrm{wt}(b_{jq+2}), \mathrm{wt}(b_{jq+2}b_{jq+3}), \dots, \mathrm{wt}(b_{jq+2} \cdots b_{(j+1)q})$ are distinct non-identity elements of H since they are equal to the respective elements in (1).

Now we define $b_1, b_{q+1}, \dots, b_{(p-1)q+1}$ as the respective matrix units $E_{i,i+1}, \dots, E_{p-1,p}$, $E_{p,1}$, $E_{1,2}, \dots, E_{i-1,i}$ in M_p.

Finally, using Lemma 3.2, one can find $h_1, \dots, h_{q-1} \in H$ such that

$$h, hh_1, hh_1h_2, \dots, hh_1 \cdots h_{q-2}$$

are distinct non-identity elements of H for h from (2). Now take $0 \neq c_i^{(p-1)} \in M_q$, $\mathrm{wt}(c_i^{(p-1)}) = h_i, i = 1, \dots, q-2$, and set

$$b_{(p-1)q+2} = c_1^{(p-1)}, \dots, b_{pq-1} = c_{q-2}^{(p-1)}.$$

The remaining element b_k will be chosen later.

Let us now check that $\mathrm{wt}(y), \mathrm{wt}(yb_1), \dots, \mathrm{wt}(yb_1 \cdots b_{k-1})$ are distinct. It is sufficient to show that $\mathrm{wt}(b_1), \mathrm{wt}(b_1b_2), \dots, \mathrm{wt}(b_1 \cdots b_{k-1})$ are distinct and not equal to e.

Suppose $\mathrm{wt}(b_1 \cdots b_r) = \mathrm{wt}(b_1 \cdots b_t)$. Then either $\mathrm{wt}(b_{jq+r} \cdots b_{jq+s}) = e$ for some j with $2 \leq r < s \leq q$ or

$$\mathrm{wt}(c_j^{(\gamma)} \cdots c_{q-1}^{(\gamma)} E_{\alpha,\alpha+1} \cdots E_{\beta,\beta+1} c_1^{(\delta)} \cdots c_s^{(\delta)}) = e \tag{3}$$

for suitable $j, s, \alpha, \beta, \gamma, \delta$ (we assume that $c_j^{(\gamma)} \cdots c_{q-1}^{(\gamma)}$ is empty if $\alpha = i$). The former equation contradicts to $\mathrm{wt}(b_{jq+2} \cdots b_{jq+r-1}) \neq \mathrm{wt}(b_{jq+2} \cdots b_{jq+s})$. Now since G is abelian, the element on the left side of (3) is a product of the type $g \,\mathrm{wt}(E_{\alpha,\beta+1})$, where $\mathrm{wt}(E_{\alpha,\beta+1}) \in \mathrm{Supp} M_p, g \in H = \mathrm{Supp}\, M_q$. As it was shown before, $H \cap \mathrm{Supp} M_p = \{e\}$, hence from (3) it follows that $\alpha = i, \beta = i-1$. But in this case the element on the left side of (3) is equal to $hh_1 \cdots h_s$, where h is defined in (2). This element is also a non-identity by the choice of $h_1, \dots, h_s$.

We have proved that $y, yb_1, \dots, yb_1 \cdots b_{k-1}$ have distinct weights with respect to the G-grading. In order to find b_k we notice that

$$b_1 \cdots b_{k-1} = E_{ii} \otimes b$$

for some homogeneous non-zero $b \in M_q$. Any non-zero homogeneous element in M_q is invertible by Lemma 2.4 and we can take $b_k = b^{-1} = I \otimes b^{-1} \in M_q$, where I is the identity matrix of M_p. In this case $yb_1 \cdots b_{k-1}b_k = y(E_{ii} \otimes b)(I \otimes b^{-1}) = yE_{ii} = y$, and the proof is complete. □

An easy consequence is as follows.

Corollary 3.1. *If $R = M_{pq}(F)$, as in Lemma 3.3, then $|G| \geq pq$.* □

4. Finite-dimensional graded algebras with polynomial codimension growth of the identity component

Lemma 4.1. *Let $A = \oplus_{g \in G} A_g$ be a finite-dimensional G-graded algebra over an algebraically closed field F, $\mathrm{char}\, F = 0$, and G a finite abelian group. Consider a decomposition $A = C + J$, where C is a G-graded semisimple subalgebra and J is* *the Jacobson radical. Let $C = C_1 \oplus \cdots \oplus C_p$ be a decomposition of C into the sum of G-graded simple subalgebras as in Lemma 2.2 and*

$$C_i = B_{i1} \oplus \cdots \oplus B_{iq_i}$$

be the sum of (non-graded) simple ideals, $i = 1, \ldots, p$. Suppose A_e has polynomially bounded codimension growth and

$$B_1 J B_2 J \cdots J B_m \neq 0, \tag{4}$$

where $B_1, \ldots, B_m$ are distinct B_{ij}, $1 \le i \le p, 1 \le j \le q_i$. Then $|G|^2 \ge \dim B_1 + \cdots + \dim B_m$.

Proof. By (4) there exist homogeneous $y_1, \ldots, y_{m-1} \in J$ and some $b_1 \in B_1, \ldots, b_m \in B_m$ such that

$$b_1 y_1 b_2 \cdots b_{m-1} y_{m-1} b_m \neq 0. \tag{5}$$

Any B_s is an ideal in C_j and is graded with respect to the G/H_j-grading for a suitable $H_j \subseteq G$. Hence we can assume that b_s in (5) are homogeneous in G/H_j-grading. Moreover, any B_s is an algebra of all $k_s \times k_s$ matrices and for any fixed s it is isomorphic to $M_r \otimes M_t$ with an elementary G/H_j-grading on M_r, with a "fine" grading on M_t, and the identity matrix $I \in B_s$ is equal to $E_{11} \otimes I' + \cdots + E_{rr} \otimes I'$, where I' is the identity matrix of M_t. It follows that we can take $b_s = b_s E_{ii} = b_s E_{ii} \otimes I'$ for some $1 \le i \le r$.

By Lemma 3.1 the identity component of B_s in G/H_j-grading has a polynomially bounded codimension growth, hence by Lemma 3.3 one can find homogeneous in G/H_j-grading elements $b_{s1}, \ldots, b_{sk_s} \in B_s$ such that $b_s b_{s1} \cdots b_{sk_s} = b_s$ and

$$\mathrm{wt}(b_s), \mathrm{wt}(b_s b_{s1}), \ldots, \mathrm{wt}(b_s b_{s1} \cdots b_{s,k_s-1}) \tag{6}$$

are pairwise distinct in G/H_j. Rewrite (5):

$$b_1 b_{11} \cdots b_{1k_1} y_1 b_2 b_{21} \cdots b_{2k_2} y_2 \cdots y_{m-1} b_m b_{m1} \cdots b_{mk_m} \neq 0. \tag{7}$$

Notice that if some $x \in B_s$ is homogeneous with respect to the G/H_j-grading, then $x = \sum_{g \in T} x_g$, where T is some coset of H_j in G. Therefore we can express all b_i, b_{ij} in (7) as sums of G-homogeneous elements and get

$$c_1 c_{11} \cdots c_{1k_1} y_1 c_2 c_{21} \cdots c_{2k_2} y_2 \cdots y_{m-1} c_m c_{m1} \cdots c_{mk_m} \neq 0, \tag{8}$$

where all c_i, c_{ij} are homogeneous with respect to the G-grading and all

$$\mathrm{wt}(c_s), \mathrm{wt}(c_s c_{s1}), \ldots, \mathrm{wt}(c_s c_{s1} \cdots c_{s,k_s-1})$$

are pairwise distinct in G for any fixed $s \in \{1, \ldots, m\}$ since all elements in (6) are pairwise distinct in G/H_j.

Now we denote by $\widetilde{B}_s$ the G-graded simple subalgebra in A generated by B_s, for each $s = 1, \ldots, m$. Then from (4) it follows that

$$\widetilde{B}_1 J \widetilde{B}_2 J \cdots J \widetilde{B}_m \neq 0 \tag{9}$$

and all $\widetilde{B}_1, \ldots, \widetilde{B}_m$ are some of $C_1, \ldots, C_p$ by the choice of $B_1, \ldots, B_m$.

Notice that $\widetilde{B}_i, \widetilde{B}_j$ can be equal in (9) even if $i \neq j$ but if, say, $C_1 = \widetilde{B}_i$ appears among $\widetilde{B}_1, \ldots, \widetilde{B}_m$ exactly t_1 times, then C_1 contains at least t_1 distinct simple summands and every summand is a matrix $k_1 \times k_1$-algebra with a G/H_1-grading for a fixed subgroup H_1. Besides, $|H_1| \ge t_1$ since $|H_1| = [\widehat{G} : \Lambda_1]$ and the index of Λ_1 in $\widehat{G}$ is exactly the number of simple (non-graded) summands in C_1. In particular, $\dim C_1 \geq t_1 k_1^2$. By Lemma 3.1 and Corollary 3.1, $|G/H_1| \geq k_1$, hence $|G| \geq t_1 k_1$.

Now we consider a set of integers $1 \leq i_1 < \ldots < i_r \leq m$ such that $\widetilde{B}_{i_1} = C_{j_1}, \ldots, \widetilde{B}_{i_r} = C_{j_r}$ are distinct in (9) but all other $\widetilde{B}_j$ are equal to one of $\widetilde{B}_{i_1}, \ldots, \widetilde{B}_{i_r}$ (of course, i_1 can be taken as 1). If $\widetilde{B}_{i_j}$ appears among $\widetilde{B}_1, \ldots, \widetilde{B}_m$ exactly t_j times and k_{i_j} is the size of the matrix algebra, which defines $\widetilde{B}_{i_j}$, then $|G| \geq t_j k_{i_j}$ as it was shown before.

Next it follows from (8) that there exist homogeneous $z_1, \ldots, z_{r-1} \in J$ such that

$$c_{i_1 1} \cdots c_{i_1 k_{i_1}} z_1 c_{i_2 1} \cdots c_{i_2 k_{i_2}} z_2 \cdots z_{r-1} c_{i_r 1} \cdots c_{i_r k_{i_r}} \neq 0 \tag{10}$$

and all $\mathrm{wt}(c_{i_j 1}), \mathrm{wt}(c_{i_j 1} c_{i_j 2}), \ldots, \mathrm{wt}(c_{i_j 1} \cdots c_{i_j k_{i_j}})$ are distinct elements of the G. Moreover, $c_{i_1 \alpha} \in C_{j_1}, \ldots, c_{i_r \beta} \in C_{j_r}$ with distinct G-graded simple $C_{j_1}, \ldots, C_{j_r}$.

Now we will show that the following $k_{i_1} + \cdots + k_{i_r}$ elements of G are pairwise distinct:

$$\begin{gathered} \mathrm{wt}(c_{i_1 1}), \ldots, \mathrm{wt}(c_{i_1 1} \cdots c_{i_1 k_{i_1}}), \mathrm{wt}(c_{i_1 1} \cdots c_{i_1 k_{i_1}} z_1 c_{i_2 1}), \ldots, \\ \mathrm{wt}(c_{i_1 1} \cdots c_{i_r k_{i_r}}). \end{gathered} \tag{11}$$

If two weights in (11) coincide, then

$$\mathrm{wt}(c_{i_\alpha j} \cdots c_{i_\beta l}) = e.$$

The case $\alpha = \beta$ is impossible, as it was mentioned before. Suppose $\alpha \neq \beta$. Then C_{j_α} and C_{j_β} are distinct graded simple subalgebras in A. In particular, $C_{j_\alpha} C_{j_\beta} = C_{j_\beta} C_{j_\alpha} = 0$. Hence for identity elements $1_\alpha, 1_\beta$ of the algebras $C_{j_\alpha}, C_{j_\beta}$, respectively, and for $z = c_{i_\alpha j} \cdots c_{i_\beta l}$ one has

$$1_\alpha z = z 1_\beta = z \neq 0, \quad 1_\beta z = z 1_\alpha = 0, \quad 1_\alpha, 1_\beta, z \in A_e. \tag{12}$$

From $\alpha \neq \beta$ it follows that $z^2 = z 1_\beta 1_\alpha z = 0$ in (12) and $1_\alpha, 1_\beta, z$ form a subalgebra isomorphic to the algebra of upper triangular 2×2 matrices. Hence, again by [5], $\mathrm{EXP}(A_e) \geq 2$, a contradiction.

We have proved that all elements in (11) are pairwise distinct. Now we set, for brevity, $k_{i_1} = m_1, \ldots, k_{i_r} = m_r$. We have obtained that

$$|G| \geq t_1 m_1, \ldots, t_r m_r, m_1 + \cdots + m_r.$$

Now let us check that

$$\dim B_1 + \cdots + \dim B_m = t_1 m_1^2 + \cdots + t_r m_r^2.$$

Since $\widetilde{B}_{i_j}$ appears among $\widetilde{B}_1, \ldots, \widetilde{B}_m$ exactly t_j times, it follows that t_1 matrix algebras among $B_1, \ldots, B_m$ have dimension $k_{i_1}^2$, t_2 have dimension $k_{i_2}^2$, and so on. Hence the sum of dimensions of $B_1, \ldots, B_m$ is equal to $t_1 m_1^2 + \cdots + t_r m_r^2$.

In order to complete the proof of our lemma we need only to check the inequality

$$t_1 m_1^2 + \cdots + t_r m_r^2 \leq \max\{t_1^2 m_1^2, \ldots, t_r^2 m_r^2, (m_1 + \cdots + m_r)^2\}. \tag{13}$$

Set $x_i = t_i m_i, i = 1, \ldots, r$. Then (13) can be written as

$$x_1 m_1 + \cdots + x_r m_r \leq \max\{x_1^2, \ldots, x_r^2, (m_1 + \cdots + m_r)^2\}. \tag{14}$$

Also set $t = m_1 + \cdots + m_r$, $\alpha_i = \frac{m_i}{t}, i = 1, \ldots, r$. After dividing by t, the latter equation takes the form

$$\alpha_1 x_1 + \cdots + \alpha_r x_r \leq \max\{\frac{1}{t} x_1^2, \ldots, \frac{1}{t} x_r^2, t\}. \tag{15}$$

Besides, $\alpha_1 + \cdots + \alpha_r = 1$ and $\alpha_1, \ldots, \alpha_r \geq 0$ in (15). Now the left-hand side of (15) does not exceed

$$x_i = \max\{x_1, \ldots, x_r\}.$$

The right-hand side of (15) is at least $\max\{t, \frac{1}{t}x_i^2\}$. Obviously, if $x_i > t, x_i > \frac{1}{t}x_i^2$, then $x_i^2 > x_i^2$, a contradiction. Thus (13) holds for all possible m_i, t_i, and the proof of the lemma is complete. □

Now we can complete the proof of the main result of this paper.

Proof of Theorem 1.1. As it was shown in Section 3, we can assume F algebraically closed. Consider the decomposition $A = C + B$, $C = C_1 \oplus \cdots \oplus C_p$ as in Lemma 2.2. If we write C_i as the sum of minimal ideals, $C_i = B_{i_1} \oplus \cdots \oplus B_{iq_i}$, then $C = \oplus_{i,j} B_{ij}$ and by Lemma 2.1 one can find $B_1, \ldots, B_m \in \{B_{11}, \ldots, B_{pq_p}\}$ such that $B_1 J B_2 J \cdots J B_m \neq 0$ and

$$\mathrm{EXP}(A) = d = \dim B_1 + \cdots + B_m.$$

It follows from Lemma 4.1 that $\mathrm{EXP}(A) = d \leq |G|^2$, and the proof of Theorem 1.1 is complete. □

From the definition of the nth codimension it follows immediately that for any commutative algebra B one has $c_n(B) \leq 1$ and $c_n(B)$ is polynomially bounded. Hence we can apply our theorem to all graded algebras with the commutative identity component.

Corollary 4.1. *Let $A = \oplus_{g\in G} A_g$ be a finite-dimensional G-graded algebra over a field of characteristic zero and G is a finite abelian group. If A_e is commutative, then* $\mathrm{EXP}(A) \leq |G|^2$.

5. Semisimple algebras with finite gradings

In this section we generalize our previous results to the case where a PI-algebra A is not necessarily finite-dimensional and the value of the exponent of A_e is arbitrary, provided that A is finitely generated and semisimple. We start with the finite-dimensional case.

Theorem 5.1. *Let $A = \oplus_{g\in G} A_g$ be a finite-dimensional semisimple algebra over F, $\mathrm{char}\, F = 0$, graded by a finite abelian group G. Then* $\mathrm{EXP}(A) \leq |G|^2\, \mathrm{EXP}(A_e)$.

Proof. As in the previous Theorem 1.1 we can assume that F is algebraically closed. We first show that it is sufficient to prove our theorem only for G-graded simple algebras.

Let $A = B \oplus C$ be the sum of two G-graded algebras. Then, clearly, $\mathrm{Id}(A) = \mathrm{Id}(B) \cap \mathrm{Id}(C)$ and

$$c_n(B), c_n(C) \leq c_n(A) \leq c_n(B) + c_n(C).$$

From these inequalities and the integrality of the exponents it follows easily that $\mathrm{EXP}(A) = \max\{\mathrm{EXP}(B), \mathrm{EXP}(C)\}$. Let $\mathrm{EXP}(B) \geq \mathrm{EXP}(C)$. Then $\mathrm{EXP}(A) = \mathrm{EXP}(B)$ and $\mathrm{EXP}(B_e) \leq \mathrm{EXP}(A_e)$, where e is the identity of G since B_e is a subalgebra of A_e. Suppose now that the inequality $\mathrm{EXP}(B) \leq |G|^2\, \mathrm{EXP}(B_e)$ is already proved. Then

$$\mathrm{EXP}(A) = \mathrm{EXP}(B) \leq |G|^2\, \mathrm{EXP}(B_e) \leq |G|^2\, \mathrm{EXP}(A_e).$$

By the hypothesis of our theorem, A is finite-dimensional and semisimple. Hence by Lemma 2.2, A is the direct sum of G-graded simple algebras and applying the previous remark we reduce our proof to the case where A is G-graded simple.

Now let A be a G-graded simple algebra. Then by Lemma 2.3 (see also the proof of Lemma 3.1) A is the direct sum of isomorphic simple algebras, $A = B_1 \oplus \cdots \oplus B_k$, $B_1 \simeq \ldots \simeq B_k$, and $B = B_1$ is a G/K-graded algebra for some subgroup $K \subseteq G$. As before, $\mathrm{EXP}(A) = \mathrm{EXP}(B)$ and $\mathrm{EXP}(A_e) = \mathrm{EXP}(B_{\bar{e}})$ since $A_e \simeq B_{\bar{e}}$ by Lemma 3.1 ($e, \bar{e}$ are identity elements of $G, G/K$, respectively). Hence it is sufficient to prove the relation $\mathrm{EXP}(B) \leq |G/K|^2\,\mathrm{EXP}(B_{\bar{e}})$. In other words we have reduced our proof to the case of a simple graded algebra.

Recall that F is algebraically closed, therefore any finite-dimensional simple F-algebra is isomorphic to the full matrix algebra $M_k(F)$. Now let $M_k(F) = A = \oplus_{g\in G} R_g$ be a $k \times k$ matrix algebra graded by G. By Lemma 2.5 we have $A \simeq M_p(F) \otimes M_q(F)$, where $M_p(F)$ has an elementary G-grading and $M_q(F)$ has a "fine" grading. Moreover, by Lemma 2.6 $\mathrm{Supp}\, M_p(F) \cap \mathrm{Supp}\, M_q(F) = \{e\}$.

First we consider $B = M_p(F)$ with an elementary G-grading defined by a p-tuple $(g_1, \ldots, g_p)$. By the definition of an elementary G-grading we have

$$B_e = \mathrm{Span}\{E_{ij} \in B |\ g_i = g_j\}.$$

If pairwise distinct $h_1, \ldots, h_s \in G$ appear among $g_1, \ldots, g_p$ exactly $k_1, \ldots, k_s$ times, respectively, then $B_e = B_1 \oplus \cdots \oplus B_s$, where

$$B_r = \mathrm{Span}\{E_{ij} \in B |\ g_i = g_j = h_r\} \quad \text{and} \quad B_r \simeq M_{k_r}(F).$$

In particular, $\mathrm{EXP}(B_e) = \max\{k_1^2, \ldots, k_s^2\}$. Set $a = \max\{k_1, \ldots, k_s\}$. Then $p = k_1 + \cdots + k_s \leq sa$. Notice also that

$$\mathrm{Supp}\, B = \{h_i^{-1} h_j |\ 1 \leq i, j \leq s\} \subset G.$$

Thus we have $s \leq |\mathrm{Supp}\, B| \leq |G|$. It follows that

$$\mathrm{EXP}(B) = p^2 \leq s^2 a^2 \leq |G|^2\,\mathrm{EXP}(B).$$

In the general case, we set $H = \mathrm{Supp}\, M_q(F)$. If $h_i H = h_j H$ for some $i \neq j$, then $e \neq h_i^{-1} h_j \in \mathrm{Supp}\, M_p(F) \cap \mathrm{Supp}\, M_q(F)$, a contradiction. It follows that $|G| \geq s|H| = sq^2$ since $|H| = \dim M_q(F) = q^2$. Finally, we get

$$\mathrm{EXP}(A) = (pq)^2 \leq s^2 a^2 q^2 \leq (sq^2)^2 a^2 \leq |G|^2\,\mathrm{EXP}(B_e) \leq |G|^2\,\mathrm{EXP}(A_e)$$

and the proof of Theorem 5.1 is complete. □

For an extension of the previous result we need the following general remark.

Lemma 5.1. *Let $A = \oplus_{g\in G} A_g$ be a finitely generated semisimple PI-algebra over an algebraically closed field of characteristic zero graded by a finite abelian group G. Then there exists a finite-dimensional semisimple algebra $B = \oplus_{g\in G} B_g$ with a G-grading which has the same graded identities as A.*

Proof. It is known that a finitely generated semisimple PI-algebra is residually n-dimensional, i.e. it is a subdirect product of finite-dimensional algebras with dimensions less than or equal to n (see, for example, [11]). In other words, the intersection of all ideals of codimension less than or equal to n is zero.

Let I be a two-sided ideal of A and $\dim A/I \leq n$. Again we use the duality between the G-gradings and the $\widehat{G}$-actions on A. Let $|G| = k$ and $\widehat{G} = \{\gamma_1, \ldots, \gamma_k\}$.

Then

$$\bar{I} = \gamma_1(I) \cap \ldots \cap \gamma_k(I)$$

is a two-sided ideal of A and $\dim A/\bar{I} \leq kn$. It follows that A is a subdirect product $\Pi_{\alpha \in M} A_\alpha$ of G-graded algebras with $\dim A_\alpha \leq kn$. Since A is semisimple, all factors A_α can be taken semisimple. From the description of all possible gradings on the full matrix algebra (see [9], Theorems 5 and 6) and graded simple algebras ([9], Theorem 7) it follows that over an algebraically closed field of characteristic zero there exist only finitely many pairwise non-isomorphic G-graded semisimple algebras of fixed dimension. Hence one can take a finite set of G-graded homomorphic images $A_1, \ldots, A_m$ of A such that $A_1, \ldots, A_m$ are semisimple and $A_1 \oplus \cdots \oplus A_m$ has the same graded identities as A. □

Applying Theorem 5.1 and replacing the base field by its algebraic closure, if necessary, we immediately obtain our last result:

Corollary 5.1. *Let $A = \oplus_{g \in G} A_g$ be a finitely generated semisimple PI-algebra over F, $\operatorname{char} F = 0$, graded by a finite abelian group G. Then $\operatorname{EXP}(A) \leq |G|^2 \operatorname{EXP}(A_e)$.* □

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