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RADON'S INVERSION FORMULAS

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ABSTRACT. Radon showed the pointwise validity of his celebrated inversion formulas for the Radon transform of a function f of two real variables (formulas (III) and (III′) in J. Radon, Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten, *Ber. Verh. Sächs. Akad. Wiss. Leipzig*, Math.-Nat. kl. 69 (1917), 262-277) under the assumption that f is continuous and satisfies two other technical conditions. In this work, using the methods of modern analysis, we show that these technical conditions can be relaxed. For example, the assumption that f be in $L^p(\mathbb{R}^2)$ for some p satisfying $4/3 < p < 2$ suffices to guarantee the almost everywhere existence of its Radon transform and the almost everywhere validity of Radon's inversion formulas.

1. INTRODUCTION

The Radon transform $Rf(\theta,t)$, $0 \le \theta < 2\pi$, $-\infty < t < \infty$, of a sufficiently well-behaved scalar valued function $f(x)$ of the variable $x = (x_1, x_2)$ in the plane $\mathbb{R}^2$ may be defined by

$$Rf(\theta,t) = \int_{-\infty}^{\infty} f(tu_\theta + sv_\theta)ds, \tag{1}$$

where $u_\theta = (\cos\theta, \sin\theta)$ and $v_\theta = u_{\theta+\pi/2} = (-\sin\theta, \cos\theta)$. An inversion formula is an expression for $f(x)$ in terms of Rf. There are many such expressions involving various hypotheses on the function f, for example, see [2, 3, 4, 6, 7, 8, 9]; Radon's article [7] is reproduced in [3] and an English translation can be found in [2]. One reason such formulas are of interest is due to applications in computed tomography; several such applications are described, for instance, in [2, 6, 8, 9].

This note concerns Radon's inversion formulas [7, formulas III and III′]. Specifically, if

$$F_x(t) = \frac{1}{2\pi}\int_0^{2\pi} Rf(\theta, \langle x, u_\theta\rangle + t)d\theta, \tag{2}$$

where $\langle x, u_\theta\rangle = x_1\cos\theta + x_2\sin\theta$ is the scalar product of x and u_θ, then these formulas are

$$f(x) = -\frac{1}{\pi}\int_0^{\infty} \frac{dF_x(t)}{t} \tag{3}$$

and

$$f(x) = \frac{1}{\pi}\lim_{\epsilon\to\infty}\left\{\frac{F_x(\epsilon)}{\epsilon} - \int_\epsilon^\infty \frac{F_x(t)}{t^2}dt\right\}. \tag{4}$$

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Radon established the validity of these formulas under the assumptions that

(a) f is continuous,
(b) $\displaystyle\int_{\mathbb{R}^2} \frac{|f(x)|}{|x|} dx$ is finite, and
(c) for every x in $\mathbb{R}^2$

$$\lim_{r\to\infty} \int_0^{2\pi} f(x + ru_\theta)d\theta = 0\,.$$

Note that the discussion in [7] indicates that (3) is to be interpreted as

$$f(x) = -\frac{1}{\pi} \lim_{\epsilon\to 0} \lim_{\rho\to\infty} \int_\epsilon^\rho \frac{dF_x(t)}{t}, \tag{5}$$

where the integral is a standard Riemann Stieltjes integral in the variable t.

While it is clear that mollified or regularized variants of these formulas are valid for wider classes of functions f, Radon's result seems to be the strongest found in the literature involving the properties of f in the inversion procedures (3) and (4).

One purpose of this note is to indicate the degree to which the restrictions (a)-(c) can be relaxed.

The restriction (b) appears to play an important role in that it guarantees the existence of (1) almost everywhere. On the other hand, the only role played by restriction (c) seems to be in establishing the equivalence of (4) and (5). Also, while condition (a) guarantees that $f(x)$ is well defined at every point x in $\mathbb{R}^2$, it is not essential for the almost everywhere existence of the Radon transform (1) nor the almost everywhere validity of Radon's inversion formulas (3) and (4).

In this article we show, among other things, that (4) holds almost everywhere whenever f is locally in L^p, $p > 4/3$, and satisfies a condition equivalent to (b). Furthermore, the requirement that p be greater than $4/3$ cannot be relaxed. On the other hand, if $F_x(\epsilon)$ in the right-hand side of (4) is replaced with $F_x(0)$, then the restriction on p can be lifted. Concerning (3) or its equivalent, (5), some global condition on f is required for its validity. We show that the restriction that f be in $L^p(\mathbb{R}^2)$, $p > 4/3$, is sufficient.

We use standard modern mathematical terminology, notation, and conventions (for example, see [10, 11]) and only remind the reader that $f * g$ denotes the convolution of the functions f and g which is defined by

$$f * g(x) = \int_{\mathbb{R}^2} f(x-y)g(y)dy$$

whenever it makes sense.

The precise statement of the main results together with some supporting material are presented in Section 2. Details, including various supporting lemmas and propositions, are given in Section 3. Section 4 is devoted to certain details which are included for completeness but which are too mundane or tedious to be included in Section 3.

2. Highlights

Given a locally integrable function f on $\mathbb{R}^2$ it is clear that some restriction on its behavior at infinity is required to guarantee the existence of its Radon transform.

One such restriction is

$$\|f\|_{LR} = \int_{\mathbb{R}^2} \frac{|f(x)|}{1+|x|} dx < \infty \,. \tag{6}$$

Indeed if f enjoys (6), then $Rf(\theta, t)$ is locally integrable on $[0, 2\pi) \times \mathbb{R}$ and thus finite for almost all (θ, t). Furthermore, for non-negative functions f the local integrability of $Rf(\theta, t)$ is equivalent to (6).

For convenience we denote the class of those locally integrable functions which satisfy condition (6) as LR. Note that Hölder's inequality implies that f is in LR whenever it is in $L^p(\mathbb{R}^2)$ for some p, $1 \le p < 2$. It may also be interesting to note that for continuous functions f, condition (6) is equivalent to the restriction labeled (b) in the Introduction.

Next consider formulas (4) and (5). These formulas are equivalent under the assumptions that the Riemann Stieltjes integral on on the right-hand side of (5) makes sense, that integration by parts is valid, and that

$$\lim_{t\to\infty} \frac{F_x(t)}{t} = 0 \,. \tag{7}$$

While these assumptions are clearly valid in the case when f is continuous and compactly supported, they are more difficult to verify under less restrictive conditions. Indeed, Radon [7] introduced the additional restriction (c) essentially so that (7) be valid. However, formula (4) by itself can make sense without these assumptions. For this reason we examine it first.

For convenience we denote the expression parametrized by ϵ on the right-hand side of (4) by $g_\epsilon(x)$. More precisely,

$$g_\epsilon(x) = \frac{1}{\pi} \left\{ \frac{F_x(\epsilon)}{\epsilon} - \int_\epsilon^\infty \frac{F_x(t)}{t^2} dt \right\},$$

where $F_x(t)$ is defined by (2). The fact that for sufficiently well-behaved functions f the transformation $f \to g_\epsilon$ is translation invariant in x implies that g_ϵ is the convolution of f with some distribution. Indeed, we have the following.

Proposition 1. *For every f in LR*

$$g_\epsilon(x) = k_\epsilon * f(x), \tag{8}$$

where

$$k_\epsilon(x) = \epsilon^{-2} k\left(x/\epsilon\right) \,, \tag{9}$$

$$k(x) = \frac{1}{\pi^2 |x|^2 \sqrt{|x|^2 - 1}} \chi(|x|) \,, \tag{10}$$

and

$$\chi(t) = \begin{cases} 0 & \text{if} \quad |t| \le 1, \\ 1 & \text{if} \quad |t| > 1 \,. \end{cases}$$

Identity (8) together with the fact the kernel k is integrable and satisfies

$$\int_{\mathbb{R}^2} k(x) dx = 1$$

imply various convergence results as $\epsilon \to 0$. We mention the following which follows from fairly routine calculations but which has an interesting corollary.

Proposition 2. *Suppose f is in LR and $f(x)$ is continuous for all x in some open set Ω. Then*

$$\lim_{\epsilon \to 0} g_\epsilon(x) = f(x)$$

for all x in Ω.

The Radon transform of any function f which satisfies the above hypotheses is well defined and thus Proposition 2 implies that Radon's inversion formula (2) is valid for such f at all points x in Ω. In particular we may conclude that Radon's hypothesis (c) is not necessary for the validity of his formula (4).

Corollary 1. *If f satisfies conditions* (a) *and* (b) *in the Introduction, then $Rf(\theta, t)$ is well defined for almost all (θ, t) in $[0, 2\pi) \times \mathbb{R}$ and the inversion formula* (4) *is valid for all x.*

On the other hand the L^p result announced in the Introduction is not as routine as Proposition 1; namely, it is not an immediate consequence of identity (8) and well-known almost everywhere convergence results for $k_\epsilon * f$ such as those outlined in [10, 11]. The reason for this is that the kernel k has a significant singularity away from the origin on the circle $|x| = 1$. Nevertheless the techniques found in [10, 11] can be used and lead us to consider the maximal function $M_k f$ defined by

$$M_k f(x) = \sup_{\epsilon > 0} |k_\epsilon * f(x)|,$$

where k_ϵ is defined by (9) and (10). An application of Bourgain's theorem [1] concerning the bivariate circular maximal function and Marcinkiewicz's interpolation theorem [12, XII.4.6] results in the following.

Proposition 3. *If f is in $L^p(\mathbb{R}^2)$ for some $p > 4/3$, then $M_k f(x)$ is finite almost everywhere and*

$$\|M_k f\|_{L^p(\mathbb{R}^2)} \le C_p \|f\|_{L^p(\mathbb{R}^2)},$$

where C_p is a constant independent of f.

As a consequence of this proposition we may make the following conclusion.

Proposition 4. *If f is in $L^p(\mathbb{R}^2)$ for some $p > 4/3$, then*

$$\lim_{\epsilon \to 0} k_\epsilon * f(x) = f(x)$$

holds for almost all x.

Finally, Proposition 4 allows us to conclude the almost everywhere convergence result alluded to in the Introduction.

Theorem 1. *If f in LR and locally in L^p for some $p > 4/3$, then $Rf(\theta, t)$ is well defined for almost all (θ, t) in $[0, 2\pi) \times \mathbb{R}$ and the inversion formula* (4) *is valid for almost all x.*

Next we consider the effect of replacing $F_x(\epsilon)$ with $F_x(0)$ in the right-hand side of formula (4). Let $G_\epsilon(x)$ be the analogue of $g_\epsilon(x)$, namely,

$$G_\epsilon(x) = \frac{1}{\pi} \left\{ \frac{F_x(0)}{\epsilon} - \int_\epsilon^\infty \frac{F_x(t)}{t^2} dt \right\}. \tag{11}$$

Again, the fact that the transformation $f \to G_\epsilon$ is translation invariant in x implies that G_ϵ is the convolution of f with some distribution.

Proposition 5. *For every f in LR*

$$G_\epsilon(x) = K_\epsilon * f(x), \tag{12}$$

where

$$K_\epsilon(x) = \epsilon^{-2} K\left(x/\epsilon\right) \ , \tag{13}$$

$$K(x) = \frac{1}{\pi^2}\left\{\frac{1}{|x|} - \frac{\sqrt{|x|^2-1}}{|x|^2}\chi(x)\right\} \tag{14}$$

and χ is the indicator function of $\{x : |x| > 1\}$ as in Proposition 1.

As in the case with g_ϵ considered earlier, representation (12) together with the properties of the kernel K_ϵ routinely imply various convergence results as $\epsilon \to 0$. Moreover, the kernel K is significantly better behaved than the k in Proposition 1. This can be seen by re-expressing K as

$$K(x) = \frac{1}{\pi^2}\left\{\frac{1}{|x|}\{1-\chi(x)\} + \frac{1}{|x|^2\left\{|x| + \sqrt{|x|^2-1}\right\}}\chi(x)\right\}$$

and observing that it is a radial function which is monotonically decreasing as a function of $|x|$, $0 < |x| < \infty$, and $K(x) = O(|x|^{-3})$ as $|x|$ tends to ∞. These properties of K allow us to apply well-established results directly; for example see [10, Theorem 2, p. 62]. In particular we may conclude that

$$\lim_{\epsilon\to 0} K_\epsilon * f(x) = f(x) \tag{15}$$

for almost all x whenever f is in $L^p(\mathbb{R}^2)$ and $1 \le p \le \infty$. The same reasoning which allowed us to infer Corollary 1 from Proposition 2 and Theorem 1 from Proposition 4 allows us to use (15) to conclude the following.

Theorem 2. *If f is in LR, then $Rf(\theta, t)$ is well defined for almost all (θ, t) in $[0, 2\pi) \times \mathbb{R}$ and the inversion formula*

$$f(x) = \frac{1}{\pi}\lim_{\epsilon\to\infty}\left\{\frac{F_x(0)}{\epsilon} - \int_\epsilon^\infty \frac{F_x(t)}{t^2}dt\right\} \tag{16}$$

is valid for almost all x.

It may be of some interest to note that $G_\epsilon(x)$ may also be expressed as

$$G_\epsilon(x) = \frac{-1}{8\pi^2}\int_0^{2\pi}\int_{|t|>\epsilon}\frac{\Delta_t^2 Rf(\theta, \langle x, u_\theta\rangle)}{t^2}dtd\theta,$$

where

$$\Delta_t^2 Rf(\theta, \langle x, u_\theta\rangle) = Rf(\theta, \langle x, u_\theta\rangle + t) - 2Rf(\theta, \langle x, u_\theta\rangle) + Rf(\theta, \langle x, u_\theta\rangle - t)\,.$$

Thus Theorem 2 may be restated as follows.

Corollary 2. *If f is in LR, then*

$$f(x) = \frac{-1}{8\pi^2}\int_0^{2\pi}\int_{-\infty}^{\infty}\frac{\Delta_t^2 Rf(\theta, \langle x, u_\theta\rangle)}{t^2}dtd\theta \tag{17}$$

for almost all x, where the integral in the t variable is interpreted in the principal value sense.

Formula (17) was originally derived in [4] by a different method under the assumption that f is continuously differentiable and compactly supported.

Finally we return to formula (5). As was noted earlier, it appears that, in addition to the conditions in Theorem 1, it may be necessary to impose additional restrictions on f. Indeed, there are continuous functions f in $L^1(\mathbb{R}^2)$ which do not enjoy (7) for any x. For such functions integration by parts of the integral on the right-hand side of (5) is permissible and results in

$$-\frac{1}{\pi}\int_\epsilon^\rho \frac{dF_x(t)}{t} = g_\epsilon(x) - \frac{1}{\pi}\frac{F_x(\rho)}{\rho}$$

so that in view of Corollary 1 the failure of (7) implies the failure of (5). It follows that, while the restriction (c) may not be absolutely necessary, some sort of additional restriction on the behaviour of $f(x)$ for large $|x|$ is required to ensure condition (7).

Such a restriction can be quite mild. For example, assume that f is in LR. Then if $2 < p \leq \infty$ and f is in $L^p(\mathbb{R}^2)$, an application of Hölder's inequality shows that $F_x(t)$ is continuous and (7) is valid for all x. Hence for such f we may conclude that

$$k_\epsilon * f(x) = g_\epsilon(x) = -\frac{1}{\pi}\lim_{\rho\to\infty}\int_\epsilon^\rho \frac{dF_x(t)}{t} \tag{18}$$

for all x. Furthermore, by applying maximal function methods similar to those used to obtain Proposition 4, that is, by considering the maximal function

$$\sup_{t>0}\frac{|F_x(t)|}{t},$$

it is possible to show that if $p > 4/3$ and f is in $L^p(\mathbb{R}^2)$, then for almost all x both $F_x(t)$ is continuous in t and identity (7) is valid. This implies (18) for such x and, in view of Proposition 4, Radon's inversion formula.

Theorem 3. *Suppose $p > 4/3$ and f is in $LR \cap L^p(\mathbb{R}^2)$. Then $Rf(\theta, t)$ is well defined for almost all (θ, t) in $[0, 2\pi) \times \mathbb{R}$ and the inversion formula* (5) *is valid for almost all x.*

We remind the reader that if $1 \leq p < 2$, then $LR \supset L^p(\mathbb{R}^2)$ so that $LR \cap L^p(\mathbb{R}^2) = L^p(\mathbb{R}^2)$. Thus in the case $4/3 < p < 2$ mentioned in the abstract, the hypothesis on f in the above theorem reduces to $f \in L^p(\mathbb{R}^2)$.

The lower bound on the parameter p in Theorems 1 and 3 is sharp in the sense that if $p \leq 4/3$, there are functions f in $L^p(\mathbb{R}^2)$ which are non-negative and such that

$$\limsup_{\epsilon\to 0} g_\epsilon(x) = \infty$$

for all x in $\mathbb{R}^2$. Also, examples show that continuity and condition (6) are not enough to ensure (7). These and related examples are considered in the next section.

3. Details

3.1. $F_x(t)$ and condition (6). To see the significance of condition (6) suppose $f(x)$ is a continuous function with compact support on $\mathbb{R}^2$. Then the Radon transform

$Rf(\theta, t)$ is a continuous function on the cylinder $[0, 2\pi) \times \mathbb{R}$ and, using an appropriate change of variables, expression $F_x(t)$ defined by (3) can be re-expressed as

$$2\pi F_x(t) = 2 \int_{|y|>|t|} \frac{f(x-y)}{\sqrt{|y|^2 - t^2}} dy \,. \tag{19}$$

Identity (19) can be found in [7] in the case $x = 0$ and follows in the general case from translation invariance; see also [6]. An explicit derivation can be found in [5, p. 83].

Setting $x = 0$ and integrating both sides of (19) over the interval $(-\delta, \delta)$ with respect to the variable t results in

$$\int_{-\delta}^{\delta} \int_0^{2\pi} Rf(\theta, t) d\theta dt = 4 \int_{\mathbb{R}^2} f(y) h(y/\delta) dy \tag{20}$$

so that

$$\int_{-\delta}^{\delta} \int_0^{2\pi} |Rf(\theta, t)| d\theta dt \leq 4 \int_{\mathbb{R}^2} |f(y)| h(y/\delta) dy \tag{21}$$

for every positive number δ, where

$$h(x) = \begin{cases} \pi/2 & \text{if} \quad |x| \leq 1, \\ \arcsin(1/|x|) & \text{if} \quad |x| > 1 \,. \end{cases}$$

Since $0 < c \leq (1 + |x|)h(x) \leq C < \infty$ for all x, inequality (21) and Fubini Tonelli theorems allow us to conclude that Rf is locally integrable whenever f satisfies condition (6), in other words, whenever f is in LR. Furthermore, for positive functions f relation (20) implies that condition (6) is equivalent to the local integrability of Rf.

3.2. **Propositions 1 and 2 and related results.** Identity (19) also implies Proposition 1. Simply replace $F_x(t)$ with the right-hand side of (19) in the definition of $g_\epsilon(x)$, interchange orders of integration, and integrate out the t variable. For more explicit details see [5, Sec. 3.2].

Proposition 2 is a routine consequence of Proposition 1. To see this let Ω_0 be a compact subset of Ω whose interior is not empty and write $f(x) = f_0(x) + f_1(x)$, where

$$f_0(x) = \begin{cases} f(x) & \text{if } x \in \Omega_0, \\ 0 & \text{otherwise,} \end{cases}$$

and verify that

$$\lim_{\epsilon \to 0} k_\epsilon * f_0(x) = f_0(x) = f(x) \quad \text{ and } \lim_{\epsilon \to 0} k_\epsilon * f_1(x) = 0$$

whenever x is in the interior of Ω_0. Since every x in Ω is an element of the interior of a compact subset of Ω, the desired result follows.

3.3. **$F_x(t)$ when f is in $L^p(\mathbb{R}^2)$, $p > 2$.** Because a significant part of the analysis required to obtain Proposition 3 is analogous to that required to make sense of the Riemann Stieltjes integral in formula (5) when f is not necessarily continuous and compactly supported, we now turn to a closer examination of $F_x(t)$.

Notice that (19) can be re-expressed as

$$\pi \frac{F_x(t)}{t} = \phi_t * f(x),$$

where

$$\phi_t(x) = t^{-2}\phi(x/t),$$
$$\phi(x) = \frac{1}{\sqrt{|x|^2-1}}\chi(x),$$

and $\chi(x)$ is the indicator function of $\{x: |x| > 1\}$ as in Proposition 1.

In what follows it is convenient to express $\phi(x)$ as a sum

$$\phi(x) = \psi(x) + \lambda(x), \tag{22}$$

where

$$\psi(x) = \begin{cases} \phi(x) & \text{if } |x| \le 2, \\ 0 & \text{if } |x| > 2, \end{cases}$$

and $\lambda(x) = \phi(x) - \psi(x)$. Thus

$$\phi_t * f(x) = \psi_t * f(x) + \lambda_t * f(x),$$

where $\psi_t(x) = t^{-2}\psi(x/t)$ and $\lambda_t(x) = t^{-2}\lambda(x/t)$.

Proposition 6. *Suppose $p > 2$ and f is in $L^p(\mathbb{R}^2)$. Then for every x in $\mathbb{R}^2$*

(i) *$\psi_t * f(x)$ is a continuous function of t, $0 < t < \infty$,*
(ii) (a) *if $p < \infty$ or*
(b) *if $p = \infty$ and f is also in LR,*
then

$$\lim_{t\to\infty} \psi_t * f(x) = 0\,.$$

Proof. Because of translation invariance it suffices to prove this proposition in the case $x = 0$. We do so for the sake of notational convenience.

To see (i) take any positive t_0 and t and, because ψ is in $L^q(\mathbb{R}^2)$ where $\frac{1}{p} + \frac{1}{q} = 1$, write

$$|\psi_t * f(0) - \psi_{t_0} * f(0)| \le \|\psi_t - \psi_{t_0}\|_{L^q(\mathbb{R}^2)} \|f\|_{L^p(\mathbb{R}^2)}\,.$$

Statement (i) now follows from

$$\lim_{t\to t_0} \|\psi_t - \psi_{t_0}\|_{L^q(\mathbb{R}^2)} = 0\,.$$

Statement (ii)(a) is an immediate consequence of

$$|\psi_t * f(0)| \le t^{-2/p}\|\psi\|_{L^q(\mathbb{R}^2)}\|f\|_{L^p(\mathbb{R}^2)}$$

which follows from Hölder's inequality and $\|\psi_t\|_{L^q(\mathbb{R}^2)} = t^{-2/p}\|\psi\|_{L^q(\mathbb{R}^2)}$.

The proof of statement (ii)(b) is only slightly more complicated. To see it take any positive ϵ, choose a positive δ so that

$$\int_{1<|x|\le 1+\delta} \frac{1}{\sqrt{|x|^2-1}}dx < \frac{\epsilon}{2\|f\|_{L^\infty(\mathbb{R}^2)}}\,,$$

and write

$$\psi_t * f(0) = I_1 + I_2,$$

where

$$I_1 = \int_{t<|x|\le(1+\delta)t} \psi_t(x)f(x)dx \quad \text{and } I_2 = \int_{(1+\delta)t<|x|\le 2t} \psi_t(x)f(x)dx\,.$$

Now, in view of our choice of δ and the fact that f is in $L^\infty(\mathbb{R}^2)$, it is clear that $|I_1| < \epsilon/2$. On the other hand,

$$|I_2| \leq \frac{1}{t}\int_{(1+\delta)t<|x|\leq 2t} \frac{|x/t|}{\sqrt{|x/t|^2-1}}\frac{|f(x)|}{|x|}dx, \leq \frac{1}{t}\frac{C}{\sqrt{\delta}}\int_{t<|x|\leq 2t}\frac{|f(x)|}{|x|}dx,$$

and it is clear, because f is in LR, that $|I_2| < \epsilon/2$ whenever t is sufficiently large. So we may conclude that $|\psi_t * f(0)| < \epsilon$ whenever t is sufficiently large, which is the desired result. □

In the case $p = \infty$ it should be clear from the proof that the additional condition $f \in LR$ is sufficient but not necessary to obtain the conclusion of statement (ii). However some sort of decay of $f(x)$ for large $|x|$ is necessary since, for instance, $\psi_t * f(x)$ is constant whenever f is. For example, the condition

$$\lim_{r\to\infty} f(ru_\theta) = 0 \quad \text{ uniformly in } \theta$$

is also sufficient.

Proposition 7. *If f is in LR, then for every x in $\mathbb{R}^2$ the convolution $\lambda_t * f(x)$ is a continuous function of t, $0 < t < \infty$, and*

$$\lim_{t\to 0}\lambda_t * f(x) = 0\,.$$

Proof. As in the above argument we consider the case $x = 0$.

Write

$$|\lambda_t * f(0) - \lambda_{t_0} * f(0)| \leq I(t,t_0)\|f\|_{LR},$$

where $I(t,t_0)$ is the $L^\infty(\mathbb{R}^2)$ norm of $(1+|x|)(\lambda_t(x)-\lambda_{t_0}(x))$ in the x variable. The continuity statement follows from the fact that

$$\lim_{t\to t_0} I(t,t_0) = 0\,.$$

The remaining statement follows from

$$|\lambda_t * f(0)| \leq I(t)\|f\|_{LR},$$

where $I(t)$ is the $L^\infty(\mathbb{R}^2)$ norm of $(1+|x|)\lambda_t(x)$ in the x variable and the fact that

$$\|\lambda_t\|_{L^\infty(\mathbb{R}^2)} \leq \frac{1}{t^2}$$

and

$$\sup_{x\in\mathbb{R}^2}|x|\lambda_t(x) = \sup_{\{x:\,|x|>2t\}}\frac{1}{t}\frac{|x/t|}{\sqrt{|x/t|^2-1}} \leq \frac{2}{t}.$$

□

The above propositions allow us to conclude that if $p > 2$ and f is in $L^p(\mathbb{R}^2)\cap LR$, then $F_x(t)$ is a continuous function of t, $0 < t < \infty$, for every x. Thus for all x in $\mathbb{R}^2$, the Riemann Stieltjes integral in (5) is well defined and integration by parts results in

$$\int_\epsilon^\rho \frac{dF_x(t)}{t} = \frac{F_x(\rho)}{\rho} - \frac{F_x(\epsilon)}{\epsilon} + \int_\epsilon^\rho \frac{F_x(t)}{t^2}dt\,.$$

These propositions also allow us to conclude that for such f relation (7) is valid for all x. Summarizing gives us the following.

Proposition 8. *Suppose $p > 2$ and f is in $L^p(\mathbb{R}^2) \cap LR$. Then for every x in $\mathbb{R}^2$ the function $F_x(t)$ is a continuous function of t, $0 < t < \infty$, and*

$$-\frac{1}{\pi}\lim_{\rho\to\infty}\int_\epsilon^\rho \frac{dF_x(t)}{t} = g_\epsilon(x) = k_\epsilon * f(x)\,.$$

3.4. $F_x(t)$ **when** f **is in** $L^p(\mathbb{R}^2)$**,** $p \le 2$**.** In the case $p \le 2$, indicated by the title of this subsection, matters are not quite as nice because, recalling the decomposition (22), ψ is not in $L^q(\mathbb{R}^2)$, where $\frac{1}{p}+\frac{1}{q}=1$, so that the relatively routine techniques involving Hölder's inequality used in the previous subsection fail for $\psi_t * f(x)$. We must use another tactic and, as suggested by the techniques described in [10, 11, 12], consider the maximal function

$$M_\psi f(x) = \sup_{t>0}\int_{\mathbb{R}^2}\psi_t(x-y)|f(y)|dy\,. \tag{23}$$

Proposition 9. *If $p > 4/3$ and f is in $L^p(\mathbb{R}^2)$, then $M_\psi f(x)$ is finite for almost all x and*

$$\|M_\psi f\|_{L^p(\mathbb{R}^2)} \le C_p\|f\|_{L^p(\mathbb{R}^2)},$$

where C_p is a constant which depends only on p.

This proposition is a consequence of a bound on yet another maximal function $T_\delta f(x)$ defined by

$$T_\delta f(x) = \sup_{t>0}\int_{1\le|y|\le 1+\delta}|f(x+ty)|dy\,. \tag{24}$$

Lemma 1. *If f is in $L^p(\mathbb{R}^2)$ for some p, $1 \le p \le \infty$, then $T_\delta f(x)$ is finite almost everywhere. If $1 < p \le \infty$, then $T_\delta f(x)$ is in $L^p(\mathbb{R}^2)$ and*

$$\|T_\delta f\|_{L^p(\mathbb{R}^2)} \le C_{p,\delta}\|f\|_{L^p(\mathbb{R}^2)}, \tag{25}$$

where $C_{p,\delta}$ is a constant which depends on p and δ but is independent of f. Specifically for $0 < \delta \le 1$

$$C_{p,\delta} \le c_p \begin{cases} \delta^{(1-1/p)(2-\epsilon)} & \text{if } 1 < p \le 2,\\ \delta & \text{if } p > 2,\end{cases} \tag{26}$$

where c_p is a constant which depends only on p and the first inequality is valid for all positive ϵ.

Proof. Note that

$$T_\delta f(x) \le CMf(x), \tag{27}$$

where

$$Mf(x) = \sup_{t>0}\frac{1}{\pi\epsilon^2}\int_{|y|\le t}|f(x-y)|dy \tag{28}$$

is the classical bivariate Hardy-Littlewood maximal function and C is a constant which is independent of δ if $0 < \delta \le 1$. Inequality (27) together with properties of $Mf(x)$ imply the first two assertions of the lemma.

To see (26) let

$$Af(x) = \sup_{t>0}\int_0^{2\pi}|f(x+tu_\theta)|d\theta$$

and recall Bourgain's result [1], that is, if f is in $L^p(\mathbb{R}^2)$ for $p > 2$, then $Af(x)$ is finite almost everywhere and

$$\|Af\|_{L^p(\mathbb{R}^2)} \leq C_p \|f\|_{L^p(\mathbb{R}^2)}, \tag{29}$$

where the constant C_p depends only on p. Clearly

$$T_\delta f(x) \leq C_\delta Af(x), \tag{30}$$

where $C_\delta = \int_1^{1+\delta} r dr$ so that

$$C_\delta \leq 2\delta \text{ if } 0 < \delta \leq 1\,. \tag{31}$$

Inequalities (29), (30), and (31) imply (26) in the case $p > 2$.

To see (26) in the case $1 < p \leq 2$ we may and do assume $\epsilon < 1$. Use the fact implied by (27) that the transformation $f \to T_\delta f$ is of weak type (1,1) with constant C_0, the bound (26) in the case $p > 2$, and the interpolation theorem of Marcinkiewicz [12, XII.4.6] to obtain the bound

$$C_{p,\delta} \leq C C_0^{1-\theta} C_{p_0,\delta}^{\theta}, \tag{32}$$

where C and C_0 are independent of δ, $p_0 > 2$, and θ satisfies

$$\frac{1}{p} = 1 - \theta + \frac{\theta}{p_0}\,.$$

This means that

$$\theta = \frac{1 - \frac{1}{p}}{1 - \frac{1}{p_0}}$$

so that the choice

$$p_0 = 2 + \frac{\epsilon}{1 - \epsilon}$$

implies the desired result and completes the proof of the lemma. □

To see the proposition let $\Omega_j = \{x : 1 + 2^{-j} < |x| \leq 1 + 2^{1-j}\}$ and write

$$\begin{aligned}
\int_{\mathbb{R}^2} \psi_t(x-y)|f(y)|dy &= \int_{\mathbb{R}^2} \psi(y)|f(x-ty)|dy \\
&= \sum_{j=1}^{\infty} \int_{\Omega_j} \psi(y)|f(x-ty)|dy \\
&\leq \sum_{j=1}^{\infty} 2^{j/2} \int_{\Omega_j} |f(x-ty)|dy
\end{aligned}$$

so it is clear that

$$M_\psi f(x) \leq \sum_{j=1}^{\infty} 2^{j/2} T_{\delta_j} f(x), \tag{33}$$

where $\delta_j = 2^{1-j}$. The lemma implies that

$$\|M_\psi f\|_{L^p(\mathbb{R}^2)} \leq C_p \left\{ \sum_1^{\infty} 2^{j/2} \gamma_j \right\} \|f\|_{L^p(\mathbb{R}^2)}, \tag{34}$$

where

$$\gamma_j = \begin{cases} 2^{-j(1-1/p)(2-\epsilon)} & \text{if } 1 < p \leq 2, \\ 2^{-j} & \text{if } p > 2 \end{cases}$$

and ϵ can be chosen to be any positive number. By choosing ϵ sufficiently small, the series in (34) converges when $p > 4/3$. Thus (33) and (34) imply Proposition 9 so its proof is complete.

As a consequence of Proposition 9 we have the following.

Proposition 10. *If $4/3 < p < \infty$ and f is in $L^p(\mathbb{R}^2)$, then for almost all x in $\mathbb{R}^2$ the convolution $\psi_t * f(x)$ is a continuous function of t, $0 < t < \infty$, and*

$$\lim_{t\to\infty} \psi_t * f(x) = 0\,.$$

Finally, using reasoning similar to that at the end of the last subsection, we may conclude the following.

Proposition 11. *Suppose that $4/3 < p < 2$ and f is in $L^p(\mathbb{R}^2)$ or that $p = 2$ and f is in $L^p(\mathbb{R}^2) \cap LR$. Then for almost all x in $\mathbb{R}^2$ the function $F_x(t)$ is a continuous function of t, $0 < t < \infty$, and*

$$-\frac{1}{\pi} \lim_{\rho\to\infty} \int_\epsilon^\rho \frac{dF_x(t)}{t} = g_\epsilon(x) = k_\epsilon * f(x)\,.$$

3.5. **Propositions 3 and 4 and Theorems 1 and 3.** To see Proposition 3 note that

$$k(x) \le C\left\{\psi(x) + \frac{\chi(x/2)}{|x|^3}\right\}, \tag{35}$$

where $\psi(x)$ is the function in decomposition (22) and $\chi(x)$ is the indicator function of $\{x : |x| > 1\}$ as in Proposition 1. Because the radial majorant of the second term on the right, namely the function of x defined by

$$\sup_{\{y:|y|>|x|\}} \frac{\chi(y/2)}{|y|^3}\,,$$

is integrable over $\mathbb{R}^2$, we may conclude that

$$M_k f(x) \le C\{M_\psi f(x) + Mf(x)\}, \tag{36}$$

where C is a constant independent of f, $M_\psi f(x)$ is the maximal function defined by (23), and $Mf(x)$ is the classical Hardy-Littlewood maximal function defined by (28). In view of (36) Proposition 3 is a consequence of Proposition 9 and the corresponding property of $Mf(x)$.

Proposition 4 follows from Proposition 3 in the same way that the Lebesgue differentiation theorem for integrals of functions f in L^p follows from the properties of $Mf(x)$. For more details see Subsection 4.1 or [10].

Theorem 1 follows from Propositions 2 and 4. To see this assume f satisfies the hypothesis of the theorem and write

$$k_\epsilon * f(x) = k_\epsilon * f_0(x) + k_\epsilon * f_1(x),$$

where $f_1(x) = f(x)\chi(x/\rho)$, $\rho > 0$, with $\chi(x)$ as above and $f_0(x) = f(x) - f_1(x)$. Then Proposition 2 implies that

$$\lim_{\epsilon\to 0} k_\epsilon * f_1(x) = 0 \quad \text{for all } x \text{ in } B_\rho = \{x : |x| < \rho\}$$

while Proposition 4 implies that

$$\lim_{\epsilon\to 0} k_\epsilon * f_0(x) = f_0(x) = f(x) \quad \text{for almost all } x \text{ in } B_\rho\,.$$

Since every x in $\mathbb{R}^2$ is in B_ρ for some positive ρ it follows that for almost all x in $\mathbb{R}^2$ both

$$\lim_{\epsilon\to 0} k_\epsilon * f(x) = f(x)$$

and Radon's inversion formula (4) are valid.

Theorem 3 of course follows from Propositions 4, 8, and 11.

3.6. **Examples.** The examples below show the following:

- that the restriction on p in Theorems 1 and 3 and in the supporting propositions cannot be relaxed,
- that there are continuous functions f for which (4) is valid for all x while (5) fails for all x, and
- that there are functions f for which both (4) and (5) are valid for almost all x while Radon's condition (c) fails for all x.

These examples are based on the following observation.

Lemma 2. *If ψ is the function defined by decomposition* (22) *and f is any non-negative radial function, then*

$$\psi_{|x|} * f(x) \geq \frac{c}{|x|\sqrt{|x|+1}} \int_D |y|^{-1/2} f(y)dy,$$

where $D = \{y:\ |y| < \min(|x|, 1)\}$ and c is a constant independent of f and x.

Since $\psi(x) \leq 4k(x)$ this lemma remains valid with ψ replaced with k.

Consider the function $h(x)$ defined by the formula

$$h(x) = \begin{cases} |x|^{-3/2}\,|\log|x||^{-1} & \text{if } |x| \leq 1/2, \\ 0 & \text{if } |x| > 1/2\,. \end{cases} \tag{37}$$

Clearly h is in $L^p(\mathbb{R}^2)$ for $p \leq 4/3$. However, using Lemma 2 to estimate $k_{|x|} * h(x)$ from below, it is also clear that both

$$\psi_{|x|} * h(x) = \infty \quad \text{and} \quad k_{|x|} * h(x) = \infty, \tag{38}$$

so that for every x in $\mathbb{R}^2$ both

$$M_\psi h(x) = \infty \quad \text{and} \quad M_k * h(x) = \infty\,.$$

Proposition 12. *Suppose that*

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} h(x - x_n),$$

where $h(x)$ is defined by (37) *and where $\{x_n\}$ is a countable dense subset of $\mathbb{R}^2$. Then f is in $L^p(\mathbb{R}^2)$ for $1 \leq p \leq 4/3$ and*

$$\limsup_{\epsilon\to 0} k_\epsilon * f(x) = \infty$$

for every point x in $\mathbb{R}^2$.

Next, suppose that α and ϵ are real parameters satisfying $0 < \alpha < 2$ and $0 < \epsilon < 1$. Let $h_{\alpha,\epsilon}(x)$ be a continuous non-negative radial function which is monotonically decreasing as a function of $|x|$ and satisfies

$$h_{\alpha,\epsilon}(x) = \begin{cases} \epsilon^{-\alpha} & \text{if } |x| \leq \epsilon, \\ |x|^{-\alpha} & \text{if } \epsilon < |x| \leq 1, \\ 0 & \text{if } |x| \geq 3/2\,. \end{cases}$$

Routine calculations show that

$$\int_{\mathbb{R}^2} |h_{\alpha,\epsilon}(x)|^p dx \le C_{\alpha,p} \begin{cases} \epsilon^{2-\alpha p} & \text{if } \alpha p > 2, \\ (1+|\log\epsilon|) & \text{if } \alpha p = 2, \\ 1 & \text{if } \alpha p < 2\,, \end{cases} \tag{39}$$

$$\int_{\mathbb{R}^2} |x|^{-1/2} h_{\alpha,\epsilon}(x) dx \ge C_{\alpha} \begin{cases} \epsilon^{3/2-\alpha} & \text{if } \alpha > 3/2, \\ (1+|\log\epsilon|) & \text{if } \alpha = 3/2, \\ 1 & \text{if } \alpha < 3/2\,, \end{cases} \tag{40}$$

and, if $|x| > 2$,

$$\frac{1}{2\pi}\int_0^{2\pi} h_{\alpha,\epsilon}(x - |x|u_\theta) d\theta \ge \frac{C_\alpha}{|x|} \begin{cases} \epsilon^{1-\alpha} & \text{if } \alpha > 1, \\ (1+|\log\epsilon|) & \text{if } \alpha = 1, \\ 1 & \text{if } \alpha < 1\,. \end{cases} \tag{41}$$

Define $f_\alpha(x)$ via

$$f_\alpha(x) = \sum_{n=1}^{\infty} \frac{h_{\alpha,\epsilon_n}(x - x_n)}{n^2}, \tag{42}$$

where $\{x_n\}$ is a sequence of points in $\mathbb{R}^2$ which satisfy $|x_n| = 4n$. The specific values of the parameters $\{\epsilon_n\}$ will be chosen later.

Note that $f_\alpha(x)$ is a non-negative continuous function on $\mathbb{R}^2$ and in view of (39)

$$\int_{\mathbb{R}^2} |f_\alpha(x)|^p dx < \infty \quad \text{ if } \alpha < 2/p\,. \tag{43}$$

In particular this means that f_α is not only in LR but also in $L^1(\mathbb{R}^2)$ for all values of the parameter α in the range $0 < \alpha < 2$.

Proposition 13. *Suppose that $f_\alpha(x)$ is defined by* (42), $3/2 < \alpha < 2$, *and* $\epsilon_n = 2^{n/(3/2-\alpha)}$, $n = 1, 2, \dots$. *Then*

$$\limsup_{t\to\infty} \psi_t * f_\alpha(x) = \infty$$

for every point x in $\mathbb{R}^2$.

Note that the functions f_α in the above proposition satisfy the hypothesis of Proposition 2, so that Radon's inversion formula (4) is valid for such functions for all x in $\mathbb{R}^2$. However the conclusion of the above proposition implies that inversion formula (5) fails for such functions for all x in $\mathbb{R}^2$.

Proposition 14. *Suppose that $f_\alpha(x)$ is defined by* (42), $1 < \alpha < 2$, *and* $\epsilon_n = 2^{n/(1-\alpha)}$, $n = 1, 2, \dots$. *Then*

$$\limsup_{t\to\infty} \int_0^{2\pi} f_\alpha(x + tu_\theta) d\theta = \infty$$

for every point x in $\mathbb{R}^2$.

Again, note that the functions f_α in the above proposition satisfy the hypothesis of Proposition 2 so that Radon's inversion formula (4) is valid for such functions for all x in $\mathbb{R}^2$. Moreover, if $\alpha < 3/2$ and $4/3 < p < 2/\alpha$, then the functions f_α are in $L^p(\mathbb{R}^n)$ so that in view of Theorem 3 Radon's inversion formula (5) is also valid for almost all x in $\mathbb{R}^2$. However, the above proposition implies that Radon's condition (c) fails for every x.

4. More details

4.1. **Proposition 10.** Proposition 10 follows from Proposition 9 essentially in the same way that Proposition 4 follows from Proposition 3. For the sake of completeness we outline the argument.

First, without loss of any generality, we may and do assume that the function f is real valued. Next, adapting *mutatis mutandis* the argument found in [10], define

$$\Omega_0 f(x) = \sup_{0<t_0<\infty} \left| \limsup_{t\to t_0} \psi_t * f(x) - \liminf_{t\to t_0} \psi_t * f(x) \right| ,$$

$$\Omega_1 f(x) = \limsup_{t\to\infty} \left|\psi_t * f(x)\right| ,$$

and observe the following:

- $\Omega_0 f(x) \le 2M_\psi f(x)$ and $\Omega_1 f(x) \le M_\psi f(x)$ so, in view of Proposition 9, if $p > 4/3$ and f is in $L^p(\mathbb{R}^2)$, then for any positive ϵ

$$\operatorname{meas}\{x : \Omega_i f(x) > \epsilon\} \le \frac{C}{\epsilon^p}\|f\|^p_{L^p(\mathbb{R}^2)} , \quad i = 0, 1,$$

  where C is a constant independent of f.
- If f_0 is continuous and compactly supported, then $\Omega_0 f_0(x) = 0$ and $\Omega_1 f_0(x) = 0$ for all x in $\mathbb{R}^2$. Thus

$$\Omega_i f \le \Omega_i(f - f_0)(x) + \Omega_i(f_0) = \Omega_i(f - f_0)(x) , \quad i = 0, 1.$$

Now suppose that $4/3 < p \le 2$ and f is in $L^p(\mathbb{R}^2)$. Then in view of the above observations, for any positive ϵ and any continuous and compactly supported f_0 we may write

$$\operatorname{meas}\{x : \Omega_i f(x) > \epsilon\} \le \frac{C}{\epsilon^p}\|f - f_0\|^p_{L^p(\mathbb{R}^2)} , \quad i = 0, 1.$$

Since such an f_0 may be chosen so that $\|f - f_0\|_{L^p(\mathbb{R}^2)}$ is arbitrarily small, we may conclude that

$$\operatorname{meas}\{x : \Omega_i f(x) > 0\} = 0 , \quad i = 0, 1.$$

This is the desired result.

4.2. **Lemma 2.** To see Lemma 2 recall that

$$\psi_t(x) = \frac{1}{t\sqrt{|x|^2 - t^2}} \quad \text{if} \quad 1 < |x/t| \le 2$$

and is $= 0$ otherwise. Also note that if $f(x)$ is a radial function, then so is $\psi_t * f(x)$ and thus

$$\psi_t * f(x) = \psi_t * f((|x|, 0)) .$$

Next observe that with $t = |x|$, $y = (y_1, y_2)$ and $x = (|x|, 0)$ we may write

$$\begin{aligned} |x - y|^2 - t^2 &= |y|^2 - 2|x|y_1 \\ &= y_1^2 + y_2^2 + 2|x||y_1| \qquad \text{if} \quad y_1 \le 0 \\ &\le (1 + 2|x|)|y_1| + |y_2| \qquad \text{if} \quad |y| \le 1 \\ &\le (1 + 2|x|)(|y_1| + |y_2|) \\ &\le 4(1 + |x|)|y| \end{aligned}$$

and, upon sketching a plot of the support of $\psi_{|x|}(x-y)$ as a function $y=(y_1,y_2)$, see that

$$\psi_{|x|}(x-y) \geq \frac{1}{2|x|\sqrt{|x|+1}}\frac{1}{\sqrt{|y|}}$$

whenever y is in $D_1=\{y=(y_1,y_1):\ y_1\leq 0 \text{ and } |y|\leq \min(|x|,1)\}$. Hence if f is a non-negative radial function, then

$$\psi_{|x|}*f(x) \geq \int_{D_1}\psi_{|x|}(x-y)f(y)dy, \geq c\int_D \frac{|y|^{-1/2}}{|x|\sqrt{|x|+1}}f(y)dy,$$

where $D=\{y:\ \ |y|<\min(|x|,1)\}$. The last inequality follows from the lower bound on $\psi(x-y)$ and the radial symmetry of the resulting integrand.

4.3. **Propositions 12, 13, and 14.** To see Proposition 12 note that

$$k_\epsilon * f(x) \geq 2^{-n}k_\epsilon * h(x-x_n)$$

which, in view of (38), is ∞ for $\epsilon=|x-x_n|$. Since $\{x_n\}$ is dense in $\mathbb{R}^2$, choosing a subsequence $\{x_{n_j}\}$ which converges to x and setting $\epsilon_{n_j}=|x-x_{n_j}|$, we see that

$$\lim_{n_j\to\infty}\epsilon_{n_j}=0 \quad \text{ and } \quad k_{\epsilon_{n_j}}*f(x)=\infty \quad \text{ for each } \epsilon_{n_j}.$$

This implies Proposition 12.

The proofs of Propositions 13 and 14 follow pretty much the same pattern.

To see Proposition 13 write

$$\psi_t * f_\alpha(x) \geq n^{-2}\psi_t * h_{\alpha,\epsilon_n}(x-x_n)$$

and, in view of Lemma 2,

$$\psi_{|x-x_n|}*h_{\alpha,\epsilon_n}(x-x_n) \geq \frac{c_1}{|x-x_n|^{3/2}}\int_{|y|<1}|y|^{-1/2}h_{\alpha,\epsilon_n}(y)dy \geq \frac{c_2\, 2^n}{n^{3/2}}$$

whenever $|x_n|=4n$ is sufficiently large, where c_1 and c_2 are positive constants independent of x and n. Thus by choosing $t_n=|x-x_n|$, it is clear that $t_n\to\infty$ as $n\to\infty$ and that

$$\lim_{t_n\to\infty}\psi_{t_n}*f_\alpha(x)=\infty$$

which implies the desired result.

To see Proposition 14 note that

$$\int_0^{2\pi} f_\alpha(x+|x-x_n|u_\theta)d\theta \geq n^{-2}\int_0^{2\pi} h_{\alpha,\epsilon_n}(x-x_n+|x-x_n|u_\theta)d\theta \geq \frac{c\,2^n}{n^3}$$

for sufficiently large $|x_n|=4n$. As above, choosing $t_n=|x-x_n|$, it is clear that $t_n\to\infty$ as $n\to\infty$ and that

$$\lim_{t_n\to\infty}\int_0^{2\pi} f_\alpha(x+t_nu_\theta)d\theta=\infty$$

which implies the desired result.

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