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SUBGROUPS OF $\mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)$ ACTING TRANSITIVELY ON 4-TUPLES

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Abstract. We consider subgroups of C^{∞}-diffeomorphisms of the circle $\mathbb{S}^1$ which act transitively on 4-tuples of points. We show, in particular, that these subgroups are dense in the group of homeomorphisms of $\mathbb{S}^1$. A stronger result concerning C^{∞}-approximations is obtained as well. The techniques employed in this paper rely on Lie algebra ideas and they also provide partial generalizations to the differentiable case of some results previously established in the analytic category.

1. Introduction

In the list of problems on Group Theory compiled by M. Bestvina, it is asked whether or not a subgroup of $\mathrm{Homeo}_{+}(\mathbb{S}^1)$ acting transitively on 4-tuples must be (C^0) dense in $\mathrm{Homeo}_{+}(\mathbb{S}^1)$. Here $\mathrm{Homeo}_{+}(\mathbb{S}^1)$ stands for the group of orientation-preserving homeomorphisms of the circle which acts diagonally on the set of 4-tuples (it seems that this question should be attributed to de la Harpe). In this paper we investigate its differentiable version, namely we deal with the group of orientation-preserving C^{∞}-diffeomorphisms of the circle.

The methods employed in this paper are based on the Lie structure of the space of C^{∞} vector fields on $\mathbb{S}^1$. They seem to be interesting by themselves, since similar methods/results have proved to be useful in the analytic setting (cf. [Na], [Re]). In fact, to make precise the extent to which the main results of [Na], [Re] require the analytic assumption was an additional motivation for the present article. In particular, we solve the question above when Γ is a group of diffeomorphisms.

Before stating our main results, recall that a group $\Gamma \subseteq \mathrm{Homeo}_{+}(\mathbb{S}^1)$ possesses a natural diagonal action on the n-dimensional torus $\mathbb{T}^n$ viewed as the product of n copies of $\mathbb{S}^1$. Precisely, a homeomorphism $h \in \Gamma$ induces a homeomorphism of $\mathbb{T}^n$ through the assignment $(x_1, \ldots, x_n) \mapsto (h(x_1), \ldots, h(x_n))$ where $x_i \in \mathbb{S}^1$ are natural coordinates for $\mathbb{T}^n$. Given Γ as above, let us denote by $\Gamma^{(n)}$ its diagonal action on $\mathbb{T}^n$. Note that $\Gamma^{(n)}$ leaves the diagonal of $\mathbb{T}^n$ invariant so that this action is never transitive in the proper sense. This leads to the adaptation of the notion of transitivity described in the sequel. Consider ordered n-tuples $(x_1, \ldots, x_n) \in \mathbb{T}^n$ and $(y_1, \ldots, y_n) \in \mathbb{T}^n$. This means that starting at x_1 (resp. y_1) and going around the circle in the sense of the fixed orientation of $\mathbb{S}^1$ we have $x_1 \leq x_2 \leq \cdots \leq x_n$ (resp. $y_1 \leq y_2 \leq \cdots \leq y_n$). Now consider ordered n-tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ satisfying the following condition: for every pair of indices $1 \leq i_0 < i_1 \leq n$ such that

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$x_{i_0} = \cdots = x_{i_1}$ one automatically has $y_{i_0} = \cdots = y_{i_1}$. A group $\Gamma \subseteq \mathrm{Homeo}_+(\mathbb{S}^1)$ is said to be transitive on n-tuples if, given $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ satisfying the condition above, there is $h \in \Gamma$ such that $h(x_i) = y_i$ for every $i = 1, \ldots, n$.

Notice that $\mathrm{PSL}(2, \mathbb{R})$ is a distinguished closed subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$ which is transitive on 3-tuples but not on 4-tuples. In fact, as is well known, the action of $\mathrm{PSL}(2, \mathbb{R})$ on 4-tuples of points preserves their cross-ratio. Hence the assumption that the initial group $\Gamma \subseteq \mathrm{Homeo}_+(\mathbb{S}^1)$ acts transitively on 4-tuples of points is necessary to exclude $\mathrm{PSL}(2, \mathbb{R})$. It may also be observed that several results in this paper require only the group to be transitive on 3-tuples. Indeed, transitivity on 3-tuples is sufficient to allow the construction of certain nontrivial vector fields naturally associated to the group Γ which are the main tools of this paper. These vector fields actually form a sort of "Lie algebra" associated to Γ. If, in addition, the group is transitive on 4-tuples, then this "algebra" is not contained in the Lie algebra of $\mathrm{PSL}(2, \mathbb{R})$. With these notations, our first statement is:

Theorem A′. *Assume that a subgroup* $\Gamma \subset \mathrm{Diff}_+^\infty(\mathbb{S}^1)$ *acts transitively on* 4*-tuples. Then* Γ *is* C^0*-dense in* $\mathrm{Homeo}_+(\mathbb{S}^1)$*.*

Since the group Γ is a group of diffeomorphisms, it is natural to consider convergence of derivatives as well. In this direction our main result is the theorem below which immediately implies Theorem A′.

Theorem A. *Consider a subgroup* Γ *of* $\mathrm{Diff}_+^\infty(\mathbb{S}^1)$ *and assume that* Γ *acts transitively on* 4*-tuples. Let* F *be a* C^∞*-diffeomorphism of* $\mathbb{S}^1$ *and consider an interval* $I \subset \mathbb{S}^1$ *of arbitrary length (i.e.,* $\mathbb{S}^1 \setminus I$ *is only supposed to be an interval containing more than one point). Then there exists a sequence* $\{h_k\} \subset \Gamma$ *such that the restrictions of the* h_k*'s to* I *converge* C^∞ *to the restriction of* F *to* I*.*

We believe that similar ideas might allow one to dispense with the "interval I" and lead to a complete characterization of closed subgroups of Diff_+^∞ acting transitively on 4-tuples.

We say that a subgroup $\Gamma \subset \mathrm{Diff}_+^\infty(\mathbb{S}^1)$ is *nondiscrete* if and only if the conditions below are verified:

- Γ contains a sequence of elements $\{h_k\}_{k \in \mathbb{N}}$ converging (C^∞) to the identity, Id, on some open (nonempty) interval $I \subset \mathbb{S}^1$.
- The set of fixed points of each h_k has empty interior on I.

With this definition, many examples of nondiscrete subgroups Γ arise in Foliation Theory and Differential Geometry. Dealing with nondiscrete subgroups of $\mathrm{Diff}_+^\infty(\mathbb{S}^1)$, our methods are flexible enough to allow extensions of the results of [Re] under reasonably weak assumptions. The theorem below is an example in which we assume that Γ is generated by diffeomorphisms close to the identity in order to make a clearer statement. Given a diffeomorphism $h \in \mathrm{Diff}_+^\infty$, denote by $\mathrm{Fix}(h)$ the set of fixed points of h.

Theorem B. *Denote by* Γ *a subgroup of* $\mathrm{Diff}_+^\infty(\mathbb{S}^1)$ *generated by elements sufficiently close to the identity which does not preserve any probability measure on* $\mathbb{S}^1$*. Let* $\{h_k\} \subset \Gamma$ *be a sequence converging* C^∞ *to the identity on some open interval* $I \neq \emptyset$*. Assume that* $\mathrm{Fix}(h_k)$ *does not have isolated points in* I *for every* k*. Then for every point* $q \in \mathbb{S}^1$ *and every* $r \in \mathbb{N}$*, there exists a nontrivial* C^r *vector field* X *defined on a neighborhood* I_0 *of* q *and having the following property: given a* *relatively compact interval $I' \subset I_0$ and $t_0 \in \mathbb{R}$ so small that the local flow ϕ_X^t associated to X is defined on I' for every $0 \leq t \leq t_0$, then the mapping $\phi^{t_0} : I' \to \mathbb{R}$ is C^r-approximated on I' by a suitable sequence of elements in Γ.*

Notice that the assumptions made on the sequence h_k imply that, for fixed k, the intersection of the set of fixed points of h_k with I is either a Cantor set or empty. Some alternative assumptions for Theorem B will be stated in Section 5.

The method of associating vector fields to certain group actions, which is exploited in this paper, was originally developed by Shcherbakov and Nakai for the case of $\mathrm{Diff}(\mathbb{C}, 0)$ (the group of germs of holomorphic diffeomorphisms; cf. [Sh], [Na]). Later I succeeded in constructing similar vector fields associated to certain groups of analytic diffeomorphisms of the circle (cf. [Re]). Finally, in [L-R], a good part of the results of [Sh], [Na] and [Re] were extended to arbitrary dimensions. One of the applications of this theory appears in [Be], where Belliart combines results similar to those of [L-R] to Cartan's classification of graded Lie algebras to approximate functions by elements of certain groups. This idea of bringing Cartan's theorem into the discussion will also be employed in Section 4 of the present paper.

To close this introduction, we want to point out that the vector fields mentioned above are intrinsically dependent on the differentiable nature of our problems. Precisely, the structure of Lie algebras exploited here naturally requires the existence of C^∞ vector fields associated to the group Γ. These C^∞ vector fields will be contructed in Proposition 4.1. In Section 4 it will be shown that they are suited to deal with the differentiable version of de la Harpe's question. As to the original topological version, they suggest a possible new approach based on *topological groups* rather than Lie groups. There is a well-developed theory of topological groups in connection with Hilbert's problem about which topological groups carry a Lie group structure. Strictly speaking there is no "Lie algebra" associated with topological groups since "vector fields" make no sense in this context. However, we can talk about 1-parameter groups or *homomorphism* from $\mathbb{R}$ into the group. These 1-parameter groups are the topological analogue of our Lie algebras (or a "topological Lie algebra") and they may provide a natural framework to adapt our techniques to the group $\mathrm{Homeo}_+(\mathbb{S}^1)$. For an excellent discussion of this subject we refer the reader to [Ka].

2. Some nondiscrete groups

In this section we are going to establish an elementary criterium to detect nondiscrete subgroups of $\mathrm{Diff}_+^\infty(\mathbb{S}^1)$. This criterium will imply that groups Γ as in the statement of Theorem A are always nondiscrete.

Recall that a subgroup $\Gamma \subset \mathrm{Diff}_+^\infty(\mathbb{S}^1)$ is said to be *nondiscrete* if it contains a sequence of elements $\{h_i\}_{i\in\mathbb{N}}$ ($h_i \neq Id$ for all i) converging C^∞ to the identity, Id, on some open (nonempty) intervals $I \subset \mathbb{S}^1$. In addition, we suppose that the set of fixed points of each h_i has empty interior in I. Let us begin with the following easy lemma.

Lemma 2.1. *Assume that Γ is as in the statement of Theorem A. Then, for a fixed point $p \in \mathbb{S}^1$, there exist elements f, g in Γ such that:*

1. *p is a hyperbolic fixed point of f;*

2. *$g(p) = p$, $g'(p) = 1$ but g is different from the identity on a (arbitrarily small) neighborhood of p.*

Proof. First let us prove item 1. To begin with, we observe that Γ is not Abelian. Actually, suppose for a contradiction that Γ is Abelian. Then Γ preserves a probability measure μ. The support of μ must coincide with the whole of $\mathbb{S}^1$ since Γ is transitive. Thus, parametrizing $\mathbb{S}^1$ by μ, we conjugate Γ to a group of rotations which cannot be transitive on pairs of points. The resulting contradiction proves that Γ is not Abelian. Since Γ is transitive on $\mathbb{S}^1$, to prove item 1, it suffices to prove the existence of an element $f \in \Gamma$ possessing a hyperbolic fixed point. According to a theorem due to Hölder, the fact that Γ is not Abelian implies the existence of a diffeomorphism (different from the identity) in Γ possessing a fixed point (see [H-H]). The existence of an element $f \in \Gamma$ possessing a *hyperbolic* fixed point then follows from a result due to Ghys (Theorem 1.2.7 of [E-T]) which relies on Sacksteder's theorem.

Next let $\mathrm{Stab}_p(\Gamma)$ denote the *stabilizer* of p, namely the subgroup of Γ consisting of diffeomorphisms fixing p. In view of the preceding, $\mathrm{Stab}_p(\Gamma)$ strictly contains the identity. To prove item 2, we suppose for a contradiction that all diffeomorphisms h in $\mathrm{Stab}_p(\Gamma)$, $h \neq Id$ on a neighborhood of p, satisfy $h'(p) \neq 1$. It immediately results that the germ of $\mathrm{Stab}_p(\Gamma)$ at p is an Abelian subgroup of $\mathrm{Diff}_+^\infty(\mathbb{R}, 0)$. In fact, if h_1, h_2 are elements in $\mathrm{Stab}_p(\Gamma)$ which do not commute (in the sense of germs), then $g = h_1 \circ h_2 \circ h_1^{-1} \circ h_2^{-1}$ fulfils our requirements.

Since f has an isolated fixed point at p, Koppel's lemma (see [Ko]) ensures the existence of a topological flow ϕ containing all germs of homeomorphisms (on a neighborhood of p) commuting with f. Hence the flow ϕ contains the group of germs induced by $\mathrm{Stab}_p(\Gamma)$. In particular, the local action of (the germs of) the group $\mathrm{Stab}_p(\Gamma)$ preserves ϕ. To obtain a contradiction we now proceed as follows.

We consider the diagonal action of Γ on the torus $\mathbb{T}^2$ (the set of pair of points). We also have the diagonal "local" action induced by $\mathrm{Stab}_p(\Gamma)$ on a neighborhood U of the point $(p, p) \in \mathbb{T}^2$. Modulo shrinking U, the flow ϕ induces a topological 1-dimensional foliation on U. The fact that $\mathrm{Stab}_p(\Gamma)$ preserves ϕ on $\mathbb{S}^1$ clearly implies that the diagonal local action of $\mathrm{Stab}_p(\Gamma)$ on U preserves the topological foliation in question. This gives the desired contradiction in view of the following claim:

Claim. Assume that we are given a pair of points (x_1, x_2) and (y_1, y_2) in a small neighborhood of $p \in \mathbb{T}^2$ which satisfy $x_1 \leq x_2$ and $y_1 \leq y_2$ ($y_1 = y_2$ if $x_1 = x_2$). Suppose also that $p < \min\{x_1, y_1\}$. Then there is $h \in \mathrm{Stab}_p(\Gamma)$ such that $h(x_1) = y_1$ and $h(x_2) = y_2$.

Proof of the Claim. By assumption the 3-tuples (p, x_1, x_2) and (p, y_1, y_2) are ordered. Furthermore, since Γ is transitive on 3-tuples, there must exist $h \in \Gamma$ such that $h(p) = p$, $h(x_1) = y_1$ and $h(x_2) = y_2$. This proves the claim, therefore, completing the proof of the lemma. $\square$

The lemma above has the following fundamental consequence.

Lemma 2.2. *Let Γ be as before. There exist an open (nonempty) interval $I \subset \mathbb{S}^1$ and a sequence $\{h_k\}$ of diffeomorphisms in Γ such that the restrictions to I of the h_k's converge C^∞ to the identity on I.*

Proof. Consider a point $p \in \mathbb{S}^1$ and diffeomorphisms $f, g \in \mathrm{Stab}_p(\Gamma)$ satisfying the conclusions of Lemma 2.1. A theorem due to Sternberg [St], ensures the existence of a C^∞-coordinate φ where f is given as $f(x) = \lambda x$ ($\lambda = f'(0)$, $0 < \lambda < 1$).

On the other hand, we define a sequence $\{h_k\}$ of elements in Γ by letting $h_k = f^{-k} \circ g \circ f^k$. In the coordinate φ we have $h_k = \lambda^{-k}\mathbf{g}(\lambda^k x)$ (where $g = \varphi^{-1} \circ \mathbf{g} \circ \varphi$). However, the homothety λx as well as $\mathbf{g}$ are defined only on a neighborhood of $0 \in \mathbb{R}$. Nonetheless, all the maps $\{h_k\}$ are still defined on some uniform open interval containing $0 \in \mathbb{R}$. Indeed, since $\mathbf{g}'(0) = 1$, it follows that $|| \, \mathbf{g}(\lambda^k x) - \lambda^k x \, || \leq Const \, || \, \lambda^{2k} x \, ||$. This estimate combined with the fact that, in the coordinate φ, h_k has the form $h_k = \lambda^{-k}\mathbf{g}(\lambda^k x)$ implies the existence of a common domain for all the h_k's.

Using φ it is also easy to check that the maps h_k's converge C^∞ to the identity on a small neighborhood of p. Indeed, one has $h_k'(x) = \mathbf{g}(\lambda^k x)$ which converges uniformly to $\mathbf{g}(0) = 1$ when k goes to infinity. Similarly, we see that the higher derivatives of h_k converge uniformly to *zero*. The lemma is proved. □

Remark 2.3. Let us point out that a group Γ as above containing a diffeomorphism with irrational rotation number, can always be perturbed into a nondiscrete group (in our sense). Actually, this can be proved by using the theorem of M. Herman about linearizations of diffeomorphisms of the circle ([He]). For free groups on two or more generators, this fact is a consequence of the lemmas above combined with the C^∞-Closing Lemma on the circle (Peixoto [Pe]) which is infinitely simpler than Herman's theorem.

To close this section, let us prove Proposition 2.4 below which will be necessary later on.

Proposition 2.4. *Assume that $\Gamma \subset \mathrm{Diff}_+^\infty(\mathbb{S}^1)$ is transitive on unordered 4-tuples. Then there exist an open interval $I_0 \subset \mathbb{S}^1$ ($I_0 \neq \emptyset$) and a sequence of diffeomorphisms $\{h_k\}_{k\in\mathbb{N}}$ contained in Γ such that*

1. *$\{h_k\}$ converges C^∞ to the identity on I_0;*
2. *each of the diffeomorphisms h_k has at most countably many fixed points in I_0.*

Proof. The proof is a refinement of the proof of Lemma 2.2. Again consider a point $p \in \mathbb{S}^1$ and a diffeomorphism $f \in \Gamma$ such that $f(p) = p$ and $f'(p) \neq 1$. Consider also a diffeomorphism $g \in \Gamma$, $g \neq Id$, such that $g(p) = p$ and $g'(p) = 1$. Next let $\mathrm{Fix}(g) \subset \mathbb{S}^1$ denote the set of the fixed points of g. Clearly, $\mathrm{Fix}(g)$ is a closed set strictly contained in $\mathbb{S}^1$.

Recall that a closed set is called *perfect* if it does not have isolated points. According to Bendixson's theorem (cf. [Mo]), $\mathrm{Fix}(g)$ is the union of a countable set and a perfect set denoted by $\mathrm{FixK}(g)$. We claim the existence of a point $q \in \mathrm{FixK}(g)$ which is not accumulated "on both sides" by points belonging to $\mathrm{FixK}(g)$ provided that $\mathrm{FixK}(g) \neq \emptyset$. In other words, we want to prove the existence of a point $q \in \mathrm{FixK}(g)$ together with an open interval I_0 having q as one extremity and such that $I_0 \cap \mathrm{FixK}(g) = \emptyset$ (note that $q \notin I_0$ since I_0 is open). To prove our claim we can suppose that $\mathrm{FixK}(g)$ is totally disconnected even though this is not strictly necessary. Indeed, to obtain a totally disconnected set out of $\mathrm{FixK}(g)$, it is enough to collapse its connected components. It follows that $\mathrm{FixK}(g)$ is a Cantor set (i.e., a totally disconnected perfect set). Therefore, there exists a homeomorphism of $\mathbb{S}^1$ taking $\mathrm{FixK}(g)$ to a "standard middle-third" Cantor set (cf. [Mo]). For this last set the existence of the point q along with the interval I_0 is clear. The claim is then established.

To complete the proof of the proposition we proceed as follows. Suppose first that $\mathrm{FixK}(g)$ is empty so that $\mathrm{Fix}(g)$ is countable. In this case the diffeomorphism $h_k = f^{-k} \circ g \circ f^k$ has only countably many fixed points for every $k \in \mathbb{N}$. Furthermore, by the argument of Lemma 2.2, the sequence formed by the h_k's, $k \in \mathbb{N}$, converges to the identity on a neighborhood of p and hence satisfies the conditions of the statement.

Now suppose that $\mathrm{FixK}(g)$ is not empty and take $q \in \mathrm{FixK}(g)$ as above. Clearly, $g(q) = q$ and $g'(q) = 1$ since hyperbolic fixed points are always isolated. Besides there is $h \in \Gamma$ such that $h(p) = q$ for Γ is transitive. Thus, modulo replacing f by $h \circ f \circ h^{-1}$, we can suppose that $f(q) = q$ and $f'(q) \neq 1$. Again the argument of Lemma 2.2 shows that the sequence $h_k = f^{-k} \circ g \circ f^k$ converges to the identity on a neighborhood of q. Moreover, the restriction of each h_k to I_0 (the open interval corresponding to q in the preceding claim) can have at most countably many fixed points since $I_0 \cap \mathrm{FixK}(g) = \emptyset$. The proposition is proved. $\square$

3. From nondiscrete groups to vector fields

In what follows we shall adapt methods originally introduced in [Sh] and [Na] for subgroups of $\mathrm{Diff}(\mathbb{C}, 0)$ and in [Re] for subgroups of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$. Following [Re] (see also [L-R]), let us consider a pseudogroup G of (orientation-preserving) maps defined on a neighborhood of $0 \in \mathbb{R}$, and taking values in $\mathbb{R}$. The pseudogroup G is supposed to verify the following assumptions:

1. There is $f \in G$ which is a contracting homothety, i.e., $f(x) = \lambda x$, $\lambda \in (0, 1)$.
2. There are an open interval I containing 0 and a sequence of elements $\{h_k\}_{k \in \mathbb{N}} \subset G$ such that all the h_k's are defined on I. Furthermore, they converge C^{∞} to the identity on I.
3. $h_k(0) \neq 0$ for all k.

The main result of this section is the following proposition.

Proposition 3.1. *Given $r \in \mathbb{N}$, there exists a (nontrivial) C^r-vector field X defined on I which has the following property: given a relatively compact interval $I' \subset I$ and $t_0 \in \mathbb{R}$ so small that the local flow ϕ_X^t associated to X is defined on I' for every $0 \leq t \leq t_0$, then the mapping $\phi^{t_0} : I' \to \mathbb{R}$ is C^r-approximated on I' by a suitable sequence of elements in G.*

Assume that G is the pseudogroup induced by the restrictions to an interval $I \subset \mathbb{S}$ of the elements of a subgroup $\Gamma \subset \mathrm{Diff}_+^{\infty}(\mathbb{S}^1)$. A vector field X defined on I is said to be *in the C^r-closure of* Γ *relative to* I (or simply in the closure of Γ when no misunderstanding is possible) if it satisfies the condition of Proposition 3.1. Namely, for any relatively compact interval $I' \subset I$ and $t_0 \in \mathbb{R}$ so small that the local flow ϕ_X^t associated to X is defined on I' for every $0 \leq t \leq t_0$, the mapping $\phi^{t_0} : I' \to \mathbb{R}$ in C^r-approximated on I' by a suitable sequence of elements in G. In particular, the map ϕ^{t_0} is C^r-approximated on I' by restrictions of suitable diffeomorphisms in Γ. We also point out that Γ naturally acts on vector fields contained in its closure. In other words, if X defined on an interval $I \subset \mathbb{S}$ is in the closure of Γ relative to I, then, for any diffeomorphism $F \in \Gamma$, the vector field $F_* X$ defined on $F(I)$ also belongs to the closure of Γ relative to $F(I)$.

Given a C^r map g from an interval I' to $\mathbb{R}$, let $\| \; g - id \; \|_{r,I'}$ stand for the C^r-distance between g and the identity on I'. The proof of Proposition 3.1 is a consequence of the lemma below.

Lemma 3.2. *Fixed $r \in \mathbb{N}$, there exist a compact interval $I_0 \subset I$ and a sequence of maps $\{g_k\} \subset G$ (g_k defined on I for all k) converging C^r to the identity on I. Also, the estimate*

$$\| \; g_k - id \; \|_{r,I_0} \leq Const \sup_{x \in I_0} \| \; g_k - id \; \| = Const \; \| \; g_k - id \; \|_{0,I_0} \tag{1}$$

holds for some uniform constant Const.

In the sequel we shall deduce Proposition 3.1 from Lemma 3.2. The argument is a variant of the one employed in [Re] (see also [L-R]).

Let $r \in \mathbb{N}$ be fixed. Using Lemma 3.2, we select a sequence $\{g_k\} \subset G$ converging C^{r+1} to the identity on I (in particular, all the g_k's are defined on I) and satisfying the estimate

$$\| \; g_k - id \; \|_{r+1,I} \leq Const \sup_{x \in I_0} \| \; g_k - id \; \| \tag{2}$$

for a suitable uniform constant $Const$.

Proof of Proposition 3.1. For each $k \in \mathbb{N}$, denote by C_k the supremum $C_k = \sup_{x \in I_0} \| \; g_k - id \; \|$. We then consider the vector field X_k defined on I by the formula

$$X_k(x) = \frac{1}{C_k} \cdot (g_k(x) - x)\partial/\partial x \, . \tag{3}$$

The C^{r+1}-norm of the vector fields X_k's on I is uniformly bounded by $Const$. Thus, by applying Ascoli-Arzela's theorem, we can suppose, without loss of generality, that the sequence of vector fields X_k converge in the C^r-topology to a C^r vector field X_∞ on I_0. The definition of C_k immediately implies that X_∞ is not identically *zero*. In fact, $\sup_{x \in I_0} \| \; X(x) \; \| = 1$.

It only remains to check that the local flow of the vector field X_∞ can be C^r-approximated by mappings in G (in the sense of Proposition 3.1). Therefore, we denote by ϕ^t_X the local flow associated to X_∞ and consider an interval $I_1 \subset I_0$ and $t_0 \in \mathbb{R}$ such that $\phi^t_X : I_1 \to \mathbb{R}$ is defined whenever $0 \leq t \leq t_0$. Precisely, we claim that the sequence $g_k^{[t_0/C_k]}$ converges in the C^r-topology to ϕ^t_X on I_1 as $k \to \infty$ (where [.] stands for the integral part). The verification of the last assertion is straightforward and left to the reader. The proof of the proposition is over. $\square$

The remainder of this section is devoted to the proof of Lemma 3.2. Fix $r \in \mathbb{N}$ and recall that we need to find a sequence $\{g_k\} \subset G$ satisfying the conditions of the lemma in question. Let us begin by considering the sequence $\{h_k\}$ which satisfies the assumption 2 in the beginning of this section. For each $n \in \mathbb{N}$, let $P^n_k(x)$ denote the Taylor polynomial $P^n_k(x) = h_k(0) + h_k^{(1)}(0)x + \cdots + h_k^{(n)}(0)x^n/n!$. We have

$$h_k(x) - P^n_k(x) \leq \frac{1}{(n+1)!} \sup_{x \in I} \| \; h_k^{(n+1)}(x) \; \| \; x^{n+1} \, . \tag{4}$$

Recall first that $h_k(0) \neq 0$ for every k. Also we consider a fixed $r \geq 2$. Since the sequence $\{h_k\}$ converges C^∞ to the identity, it follows that

$$\sup_{x \in I} \| \, h_k^{(n)}(x) \, \| \ , \quad k \in \mathbb{N},$$

converges to *zero* (for $2 \leq n \leq r+1$).

Next we fix a sufficiently small positive constant $\delta > 0$. Fixing $k \in \mathbb{N}$, consider the sequence $f^{-i} \circ h_k \circ f^i = \lambda^{-i} h_k(\lambda^i x)$ where $i \in \mathbb{N}$.

Lemma 3.3. *To each $k \in \mathbb{N}$ there corresponds a minimum positive integer $i(k) \in \mathbb{N}$ such that:*

1. *All the maps $g_k = \lambda^{-i(k)} h_k(\lambda^{i(k)} x)$, $k \in \mathbb{N}$, are defined on a uniform interval $I_0 \subset I$ which contains $0 \in \mathbb{R}$.*
2. $\sup_{x \in I_0} \| \, g_k(x) - x \, \| > \delta$.

Proof. Assume that we are given a map H defined on I. If $\sup_{x \in I} \| \, F(x) - x \, \| < \delta$ and $\delta > 0$ was chosen small enough, then the map $\lambda^{-1} F(\lambda x)$ is defined on a uniform interval I_0 containing $0 \in \mathbb{R}$. To prove the lemma is, therefore, enough to ensure the existence of a minimum positive integer $i(k)$ verifying the second assertion above.

For each $k \in \mathbb{N}$ fixed, let us consider the sequence $g_{i,k}(x) = \lambda^{-i} h_k(\lambda^i x)$. Note that $g_{i,k}(0) = \lambda^{-i} h_k(0)$. Thus

$$\| \, g_{i,k}(0) - 0 \, \| = \lambda^{-i} \, \| \, h_k(0) - 0 \, \| \ .$$

Since $h_k(0) \neq 0$ and $0 < \lambda < 1$, it follows that $\| \, g_{i,k}(0) - 0 \, \| > \delta$ for i very large. Therefore, there exists a minimum positive integer $i(k) \in \mathbb{N}$ so that g_k fulfils the desired conditions. □

Proof of Lemma 3.2. It is enough to check that the sequence $\{g_k\}$ constructed above verifies the estimate (1) for some appropriate uniform constant $Const$. Since $\| \, g_k - id \, \|_{0,I_0} \geq \delta > 0$, it suffices to check that

$$\sup_{x \in I_0} \| \, g_k^{(1)}(x) - 1 \, \| \quad \text{and} \quad \sup_{x \in I_0} \| \, g_k^{(n)}(x) \, \|$$

$(2 \leq n \leq r+1)$ both converge to *zero* when $k \to \infty$. However, the derivative of g_k is given by $g_k^{(1)}(x) = h_k^{(1)}(\lambda^{i(k)} x)$. Hence

$$\sup_{x \in I_0} \| \, g_k^{(1)}(x) - 1 \, \| \leq \sup_{x \in I} \| \, h_k^{(1)}(x) - 1 \, \|$$

which converges to *zero* by assumption. Similarly, we can see that $\sup_{x \in I_0} \| g_k^{(n)}(x) \, \|$ goes to *zero* when $k \to \infty$. For instance, for $n = 2$, one clearly has $g_k^{(2)}(x) = \lambda^{i(k)} . h_k^{(2)}(\lambda^{i(k)} x)$. The desired convergence follows at once. □

Remark 3.4. Consider again a pseudogroup G consisting of (orientation-preserving) maps defined on a neighborhood of $0 \in \mathbb{R}$, and taking values on $\mathbb{R}$. The reader will easily check that the conclusions of Proposition 3.1 also hold if G satisfies only the following conditions:

1. There is $f \in G$ which is a contracting homothety, i.e., $f(x) = \lambda x$, $\lambda \in (0, 1)$.
2. There is $g \in G$, $g(0) = 0$ and $g'(0) = 1$, which is not C^∞-flat at $0 \in \mathbb{R}$.

4. Proof of Theorem A

Once again let $\Gamma \subset \mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)$ be a group as in the statement of Theorem A. Using Propositions 2.4 and 3.1, it is easy to construct a (nontrivial) C^r vector field X defined on an open (nonempty) interval $I \subset \mathbb{S}$ which is in the C^r-closure of Γ relative to I (in the sense of Section 3). Letting Γ act on the vector field X, we produce new ones which also belong to the closure of Γ (relative to appropriate domains) and the existence of "many" such vector fields has several additional consequences on the dynamics of Γ itself as pointed out in [L-R] (see also [R-S]). We are then led to think of Γ as having an associated nontrivial "Lie algebra". Finally, we shall be able to combine the fact that the "Lie algebra" associated to Γ is "large" with Cartan's classification of graded Lie algebras to approximate certain functions by elements of Γ. Let us make this discussion precise.

In order to consider Lie algebras of vector fields, we clearly need to have C^∞ vector fields because the commutator of two vector fields depends on their derivatives. Besides, to be coherent, these vector fields have to belong to the C^∞-closure of Γ; that is, their local flows must be C^∞-approximated by diffeomorphisms in Γ (restricted to appropriate domains). The existence of these vector fields do not immediately follow from Proposition 3.1. Indeed, to construct C^∞-vector fields we shall employ a method of "promoting" C^r-vector fields which relies again on the local expansion associated to a hyperbolic fixed point. This is the contents of Proposition 4.1 below.

Proposition 4.1. *Let $\Gamma \subset \mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)$ be a group acting transitively on 4-tuples. Then there exists an open interval $I \subset \mathbb{S}^1$ ($I \neq \emptyset$) endowed with a nowhere vanishing C^∞ vector field X which is in the C^∞-closure of Γ relative to I. Furthermore, for a fixed point $p \in I$, there is a local coordinate x defined on I and a diffeomorphism $f \in \Gamma$ satisfying the following conditions:*

1. *The coordinate x identifies p with $0 \in \mathbb{R}$ and, in this coordinate, the vector field X is constant.*
2. *$f(p) = p$ and, in the coordinate x ($p \simeq 0$), f is given by $f(x) = \lambda x$ with $0 < \lambda < 1$.*

Proof. First note that Proposition 2.4 provides us a point $p \in \mathbb{S}$, an interval I_0 (having p as one of its extremities) and a sequence of diffeomorphisms $\{h_k\} \subset \Gamma$ verifying all the conditions below.

1. There exists $f \in \Gamma$ such that $f(p) = p$ and $f'(p) \neq 1$.
2. The sequence $\{h_k\}$ converges to the identity on I_0 in the C^∞-topology.
3. Each h_k possesses only countably many fixed points in I_0.

Since each h_k has only countably many fixed points, there is a point $q \in I_0$ such that $h_k(q) \neq q$ for every q. Also, there is $g \in \Gamma$ such that $g(p) = q$ (here we are using the transitivity of Γ). We then consider the sequence $H_k = g^{-1} \circ h_k \circ g$ which converges in the C^∞-topology to the identity on some small interval I_1 containing p. Clearly, $H_k(p) \neq p$ for all k. Next, in view of Sternberg's theorem ([St]), there is a local coordinate around p ($\simeq 0$) where $f(x) = \lambda x$ for $0 < \lambda < 1$.

Now we fix $r \in \mathbb{N}$. According to Proposition 3.1 there is a nontrivial C^r vector field X defined on I_1 and contained in the closure of Γ relative to I_1. Thanks again to the transitivity of Γ, we can, in fact, suppose that $X(p) \neq 0$. By definition, this vector field is C^r-approximated by suitable restrictions of elements in Γ.

The rest of the proof of the proposition consists of two steps. First we shall obtain a C^∞ vector field in the C^r-closure of Γ. Second we are going to see that the C^∞ vector field in question is, in fact, in the C^∞-closure of Γ.

Consider the sequence of vector fields $X_n = (f^n)^*(\alpha_n X)$. Clearly, all the X_n are (defined on I_1 and) contained in the C^r-closure of Γ relative to I_1. Furthermore, if we set $\alpha_n = \lambda^n$, $\{X_n\}$ converges in the C^r topology towards the constant vector field $X_\infty(x) = X(p)\partial/\partial x$ (which is obviously of class C^∞ since the linearizing coordinate previously fixed is so). It follows that X_∞ is also in the C^r-closure of Γ relative to I_1. To conclude that X_∞ belongs to the C^∞-closure of Γ we just need to notice that r was arbitrarily chosen. In other words, if we replace r by, say, $r+1$, the same procedure carried out above would lead us to the same "constant" vector field X_∞. This implies that X_∞ is also in the C^{r+1}-closure of Γ. Since the conclusion holds for every $r \in \mathbb{N}$, it follows that X_∞ indeed belongs to the C^∞-closure of Γ. The proof of the proposition is over. $\square$

As a by-product of the discussion above, we obtain the following:

Corollary 4.2. *There is a finite collection $I_1, \cdots, I_r$ of intervals covering $\mathbb{S}^1$ such that each I_i is equipped with a nowhere vanishing vector field X_i in the closure of Γ relative to I_i. Besides each of these vector fields become constant in the linearizing coordinate of a hyperbolic fixed point of an appropriate diffeomorphism in Γ.*

Proof. In the proof of Proposition 4.1 it was shown that a neighborhood of $p \in \mathbb{S}^1$ is equipped with a "constant" vector field X in the closure of Γ, where $p \in \mathbb{S}^1$ is a hyperbolic fixed point of a suitable element $f \in \Gamma$. Using the transitivity of Γ, we see that any point of $\mathbb{S}^1$ has a neighborhood equipped with a "constant" (nonvanishing) vector field in the closure of Γ. Indeed, any point of $\mathbb{S}^1$ is a hyperbolic fixed point for an appropriate diffeomorphism in Γ. The corollary then follows from the compactness of $\mathbb{S}^1$. $\square$

The proof of Theorem A is an immediate consequence of Proposition 4.3 below. Denote by p^N the north pole of $\mathbb{S}^1$.

Proposition 4.3. *There is a small neighborhood V of p^N with the following property: given a C^∞-function F from V into $\mathbb{R}$ satisfying $\mathsf{F}(p^N) = p^N$, there exists a sequence $\{g_j\} \subset \Gamma$ converging C^∞ to F on V.*

Now let us prove Theorem A.

Proof of Theorem A. Consider a C^∞-function f defined on an interval $\mathbf{I} \subset \mathbb{S}^1$ as in the statement of the theorem in question. Let p^N be the north pole of $\mathbb{S}$ and V a neighborhood of p^N as in the statement of Proposition 4.3. Because of the transitivity of Γ on pair of points, we can find an element $h^I \in \Gamma$ such that $h^I(\mathbf{I}) = V$.

Next let us consider the diffeomorphism $\widetilde{\mathsf{f}} = \mathsf{f} \circ (h^I)^{-1}$ from V onto its image. It suffices to check that $\widetilde{\mathsf{f}}$ can be C^∞-approximated by elements of Γ on V. Clearly we can assume without loss of generality that $\widetilde{\mathsf{f}}(p^N) = p^N$. Finally the diffeomorphism $\widetilde{\mathsf{f}}$ restricted to V can be identified with a real-valued function from V to $\mathbb{R}$ fixing $0 \in \mathbb{R}$ (where V should be thought of as a neighborhood of $0 \in \mathbb{R}$). Theorem A then follows from Proposition 4.3. $\square$

The rest of the section is devoted to establishing Proposition 4.3.

Consider the collection $\{\mathcal{G}\}$ of all C^{∞} vector fields $\{X\}$ defined on a neighborhood of p^N and contained in the C^{∞}-closure of Γ. It is clear that $\mathcal{G}$ is *closed* under C^{∞}-convergence of vector fields. Furthermore, $\mathcal{G}$ is actually a (local) Lie algebra in the sense that it is a vector space over $\mathbb{R}$ and the bracket of two elements of $\mathcal{G}$ still belongs to $\mathcal{G}$. Notice that, in order to consider $\mathcal{G}$ as a Lie algebra, it is intrinsically important to work with C^{∞} vector fields (rather than C^k ones) since the commutator of two vector fields involves their derivatives.

On the other hand, Proposition 4.1 guarantees that $\mathcal{G}$ is not trivial and, in fact, contains nowhere vanishing vector fields (which become constant in suitable coordinates). We refer to the dimension of $\mathcal{G}$ as being the dimension of $\mathcal{G}$ as a vector space over $\mathbb{R}$. We are going to see that $\mathcal{G}$ is an infinite-dimensional Lie algebra.

We consider $\mathbb{S}^1$ as the unit circle in $\mathbb{C} \simeq \mathbb{R}^2$ equipped with the Euclidean metric. The group $\mathrm{PSL}(2,\mathbb{R})$ will be identified with the corresponding group of diffeomorphisms of $\mathbb{S}^1$ induced by the standard projective action. Similarly, the Lie algebra $\mathsf{psl}(2,\mathbb{R})$ of $\mathrm{PSL}(2,\mathbb{R})$ is identified with its projective representation in the space of (analytic) vector fields on $\mathbb{S}^1$.

Finally, if Π_K is a covering map of degree $K \in \mathbb{N}$, the group $\mathrm{PSL}(2,\mathbb{R}) \subset \mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)$ can be lifted through Π_K to a new group of diffeomorphisms $\mathrm{PSL}_K(2,\mathbb{R}) \subset \mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)$ referred to as the *degree-K covering of* $\mathrm{PSL}(2,\mathbb{R})$.

Proposition 4.4. *The dimension of $\mathcal{G}$ over $\mathbb{R}$ is infinite.*

To prove the proposition above, we suppose for a contradiction that $\mathcal{G}$ is a finite-dimensional Lie algebra over $\mathbb{R}$. Under this assumption we have:

Lemma 4.5. *Γ is contained in $\mathrm{Diff}^{\omega}_{+}(\mathbb{S}^1)$, i.e., Γ consists of analytic diffeomorphisms.*

Proof. Since $\mathcal{G}$ has finite dimension as a real vector space, it follows that the vector fields in $\mathcal{G}$ verify a differential equation with constant coefficients. Thus the vector fields belonging to $\mathcal{G}$ are, in fact, real analytic. Similarly, we see that any vector field in the closure of Γ is analytic in its domain of definition. Next suppose that we are given a point $q \in \mathbb{S}^1$ and a diffeomorphism $h \in \Gamma$. Let Y denote a nontrivial vector field defined on a neighborhood of $h(q)$ and contained in the closure of Γ. The vector field h^*Y, defined on a neighborhood of q, belongs to the closure of Γ and therefore is analytic. Since, by construction h conjugates Y and Z, it results that h itself is analytic on a neighborhood of q. Since q and h are arbitrary, the lemma results. $\square$

Note that Γ is not Abelian. Indeed, if Γ were Abelian, it would preserve the constant vector fields provided by Corollary 4.2. Thus Γ would consist of rotations. This is obviously impossible since Γ must act transitively on pair of points. As a consequence, $\mathcal{G}$ contains nontrivial vector fields vanishing at p^N (for this it is enough to subtract a nonconstant vector field from a suitable constant one).

Lemma 4.6. *$\mathcal{G}$ is isomorphic to the Lie algebra $\mathsf{psl}(2,\mathbb{R})$ (i.e., to $\partial/\partial x$, $x\partial/\partial x$, $x^2\partial/\partial x$).*

Proof. Because of Lie's theorem (cf. [Lie]), we just need to check that $\mathcal{G}$ is not isomorphic to $\partial/\partial x$, $x\partial/\partial x$. However, this would imply that, up to a constant factor, $\mathcal{G}$ contains only one nontrivial vector field Y vanishing at p. Thus the stabilizer Γ_0 of p^N in Γ is Abelian and locally contained in the flow of Y. Using

Corollary 4.2, we see that the derivative of Y is constant of $\mathbb{S}^1$. Thus Y must be trivial. The resulting contradiction proves the lemma. $\square$

Proof of Proposition 4.4. The argument is now essentially the same as in Section 5 of [R-S]. We summarize the discussion. Let ξ be an analytic coordinate defined on a neighborhood of p^N and conjugating $\mathcal{G}$ to $\mathsf{psl}(2,\mathbb{R})$ (i.e., to the restriction of $\mathsf{psl}(2,\mathbb{R})$ to a neighborhood of $0 \in \mathbb{R}$). The existence of ξ is part of the Lie's theorem mentioned above. Assume that ξ is defined on the interval $[p^N, q) \subset \mathbb{S}^1$. Thanks to Lemma 4.1, we see that ξ conjugates (where ξ is defined) a constant vector field defined on a neighborhood of q to a vector field defined on a neighborhood of $0 \in \mathbb{R}$ (and given as the restriction of a vector field in $\mathsf{psl}(2,\mathbb{R})$). The local flows of these vector fields allow us to extend ξ to a neighborhood of q. It results that ξ has an analytic continuation $\check{\xi}$ around $\mathbb{S}^1$. Furthermore, the fact that both the original ξ and its analytic continuation $\check{\xi}$ obtained after one tour around $\mathbb{S}^1$ must preserve $\mathcal{G}$ implies that they glue together. In other words, ξ has an analytic extension to the whole of $\mathbb{S}^1$. This analytic extension is, indeed, a local diffeomorphism from $\mathbb{S}^1$ to $\mathbb{S}^1$ and therefore a covering map.

Recall that Γ_0 is the stabilizer of p^N. In the local coordinate ξ the elements of Γ_0 are given by Möebius transformations since they preserve $\mathcal{G}$. Fixing $h \in \Gamma_0$ and $M \in \mathrm{PSL}(2,\mathbb{R})$ so that $\xi \circ h \circ \xi^{-1} = M$ on a neighborhood of p^N, we easily see that h coincides with one lift of M through the covering $\check{\xi}$. If K stands for the degree of $\check{\xi}$, then it results that Γ is contained in $\mathrm{PSL}_K(2,\mathbb{R})$ (see [R-S] for further details).

Finally, the diagonal action of $\mathrm{PSL}(2,\mathbb{R})$ on 4-tuples (thought of as the torus $\mathbb{T}^4$) preserves a codimension 1 foliation whose leaves are the level set of the cross-ratio (recall that the cross-ratio may be viewed as a function from $\mathbb{T}^4$ to $\mathbb{S}^1$). Hence the action of $\mathrm{PSL}_K(2,\mathbb{R})$ on $\mathbb{T}^4$ preserves the corresponding lift of this foliation. This contradicts the fact that Γ is transitive on 4-tuples. The proposition is proved. $\square$

Proof of Proposition 4.3. We keep the preceding notation. The dimension of the Lie algebra $\mathcal{G}$ is infinite after Proposition 4.4. To prove our statement it is clearly enough to check that $\mathcal{G}$ coincides with the Lie algebra of all C^∞-vector fields defined on a neighborhood of p^N. In order to do this, it suffices to verify that $\mathcal{G}$ is *graded* (see the definition in [De]). In fact, all these Lie algebras were classified by E. Cartan (cf. [De]) and, in view of his classification, $\mathcal{G}$ must be the whole of the Lie algebra of all C^∞ vector fields since its dimension is infinite. The proof of the fact that $\mathcal{G}$ is graded amounts to verify the following claim:

Claim. Assume that Y belongs to $\mathcal{G}$ and that Y^n is the polynomial vector field of degree $n \in \mathbb{N}$ obtained by truncating the Taylor series of Y at p^N. Then Y^n belongs to $\mathcal{G}$ as well.

Proof of the Claim. Let Y be a vector field in $\mathcal{G}$. Since $\mathcal{G}$ contains the constant vector fields, we can suppose that $Y(p^N) = 0$ (otherwise we would replace Y by $Y - Y(p^N)\partial/\partial x$). Set

$$Y^n(x) = (C_1 x + C_2 x^2 + \cdots + C_n x^n)\partial/\partial x\,.$$

In addition, we can suppose that Γ contains the homotethy $x \mapsto \lambda x$. Clearly, the sequence of vector fields $(\lambda^k)^* Y$ belongs to $\mathcal{G}$ and converges to $C_1 x\partial/\partial x$ in the C^∞-topology (cf. the proof of Lemma 3.2). Thus $C_1 x\partial/\partial x$ belongs to $\mathcal{G}$. Next the vector field $Y_2(x) = Y(x) - C_1 x\partial/\partial x = C_2 x^2 + \cdots + C_n x^n + \text{h.o.t.}$ also belongs to $\mathcal{G}$. A sequence of appropriate constant multiples of the vector fields $(\lambda^k)^* Y_2$ is contained in $\mathcal{G}$ and converges to $C_2x^2\partial/\partial x$ in the C^∞-topology. Thus $C_2x^2\partial/\partial x$ belongs to $\mathcal{G}$. Proceeding inductively we eventually conclude that Y^n belongs to $\mathcal{G}$ proving the claim. The proof of the proposition is also over. □

5. Complementary results

In this last section we shall establish Theorem B as well as complement our results with a few remarks.

Consider a group Γ as in the statement of Theorem B. Our main tool to construct vector fields in the closure of Γ is Proposition 3.1. In the sequel we shall discuss when this proposition can be applied to Γ so as to produce nontrivial vector fields in its closure (in particular, proving Theorem B).

First note that Γ acts minimally on $\mathbb{S}^1$ (i.e., all orbits are dense). Otherwise an unpublished but well-known theorem due to G. Duminy would imply that Γ has a finite orbit. Therefore, Γ would preserve an atomic measure associated to this finite orbit which is impossible in view of our assumptions. Besides Γ is not Abelian since Abelian groups acting on $\mathbb{S}^1$ always preserve a probability measure. From the discussion in Lemma 2.1, we obtain the following fact:

$(*)$ there is a point $p \in \mathbb{S}^1$ and a diffeomorphism $f \in \Gamma$ such that $f(p) = p$ and $f'(p) = \lambda$ with $0 < \lambda < 1$.

Let $\{h_k\} \subset \Gamma$ be a sequence of elements in Γ whose restriction to an appropriate interval I converges C^∞ to the identity. Suppose also that the intersection of the set of fixed points of h_k with I has empty interior. The existence of $\{h_k\}$ and I is part of the assumptions of Theorem B. Because of the denseness of the orbits of Γ, we can suppose without loss of generality that p belongs to I. Therefore, if for infinitely many h_k's, we have $h_k(p) \neq p$, the Theorem B results at once from Proposition 3.1.

Fixed $k \in \mathbb{N}$, let $\mathrm{Fix}_I(h_k)$ denote the intersection with I of the set of fixed points of h_k.

Proof of Theorem B. As observed above, we just need to find a sequence $\{H_k\} \subset \Gamma$ converging C^∞ to the identity on I and satisfying $H_k(p) \neq p$ for all $k \in \mathbb{N}$. Thus we can suppose that the original sequence $\{h_k\}$ is such that $h_k(p) = p$ for every k. Fix k_0 such that p belongs to the set of fixed points, $\mathrm{Fix}(h_{k_0})$, of h_{k_0}. Note that $\mathrm{Fix}(h_{k_0})$ is a Cantor set and, in particular, $h'_{k_0}(p) = 1$. Hence the sequence $H_k = f^{-k} \circ h_{k_0} \circ f^k$ converges to the identity on a neighborhood of p (cf. Section 2). Besides using the fact that all Cantor sets in a line are homeomorphic to the standard "middle-third" Cantor set by a homeomorphism of the entire line (as in the proof of Proposition 2.4), we conclude the existence of an open interval $J \neq \emptyset$ where H_k has no fixed points for every k. Finally, the denseness of the orbit of p implies the existence of $g \in \Gamma$ such that $g(p)$ belongs to J. Clearly, $g(p) \in J$ is a hyperbolic fixed point of $g \circ f \circ g^{-1}$. In turn, Proposition 3.1 allows one to construct a nontrivial vector field X defined on J and contained in the closure of Γ. The minimality of Γ then implies the final statement. □

Let us point out that our assumption that the diffeomorphisms h_k do not have isolated points can actually be significantly weakened. Indeed, denoting by

$\mathrm{Fixiso}(h_k)$ the set of isolated points of $\mathrm{Fix}(h_k)$, we just need to suppose that

$$\bigcup_{i=1}^{\infty} \mathrm{Fixiso}(h_k)$$

is not dense in I for some sequence h_k as above.

Consider again $p \in \mathbb{S}^1$ and $f \in \Gamma$ such that p is a hyperbolic fixed point of f. Even though we cannot obtain a sequence $\{h_k\} \subset \Gamma$ converging to the identity on a neighborhood of p and satisfying $h_k(p) \neq p$ for all $k \in \mathbb{N}$, it may still be possible to construct a nontrivial vector field in the closure of Γ. Namely, suppose that $h_0(p) = p$ but one of the following possibilities hold:

- $h_0'(p) = 1$ but h_0 is not C^∞-flat at p.
- $h_0'(p) \neq 1$ but the (formal) Taylor series of h_0 and f at p do not commute.

In the first case, the argument mentioned in Remark 3.4 guarantees the existence of a nontrivial vector field in the closure of Γ. In the second case the diffeomorphism $h_1 = f \circ h_0 \circ f^{-1} \circ h_0^{-1}$ is not C^∞-flat and satisfies $h_1'(p) = 1$. Thus we again obtain the desired vector field. The presence of vector fields has further implications on the dynamics of Γ. For instance, they allow one to derive the *ergodicity* of Γ with respect to the Lebesgue measure from Duminy's theorem on denseness of orbits (a result that does not seem to be accessible with Duminy's techniques).

Summarizing, whenever we have a nondiscrete subgroup $\Gamma \subset \mathrm{Diff}_+^\infty(\mathbb{S}^1)$, the existence of a nontrivial flow in the closure of Γ is a very common phenomenon. The structure of discrete and nondiscrete subgroups of $\mathrm{Diff}_+^\infty(\mathbb{S}^1)$ will be discussed further somewhere else.

The reader will also observe that Theorem B admits a natural C^r version. Namely, it holds for $\mathrm{Diff}_+^r(\mathbb{S}^1)$ where $r \geq 2$ but the approximation is only of class $r-2$. Here is a precise statement.

Theorem 5.1. *Denote by Γ a subgroup of $\mathrm{Diff}_+^r(\mathbb{S}^1)$, $r \geq 2$, generated by elements sufficiently close to the identity which does not preserve any probability measure on $\mathbb{S}^1$. Let $\{h_k\} \subset \Gamma$ be a sequence converging C^r to the identity on some open interval $I \neq \emptyset$. Assume that $\mathrm{Fix}(h_k)$ does not have isolated points in I for every k. Then for every point $q \in \mathbb{S}^1$, there exists a nontrivial C^{r-2} vector field X defined on a neighborhood I_0 of q and having the following property: given a relatively compact interval $I' \subset I_0$ and $t_0 \in \mathbb{R}$ so small that the local flow ϕ_X^t associated to X is defined on I' for every $0 \leq t \leq t_0$, then the mapping $\phi^{t_0} : I' \to \mathbb{R}$ is C^{r-2}-approximated on I' by a suitable sequence of elements in Γ.*

Proof. We just need to check that the proof of Theorem B applies to the present case. Clearly, we still have $f \in \Gamma$ which possesses a hyperbolic fixed point $p \in \mathbb{S}^1$. However, in the C^r-case the theorem of linearization of Sternberg [St] asserts the existence of a C^{r-1} coordinate where the f is locally conjugate to the homothety $x \mapsto \lambda x$ ($\lambda = f'(p)$).

The rest of the proof is essentially the same, the only important difference is the following: when we apply the Ascoli-Arzela compactness criterium, we lose one derivative of our sequence of C^{r-1} vector fields. This explains why the final convergence is only in C^{r-2}-topology. The proof of the theorem is over. □

A similar C^r version of Theorem A is unclear since our proof relies on Cartan's theorem and the space of C^r vector fields on $\mathbb{S}^1$ does not constitute a Lie algebra. This emphasizes the interest of finding a more combinatoric (or topological) proof of our main result (or more precisely of the Cartan's theorem mentioned in Section 4). This is probably a reasonably accessible question, but I did not attempt at proving it. The case of C^r-diffeomorphism might, indeed, have additional subtleties which are not present in $\mathrm{Diff}_+^\infty(\mathbb{S}^1)$ or $\mathrm{Homeo}_+(\mathbb{S}^1)$.

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