# DUPIN INDICATRICES
# AND FAMILIES OF CURVE CONGRUENCES

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ABSTRACT. We study a number of natural families of binary differential equations (BDE's) on a smooth surface $M$ in $\mathbb{R}^3$. One, introduced by G. J. Fletcher in 1996, interpolates between the asymptotic and principal BDE's, another between the characteristic and principal BDE's. The locus of singular points of the members of these families determine curves on the surface. In these two cases they are the tangency points of the discriminant sets (given by a fixed ratio of principle curvatures) with the characteristic (resp. asymptotic) BDE.

More generally, we consider a natural class of BDE's on such a surface $M$, and show how the pencil of BDE's joining certain pairs are related to a third BDE of the given class, the so-called polar BDE. This explains, in particular, why the principal, asymptotic and characteristic BDE's are intimately related.

## 1. INTRODUCTION

We shall deal in this paper with Binary Differential Equations (BDE's) defined on surfaces in $\mathbb{R}^3$. A BDE is an implicit differential equation of the form

$$a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2 = 0$$

where $a$, $b$, $c$ are smooth real functions in $(x,y)$ (in this paper smooth means infinitely differentiable). A BDE defines two directions in the region where $\delta = (b^2 - ac)(x,y) > 0$, a double direction on the set $\Delta = \{\delta = 0\}$ and no direction where $\delta < 0$. The set $\Delta$ is the *discriminant* of the BDE. The local behaviour of these equations have been studied by several authors (see, for example, [1]–[14], [17]–[31]) and, in particular, all codimension $\leq 1$ singularities and their bifurcations have been classified. Global aspects of certain types of BDE's have also been studied ([18], [29]).

Asymptotic curves and lines of curvature are given by BDE's and have been considered as separate objects. However, in [16] a natural 1-parameter family of BDE's is constructed interpolating between the asymptotic BDE and that of the lines of curvature. This family is referred to as the *conjugate congruence* and is denoted by $\mathcal{C}_\alpha$.

There is another BDE on the surface, namely that determining the characteristic directions (those conjugate directions at an elliptic point which are inclined at a minimum angle). This is a classical BDE which does not appear to have been much studied, and plays a similar role to the asymptotic BDE but in the elliptic region. We exhibit in this paper a second natural congruence, labelled *reflected*

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*conjugate congruence* and denoted by $\mathcal{R}_\alpha$, that interpolates between the principal and characteristic BDE's.

The discriminants of the $\mathcal{C}_\alpha$ (resp. $\mathcal{R}_\alpha$) foliate the elliptic (resp. hyperbolic) region of the surface. There are points on these discriminants where $\mathcal{C}_\alpha$ (resp. $\mathcal{R}_\alpha$) has a well-folded singularity (these are the stable singularities of BDE's, see [11]) or worse. We shall call the set of such points *the conjugate* (resp. *reflected*) *zero curve* of $\mathcal{C}_\alpha$ (resp. $\mathcal{R}_\alpha$). These are new robust features (in the sense that they can be marked on an evolving surface), and are studied in detail in [3]. Interestingly these zero curves have another characterisation. For the conjugate curve congruence the zeros are the points of tangency of the characteristic BDE and the natural foliation given by the constant eccentricity of the Dupin ellipses or hyperbola. For the reflected curve congruence it is the set of tangency points of this foliation and the asymptotic BDE. So in these examples the roles of the asymptotic and characteristic BDE's are interchanged. The families $\mathcal{C}_\alpha$ and $\mathcal{R}_\alpha$ originate with the pair of involutions on the tangent directions at a non-umbilic point on a surface, given by conjugation or reflection in either principal direction. There is a third family (indeed a variety of them) which interpolates between the asymptotic and characteristic BDE's with the auxilliary role played by the principal BDE's, showing that these three BDE's are intimately and symmetrically related.

These families are examples of a more general phenomena which we study. Indeed, we consider *curvature* BDE's which, in a principal co-ordinate system, can be written (in an invariant way) in terms of the principal curvatures. We show that they extend across isolated umbilic points and so determine globally defined BDE's. However, we can also view them as points in the projective plane over the field of rational functions in the curvatures. We use some elementary plane geometry to explain how the characteristic, asymptotic and principal BDE's (the three classically best known natural BDE's) are, in fact, intimately related. We produce further examples of such triples.

The paper is arranged as follows. Section 2 reviews some basic properties of conjugacy, introduces the characteristic BDE and establishes its properties. Section 3 introduces the conjugate and reflection curve congruences $\mathcal{C}_\alpha$ and $\mathcal{R}_\alpha$. In Section 4 we discuss binary quadratic forms and relate their properties to binary differential equations. In Section 5 we define curvature BDE's, show that they determine global BDE's on the surface and determine some of their elementary properties. Finally, in Section 6 we give some further examples of natural families of BDE's on surfaces.

## 2. Conjugate and characteristic directions

We start with a discussion of conjugate directions. Suppose given an (affine) conic section, and a direction. The lines parallel to this direction will meet the conic in 0 or 2 points, or be tangent (one repeated point), or pass through a singular point in the case of a line pair. In the case of a pair of intersection points consider their midpoint. These midpoints are collinear and determine the *conjugate direction*. If $l$ is conjugate to $l'$ it is an elementary fact that $l'$ is the conjugate of $l$. Given a surface $M$ in $\mathbb{R}^3$, at each point $p$ we have a Dupin indicatrix. This is the conic approximation of the intersection of the surface at the point with a small translate of the tangent plane in the normal direction (at an elliptic point this has to be the right direction). Two directions are conjugate if they are conjugate with respect to the Dupin indicatrix. For the ellipse $x^2/a^2 + y^2/b^2 = 1$ (respectively the hyperbola

$x^2/a^2 - y^2/b^2 = 1$) the directions $y = m_1 x$, $y = m_2 x$ are conjugate if and only if $m_1 m_2 = -b^2/a^2$ (respectively $m_1 m_2 = b^2/a^2$). At hyperbolic points it is sometimes traditional to consider the indicatrix to be the union of hyperbolas obtained by translating in both directions, i.e., $x^2/a^2 - y^2/b^2 = \pm 1$. At parabolic points where the second fundamental form is not identically zero, the indicatrix is a pair of parallel lines and each direction is conjugate to that determined by these lines (other than the line direction itself which has no well-defined conjugate direction).

We need some notation. Given an oriented surface $M$ in $\mathbb{R}^3$ with a family of normals $N$ we have a Gauss map $N : M \to S^2$. At a point $p$ the map $-dN(p) : T_p M \to T_{N(p)} S^2$ can be thought of as an automorphism of $T_p M$, this is the classical shape operator $S_p$, or simply $S$. If $M$ is parametrised by $\mathbf{r}(x, y)$ with shape operator $S$, the coefficients of the first (resp. second) fundamental form $E, F, G$ (resp. $l, m, n$) are given by

$$E = \mathbf{r}_x . \mathbf{r}_x, \ F = \mathbf{r}_x . \mathbf{r}_y, \ G = \mathbf{r}_y . \mathbf{r}_y,$$
$$l = S(\mathbf{r}_x) . \mathbf{r}_x = N . \mathbf{r}_{xx},$$
$$m = S(\mathbf{r}_x) . \mathbf{r}_y = S(\mathbf{r}_y) . \mathbf{r}_x = N . \mathbf{r}_{xy},$$
$$n = S(\mathbf{r}_y) . \mathbf{r}_y = N . \mathbf{r}_{yy}.$$

*Remarks* 2.1. Here are some properties and interpretations of conjugacy:

(1) Two vectors $v$ and $w \in T_p M$ are conjugate if and only if $II(v, w) = 0$, where $II$ is the second fundamental form at $p$.

(2) The notion of conjugacy is invariant under affine and inversive transformations.

(3) Let $v$ be a direction in the tangent plane $T_p M$ at $p$ and consider a curve on $M$ through $p$ in the direction of $v$. Then the characteristic at $p$ of the envelope of planes determined by this curve is a line in the direction conjugate to $v$.

(4) If we parallel project a surface $M$ in the direction $v$, then the conjugate direction at a singular point $p$ of the projection is the tangent to the singular set of the projection. (This is a consequence of the characterisation given in (3).)

(5) If $v, \overline{v}$ are conjugate directions at $p \in M$, then $S(v)$ and $\overline{v}$ are orthogonal. (This is just the definition of conjugacy in another guise.)

(6) The angle between conjugate directions $v, \overline{v}$ and the vectors $S(v)$, $S(\overline{v})$ are equal or supplementary according to whether the point is hyperbolic or elliptic.

(7) For any $v \in T_p M$ (that is not an asymptotic direction at a parabolic point) let $\alpha$ be the signed angle between $v$ and $\overline{v}$. Then from (5)

$$\sin \alpha = \frac{S(v).v}{\|S(v)\| \|v\|}.$$

(8) The sum of the radii of curvature in conjugate (non-asymptotic) directions is constant, and consequently equal to the sum of the principal radii of curvature.

(9) The only orthogonal conjugate directions are those given by the principal directions.

(10) At non-parabolic points the only self-conjugate directions are the asymptotic directions.

(11) Two direction fields on a surface $M$ are said to be *conjugate* if the corresponding pairs of directions are conjugate everywhere. So, for example (away from umbilics), the direction fields determined by the principal directions are conjugate. In a rather degenerate way the direction field in the hyperbolic region given locally by (one of the) families of asymptotic directions are self-conjugate. One way of

finding conjugate fields (indeed, families of integral curves of conjugate fields) on a surface is by fixing a line $L$ and considering (a) the planar sections of the surface by the pencil determined by $L$, and (b) the set of singular points for central projections of $M$ from points of $L$. Of course, any direction field on a surface determines a conjugate field.

Given a direction $v$ in $T_pM$ consider the conjugate direction $\overline{v}$. At hyperbolic points these directions can coincide (the asymptotic directions). At elliptic points, however, there is a unique pair of conjugate directions for which the included angle (i.e., the angle between these directions) is minimal. These directions are called *characteristic directions*, and their integral curves are called the *characteristic curves*.

**Proposition 2.2.** *At an elliptic point $p$ of $M$ there is a unique conjugate pair of characteristic directions. These directions are reflections of each other in either of the lines of symmetry of the Dupin ellipse (corresponding to the principal directions at $p$).*

*The BDE determining the above pair is given, in terms of the coefficients of the first and second fundamental forms, by*

$$(2m(Gm - Fn) - n(Gl - En))dy^2 +$$
$$2(m(Gl + En) - 2Fln)dydx +$$
$$(l(Gl - En) - 2m(Fl - Em))dx^2 = 0.$$

*Away from umbilics this can be written, with respect to a co-ordinate system given by lines of curvature, as*

$$\kappa_2 dy^2 - \kappa_1 dx^2 = 0$$

*where $\kappa_1$ (resp. $\kappa_2$) is the principal curvature in the $x$- (resp. $y$-) direction.*

*Proof.* See, for example, [15], p. 129, or Proposition 3.6.                    □

*Remarks* 2.3. (1) Each elliptic point has two characteristic directions; each parabolic point one (coinciding with the asymptotic direction there); each hyperbolic point none.

(2) The characteristic directions are, in many ways, the analogue of the asymptotic directions for the elliptic region of a surface. (Recall that in the above co-ordinate system the asymptotic directions are given by $\kappa_2 dy^2 + \kappa_1 dx^2 = 0$, a switch of signs from that for the equation for the characteristic directions.)

(3) The characteristic directions are also characterised as the conjugate directions where the radii of normal curvature are equal, or as stated above the conjugate directions bisected by the principal directions.

(4) The solution curves of the characteristic BDE, as for the asymptotic BDE, yield some interesting information. Indeed the angle between them determines the eccentricity of the Dupin indicatrix (the ratio of principal curvatures), and the lines themselves determine its orientation. The same holds for the asymptotic lines in the hyperbolic region. (Note that the principal directions, which are the bisectors of the characteristic and asymptotic directions, only determine the orientation of the Dupin indicatrix.)

(5) Other BDE's defined in the non-hyperbolic region of the surface are studied in [19] and [20].
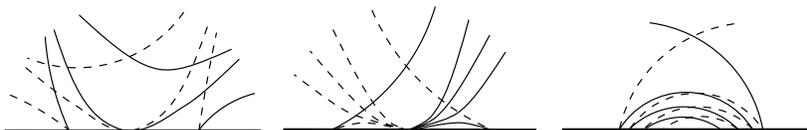
FIGURE 1. Configurations of the well-folded singularities: saddle (left), node (centre), focus (right).

**Corollary 2.4.** *The discriminant of the characteristic BDE consists of the parabolic set and umbilic points, and the singular points of the characteristic equation are given by the cusps of Gauss and the umbilics.*

Next we need to analyse the zeros of the characteristic BDE. We relate below the nature of the zeros of the characteristic and asymptotic BDE's at cusps of Gauss. Recall that a well-folded saddle singularity has index $-\frac{1}{2}$ and a well-folded node and focus have index $+\frac{1}{2}$ (Figure 1).

**Proposition 2.5.** *At a cusp of Gauss the asymptotic and characteristic BDE's generically have well-folded singularities of opposite indices, that is, on one side of the parabolic curve we have a well-folded saddle and on the other a well-folded node or focus.*

*Proof.* We can write the surface at a cusp of Gauss in the form $z = h(x,y) = y^2 + ax^2y + bxy^2 + cy^3 + dx^4 + \ldots$ and assume that the parabolic set is smooth, so that $b \neq 0$. Then the equations of the asymptotic and characteristic directions are both locally smoothly equivalent to a BDE of the form $dy^2 - (y - \lambda x^2) = 0$, with $\lambda = \lambda_1 = 3(-b^2 + 4d)/(2b^2)$ for the asymptotic BDE and $\lambda = \lambda_2 = -3(-b^2 + 4d)/(2b^2)$ for the characteristic BDE. We have a well-folded singularity provided $\lambda \neq 0, \frac{1}{4}$, and the singularity is of type well-folded saddle (resp. node or focus) if $\lambda < 0$ (resp. $0 < \lambda < \frac{1}{4}$ or $\frac{1}{4} < \lambda$) (see [11]). Considering $\lambda_1$ and $\lambda_2$ in the $(b, d)$-plane we find that the only combinations for the types of singularities of asymptotic and characteristic BDE's at a cusp of Gauss are (well-folded saddle, well-folded node), (well-folded saddle, well-folded focus) or vice-versa. □

At umbilics we can consider the surface in Monge form $h = \frac{\kappa}{2}(x^2 + y^2) + C(x,y) + O(4)$, in such a way that the cubic part $C(x,y)$ is given in complex form $\phi = Re(z^3 + \beta z^2 \overline{z})$, where $\beta$ is a complex number. Then the configuration of the characteristic directions at the umbilic is generically either of type star, monstar or lemon (Figure 2).

**Proposition 2.6.** *The umbilic of the characteristic BDE is of type star if $\beta$ is inside the circle $|\beta| = 3$, of type lemon if $\beta$ is outside the hypercycloid $\beta = 3(2e^{i\theta} + e^{-2i\theta})$ and of type monstar in the remaining regions of the complex plane; see Figure 3.*

In the next section we construct families of BDE's on a surface $M$. The key idea is the following. Consider the projective space $PT_pM$ of all tangent directions through a point $p$ of $M$ which is neither an umbilic nor parabolic point. Conjugation gives an involution on $PT_pM$, $v \mapsto \overline{v} = C(v)$. There is another involution on $PT_pM$ which is simply reflection in (either of) the principal directions, $v \mapsto R(v)$. These determine a mapping on the set of directions through $p$. We use $C$ and $C \circ R$ to determine families of BDE's by asking that the angle between a direction $v$
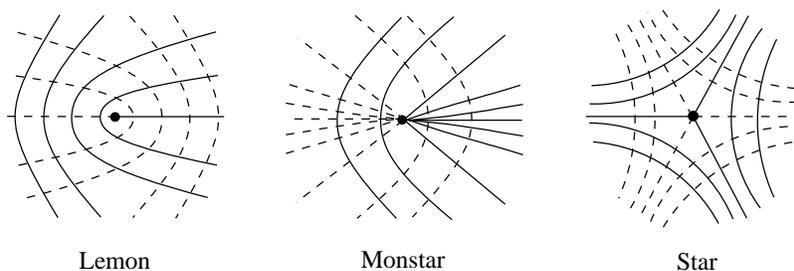
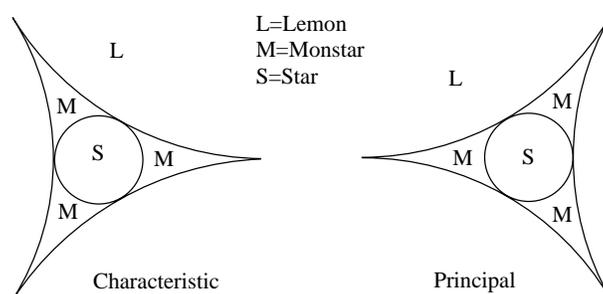FIGURE 2. Configurations of the characteristic directions at an umbilic point.



FIGURE 3. Partition of the $\beta$-plane: characteristic directions (left), principal directions (right).

and the image of $v$ under one of these mappings is constant. The following is straightforward.

**Proposition 2.7.** (1) $R \circ C = C \circ R$, so $\overline{R(v)} = R(\overline{v})$.

(2) *Let $v$ be a direction; then the set $\{v,\ C(v),\ R(v),\ C(R(v))\}$ has less than four elements if and only if $v$ is either a principal direction $(v = R(v))$, an asymptotic direction $(v = C(v))$, or a characteristic direction $(v = R(C(v)))$.*

(3) *Given a line field $\xi$, we can create 3 other fields from it using this pair of involutions. Naturally these can be arranged in three ways as pairs of fields, or BDE's. These pairs will be closed under $C$, $R$ or $R \circ C = C \circ R$.*

(4) *The angle between $v$ and resp. $C(v)$, $C \circ R(v)$, $R(v)$ is zero (resp. $\frac{\pi}{2}$) if and only if $v$ is resp. asymptotic, characteristic, principal (resp. principal, principal, bisector of the principal directions).*

*Remark* 2.8. We are grateful to Terry Wall for the following observation. Given an orientable surface, we can consider, at each point, automorphisms of the unit tangent vectors given by reflections, $R_1$, $R_2$ in the two principal directions and conjugation $C$, where $C(v)$ is chosen so that $v, C(v)$ is positively oriented. These generate a quaternion group acting on the space of unit vectors.

## 3. THE CONJUGATE AND REFLECTED CONGRUENCES

We obtain in this section one-parameter families of BDE's by orienting the surface $M$ and considering those directions that make an oriented angle $\alpha$ between

$v$ and $C(v)$ or between $v$ and $R \circ C(v)$. The first family, labelled *conjugate congruence*, interpolates between the asymptotic and principal BDE's and was first introduced in [16]; the second, labelled *reflected congruence*, interpolates between the characteristic and principal BDE's. The first set of BDE's is closed under $R$; the second under $C$. In what follows we define the angle between two lines to be between $0$ and $\pi/2$. If the plane is oriented we attach a sign to the angle between two ordered lines, $+$ (resp. $-$) according to whether the ordered half lines inclined at an angle of $< \pi/2$ yield the given orientation or not. This is well defined, except when the lines are orthogonal, i.e., we cannot distinguish between $\pm\pi/2$. Note that if we reverse the orientation, the signed angle changes sign.

**Definition 3.1** ([16])**.** Let $PTM$ denote the projective tangent bundle to $M$, and define $\Theta : PTM \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ by $\Theta(p, v) = \alpha$ where $\alpha$ denotes the oriented angle between a direction $v$ and the corresponding conjugate direction $\overline{v} = C(v)$. Note that $\Theta$ is not well defined at points corresponding to asymptotic directions at parabolic points. The conjugate curve congruence, for a fixed $\alpha$, is defined to be $\Theta^{-1}(\alpha)$ which we denote $\mathcal{C}_\alpha$.

It is clear that the set of all asymptotic directions is $\mathcal{C}_0$ and $\mathcal{C}_{\pm\frac{\pi}{2}}$ is the set of all principal directions. Note that, for any $\alpha$, $\mathcal{C}_\alpha$ contains the asymptotic directions at parabolic points. It is shown in [16] that $\mathcal{C}_\alpha$ is given by a BDE.

**Proposition 3.2** ([16])**.** (1) *The conjugate curve congruence $\mathcal{C}_\alpha$ of a parametrised surface is given by the BDE*

$$(\sin \alpha(Gm - Fn) - n \cos \alpha \sqrt{EG - F^2})dy^2 +$$
$$(\sin \alpha(Gl - En) - 2m \cos \alpha \sqrt{EG - F^2})dydx +$$
$$(\sin \alpha(Fl - Em) - l \cos \alpha \sqrt{EG - F^2})dx^2 = 0.$$

(2) *The discriminant of $\mathcal{C}_\alpha$, which we denote $\Delta_C^\alpha$, is given by*

$$H^2(x, y) \sin^2 \alpha - K(x, y) = 0,$$

*where $H$ is the mean curvature and $K$ is the Gauss curvature.*

(3) *Away from umbilics the BDE $\mathcal{C}_\alpha$ can be written, with respect to a co-ordinate system given by lines of curvature, by*

$$\kappa_2 \cos \alpha dy^2 + (\kappa_2 - \kappa_1) \sin \alpha dydx + \kappa_1 \cos \alpha dx^2 = 0.$$

*Remarks* 3.3. (1) It is clear that the discriminant $\Delta_C^0$ is the parabolic curve and $\Delta_C^{\pm\frac{\pi}{2}}$ consists of the umbilic points. For any $\alpha$ the discriminant curves for $\mathcal{C}_\alpha$ occur in the non-hyperbolic region of the surface, and clearly foliate this region.

(2) Note that whilst $\mathcal{C}_\alpha \neq \mathcal{C}_{-\alpha}$ generally, we have $\Delta_C^\alpha = \Delta_C^{-\alpha}$. Indeed it is not hard to see that the pairs of directions determined by $\mathcal{C}_\alpha$ have as their conjugates the pairs of directions determined by $\mathcal{C}_{-\alpha}$. In other words, these are conjugate BDE's. On the discriminant $\Delta_C^\alpha = \Delta_C^{-\alpha}$ the BDE $\mathcal{C}_\alpha$ determines a repeated direction which is one of the chacteristic directions. The other corresponds to the repeated direction determined by $\mathcal{C}_{-\alpha}$.

(3) A well-folded singularity occurs at points where the unique direction on the discriminant determined by the BDE is tangent to the discriminant. So from (2) above the set of zeros of the family $\mathcal{C}_\alpha$ is the locus of tangency points of the pair of foliations determined by the characteristic BDE and that given by equal eccentricity

curves of the Dupin ellipses. This relation between the principal, asymptotic and characteristic directions is investigated in the next section.

(4) The curve $\Delta_C^\alpha$ can also be characterised as the set of points where the characteristic directions make an angle $\pm\frac{\alpha}{2}$ with one of the principal directions.

(5) It is not difficult to see that $H^2/K = constant$ is (away from umbilics) equivalent to the ratio of the principal curvatures $\kappa_1/\kappa_2 = \mu$ being constant. Indeed, one easily checks that $4H^2/K = \mu + 2 + \mu^{-1}$, and such an equation has two solutions $\mu_1, \mu_2$ which are mutually inverse. Of course, the umbilic points correspond to $H^2/K = 1$, and in the elliptic region this is a minimum value for $H^2/K$.

(6) At heart, much of this paper is concerned with indicatrices of Dupin. Another interpretation of the invariant $H^2/K$ is that it determines the eccentricity of these ellipses and hyperbola, that is their type up to a Euclidean motion and dilatation. Of course, any relation concerning angles between conjugate directions will be invariant under these changes.

We now consider the congruence with $R \circ C$ replacing $C$.

**Definition 3.4.** Let $\Phi : PTM \to [-\pi/2, \pi/2]$ be given by $\Phi(p,v) = \alpha$ where $\alpha$ is the signed angle between $v$ and $R(\overline{v})(= R \circ C(v))$. Note that $\Phi$ is not well defined at umbilics. Then the reflected conjugate curve congruence, for a fixed $\alpha$, is defined to be $\Phi^{-1}(\alpha)$, which we denote $\mathcal{R}_\alpha$.

*Remarks* 3.5. (1) An alternative way of defining $\Phi$ at $(p,v)$ is as the sum of the signed angles between $v$ and a principal direction $e$ and $\overline{v}$ and $e$ (ignoring multiples of $\pi$). This does not depend on the particular principle direction chosen.

(2) It is clear that the set $\mathcal{R}_{\frac{\pi}{2}}$ is the set of principal directions and the set $\mathcal{R}_0$ is the set of characteristic directions. So $\mathcal{R}_\alpha$ interpolates between the characteristic BDE and the principal BDE and indeed interchanges the roles played by the characteristic and asymptotic BDE's.

**Proposition 3.6.** (1) *The reflected conjugate congruence $\mathcal{R}_\alpha$ is given by the BDE*

$$\{(2m(Gm - Fn) - n(Gl - En))\cos\alpha + (Fn - Gm)\frac{2Fm - Gl - En}{\sqrt{(EG - F^2)}}\sin\alpha\}dy^2 +$$

$$\{2(m(Gl + En) - 2Fln)\cos\alpha + (En - Gl)\frac{2Fm - Gl - En}{\sqrt{(EG - F^2)}}\sin\alpha\}dydx +$$

$$\{(l(Gl - En) - 2m(Fl - Em))\cos\alpha + (Em - Fl)\frac{2Fm - Gl - En}{\sqrt{(EG - F^2)}}\sin\alpha\}dx^2 = 0.$$

(2) *The discriminant consists of umbilic points together with the set*

$$K\cos^2\alpha + H^2\sin^2\alpha = 0,$$

*which we denote by $\Delta_R^\alpha$.*

(3) *Away from umbilics the equation for $\mathcal{R}_\alpha$ is given, in the principal co-ordinate system, by*

$$\kappa_2\cos\alpha\, dy^2 - (\kappa_1 + \kappa_2)\sin\alpha\, dydx - \kappa_1\cos\alpha\, dx^2 = 0.$$

*Proof.* In the principal co-ordinate system $e_1, e_2$ two directions $u = e_1 + pe_2$, $\overline{u} = e_1 + re_2$ are conjugate if and only if $\kappa_1 + \kappa_2 pr = 0$, so $r = -\kappa_1/\kappa_2 p$. Now if we measure angles from the first principal direction, then these two directions are at angles $\theta_1$, $\theta_2$ where $\tan\theta_1 = p$, $\tan\theta_2 = r$. Setting the formula for $\tan(\theta_1 + \theta_2)$

equal to $\tan \alpha$ we obtain the equation of the conjugate curve congruence in the principal co-ordinate system.

We extend the equation of the reflected conjugate curve congruence to umbilics following the same argument in the proof of Proposition 5.4 below.                    $\square$

*Remarks* 3.7. (1) Clearly $\mathcal{R}_\alpha$ is invariant under conjugation (it simply interchanges the two directions). In the previous case we saw that the BDE was not closed under conjugation; indeed the BDE conjugate to $\mathcal{C}_\alpha$ is $\mathcal{C}_{-\alpha}$.

(2) There are consequently two ways of moving from the principal BDE to the characteristic BDE: as $\alpha$ moves from $\frac{\pi}{2}$ to 0 or from $-\frac{\pi}{2}$ to 0.

(3) Following the same argument as in Remark 3.3(5), the curves $\Delta_R^\alpha$ coincide with the curves $\kappa_1/\kappa_2 = constant$. The set $\Delta_R^\alpha$ foliates the hyperbolic region of the surface, and $\Delta_R^\alpha = \Delta_R^{-\alpha}$. Note that $\Delta_R^0$ is the parabolic curve, and $\Delta_R^{\frac{\pi}{2}}$ is the curve $H = 0$.

(4) On the discriminant $\Delta_R^\alpha$ the pair defined by $\mathcal{R}_\alpha$ reduces to an asymptotic direction. Consequently, the discriminant $\Delta_R^\alpha$ can also be characterised as the set of points where the angle between an asymptotic direction and the chosen principal direction is $\pm\frac{\alpha}{2}$.

(5) It follows from (4) above that the set of the zeros of the family $\mathcal{R}_\alpha$ is the locus of tangency points of the pair of foliations determined by the asymptotic BDE and that given by equal eccentricity curves of the Dupin hyperbola. (See the next section for a detailed explanation.)

(6) Clearly at points where $H = 0$ we cannot canonically order the principal directions. So in a neighbourhood of the curve $H = 0$ there are generally two curves of points whose indicatrices have the same eccentricity.

(7) The angle between any two conjugate directions at an umbilic is always $\frac{\pi}{2}$, and these were our special points for our first family (they only appeared in the single member of the family corresponding to principal curves). Umbilics are also special points for $\mathcal{R}_\alpha$. At such points all directions are principal, so for any given $\alpha$ and any direction $v$, we can find a principal direction $e$ so that the sum of the signed angles between $v$ and this principal direction $e$ and $\overline{v}$ and $e$ equals $\alpha$. At a point where the mean curvature is zero the Dupin indicatrix is a rectangular hyperbola, and here the angle constructed above is always $\frac{\pi}{2}$. So these points too are exceptional for the new family. (Again they only appear in the single member of the family corresponding to principal curves.) Note that the equation for $\mathcal{R}_{\frac{\pi}{2}}$ above is the standard equation for the principal directions multiplied by $H$.

## 4. Pencils of forms and BDE's

Our intention in the next two sections is to understand why the classical triple of BDE's (the asymptotic, characteristic and principal BDE's) are intimately related (Remark 3.3(2) and 3.7(4)) and to seek to generalise this example.

Clearly, we can write down the BDE's on a surface in any co-ordinate system. However, as we have seen the interpretation of the coefficients is then less than clear. The approach we have taken here is to consider (oriented) surfaces which have isolated umbilic points. Those surfaces in $\mathbb{R}^3$ without this property form a set of infinite codimension in a very natural sense. Away from the umbilics we may select a principal co-ordinate system. This is simply a co-ordinate system with the lines $x = $ constant and $y = $ constant the lines of curvature. We are particularly

interested in those BDE's which, in such a system, can be expressed in the form

$$a(\kappa)p^2 + 2b(\kappa)pq + c(\kappa)q^2 = 0$$

where $a$, $b$, $c$ are functions of the principal curvatures $\kappa_1, \kappa_2$ in the $x, y$-directions respectively, and the ratio $p/q$ determines the slope of the given direction with repect to the given axes.

It will be helpful to recall some elementary facts concerning binary quadratic forms and plane conics. So let $k$ denote the field of real or complex numbers. Write $L$ for a field of characteristic zero. The examples we are interested in are the field of rational functions in $x_1, \ldots, x_n$ with coefficients in $k = \mathbb{R}, \mathbb{C}$, in particular, where $n = 1, 2$, which we denote by $k(x)$. We shall be considering binary quadratic forms with coefficients in $L$.

*Remarks* 4.1. (1) We do not distinguish non-zero $L$ multiples of such forms. Consequently, we are considering elements of the projective plane $PL^2$, where $f = ap^2 + bpq + cq^2$ corresponds to the point $(a : b : c)$. Clearly, the plane contains the conic $\Delta$ of singular forms given by $b^2 = 4ac$. A singular form is one of the type $(rp + sq)^2$. If $(r : s) \in PL^1$, then clearly $(r : s) \mapsto (r^2 : 2rs : s^2)$ is the usual parametrisation of the conic $\Delta$.

(2) Motivated by our previous work we will be considering pairs of forms and the pencils determined by them. These can be viewed as lines in $PL^2$. Any such line will have 0, 1 or 2 points on the pencil determining singular forms. (Clearly, even when $k = \mathbb{C}$ the field of rational functions $L = k(x)$ is not algebraically closed.)

Given a conic in the projective plane $PL^2$, then any point in $PL^2$ determines a polar line, and given a line there is a corresponding polar point. Geometrically, if the line meets the conic then the tangents at the points of intersection meet in the polar point. Three points in the plane are said to be *self-polar* (as is the triangle determined by them) if the polar of any vertex is the line through the remaining two points. The next series of results are well known and elementary, but very useful. They relate some of the invariants of pairs of binary forms to the geometry of the conic $\Delta$ of singular forms.

**Proposition 4.2.** (1) *Let $\omega$ be a binary quadratic form, with distinct roots in $L$, determining a point in the plane $PL^2$. Then the polar line of $\omega$ with respect to the conic of singular quadratics $\Delta$ consists of the line through the two forms which are the squares of the factors of $\omega$. In other words, the tangents to the conic at these two points pass through $\omega$. We refer to this intersection point as the polar form of the pencil. Conversely, given any pencil meeting the conic $\Delta$, the corresponding polar form is the binary form whose factors are the repeated factors at the two singular members of the pencil.*

(2) *This polar form of the pencil is given by the classical Jacobian of any two of the forms in the pencil, that is the $2 \times 2$ determinant of the matrix of partial derivatives of the forms with respect to $p$ and $q$. The Jacobian is non-zero provided we have a genuine pencil, and is a square if and only if the forms have a factor in common.*

(3) *Fixing two forms $\omega = ap^2 + bpq + cq^2$, $\Omega = Ap^2 + Bpq + Cq^2$ we write $D(\alpha : \beta)$ for the discriminant of $\alpha\omega + \beta\Omega$; this is another binary quadratic form. We can write it as $D(\omega)\alpha^2 + E(\omega, \Omega)\alpha\beta + D(\Omega)\beta^2$, where $D(\omega) = (b^2 - 4ac)$, $D(\Omega) = (B^2 - 4AC)$, $E(\omega, \Omega) = 2(bB - 2aC - 2Ac)$. The associated polar point of the*

*pencil, the Jacobian, determined by $\omega$, $\Omega$ is*

$$Jac(\omega, \Omega) = (aB - Ab)p^2 + 2(aC - Ac)pq + (bC - Bc)q^2.$$

*Note, in particular, that if the two forms $\omega$ and $\Omega$ have their coefficients in some ring $R$, then so does their Jacobian. (So if $f$ and $F$ have entries in $k[\kappa_1, \kappa_2]$ or $k(\kappa_1, \kappa_2)$, the same is true of their Jacobian.)*

*(4) Pairs of forms $\omega, \Omega$ with the term $E(\omega, \Omega)$ above zero are said to be* apolar. *This is equivalent to the condition that the corresponding four roots harmonically separate each other, or that the forms lie on each others polars with respect to the conic $\Delta$, that is, are conjugate. The Jacobian of any two forms is apolar with respect to all the elements of the pencil determined by them.*

*(5) Three forms determine a self-polar triangle with respect to the conic $\Delta$ if and only if each is the Jacobian of the other two. There are a variety of ways of obtaining self-polar triples. Any form $\omega$ determines a polar line. Choose an arbitrary form say $\Omega$ on the line; this has a polar line which passes through $\omega$. Consider the intersection point of these two polar lines; this gives a third form $\mu$, with $\omega, \Omega, \mu$ self-polar. Any self-polar triple arises in this way. Also if $\omega, \Omega$ are conjugate, then the triple $\omega, \Omega, Jac(\omega, \Omega)$ is self polar. Finally if the vertices of a quadrangle lie on $\Delta$, then the diagonal triangle (the triangle whose vertices are intersections of the lines joining distinct pairs of distinct points) is self-polar.*

*(6) The discriminants, the invariant $E$, and the Jacobian of a pair of forms $\omega, \Omega$ are related as follows:*

$$Jac^2(\omega, \Omega) - 4D(\Omega)\omega^2 - 4D(\omega)\Omega^2 + 4E(\omega, \Omega)\omega\Omega = 0.$$

We now suppose that $L$ is the field of rational functions $k(x)$ in $n$-variables, $U$ is an open subset of $k^2$ and that we have a smooth map $X : U \to k^n$. Given a form $\omega = ap^2 + bpq + cq^2$, with $a, b, c \in k[x]$ we can define a BDE on $U$ by considering

$$a(X(x, y))p^2 + b(X(x, y))pq + c(X(x, y))q^2 = 0$$

where at $(x, y) \in U$ the solution $(p : q) = (p_0, q_0)$ is the direction with slope $p_0/q_0$; this BDE is denoted $X^*\omega$. As usual we suppose given a pair of forms $\omega$, $\Omega$ and consider the pencil of BDE's, $X^*(\alpha\omega + \beta\Omega) = \alpha X^*\omega + \beta X^*\Omega$, but with $\alpha$, $\beta \in k$. Each $(\alpha : \beta) \in Pk^1$ determines a BDE, and this will have a discriminant $D_{\alpha,\beta}$ in $U$.

**Proposition 4.3.** (1) *If we fix a point in $U$, there are 2 or 1 (respectively 2, 1 or 0) discriminant curves passing through it when $k = \mathbb{C}$ (respectively $\mathbb{R}$). The condition that there is just one is $E^2(\omega, \Omega) = 4D(\Omega)D(\omega)$.*

*(2) Suppose we now consider a point $p$ of $U$ through which two discriminant curves pass, so there are two values of $(\alpha : \beta)$ for which $D$ vanishes at the point. Each corresponds to a BDE with a repeated direction through $p$; putting these two directions together at each point yields a new BDE, which is just the pull-back by $X$ of the Jacobian or polar form*

$$X^*((aB - Ab)p^2 + 2(aC - Ac)pq + (bC - Bc)q^2) = 0.$$

*(It can be written in a number of other ways, e.g., if $(aB - Ab) \neq 0$, as $X^*(D(\omega)(Ap + Bq)^2 - 2E(\omega, \Omega)(ap + bq)(Ap + Bq) + D(\Omega)(ap + bq)^2) = 0$.)*

*Proof.* Part (1) is clear. For (2) we write $f = X^*(\omega)$ and $F = X^*(\Omega)$, and simply write $a$ for $X^*a = a \circ X$, etc. If we have a point through which only one discriminant curve passes, then it satisfies $\alpha^2 D_1 + E\alpha\beta + \beta^2 D_2 = 0$ with one of $\alpha$ and $\beta$

non-zero. Now if the roots are $(\alpha_j, \beta_j)$, $j = 1, 2$, then $\alpha_j(ap^2 + 2bpq + cq^2) + \beta_j(Ap^2 + 2Bpq + Cq^2)$ has a repeated root, so it must be of the form $p/q = (-b\alpha_j - B\beta_j)/(a\alpha_j + A\beta_j)$. A little manipulation gives the resulting BDE, in the second form above.

Alternatively, and more directly the pencil meets the conic of singular quadratics in two points. We are simply considering the quadratic whose roots are each of the repeated roots, and the result follows from the elementary properties above. $\square$

**Corollary 4.4.** (1) *The set of zeros of the pencil determined by $f = X^*(\omega)$ and $F = X^*(\Omega)$ coincides with the set of tangency points of the polar BDE of the pencil with the discriminants. (In the case when $n = 1$, the field is $K = k(\lambda)$, and the map $X$ is given by $X(x, y) = \kappa_2(x, y)/\kappa_1(x, y)$, where the $\kappa_j$ are the principal curvatures in some order at a non-umbilic point, then the discriminants are given by $\kappa_2/\kappa_1 = constant$.)*

*(2) The set of points through which only one discriminant curve passes (roughly the envelope of the discriminant curves) is given by $E^2 = 4D_1D_2$, that is, $(aC - Ac)^2 = (aB - Ab)(bC - cB)$. This is the discriminant of the polar BDE to the pencil (the Jacobian form). (The statement can then be read as: the envelope of the discriminants of the pencil is the discriminant of the polar or Jacobian form.)*

*(3) In our previous work we considered families of BDE's $F_\alpha$ whose discriminants $D_\alpha$ actually satisfy $D_\alpha = D_{-\alpha}$. So it is natural to consider those forms $f$ and $F$ which have the property that $D_{\alpha,\beta} = D_{-\alpha,\beta}$. The BDE's $f$ and $F$ arising from the forms $\omega$, $\Omega$ have the property that the discriminant of $\alpha f + \beta F$ coincides with that of $\alpha f - \beta F$ if $\omega, \Omega$ are apolar (or equivalently conjugate with respect to the conic $\Delta$ of singular forms). Conversely, if any pencil of BDE's induced from $\omega$, $\Omega$ has this property, then these forms are apolar.*

*(4) We can obtain a triple of BDE's with the property that each is the polar BDE of the pencil determined by the other two using the constructions above for self-polar triples.*

*Remarks* 4.5. (1) The first result tells us that the zero points of the families of the pencil will pass through the zeros of the polar form, since in any neighbourhood of a singular point of a BDE there will be tangency points with any foliation.

(2) We extend the terminology for forms (polar, Jacobian, apolar) to BDE's in the natural way.

(3) Note that if we were considering the full pencil joining $f$ and $F$, then each point of the pencil would determine a second conjugate point of the pencil. However, if we are only considering points $\alpha f + \beta F$ with $\alpha$, $\beta \in k$ (rather than $L = k(x)$), then this is not necessarily true.

**Example 4.6.** The set of asymptotic, characteristic and principal BDE's ($F = \kappa_2 p^2 + \kappa_1 q^2$, $G = pq$, $H = \kappa_2 p^2 - \kappa_1 q^2$) is a self-polar triple.

The polars of certain forms share some interesting properties; in what follows the angle $\theta$ associated to $(p : q)$ satisfies, as usual, $\tan \theta = p/q$ .

**Example 4.7.** (1) The polar of $p^2 + q^2$ is the set of forms whose roots have product $-1$ (directions are orthogonal).

(2) The polar of $p^2 - q^2$ is the set of forms whose roots have product 1 (angles sum to $\pi/2$).

(3) The polar of $pq$ is the set of forms whose roots have sum 0 (angles sum to 0).

(4) The polar of $\kappa_1 p^2 + \kappa_2 q^2$ is the set of forms whose roots determine conjugate directions.

Indeed, it is easy to check that the polar of a point $(a : b : c) \in Pk^2$ is the set of forms for which the sum of the angles is constant if and only if $a + c = 0$.

## 5. CURVATURE-BDE'S

In our examples the coefficients of the BDE's we are considering are given, in a principal co-ordinate system, by polynomials in the principal curvatures. In other words, we have taken $n = 2$ above and the map $X$ given by $x \mapsto (\kappa_1(x), \kappa_2(x))$. A key issue is that at each point we cannot canonically select a pair of ordered principal directions. For this reason we need the following discussion. At a point $z$ of our surface we can consider an ordered pair (consistent with the orientation) of principal directions $e_1$, $e_2$, yielding an ordered pair of principal vector fields nearby. Suppose given a BDE near $z \in M$ which in a principal co-ordinate system can be written in the form $a(\kappa)p^2 + 2b(\kappa)pq + c(\kappa)q^2 = 0$, with $a, b, c \in k[\kappa_1, \kappa_2]$. Here the solution $(p : q)$ at $(x, y)$ corresponds to the tangent direction, making an angle of $\arctan p/q$ to the principle direction $e_1$. If this is to be well defined in a global sense, then we need the same solutions if we replace $e_1$, $e_2$ by $e_2, -e_1$ (and $\kappa_1, \kappa_2$ by $\kappa_2, \kappa_1$). Label the alternative pair of principal directions $d_1, d_2$, with $Q, -P$ replacing $p, q$. Then the new BDE is given by

$$\overline{a}(\kappa)Q^2 - 2\overline{b}(\kappa)PQ + \overline{c}(\kappa)P^2 = 0$$

where $\overline{h}(\kappa_1, \kappa_2) = h(\kappa_2, \kappa_1)$. So we need $(a, b, c)$ to be a non-zero multiple of $(\overline{c}, -\overline{b}, \overline{a})$. In what follows we write $\tau$ for $\kappa_1 - \kappa_2$.

**Proposition 5.1.** (1) *We have $(a, b, c)$ a non-zero multiple of $(\overline{c}, -\overline{b}, \overline{a})$ if and only if the BDE is in one of the following forms:*
  (a) $a(\kappa)p^2 + b(\kappa)pq + \overline{a}(\kappa)q^2$, *with $b + \overline{b} = 0$ (type I), or*
  (b) $a(\kappa)p^2 + b(\kappa)pq - \overline{a}(\kappa)q^2$, *with $b = \overline{b}$ (type II).*
  (2) *Multiplication by $\tau$ interchanges type. In the first case $b$ is of the form $B - \overline{B}$, in the second $B + \overline{B}$ for some $B$. Alternatively, in the second case we can write $b = h(H, K)$ and in the first $b = \tau h(H, K)$ for some $h$, where $H = (\kappa_1 + \kappa_2)/2$ is the mean curvature and $K = \kappa_1\kappa_2$ is the Gauss curvature.*
  (3) *The BDE corresponding to the directions conjugate to those determined by (1) and (2) above is obtained by replacing $(a, b)$ by $(\overline{a}\kappa_2^2, \overline{b}\kappa_1\kappa_2)$.*
  (4) *Given two BDE's of type I (or II), then an $\mathbb{R}$ or $\mathbb{C}$ linear combination of them is of the same type. Indeed, if $k_s[\kappa_1, \kappa_2]$ (respectively $K_s = k_s(\kappa_1, \kappa_2)$) is the subring (subfield) of $k[\kappa_1, \kappa_2]$ ($k(\kappa_1, \kappa_2)$) consisting of symmetric functions of $\kappa_1, \kappa_2$; in other words, polynomial (rational) functions in $H$ and $K$, then $k_s[\kappa_1, \kappa_2]$-($K_s$-)linear combinations of BDE's of a given type are of the same type.*

*Proof.* This is easy to prove. For the second part of (2) note that if $b = \overline{b}$, then $b$ is symmetric in $\kappa_1$ and $\kappa_2$ and any such function can be written as a function of their sum and product. If $b + \overline{b} = 0$, then $b(t, t)$ is identically zero and so we can write $b = (\kappa_2 - \kappa_1)b'$ and one easily checks that $b'$ is symmetric.  $\square$

**Definition 5.2.** We say that a BDE of one of the above types is a curvature BDE, or CBDE for short.

*Remarks* 5.3. (1) We shall invariably select $a, b, c$ to be polynomials. Note that if the CBDE only depends on the ratio of the curvatures (that is the indicatrix up to similarity), then $a, b, c$ are homogeneous polynomials in $\kappa_1, \kappa_2$ of a given degree.

(2) Note also that the map $a \mapsto \overline{a}$ is an automorphism of order 2 of the field $K = k(\kappa_1, \kappa_2)$, with fixed set $K_s$.

(3) The set of CBDE's do not sit inside $PK^2$; a $K = k(\kappa_1, \kappa_2)$-multiple of a CBDE is not necessarily a CBDE, but a $K_s$-multiple is.

(4) Multiplication by $\tau$ interchanges BDE's of types I and II, and since $\kappa_1 = \kappa_2$ only at umbilics, it does not change the integral curves of the BDE's. For this reason it is sufficient to study BDE's of type I, say.

We now show that a CBDE extends across the umbilic points, justifying their name. (See [23], Theorem A, for an alternative proof.)

**Proposition 5.4.** *Let $M$ be an oriented surface in $\mathbb{R}^3$ with isolated umbilic points. Then any CBDE with polynomial entries determines a smooth BDE on the whole of $M$; that is, extends smoothly across umbilic points.*

*Proof.* Obviously we cannot take a special principal co-ordinate system at an umbilic; this is the key problem! So we take a general parametrisation $\mathbf{r}(x, y)$ at an umbilic and change coordinates.

Let $ap^2 + 2bpq + cq^2 = 0$ be a BDE with 2 solutions $u_1, u_2$ at a point in the plane. Suppose that we have a (parametrised) linear action on the vectors $u_1, u_2$ by a matrix

$$M = \begin{pmatrix} M_1 & N_1 \\ M_2 & N_2 \end{pmatrix}$$

and let $v_1 = Mu_1, v_2 = Mu_2$. Then $v_1, v_2$ are solutions at this point of the BDE $Ap^2 + 2Bpq + Cq^2 = 0$ with

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} M_1^2 & -2M_1N_1 & N_1^2 \\ -M_1M_2 & M_1N_2 + M_2N_1 & -N_1N_2 \\ M_2^2 & -2M_2N_2 & N_2^2 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Now we work in a neighbourhood of an umbilic and suppose that $ap^2 + 2bpq + cq^2 = 0$ is a CBDE, with the map $M$ taking the co-ordinate directions with respect to the parametrisation $\mathbf{r}$ to the principal directions. So $M$ has entries as above where $M_1\mathbf{r}_x + M_2\mathbf{r}_y$ and $N_1\mathbf{r}_x + N_2\mathbf{r}_y$ are the principal directions. These in turn are obtained by solving the principal directions BDE

$$(Fn - Gm)dy^2 + (En - Gl)dydx + (Em - Fl)dx^2 = 0.$$

So we replace $dy$ by $M_2$ (respectively $M_1$) and $dx$ by $N_2$ (respectively $N_1$) and solve the corresponding equations. If, say, $Fn - Gm \neq 0$, then the expressions for $M_2/M_1$, $N_2/N_1$ are $\{-(En - Gl) \pm 2(EG - F^2)\sqrt{H^2 - K}\}/2(Fn - Gm)$. Now the BDE $ap^2 + 2bpq + cq^2$ starts out with entries which are polynomials in the curvatures, satisfying those conditions which make them CBDE's (of type I or II). We can extend smoothly to umbilics if after changing coordinates, we can get rid of the "discriminant" $\sqrt{H^2 - K}$ in the new coefficients $(A, B, C)$. Of course, if $\kappa_1$, $\kappa_2$ are the principal curvatures, then $\tau = \kappa_1 - \kappa_2 = \pm 2\sqrt{H^2 - K}$. Since our BDE is of the form I or II, in one case the expressions for $A$, $B$, $C$ do not involve $\tau$, and in the other, $\tau$ is a factor of each of them, so it can be discounted.  $\square$

We now wish to study CBDE's; we note that when, for example, we refer to the CBDE $\kappa_2 p^2 + \kappa_1 q^2$ we will be thinking of the BDE determined by this on the whole surface $M$ (in this case the asymptotic BDE). As remarked above, if $h \in k[\kappa_1, \kappa_2]$ and $f$ is a CBDE, then the product $h.f$ need not be unless $h \in K_s$. Working over $K_s$, however, we can think of our two types of CBDE's as each determining a projective plane over $K_s$.

**Proposition 5.5.** (1) *Let* $\tau = \kappa_1 - \kappa_2$. *The map* $K_s^3 \to K^3$, $(u, v, w) \mapsto (u + \tau w, \tau v, u - \tau v)$ *is injective and has image the set of CBDE's of type I. The map* $(u, v, w) \mapsto (u + \tau w, v, -u + \tau w)$ *is injective and has image the set of CBDE's of type II. We abuse notation and write* $P_I$ *(respectively* $P_{II}$*) for the projective planes* $PK_s^3$ *with the above maps into* $PK^3$.

(2) *The pull back of the discriminant conic in* $P_I$ *(respectively* $P_{II}$*) is given by* $u^2 - \tau^2(v^2 + w^2) = 0$ *(respectively* $u^2 + v^2 - \tau^2 w^2$*). (Here* $u$, $v$, $w$ *are rational functions of* $H$ *and* $K$ *and* $\tau^2 = 4(H^2 - 4K)$.*)*

(3) *The polar BDE of two CBDE's of type I (respectively of type II, of types I and II) are CBDE's of type I (respectively I, II).*

The discussion above now works in the projective planes $P_I$ and $P_{II}$. Note, however, that if we wish to discuss self-polar triples, then the natural types of CBDE's to consider are type I, since they are closed under this operation. Since the families are essentially equivalent this is no real restriction.

One final note. The projective line of directions through a point of our surface clearly has further structure inherited from the Euclidean metric on the tangent space. Indeed, the metric $p^2 + q^2$ means we can identify the projective line $P^1$ with its dual $P^{1*}$. Elements of the dual, of course, also determine directions at the origin and we have the following result.

**Proposition 5.6.** *If* $L$ *is any field, then the form* $ap^2 + 2bpq + cq^2$ *determines* $0, 1$ *or* $2$ *directions in the plane. The dual directions are "orthogonal" to these and determined by* $cp^2 - 2bpq + aq^2$. *We refer to this as the* dual equation. *Replacing* $a, b$ *by* $\overline{a}, \overline{b}$ *in cases I or II above we obtain the corresponding dual BDE.*

## 6. Angles, maps, involutions and further families

There are a variety of other possible families (pencils) one can consider.

**Example 6.1.** As pointed out above it is natural to consider the pencil containing the characteristic and asymptotic BDE's; we now know that the principal directions will be the solutions to the associated polar BDE. It is less clear here how to choose the right family which has a nice geometric interpretation. Again this highlights the fact that the pencils we naturally consider are over $k$; since we can multiply our base CBDE's by any element of $K_s$ there are a number of possible $k$-pencils corresponding to a given $K_s$-pencil. One way that we can obtain a family is as follows. Consider a direction $v$ and its conjugate $\overline{v}$. If these make angles $\theta$ and $\phi$ with one of the principal directions, then we can consider those $v$ with $\tan(\theta + \phi)/\tan(\theta - \phi)$ constant, say $\tan(\alpha)$. It is not hard to see that this is given (in a principal co-ordinate system) by

$$F_\alpha = (\kappa_1 - \kappa_2)(\kappa_2 p^2 - \kappa_1 q^2)\sin\alpha + (\kappa_1 + \kappa_2)(\kappa_2 p^2 + \kappa_1 q^2)\cos\alpha = 0.$$

This pencil has the following properties:

(1) It interpolates between the asymptotic and characteristic BDE's, and all members are CBDE's.

(2) The discriminants $D_\alpha$ satisfy $D_\alpha = D_{-\alpha}$ (this follows because the pair of forms are apolar).

(3) If a tangent direction $v$ at a point satisfies $F_\alpha = 0$, so does $\overline{v}$. Moreover, $Rv$, $R\overline{v}$ are the two directions at the same point determined by $F_{-\alpha}$.

(4) The discriminants are given by $\kappa_1/\kappa_2 = $ constant, and on the $D_\alpha$ the repeated direction is principal.

Note, to illustrate the point about the choices of $k$-pencil available, that we could determine another suitable family by asking that the quotient $\sin 2\phi/\sin 2\theta$ is constant. This again joins the principal and characteristic BDE's (or, alternatively, the BDE's $p^2 + q^2$ and $\kappa_2^2 p^2 + \kappa_1^2 q^2$).

Of course, the most obvious family to choose is $(\kappa_2 p^2 + \kappa_1 q^2)\cos\alpha + (\kappa_2 p^2 - \kappa_1 q^2)\sin\alpha$. This $k$-pencil has the obvious disadvantage that it actually meets the singular conic (when $\tan\alpha = \pm 1$). So for these two values of $\alpha$ we obtain a line field (naturally, the principal directions), and for these values of $\alpha$ the discriminant $D_\alpha$ is the whole plane.

We now describe a general class of examples. We first recall that an invertible linear map $(p : q) \mapsto (ap+bq : cp+dq)$ yields an automorphism of the projective line, and the fixed points (or double points) of the map are given by $cp^2 + (d-a)pq - bq^2 = 0$. The map is an involution if and only if $a + d = 0$, and its double points are distinct. The involution is said to be hyperbolic if the fixed points are in $L$ and elliptic otherwise.

**Example 6.2.** Consider an element $\Gamma$ of $PGL(2, K)$, that is, an invertible $2 \times 2$ matrix with entries in $K$ up to projective equivalence which we can view as acting on the set of tangent directions at points of the surface. (It actually acts on the tangent directions in the $\kappa$-space.) With the usual conventions on entries we need then to consider the map of directions $p/q \mapsto (ap + bq)/(cp + dq) = P/Q$, which we write as $\Gamma(p, q)$. We can take the angle between the lines with slopes $(p : q)$ and $(P : Q)$ and ask that this is a given constant. As this varies we obtain the pencil joining $F = ap^2 + (b + c)pq + dq^2$ and $G = cp^2 + (d - a)pq - bq^2$. The first of these gives the directions for which $(p, q), \Gamma(p, q)$ are orthogonal, the second, as noted above, the fixed directions of the map. Note that $F$ and $G$ are apolar (so that the discriminants of $F_{\alpha,\beta} = \alpha F + \beta G$ and $F_{\alpha,-\beta}$ coincide) if and only if $b = c$ or $a + d = 0$. In the latter case the element of $PGL(2, K)$ is an involution. In either case we get a self-polar triple in the usual way. Interestingly, if $F$ and $G$ are distinct, then $b = c$ if and only if $p^2 + q^2$ lies on the line joining $F$ and the polar BDE, and $a + d = 0$ if and only if it lies on the line joining $G$ and the polar BDE. Finally, note that $F$ and $G$ are not arbitrary; indeed, with the usual conventions, they are precisely those pairs of forms conjugate with respect to the conic $(a - c)^2 + b^2 = 0$.

As noted above we are largely interested in the case of CBDE's.

**Proposition 6.3.** (1) *We get CBDE's above if and only if the matrix $\Gamma$ is of the form $(a, b, -\overline{b}, \overline{a})$, with both CBDE's of type I, and $(a, b, \overline{b}, -\overline{a})$ which makes them both of type II. We refer to these matrices as type I and II respectively. Again, multiplication by $\tau$ interchanges type, and we may restrict to matrices of type I from now on.*

(2) *Any CBDE of type I can be obtained from a matrix of type I, and this then determines a second CBDE of type I. If $f$ and $F_1$ and $f$ and $F_2$ both come from a matrix of type I, then $F_1$ differs from $F_2$ by a $K_s$-multiple of $p^2 + q^2$.*

(3) *A matrix of either form gives the same pencil as that obtained by replacing $(a, b)$ by $(\bar{b}, -\bar{a})$.*

(4) *Both sets of matrices (dropping the invertibility hypothesis) of the given form are k-convex sets. Indeed if $\alpha, \beta \in K_s$, and $\Gamma_1, \Gamma_2$ are both of type I (or II), then so is $\alpha\Gamma_1 + \beta\Gamma_2$.*

(5) *The matrices of type I form a subgroup of $PGL(2, K)$. When $k = \mathbb{R}$ the set of matrices in the form I but of determinant zero just consists of the zero matrix.*

(6) *The discriminants of the two BDE's of type I constructed above over $k = \mathbb{R}$ are given by $a\bar{a} = 0$, $b = \bar{b}$ and $b\bar{b} = 0$, $a = \bar{a}$.*

(7) *The BDE's $F$ and $G$ are apolar (so that the discriminants of $F_{\alpha,\beta} = \alpha F + \beta G$ and $F_{-\alpha,\beta}$ are coincident) if for matrices of type I $a + \bar{a} = 0$ or $b + \bar{b} = 0$.*

*Proof.* Generally, these are straightforward calculations. We do, however, prove part (5), that when $k = \mathbb{R}$ if $a\bar{a} + b\bar{b} = 0$, then $a = b = 0$. To see this, first clear the denominators so that we may suppose that $a$, $b$ lie in $\mathbb{R}[\kappa_1, \kappa_2]$. Write $a = \tau^m a_1$ and $b = \tau^n b_1$, with $\tau = (\kappa_1 - \kappa_2)$ not dividing $a_1$ or $b_1$ and suppose, without loss of generality, that $m \geq n$. Then $a\bar{a} + b\bar{b} = 0$ implies that $(-1)^{(m-n)}\tau^{2(m-n)}a_1\bar{a}_1 + b_1\bar{b}_1 = 0$. Since $\mathbb{R}[\kappa_1, \kappa_2]$ is a unique factorisation domain, $m = n$. Now evaluate at $(t, t)$ to obtain $(a_1^2 + b_1^2)(t, t) = 0$. Clearly, we deduce that $a_1(t, t) = b_1(t, t) = 0$, and given our hypotheses, $a_1 = b_1 = 0$. This is used in part (6) where each of the discriminants is an expression of the form $A\bar{A} + B\bar{B}$. $\square$

**Example 6.4.** We have already considered several cases of this type.

(1) The map $p \mapsto -\kappa_1/\kappa_2 p$ yields the first of our families.

(2) The map $p \mapsto \kappa_1/\kappa_2 p$ yields the second.

(3) There is no choice of $\Gamma$ that will yield a family joining (any multiple of) the asymptotic BDE to (any multiple of) the characteristic BDE. (Indeed, two BDE's of the form $ap^2 + cq^2$ arise as $F$ and $G$ above if and only if one of them is $p^2 + q^2$.)

(4) Consider the map $p \mapsto -p$. This yields the pair of forms $p^2 - q^2$ and $pq$. The associated family is simply that obtained from the principal BDE's by rotating through a fixed angle. The polar BDE to this pencil is $p^2 + q^2$ (which emphasises that the only zeros of this family are the umbilics). This suggests we consider the pencils joining $pq$ to $p^2 + q^2$ and $p^2 - q^2$ to $p^2 + q^2$. It is easy to see that neither can arise from this construction. In fact, it is clear that in the first case we are considering those BDE's with directions inclined at angles $\theta$ and $\pi/2 - \theta$ to either of the principle directions, $\theta$ parameterising the family, and in the second case the angles are $\theta$ and $-\theta$, again with $\theta$ parameterising the family.

(5) Consider the map $p \mapsto 1/p$. This actually gives the same family as $p \mapsto -p$, illustrating part (3) of Proposition 6.3.

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