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THE α-INVARIANT ON CERTAIN SURFACES WITH SYMMETRY GROUPS

JIAN SONG

Abstract. The global holomorphic α-invariant introduced by Tian is closely related to the existence of Kähler-Einstein metrics. We apply the result of Tian, Yau and Zelditch on polarized Kähler metrics to approximate plurisubharmonic functions and compute the α-invariant on $CP^2\#n\overline{CP^2}$ for $n = 1, 2, 3$.

1. Introduction

The global holomorphic invariant $\alpha_G(M)$ introduced by Tian [7], Tian and Yau [6] is closely related to the existence of Kähler-Einstein metrics. In his solution of the Calabi conjecture, Yau [12] proved the existence of a Kähler-Einstein metric on compact Kähler manifolds with negative or zero first Chern class. Kähler-Einstein metrics do not always exist in the case when the first Chern class is positive, for there are known obstructions such as the Futaki invariant. For a compact Kähler manifold M with positive frist Chern class, Tian [7] proved that M admits a Kähler-Einstein metric if $\alpha_G(M) > \frac{n}{n+1}$, where $n = \dim M$. In the case of compact complex surfaces, he proved that any compact complex surface with positive first Chern class admits a Kähler-Einstein metric except $CP^2\#1\overline{CP^2}$ and $CP^2\#2\overline{CP^2}$ [9]. Nevertheless, it would also be interesting to find the estimate of the α invariant for $CP^2\#1\overline{CP^2}$ and $CP^2\#2\overline{CP^2}$. In this paper, we apply the Tian-Yau-Zelditch expansion of the Bergman potential on polarized Kähler manifolds to approximate plurisubharmonic functions and compute the α-invariant of $CP^2\#n\overline{CP^2}$ for $n = 1, 2, 3$. In the case of $CP^2\#2\overline{CP^2}$, it gives an improvement of Abdesselem's result [1]. More precisely, we shall show that:

Theorem 1. $\alpha_G(CP^2\#2\overline{CP^2}) = \frac{1}{3}$.

We will give the definitions of the automorphism group G and the α_G-invariant in Section 3.

Let (M, ω) be a compact Kähler manifold, where $\omega = \sqrt{-1} g_{i\bar{j}} dz_i \wedge d\overline{z}_j$. We will also prove Tian's conjecture on the generalized Moser-Trudinger inequality in the special case where $\alpha_G(M) > \frac{n}{n+1}$, for $n = \dim M$. Let

$$P(M,\omega) = \left\{ \phi \mid \omega_\phi = \omega + \sqrt{-1}\partial\overline{\partial}\phi > 0,\ \sup_M \phi = 0 \right\}.$$

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Let F_ω and J_ω be the functionals defined on $P(M,\omega)$ by

$$F_\omega(\phi) = J_\omega(\phi) - \frac{1}{V}\int_M \phi\omega^n - \log(\frac{1}{V}\int_M e^{h_\omega - \phi}\omega^n),$$

$$J_\omega(\phi) = \frac{\sqrt{-1}}{V}\sum_{i=0}^{n-1}\frac{i+1}{n+1}\int_M \partial\phi\wedge\overline{\partial}\phi\wedge\omega^i\wedge\omega_\phi^{n-i-1}.$$

Assume (M,ω_{KE}) is a Kähler-Einstein manifold with positive first Chern class and $Ric(\omega_{KE}) = \omega_{KE}$. Then for any $\phi\in P(M,\omega_{KE})$, Ding and Tian [2] proved the following inequality of Moser-Trudinger type:

$$\frac{1}{V}\int_M e^{-\phi}\omega^n \leq Ce^{J_\omega(\phi) - \frac{1}{V}\int_M \phi\omega^n}.$$

Tian [10] also conjectured that $\frac{1}{V}\int_M e^{-\phi}\omega^n \leq Ce^{(1-\delta)J_\omega(\phi)-\frac{1}{V}\int_M\phi\omega^n}$ for $\delta > 0$ sufficiently small, if ϕ is perpendicular to Λ_1, the space of eigenfunctions of ω_{KE} with eigenvalue one.

We shall prove:

Theorem 2. *Let (M,ω) be a Kähler manifold with positive first Chern class. Assume that $\alpha(M) > \frac{n}{n+1}$, so that M admits a Kähler-Einstein metric ω_{KE}, and there exist constants $\delta = \delta(n,\alpha(M))$ and $C = C(n,\lambda_2(\omega_{KE})-1,\alpha(M))$ such that for any $\phi\in P(M,\omega_{KE})$ which satisfies $\phi\perp\Lambda_1$, we have*

$$F_{\omega_{KE}}(\phi) \geq \delta J_{\omega_{KE}}(\phi) - C.$$

Here $\lambda_2(\omega_{KE})$ is the least eigenvalue of ω_{KE} which is bigger than 1.

2. Holomorphic approximation of plurisubharmonic functions

In this section, we will employ the technique in [8, 13] to obtain the approximation of plurisubharmonic functions by logarithms of holomorphic sections of line bundles. The Tian-Yau-Zelditch asymptotic expansion of the potential of the Bergman metric is given by the following theorem [13].

Theorem 2.1. *Let M be a compact complex manifold of dimension n and let $(L,h)\to M$ be a positive Hermitian holomorphic line bundle. Let g be the Kähler metric on M corresponding to the Kähler form $\omega_g = Ric(h)$. For each $m\in N$, h induces a Hermitian metric h_m on L^m. Let $\{S_0^m, S_1^m, ..., S_{d_m-1}^m\}$ be any orthonormal basis of $H^0(M,L^m)$, $d_m = \dim H^0(M,L^m)$, with respect to the inner product:*

$$(S_1,S_2)_{h_m} = \int_M h_m(S_1(x),S_2(x))dV_g,$$

where $dV_g = \frac{1}{n!}\omega_g^n$ is the volume form of g. Then there is a complete asymptotic expansion

$$\sum_{i=0}^{d_m-1}||S_i^m(x)||_{h_m}^2 \sim a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + ...$$

for some smooth coefficients $a_j(x)$ with $a_0 = 1$. More precisely, for any k,

$$\Big|\Big|\sum_{i=0}^{d_m-1}||S_i^m(x)||_{h_m}^2 - \sum_{j<R}a_j(x)m^{n-j}\Big|\Big|_{C^k} \leq C_{R,k}m^{n-R}$$

where $C_{R,k}$ depends on R,k and the manifold M.

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Let

$$\begin{aligned}\tilde{\omega}_g &= \omega_g + \sqrt{-1}\partial\overline{\partial}\phi > 0,\\ \widetilde{h} &= he^{-\phi}.\end{aligned}$$

Let $\widetilde{h}_m$ be the induced Hermitian metric of $\widetilde{h}$ on L^m, and let $\{\widetilde{S}_0^m, \widetilde{S}_{1,\ldots,}^m \widetilde{S}_{d_m-1}^m\}$ be any orthonormal basis of $H^0(M, L^m)$, where $d_m = \dim H^0(M, L^m)$, with respect to the inner product

$$(S_1, S_2)_{\widetilde{h}_m} = \int_M \widetilde{h}_m(S_1(x), S_2(x))dV_{\widetilde{g}} \ .$$

By Theorem 2.1, we have

$$\sum_{i=0}^{d_m-1} ||\widetilde{S}_i^m(x)||^2_{\widetilde{h}_m} = \left(\sum_{i=0}^{d_m-1} ||\widetilde{S}_i^m(x)||^2_{h_m}\right) e^{-m\phi}.$$

Thus

$$\phi - \frac{1}{m}\log\left(\sum_{i=0}^{d_m-1} ||\widetilde{S}_i^m(x)||^2_{\widetilde{h}_m}\right) = -\frac{1}{m}\log\left(\sum_{i=0}^{d_m-1} ||\widetilde{S}_i^m(x)||^2_{\widetilde{h}_m}\right).$$

As $m \to +\infty$, we obtain for any positive integer R

$$\begin{aligned} & \frac{1}{m}\log\left(\sum_{j<R} \widetilde{a}_j(x)m^{n-j}\right)\\ = & \frac{1}{m}\log m^n(\sum_{j<R} \widetilde{a}_j(x)m^{-j})\\ = & \frac{n}{m}\log m + \frac{1}{m}\log(1 + O(\frac{1}{m})) \to 0.\end{aligned}$$

Thus we have the following corollary of the Tian-Yau-Zelditch expansion.

Corollary 2.1.

$$\left\|\phi - \frac{1}{m}\log\left(\sum_{i=0}^{d_m-1} ||\widetilde{S}_i^m(x)||^2_{\widetilde{h}_m}\right)\right\|_{C^k} \to 0, \ as\ m \to +\infty.$$

In other words, any plurisubharmonic function can be approximated by the logarithms of holomorphic sections of L^m.

3. Proof of Theorem 1

Let M be the blow-up of CP^2 at two points and π be its natural projection. Without loss of generality, we may assume the two points are $p_1 = [0, 1, 0]$ and $p_2 = [0, 0, 1]$. Then M is a subvariety of $CP^2 \times CP^1 \times CP^1$ defined by the equations

$$Z_0X_1 = Z_1X_0, \ \ Z_0Y_2 = Z_2Y_0,$$

where Z_i, X_j, Y_k are the homogeneous coordinates on CP^2, CP^1 and CP^1, respectively.

Let G be the automorphism group acting on $CP^2 \times CP^1 \times CP^1$ generated by θ_j and permutations τ $(0 \le j \le 2)$,

$$\theta_j : [Z_0, Z_j, Z_2] \times [X_0, X_1] \times [Y_0, Y_2] \to [Z_0, Z_je^{i\theta}, Z_2] \times [X_0, X_1] \times [Y_0, Y_2]$$

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for $\theta \in [0, 2\pi)$, and

$$\tau : [Z_0, Z_1, Z_2] \times [X_0, X_1] \times [Y_0, Y_2] \to [Z_0, Z_2, Z_1] \times [Y_0, Y_2] \times [X_0, X_1]\ .$$

Let π_0, π_1, π_2 be the projection from $CP^2 \times CP^1 \times CP^1$ onto CP^2, CP^1 and CP^1. Respectively define ω by

$$\begin{aligned}\omega &= \pi_0^*\omega_0 + \pi_1^*\omega_1 + \pi_2^*\omega_2 \\ &= \sqrt{-1}\partial\overline{\partial}\log(|Z_0|^2 + |Z_1|^2 + |Z_2|^2) + \sqrt{-1}\partial\overline{\partial}\log(|X_0|^2 + |X_1|^2) \\ &\quad + \sqrt{-1}\partial\overline{\partial}\log(|Y_0|^2 + |Y_2|^2),\end{aligned}$$

where ω_0, ω_1, ω_2 are the Fubini-Study metrics in CP^2, CP^1 and CP^1. By explicit calculation, it can be shown that the cohomological class of $\omega|_M$ is in the first Chern class of M (see [1]).

Consider the divisor

$$\{[0, Z_1, Z_2] \times CP^1 \times CP^1\} + \{CP^2 \times [1, 0] \times CP^1\} + \{CP^2 \times CP^1 \times [1, 0]\}$$

which defines a line bundle (L, h) on $CP^2 \times CP^1 \times CP^1$. The hermitian metric h is defined by

$$h = \frac{1}{(|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|X_0|^2 + |X_1|^2)(|Y_0|^2 + |Y_2|^2)};$$

then $(L, h)|_M \to M$ defines the anticanonical line bundle on M whose curvature form $-\sqrt{-1}\partial\overline{\partial}\log h$ gives the first Chern class on M.

Since $M\backslash\{\pi^{-1}\{p_1\} \cup \pi^{-1}\{p_2\}\}$ is isomorphic to $CP^2\backslash\{p_1, p_2\}$, if we choose the inhomogeneous coordinates $(z_1, z_2) = [1, z_1, z_2]$ on CP^2, the Kähler metric

$$\omega_{g_0} = \sqrt{-1}\partial\overline{\partial}\log(1 + |z_1|^2 + |z_2|^2) + \sqrt{-1}\partial\overline{\partial}\log(1 + |z_1|^2) + \sqrt{-1}\partial\overline{\partial}\log(1 + |z_2|^2)$$

can be extended to a Kähler metric g_0 on M which belongs to $c_1(M)$. If we take different inhomogeneous coordinates $(w_0, w_1) = [w_0, w_1, 1]$, the corresponding Kähler metric is

$$\omega_{g_1} = \sqrt{-1}\partial\overline{\partial}\log(1+|w_0|^2+|w_1|^2)+\sqrt{-1}\partial\overline{\partial}\log(1+|w_0|^2)+\sqrt{-1}\partial\overline{\partial}\log(|w_0|^2+|w_1|^2)$$

and we have

$$\begin{aligned}\det g_0 &= \frac{1}{(1 + |z_1|^2 + |z_2|^2)^3} + \frac{1}{(1 + |z_1|^2 + |z_2|^2)^2(1 + |z_1|^2)} \\ &\quad + \frac{1}{(1 + |z_1|^2 + |z_2|^2)^2(1 + |z_2|^2)} + \frac{1}{(1 + |z_1|^2)^2(1 + |z_2|^2)^2}, \\ \det g_1 &= \frac{1}{(1 + |w_0|^2 + |w_1|^2)^3} + \frac{1}{(1 + |w_0|^2 + |w_1|^2)^2(|w_0|^2 + |w_1|^2)} \\ &\quad + \frac{1}{(1 + |w_0|^2 + |w_1|^2)^2(1 + |w_0|^2)} + \frac{|w_0|^2}{(1 + |w_0|^2)^2(|w_0|^2 + |w_1|^2)^2}.\end{aligned}$$

Consider the line bundle $(L^N, h_N) \to CP^2 \times CP^1 \times CP^1$. Then

$$\dim H^0(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N)) = \frac{(N + 1)^3(N + 2)}{2}$$

and $\{Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} X_0^{j_0} X_1^{j_1} Y_0^{k_0} Y_2^{k_2}\}_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=N}$ is an orthogonal basis for $H^0(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N))$.

Let M_1 be the hypersurface of $CP^2 \times CP^1 \times CP^1$ defined by the equations

$$Z_0X_1 = Z_1X_0,$$

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and M_2 the hypersurface of $CP^2 \times CP^1 \times CP^1$ defined by the equations

$$Z_0 Y_2 = Z_2 Y_0.$$

Then $M = M_1 \cap M_2$.

In view of the short exact sequences

$$\begin{aligned} 0 &\to \mathcal{O}(L^N - [M_1]) \to \mathcal{O}(L^N) \to \mathcal{O}(L^N|_{M_1}) \to 0, \\ 0 &\to \mathcal{O}(L^N|_{M_1} - [M]) \to \mathcal{O}(L^N|_{M_1}) \to \mathcal{O}(L^N|_M) \to 0 \end{aligned}$$

we can choose N sufficiently large so that

$$H^1(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N - [M_1])) = H^1(M_1, \mathcal{O}(L^N|_{M_1} - [M])) = 0.$$

Then $H^0(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N)) \to H^0(M_1, \mathcal{O}(L^N|_{M_1})) \to 0$,

$$H^0(M_1, \mathcal{O}(L^N|_{M_1})) \to H^0(M, \mathcal{O}(L^N|_M)) \to 0$$

and thus

$$H^0(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N)) \to H^0(M, \mathcal{O}(L^N|_M)) \to 0.$$

Also we have $Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} X_0^{j_0} X_1^{j_1} Y_0^{k_0} Y_2^{k_1}|_M = Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}$ and

$$\begin{aligned} &||Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} X_0^{j_0} X_1^{j_1} Y_0^{k_0} Y_2^{k_2}||^2_{h_N} \\ &= \frac{\left| Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} Z_0^{j_0} Z_1^{j_1} Z_0^{k_0} Z_2^{k_2} \right|^2}{((|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(Z_0|^2 + |Z_1|^2)(Z_0|^2 + |Z_2|^2))^N} \end{aligned}$$

on $CP^2 \backslash \{p_1, p_2\}$. Therefore, $\{Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} X_0^{j_0} X_1^{j_1} Y_0^{k_0} Y_2^{k_2}|_M\}_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=N}$ contains an orthogonal basis for $H^0(M, \mathcal{O}(L^N|_M))$ with respect to h^N and the G-invariant Kähler metric g on M.

By Corollary 2.1, for any φ in $P_G(M, \omega_g)$, we have on $CP^2 \backslash \{p_1, p_2\}$,

$$\begin{aligned} &\varphi([Z_0, Z_1, Z_2]) \\ = &\lim_{N\to\infty} \frac{1}{N} \log \frac{\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=N} |a^{(N)}_{(\varphi)i_0 i_1 i_2 j_0 j_1 k_0 k_2} Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{((|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_2|^2))^N} \end{aligned}$$

for some coefficients $a^{(N)}_{(\varphi)i_0 i_1 i_2 j_0 j_1 k_0 k_2}$ satisfying $a^{(N)}_{(\varphi)i_0 i_1 i_2 j_0 j_1 k_0 k_2} = a^{(N)}_{(\varphi)i_0 i_2 i_1 k_0 k_2 j_0 j_1}$ due to the group action by G.

Lemma 3.1. *Using the notations above we have*

$$\frac{1}{n} \log \frac{\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} |Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{((|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_2|^2))^n} \leq 4$$

for any positive integer n.

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Proof. On the patch $U_0 = \{Z_0 \neq 0\}$, let $z_1 = \frac{Z_1}{Z_0}$ and $z_2 = \frac{Z_2}{Z_0}$,

$$\begin{aligned}
&\frac{1}{n}\log \frac{\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} |Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^n}\\
\leq\ &\frac{1}{n}\log\left(\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} \frac{|z_1^{i_1+j_1} z_2^{i_2+k_2}|^2}{(1+|z_1|^2+|z_2|^2)^n(1+|z_1|^2)^n(1+|z_2|^2)^n}\right)\\
\leq\ &\frac{1}{n}\log\left(\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} \frac{|z_1^{i_1+j_1} z_2^{i_2+k_2}|^2}{1+|z_1^{i_1+j_1} z_2^{i_2+k_2}|^2}\right)\\
\leq\ &\frac{1}{n}\log\left(\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} 1\right)\\
=\ &\frac{1}{n}\log\frac{(n+1)^3(n+2)}{2} \leq 4.
\end{aligned}$$

This inequality also holds on the patch $U_1 = \{Z_1 \neq 0\}$ by continuity, and so the lemma is proved. □

Lemma 3.2. *There exists $\varepsilon > 0$ such that for any $\varphi \in P_G(M, \omega_g)$ and N, there exist $n > N$, $i_0, i_1, i_2, j_0, j_1, k_0, k_2$ with $i_0 + i_1 + i_2 = j_0 + j_1 = k_0 + k_2 = n$, and $(a^{(n)}_{(\varphi)i_0i_1i_2j_0j_1k_0k_2})^{\frac{1}{n}} > \varepsilon$.*

Proof. Otherwise, for any $\varepsilon > 0$, there exist φ and N, such that for any $n > N$ and any $i_0, i_1, i_2, j_0, j_1, k_0, k_2$ satisfying $i_0+i_1+i_2=j_0+j_1=k_0+k_2=n$, we have $(a^{(n)}_{(\varphi)i_0i_1i_2j_0j_1k_0k_2})^{\frac{1}{n}} < \varepsilon$. By choosing n large enough and with the lemma above, we have

$$\begin{aligned}
&\varphi([Z_0, Z_1, Z_2])\\
&\leq \frac{1}{n}\log \frac{\max|a^{(n)}_{(\varphi)i_0i_1i_2j_0j_1k_0k_2}|^2 \sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} |Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^n} + \varepsilon\\
&\leq \frac{1}{n}\log \frac{\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} |Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^n} + 2\log\varepsilon + \varepsilon\\
&\leq \log\varepsilon + 4.
\end{aligned}$$

Since ε could be arbitrarily small, the above inequality would imply that $\varphi \to -\infty$ uniformly, which contradicts the fact that $\sup_M \varphi = 0$. □

Proof of Theorem 1. We use notations as above; since $(a^{(n)}_{(\varphi)i_0i_1i_2j_0j_1k_0k_2})^{\frac{1}{n}} > \varepsilon$, we have

$$\begin{aligned}
&\varphi([Z_0, Z_1, Z_2])\\
=\ &\lim_{N\to\infty} \frac{1}{N}\log \frac{\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=N} |a^{(N)}_{(\varphi)i_0i_1i_2j_0j_1k_0k_2} Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^N}\\
\geq\ &\frac{1}{N}\log \frac{|Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2 + |Z_0^{i_0+j_0+k_0} Z_1^{i_2+k_2} Z_2^{i_1+j_1}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^N} + \log\epsilon
\end{aligned}$$

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$$\geq \frac{1}{N}\log\frac{|Z_0{}^m Z_1^{\frac{3}{2}N-\frac{m}{2}} Z_2^{\frac{3}{2}N-\frac{m}{2}}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^N}+\log\epsilon$$
$$\geq \log\frac{|Z_0{}^{\frac{2m}{N}} Z_1^{3-\frac{m}{N}} Z_2^{3-\frac{m}{N}}|^2}{(|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2)}+\log\epsilon,$$

where $i_0+j_0+k_0=m, i_1+j_1+i_2+k_2=3N-m$.

On the patch $U_0=\{Z_0\neq 0\}$,

$$\int_{U_0\cap\{0<|z_1|,|z_2|<1\}} e^{-\alpha\varphi}\omega_{g_0}^2$$
$$\leq C_1\int_{0<|z_1|,|z_2|<1} e^{-\alpha\log\frac{|Z_0|^{\frac{2m}{N}}|Z_1|^{3-\frac{m}{N}}|Z_2|^{3-\frac{m}{N}}}{(|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2)}}\omega_{g_0}^2$$
$$= C_1\int_{0<|z_1|,|z_2|<1}\frac{(|Z_0|^2+|Z_1|^2+|Z_2|^2)^\alpha(|Z_0|^2+|Z_1|^2)^\alpha(|Z_0|^2+|Z_2|^2)^\alpha}{|Z_0|^{\frac{2\alpha m}{N}}|Z_1|^{3\alpha-\frac{\alpha m}{N}}|Z_2|^{3\alpha-\frac{\alpha m}{N}}}\omega_{g_0}^2$$
$$\leq C_2\int_{0<|z_1|,|z_2|<1}\frac{(1+|z_1|^2+|z_2|^2)^\alpha(1+|z_1|^2)^\alpha(|1+|z_2|^2)^\alpha}{|z_1|^{3\alpha-\frac{\alpha m}{N}}|z_2|^{3\alpha-\frac{\alpha m}{N}}}dz_1\wedge d\overline{z}_1 \wedge dz_2\wedge d\overline{z}_2$$
$$\leq C_3\int_{0<|z_1|,|z_2|<1}\frac{1}{|z_1|^{3\alpha-\frac{\alpha m}{N}}|z_2|^{3\alpha-\frac{\alpha m}{N}}}dz_1\wedge d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2$$
$$\leq C_3\int_{0<|z_1|,|z_2|<1}\frac{1}{|z_1|^{3\alpha}|z_2|^{3\alpha}}dz_1\wedge d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2,$$

where C_1, C_2 and C_3 are constants depending only on α and ϵ.

On the patch $U_2=\{Z_2\neq 0\}$,

$$\int_{U_1\cap\{0<|w_0|,|w_1|\leq 1\}} e^{-\alpha\varphi}\omega_{g_1}^2$$
$$\leq C_4\int_{0<|w_0|,|w_1|\leq 1} e^{-\alpha\log\frac{|Z_0|^{\frac{2m}{N}}|Z_1|^{3-\frac{m}{N}}|Z_2|^{3-\frac{m}{N}}}{(|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2)}}\omega_{g_1}^2$$
$$= C_4\int_{0<|w_0|,|w_1|\leq 1}\frac{(1+|w_0|^2+|w_1|^2)^\alpha(1+|w_0|^2)^\alpha(|w_0|^2+|w_1|^2)^\alpha}{|w_0|^{\frac{2\alpha m}{N}}|w_1|^{3\alpha-\frac{\alpha m}{N}}}\omega_{g_1}^2$$
$$\leq C_5\int_{0<|w_0|,|w_1|\leq 1}\frac{(1+|w_0|^2+|w_1|^2)^\alpha(1+|w_0|^2)^\alpha(|w_0|^2+|w_1|^2)^\alpha}{|w_0|^{\frac{2\alpha m}{N}}|w_1|^{3\alpha-\frac{\alpha m}{N}}(|w_0|^2+|w_1|^2)}dw_0\wedge d\overline{w}_0 \wedge dw_1\wedge d\overline{w}_1$$
$$\leq C_6\int_{0<|w_0|,|w_1|\leq 1}\frac{1}{|w_0|^{\frac{2\alpha m}{N}}|w_1|^{3\alpha-\frac{\alpha m}{N}}(|w_0|^2+|w_1|^2)^{1-\alpha}}dw_0\wedge d\overline{w}_0\wedge dw_1\wedge d\overline{w}_1$$
$$\leq C_6\int_{t=0}^1\int_{s=0}^1\frac{1}{s^{\frac{\alpha m}{N}}t^{\frac{3}{2}\alpha-\frac{\alpha m}{2N}}(s+t)^{1-\alpha}}dsdt$$
$$\leq C_6\int_{s=0}^1\frac{1}{s^{\frac{\alpha m}{N}}t^{\frac{3}{2}\alpha-\frac{\alpha m}{2N}}s^{(1-\alpha)p}t^{(1-\alpha)q}}dsdt,$$

where $p+q=1$ and C_4, C_5, C_6 are constants depending only on α and ϵ.

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Case 1: If $1 \leq \frac{m}{N} \leq 3$, we can choose $\alpha < \min(\frac{2}{3}, \frac{1-p}{3-p})$ so that

$$\begin{aligned} \frac{\alpha m}{N} + (1-\alpha)p &< 1, \\ 3\alpha - 1 &< 1, \\ \frac{3}{2}\alpha - \frac{\alpha m}{2N} + (1-\alpha)q &< 1. \end{aligned}$$

Case 2: If $0 < \frac{m}{N} < 1$, we can choose $\alpha < \min(\frac{2}{3}, \frac{1-q}{3/2-q})$ so that

$$\begin{aligned} \frac{\alpha m}{N} + (1-\alpha)p &< 1, \\ 3\alpha - 1 &< 1, \\ \frac{3}{2}\alpha - \frac{\alpha m}{2N} + (1-\alpha)q &< 1. \end{aligned}$$

So we could choose any $\alpha < \frac{1}{3}$, which implies that $\alpha_G(M, \omega) \geq \frac{1}{3}$.
Conversely, we choose

$$\begin{aligned} \varphi_\varepsilon &= \log\left(\frac{|Z_0|^6}{(|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_2|^2)} + \varepsilon\right) \\ &\quad - \log(1+\varepsilon) \\ &\in P_G(M, \omega). \end{aligned}$$

Then we have $\sup_M \varphi_\varepsilon = 0$ and $\varphi_\varepsilon = \log \frac{\varepsilon}{1+\varepsilon}$ on the exceptional divisors. Furthermore, we have

$$\lim_{\varepsilon \to 0} \int_M e^{-\alpha\varphi_\varepsilon}\omega^2 = \infty, \ for\ any\ \alpha > \frac{1}{3}.$$

Hence we have shown $\alpha_G(M, \omega) = \frac{1}{3}$.

We can also apply the above arguments for CP^n $(n \geq 2)$, $CP^2\#1\overline{CP^2}$ and $CP^2\#3\overline{CP^2}$.

(i) Let $M = CP^n$ and let G_n be the automorphism group acting on M, generated by θ_j and permutations $\tau_{i,j}$ $(0 \leq i < j \leq n)$,

$$\theta_j : [Z_0, ..., Z_j, ..., Z_n] \to [Z_0, ..., Z_j e^{i\theta}, ..., Z_2]$$

for $\theta \in [0, 2\pi)$, and

$$\tau_{i,j} : [Z_0, ..., Z_i, ..., Z_j, ..., Z_n] \to [Z_0, ..., Z_j, ..., Z_i, ..., Z_n].$$

Theorem 3.1. $\alpha_{G_n}(CP^n) = 1$.

(ii) Let M be the blow-up of CP^2 at 3 points which are not collinear. Then we can assume that these 3 points are $[1,0,0]$, $[0,1,0]$ and $[0,0,1]$. Let $G(3)$ be the automorphism group acting on M, generated by θ_j and permutations $\tau_{i,j}$ $(0 \leq i < j \leq 2)$,

$$\theta_j : [Z_0, Z_j, Z_2] \to [Z_0, Z_j e^{i\theta}, Z_2]$$

for $\theta \in [0, 2\pi)$, and

$$\tau_{i,j} : [..., Z_i, ..., Z_j, ...] \to [..., Z_j, ..., Z_i, ...].$$

Theorem 3.2. $\alpha_{G(3)}(CP^2\#3\overline{CP^2}) = 1$.

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(iii) Let M be the blow-up of CP^2 at one point $[1,0,0]$ and $G(1)$ be the automorphism group acting on M, generated by θ_j and permutations τ $(0 \leq i \leq 2)$,

$$\theta_j : [Z_0, Z_j, Z_2] \to [Z_0, Z_j e^{i\theta}, Z_2]$$

for $\theta \in [0, 2\pi)$, and

$$\tau : [Z_0, Z_1, Z_2] \to [Z_0, Z_2, Z_1].$$

Theorem 3.3. $\alpha_{G(1)}(CP^2 \# 1\overline{CP^2}) = \frac{1}{2}$.

Also the proof above shows that the sequence of the holomorphic invariants $\{\alpha_{G,m}(M)\}_m$ defined by Tian [8] on CP^n $(n \geq 2)$, $CP^2 \# k\overline{CP^2}$ $(k = 1, 2, 3)$ is stationary.

4. Proof of Theorem 2

In this section, we will prove the generalized Moser-Trudinger inequality on any Kähler manifold M of dimension n whose $\alpha(M)$ is greater than $\frac{n}{n+1}$. The following theorem is due to Tian and Zhu [11].

Theorem 4.1. *Let (M, ω) be a Kähler-Einstein manifold with $Ric(\omega) = \omega$; then there exist constants $\delta = \delta(n)$ and $C = C\left(n, \lambda_2(\omega) - 1\right) \geq 0$ such that for any $\phi \in P(M, \omega)$ which satisfies $\phi \perp \Lambda_1$, we have*

$$F_\omega(\phi) \geq J_\omega(\phi)^\delta - C,$$

which is the same as

$$\frac{1}{V}\int_M e^{-\phi}\omega^n \leq C e^{J_\omega(\phi) - \frac{1}{V}\int_M \phi\omega^n - J_\omega(\phi)^\delta}.$$

This implies in particular the Moser-Trudinger inequality on S^2, which reads

$$\frac{1}{4\pi}\int_{S^2} e^{-\phi}\omega \leq e^{\frac{1}{8\pi}\int_{S^2}|\nabla\phi|^2\omega - \frac{1}{4\pi}\int_{S^2}\phi}.$$

For any $\phi \in P(M, \omega)$, put $\omega' = \omega_\phi = \omega + \sqrt{-1}\partial\overline{\partial}\phi$ and $Ric(\omega) = \omega + \sqrt{-1}\partial\overline{\partial}h_\omega$. Consider the Monge-Ampère equation

$$(\omega' + \sqrt{-1}\partial\overline{\partial}\psi)^n = e^{h_\omega - t\psi}\omega'^n.$$

We will use the continuity method backwards and let ϕ_t be a smooth family which solve the above equation.

The following lemmas are well known [10], but we add the proofs for the sake of completeness.

Lemma 4.1. *$Ric(\omega_t) \geq t\omega_t$ and we have equality if and only if $t = 1$, where $\omega_t = \omega + \phi_t$ and ϕ_t solves the Monge-Ampère equation at t.*

Proof.

$$\begin{aligned} Ric(\omega_t) &= -\sqrt{-1}\partial\overline{\partial}\log\omega_t^n = -\sqrt{-1}\partial\overline{\partial}\log\frac{\omega_t^n}{\omega^n} + Ric(\omega) \\ &= -\sqrt{-1}\partial\overline{\partial}\left(h_\omega - t\phi_t\right) + \omega + \sqrt{-1}\partial\overline{\partial}h_\omega \\ &= \omega + t\phi_t = t\omega_t + (1-t)\omega \geq t\omega_t. \end{aligned}$$

□

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Lemma 4.2. *For any $\phi \in P(M,\omega)$, if the Green's function of $\omega' = \omega + \sqrt{-1}\partial\overline{\partial}\phi$ is bounded from below, we have:*

$$-\inf_M \phi \leq \frac{1}{V}\int_M (-\phi)\omega'^n + C.$$

Proof. Since $\omega + \sqrt{-1}\partial\overline{\partial}\phi = \omega'$ and $\omega' - \sqrt{-1}\partial\overline{\partial}\phi > 0$, we have $\Delta_{\omega'}\phi \leq n$, and

$$\begin{aligned}
-\phi &= \frac{1}{V}\int_M (-\phi)\omega'^n + \frac{1}{V}\int_M \Delta_{\omega'}\phi(y) G_{\omega'}(x,y)\omega'^n \\
&\leq \frac{1}{V}\int_M (-\phi)\omega'^n + \frac{1}{V}\int_M n(G_{\omega'}(x,y) - \inf G_{\omega'}(x,y))\omega'^n \\
&\leq \frac{1}{V}\int_M (-\phi)\omega'^n + C.
\end{aligned}$$

□

Let (M,ω) be a Kähler-Einstein manifold with $\mathrm{Ric}(\omega) = \omega$ and let $P(M,\omega,K) = \{\phi \in P(M,\omega) \mid G_{\omega+\sqrt{-1}\partial\overline{\partial}\phi}(x,y) \geq -K)$. Then we have:

Proposition 4.1. *Let (M,ω) be a Kähler-Einstein manifold with $Ric(\omega) = \omega$. If $\alpha(M) > \frac{n}{n+1}$, then there exist constants $\delta(n,\alpha,K)$ and $C(n,\alpha,\lambda_2(\omega)-1,K)$ such that for any $\phi \in P(M,\omega,K)$, we have*

$$F_\omega(\phi) \geq \delta J_\omega(\phi) - C.$$

Proof. Let $\omega' = \omega + \partial\overline{\partial}\phi$, where $\phi \in P(M,\omega,K)$. We have

$$\begin{aligned}
\frac{1}{V}\int_M e^{-\alpha\phi}\omega^n &= \frac{1}{V}\int_M e^{-(\alpha_1+\alpha_2+\varepsilon)\phi}\omega^n \\
&\leq \frac{1}{V}\int_M e^{-(\alpha_1+\alpha_2)\phi}\omega^n e^{-\varepsilon\inf_M \phi};
\end{aligned}$$

taking $p = \frac{1}{\alpha_1}, q = \frac{1}{1-\alpha_1}$, we have

$$\begin{aligned}
\frac{1}{V}\int_M e^{-(\alpha_1+\alpha_2)\phi}\omega^n &\leq \frac{1}{V}(\int_M e^{-\alpha_1 p\phi}\omega^n)^{1/p}(\int_M e^{-\alpha_2 q\phi}\omega^n)^{1/q} \\
&= \frac{1}{V}(\int_M e^{-\phi}\omega^n)^{\alpha_1}(\int_M e^{-\frac{\alpha_2}{1-\alpha_1}\phi}\omega^n)^{1-\alpha_1} \\
&\leq Ce^{\alpha_1 J_\omega(\phi) - \frac{\alpha_1}{V}\int_M \phi\omega^n}(\int_M e^{-\frac{\alpha_2}{1-\alpha_1}\phi}\omega^n)^{1-\alpha_1}.
\end{aligned}$$

By Lemma 4.2,

$$\begin{aligned}
e^{-\varepsilon\inf_M\phi} &\leq e^{\frac{\varepsilon}{V}\int_M(-\phi)\omega'^n + C} \\
&= e^{\varepsilon I_\omega(\phi) - \frac{\varepsilon}{V}\int_M \phi\omega^n + C} \\
&\leq e^{\varepsilon(n+1)J_\omega(\phi) - \frac{\varepsilon}{V}\int_M \phi\omega^n + C}.
\end{aligned}$$

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By Hölder's inequality,

$$\begin{aligned}
\frac{1}{V}\int_M e^{-\phi}\omega^n &\le (\frac{1}{V}\int_M e^{-\alpha\phi}\omega^n)^{\frac{1}{\alpha}} \\
&\le Ce^{\frac{\alpha_1+(n+1)\varepsilon}{\alpha}J_\omega(\phi)-\frac{\alpha_1+\varepsilon}{\alpha V}\int_M \phi\omega^n}(\int_M e^{-\frac{\alpha_2}{1-\alpha_1}\phi}\omega^n)^{\frac{1-\alpha_1}{\alpha}} \\
&= Ce^{\frac{\alpha_1+(n+1)\varepsilon}{\alpha}J_\omega(\phi)-\frac{1}{V}\int_M \phi\omega^n+\frac{\alpha_2}{V}\int_M(\phi-\sup\phi)\omega^n}(\int_M e^{-\frac{\alpha_2}{1-\alpha_1}(\phi-\sup\phi)}\omega^n)^{\frac{1-\alpha_1}{\alpha}} \\
&\le Ce^{\frac{\alpha_1+(n+1)\varepsilon}{\alpha}J_\omega(\phi)-\frac{1}{V}\int_M \phi\omega^n}(\int_M e^{-\frac{\alpha_2}{1-\alpha_1}(\phi-\sup\phi)}\omega^n)^{\frac{1-\alpha_1}{\alpha}}.
\end{aligned}$$

We need to determine α_1, α_2, ε which satisfy the following conditions:

$$\begin{aligned}
\alpha &= \alpha_1+\alpha_2+\varepsilon > 1, \\
\alpha &> \alpha_1+(n+1)\varepsilon, \\
1 &> \alpha_1.
\end{aligned}$$

So we will choose

$$\begin{aligned}
\alpha_2 &= n\varepsilon+\varepsilon', \\
\alpha_1 &= 1-\alpha_2-\varepsilon+\varepsilon'' = 1-(n+1)\varepsilon-\varepsilon'+\varepsilon'',
\end{aligned}$$

where ε , ε' , $\varepsilon'' << 1$, and $\varepsilon' = o(\varepsilon)$, $\varepsilon'' = o(\varepsilon')$.

Since $\alpha(M) > \frac{n}{n+1}$, we can choose ε, ε' , ε'' small enough; then we have

$$\frac{\alpha_2}{1-\alpha_1} = \frac{n\varepsilon+\varepsilon'}{(n+1)\varepsilon+\varepsilon'-\varepsilon''} < \alpha(M)$$

and

$$\int_M e^{-\frac{\alpha_2}{1-\alpha_1}(\phi-\sup\phi)}\omega^n < Const.$$

Combined with the inequalities above, we have

$$\frac{1}{V}\int_M e^{-\phi}\omega^n \le Ce^{(1-\delta)J_\omega(\phi)-\frac{1}{V}\int_M \phi\omega^n}.$$

This proves the lemma. □

Proof of Theorem 2. We assume ω is the Kähler-Einstein metric of M. For any $\phi \in P(M,\omega)$, put $\omega' = \omega+\sqrt{-1}\partial\overline{\partial}\phi$. Consider $(\omega'+\sqrt{-1}\partial\overline{\partial}\psi) = e^{h_{\omega'}+t\psi}$. By solving the Monge-Ampère equation backwards, we get the solutions ϕ_t, and $\phi_1 = -\phi$.

For $t > \frac{1}{2}$, let $\omega_t = \omega'+\sqrt{-1}\partial\overline{\partial}\phi_t = \omega+\sqrt{-1}\partial\overline{\partial}(\phi_t-\phi_1)$; by Lemma 4.1,

$$Ric(\omega_t) \ge \frac{1}{2}\omega_t,$$

which shows that the Green function of ω_t is uniformly bounded from below. Thus by Proposition 4.1 and the calculation in [11] we have

$$\begin{aligned}
F_\omega(\phi_t-\phi_1) &\ge \delta J_\omega(\phi_t-\phi_1)-C \\
&\ge C_1 osc_M(\phi_t-\phi_1)-C_2,
\end{aligned}$$

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and consequently,

$$\begin{aligned} n(1-t)J_\omega(\phi) &= n(1-t)J_{\omega'}(\phi_1) \\ &\geq (1-t)(I_{\omega'}(\phi_1) - J_{\omega'}(\phi_1)) \\ &\geq F_{\omega'}(\phi_t) - F_{\omega'}(\phi_1) \\ &= F_\omega(\phi_t - \phi_1) \\ &\geq C_1 osc_M(\phi_t - \phi_1) - C_2. \end{aligned}$$

Thus we have

$$\begin{aligned} F_\omega(\phi) &= -F_{\omega'}(-\phi) \\ &= \int_0^1 (I_{\omega'}(\phi_t) - J_{\omega'}(\phi_t))dt \\ &\geq (1-t)(I_{\omega'}(\phi_t) - J_{\omega'}(\phi_t)) \\ &\geq \frac{1-t}{n} J_{\omega'}(\phi_t) \\ &\geq \frac{1-t}{n} J_{\omega'}(\phi_1) - 2(1-t)(C_1 osc_M(\phi_t - \phi_1) - C_2) \\ &\geq \frac{1-t}{n} J_\omega(\phi) - 2(1-t)^2 n C_1 J_\omega(\phi) - C_3. \end{aligned}$$

The theorem follows by choosing $(1-t) < \frac{1}{2n^2 C_1}$.

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Department of Mathematics, Columbia University, New York, New York 10027
E-mail address: jsong@math.columbia.edu

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