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# STEIN'S METHOD AND PLANCHEREL MEASURE OF THE SYMMETRIC GROUP

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Abstract. We initiate a Stein's method approach to the study of the Plancherel measure of the symmetric group. A new proof of Kerov's central limit theorem for character ratios of random representations of the symmetric group on transpositions is obtained; the proof gives an error term. The construction of an exchangeable pair needed for applying Stein's method arises from the theory of harmonic functions on Bratelli diagrams. We also find the spectrum of the Markov chain on partitions underlying the construction of the exchangeable pair. This yields an intriguing method for studying the asymptotic decomposition of tensor powers of some representations of the symmetric group.

## 1. Introduction

It is accurate to state that one of the most important developments in probability theory in the last century was Stein's method. Stein's method is a highly original technique and has been useful in proving normal and Poisson approximation theorems in probability problems with limited information such as the knowledge of only a few moments of the random variable. Stein's method can be difficult to work with and often the bounds arising are not sharp, even in simple problems. But it sometimes is the only option available and has been a smashing success in Poisson approximation in computational biology. Good surveys of Stein's method (two of them books) are [ArGG], [BHJ], [Stn1], [Stn2].

Next let us recall the Plancherel measure of the symmetric group. This is a probability measure on the irreducible representations of the symmetric group which chooses a representation with probability proportional to the square of its dimension. Equivalently, the irreducible representations of the symmetric group are parameterized by partitions $\lambda$ of $n$, and the Plancherel measure chooses a partition $\lambda$ with probability $\frac{n!}{\prod_{x \in \lambda} h(x)^2}$ where the product is over boxes in the partition and $h(x)$ is the hooklength of a box. The hooklength of a box $x$ is defined as 1 + number of boxes in the same row as $x$ and to the right of $x$ + number of boxes in the same column of $x$ and below $x$. For example, we have filled in each box in the

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partition of 7 below with its hooklength

$$\begin{array}{cccc} \boxed{6} & \boxed{4} & \boxed{2} & \boxed{1} \\ \boxed{3} & \boxed{1} & & \\ \boxed{1} & & & \end{array} \quad ,$$

and the Plancherel measure would choose this partition with probability $\frac{7!}{(6*4*3*2)^2}$. Recently there has been significant interest in the statistical properties of partitions chosen from Plancherel measure. As it is beyond the scope of this paper to survey the topic, we refer the reader to the surveys [AlD], [De] and the seminal papers [J], [O], [BOO] for a glimpse of the remarkable recent work on Plancherel measure.

A purpose of the present paper is to begin the study of Plancherel measure by Stein's method. A long term goal of this program is to use Stein's method to understand the Baik-Deift-Johansson theorem, giving explicit bounds on the convergence of the first row of a Plancherel distributed partition to the Tracy-Widom distribution. We cannot at present do this but are confident that the exchangeable pair in this paper is the right one. We do attain a more modest goal of a Stein's method approach to the following result of Kerov.

**Theorem 1.1** ([K1])**.** *Let $\lambda$ be a partition of $n$ chosen from the Plancherel measure of the symmetric group $S_n$. Let $\chi^{\lambda}(12)$ be the irreducible character of the symmetric group parameterized by $\lambda$ evaluated on the transposition* (12)*. Let $dim(\lambda)$ be the dimension of the irreducible representation parameterized by $\lambda$. Then the random variable $\frac{n-1}{\sqrt{2}} \frac{\chi^{\lambda}(12)}{dim(\lambda)}$ is asymptotically normal with mean* 0 *and variance* 1.

Let us make some remarks about Theorem 1.1. The quantity $\frac{\chi^{\lambda}(12)}{dim(\lambda)}$ is called a character ratio and is crucial for analyzing the random walk on the symmetric group generated by transpositions [DSh]. In fact, Diaconis and Shahshahani prove that the eigenvalues for this random walk are the character ratios $\frac{\chi^{\lambda}(12)}{dim(\lambda)}$ each occurring with multiplicity $dim(\lambda)^2$. Hence Theorem 1.1 says that the spectrum of this random walk is asymptotically normal. Character ratios on transpositions also appear in work on the moduli space of curves [EO], [OP]. In fact, Kerov outlines a proof of the result. A full proof of the result appears in the marvelous paper [IO]. Another approach is due to Hora [H], who exploited the fact that the $k$th moment of a Plancherel distributed character ratio is the chance that the random walk generated by random transpositions is at the identity after $k$ steps (this follows from Lemma 3.4 below). Both of these proofs establish asymptotic normality by the method of moments and use combinatorial methods to estimate the moments. Note also that there is no error term in Theorem 1.1; in this paper one will be attained. Finally, we remark that Kerov (and then Hora) proves a much more general result—a multidimensional central limit theorem showing that character ratios evaluated on cycles of various lengths are asymptotically independent normal random variables.

In this paper we prove the following result. Here $P(\cdot)$ denotes the probability of an event.

**Theorem 1.2.** *For $n \geq 2$ and all real $x_0$,*

$$|P\left(\frac{n-1}{\sqrt{2}} \frac{\chi^{\lambda}(12)}{dim(\lambda)} \leq x_0\right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_0} e^{-\frac{x^2}{2}} dx| \leq 40.1 n^{-1/4}.$$

We conjecture that an upper bound of the form $Cn^{-1/2}$ holds (here $C$ is a constant). A follow-up paper [Fu2] proves an analog of Theorem 1.2 for Jack measure, an important deformation of Plancherel measure.

Theorem 1.2 will be a consequence of the following bound of Stein. Recall that if $W, W^*$ are random variables, they are called exchangeable if for all $w_1, w_2$, $P(W = w_1, W^* = w_2)$ is equal to $P(W = w_2, W^* = w_1)$. The notation $E^W(\cdot)$ means the expected value given $W$. Note from [Stn1] that there are minor variations on Theorem 1.3 (and thus for Theorem 1.2) for $h(W)$ where $h$ is a bounded continuous function with bounded piecewise continuous derivative. For simplicity we only state the result when $h$ is the indicator function of an interval.

**Theorem 1.3** ([Stn1])**.** *Let $(W, W^*)$ be an exchangeable pair of real random variables such that $E^W(W^*) = (1-\tau)W$ with $0 < \tau < 1$. Then for all real $x_0$,*

$$\begin{aligned} & |P(W \leq x_0) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_0} e^{-\frac{x^2}{2}} dx| \\ \leq \quad & 2\sqrt{E[1 - \frac{1}{2\tau} E^W (W^* - W)^2]^2} + (2\pi)^{-\frac{1}{4}} \sqrt{\frac{1}{\tau} E|W^* - W|^3}. \end{aligned}$$

In order to apply Theorem 1.3 to study a statistic $W$, one clearly needs an exchangeable pair $(W, W^*)$ such that $E^W(W^*) = (1-\tau)W$ (this second condition can sometimes be weakened in using Stein's method [RR]). Section 2 discusses the theory of harmonic functions on Bratelli diagrams and shows how it can be applied to generate a "natural" exchangeable pair $(W, W^*)$. The idea is to use a reversible Markov chain on the set of partitions of size $n$ whose stationary distribution is Plancherel measure, to let $\lambda^*$ be obtained from $\lambda$ by one step in the chain, and then set $(W, W^*) = (W(\lambda), W(\lambda^*))$. This construction also has the merit of being applicable to more general groups [Fu1] and to measures arising from symmetric functions [Fu2].

As we shall see in Section 4, we are quite fortunate in that when $W = \frac{n-1}{\sqrt{2}} \frac{\chi^\lambda(12)}{dim(\lambda)}$, it does happen that $E^W(W^*) = (1-\tau)W$ for some $\tau$ (in fact, $\tau = \frac{2}{n+1}$). There are some other simplifications which occur. For instance, we give a simple upper bound for $2\sqrt{E[1 - \frac{1}{2\tau} E^W(W^* - W)^2]^2}$, which is part of the error term and consistent with the $Cn^{-1/2}$ conjecture. To appreciate the beauty of Stein's method, we note that there is only one point in the proof of Theorem 1.2 where we even use an explicit formula for $W$, and this is in bounding $E|W^* - W|^3$. It would be interesting to study character ratios on classes other than transpositions by Stein's method; this leads to difficult questions about multiplication in the class algebra of the symmetric group.

Section 5 finds the eigenvalues and eigenvectors for the Markov chain underlying the construction of the exchangeable pair in Section 2. This leads to a curious method for studying the decomposition of tensor products in the symmetric group. For example, let $V$ be the standard $n$-dimensional representation of the symmetric group $S_n$. We deduce that for $r$ sufficiently large (roughly $\frac{n^2 log(n)}{4}$), that for a Plancherel distributed $\lambda$, the multiplicity of the irreducible representation of type $\lambda$ of $S_n$ in the $r$-fold tensor product $V \otimes \cdots \otimes V$ is very close to $\frac{dim(\lambda) n^r}{n!}$. A follow-up paper [Fu1] stengthens this using card shuffling to show that $r$ roughly $nlog(n)$ is sufficient and that $r$ of order $\frac{nlog(n)}{2}$ is necessary.

The precise organization of this paper is as follows. Section 2 uses the theory of harmonic functions to construct an exchangeable pair $(W, W^*)$. Section 3 collects some lemmas we shall need from representation theory. We have done this for two reasons: first, to make the paper more readable by probabilists who work on Stein's method, and second, because there are a few new results and some non-standard facts. Section 4 puts the pieces together and proves Theorem 1.2. Section 5 finds the eigenvalues and eigenvectors for the Markov chain underlying the construction of the exchangeable pair in Section 2, and applies it to obtain an asymptotic result about the decomposition of tensor products in the symmetric group.

## 2. Harmonic functions and exchangeable pairs

To begin we recall the theory of harmonic functions on Bratelli diagrams. This is a beautiful subject with deep connections to probability theory and representation theory. Two excellent surveys are [K2] and [BO].

The basic set-up is as follows. One starts with a Bratteli diagram; that is an oriented graded graph $\Gamma = \bigcup_{n \geq 0} \Gamma_n$ such that:

(1) $\Gamma_0$ is a single vertex $\emptyset$.
(2) If the starting vertex of an edge is in $\Gamma_i$, then its end vertex is in $\Gamma_{i+1}$.
(3) Every vertex has at least one outgoing edge.
(4) All $\Gamma_i$ are finite.

For two vertices $\lambda, \Lambda \in \Gamma$, one writes $\lambda \nearrow \Lambda$ if there is an edge from $\lambda$ to $\Lambda$. Part of the underlying data is a multiplicity function $\kappa(\lambda, \Lambda)$. Letting the weight of a path in $\Gamma$ be the product of the multiplicities of its edges, one defines the dimension $dim(\Lambda)$ of a vertex $\Lambda$ to be the sum of the weights over all maximal length paths from $\emptyset$ to $\Lambda$; $dim(\emptyset)$ is taken to be 1. Given a Bratteli diagram with a multiplicity function, one calls a function $\phi$ *harmonic* if $\phi(\emptyset) = 1$, $\phi(\lambda) \geq 0$ for all $\lambda \in \Gamma$, and

$$\phi(\lambda) = \sum_{\Lambda : \lambda \nearrow \Lambda} \kappa(\lambda, \Lambda)\phi(\Lambda).$$

An equivalent concept is that of coherent probability distributions. Namely a set $\{M_n\}$ of probability distributions $M_n$ on $\Gamma_n$ is called *coherent* if

$$M_n(\lambda) = \sum_{\Lambda : \lambda \nearrow \Lambda} \frac{dim(\lambda)\kappa(\lambda, \Lambda)}{dim(\Lambda)} M_{n+1}(\Lambda).$$

The formula allowing one to move between the definitions is $\phi(\lambda) = \frac{M_n(\lambda)}{dim(\lambda)}$.

One reason the set-up is interesting from the viewpoint of probability theory is the fact that typically every harmonic function can be written as a Poisson integral over the set of extreme harmonic functions (i.e. the boundary). For the Pascal lattice (vertices of $\Gamma_n$ are pairs $(k, n)$ with $k = 0, 1, \cdots, n$ and $(k, n)$ is connected to $(k, n+1)$ and $(k+1, n+1)$), this fact is the simplest instance of de Finetti's theorem, which says that an infinite exchangeable sequence of $0 - 1$ random variables is a mixture of coin toss sequences for different probabilities of heads. One of Kerov's insights in [K2] is that one can prove Selberg type integral formulas by expressing interesting harmonic functions as integrals over the boundary. The paper [BO] carries out this program in many cases.

Before proceeding, let us indicate that Plancherel measure is a special case of this set-up. Here the lattice which one uses is the Young lattice, that is $\Gamma_n$ consists of all partitions of size $n$, and a partition of size $n$ is adjoined to a partition of size

$n+1$ in $\Gamma_{n+1}$ if the partition of size $n+1$ can be obtained from the partition of size $n$ by adding a box. The multiplicity function $\kappa(\lambda,\Lambda)$ is equal to 1 on each edge. The dimension function $dim(\lambda)$ is simply $\frac{n!}{\prod_x h(x)}$, the dimension of the irreducible representation of the symmetric group parameterized by the partition $\lambda$. Then one can show that if $M_n$ is the Plancherel measure, the family of measures $\{M_n\}$ is coherent [K2]. More generally it follows from Frobenius reciprocity that if $H$ is a subgroup of $G$, $M_H$ is Plancherel measure of $H$ and $M_G$ is Plancherel measure of $G$, and $\kappa(\lambda,\Lambda)$ is the multiplicity of $\lambda$ in the restriction of $\Lambda$ from $G$ to $H$, then

$$M_H(\lambda) = \sum_{\Lambda:\lambda\nearrow\Lambda} \frac{dim(\lambda)\kappa(\lambda,\Lambda)}{dim(\Lambda)} M_G(\Lambda).$$

The relevance of these considerations is Proposition 2.1. Parts 1 and 2 are implicit in [K2]. Recall that a Markov chain $J$ on a finite set $X$ is said to be reversible with respect to $\pi$ if $\pi(x)J(x,y) = \pi(y)J(y,x)$ for all $x,y$. It is easy to see that if $J$ is reversible with respect to $\pi$, then $\pi$ is stationary for $J$ (i.e. that $\pi(y) = \sum_{x\in X} \pi(x)J(x,y)$). Note that if $W$ is any statistic on $X$ and $W^*$ is obtained by evaluating the statistic after taking a step according to a Markov chain which is reversible with respect to $\pi$, then $(W,W^*)$ is an exchangeable pair under the probability measure $\pi$.

**Proposition 2.1.** *Suppose that the family of measures $\{M_n\}$ is coherent for the Bratelli diagram.*

(1) *If $\lambda$ is chosen from the measure $M_n$, and one moves from $\lambda$ to $\Lambda$ with probability $\frac{dim(\lambda)M_{n+1}(\Lambda)\kappa(\lambda,\Lambda)}{dim(\Lambda)M_n(\lambda)}$, then $\Lambda$ is distributed according to the measure $M_{n+1}$.*
(2) *If $\Lambda$ is chosen from the measure $M_{n+1}$, and one moves from $\Lambda$ to $\mu$ with probability $\frac{dim(\mu)\kappa(\mu,\Lambda)}{dim(\Lambda)}$, then $\mu$ is distributed according to the measure $M_n$.*
(3) *The Markov chain $J$ on vertices in level $\Gamma_n$ of the Bratelli diagram given by moving from $\lambda$ to $\mu$ with probability*

$$J(\lambda,\mu) = \frac{dim(\lambda)dim(\mu)}{M_n(\lambda)} \sum_{\Lambda\in\Gamma_{n+1}} \frac{M_{n+1}(\Lambda)\kappa(\lambda,\Lambda)\kappa(\mu,\Lambda)}{dim(\Lambda)^2}$$

*is reversible with stationary distribution $M_n$.*
(4) *The Markov chain $J$ on vertices in level $\Gamma_n$ of the Bratelli diagram given by moving from $\lambda$ to $\mu$ with probability*

$$J(\lambda,\mu) = \frac{M_n(\mu)}{dim(\lambda)dim(\mu)} \sum_{\tau\in\Gamma_{n-1}} \frac{dim(\tau)^2\kappa(\tau,\lambda)\kappa(\tau,\mu)}{M_{n-1}(\tau)}$$

*is reversible with stationary distribution $M_n$.*

*Proof.* For part (1), observe that

$$\begin{aligned} & \sum_{\lambda\in\Gamma_n} M_n(\lambda)\frac{dim(\lambda)M_{n+1}(\Lambda)\kappa(\lambda,\Lambda)}{dim(\Lambda)M_n(\lambda)} \\ = & \frac{M_{n+1}(\Lambda)}{dim(\Lambda)} \sum_{\lambda\in\Gamma_n} dim(\lambda)\kappa(\lambda,\Lambda) \\ = & M_{n+1}(\Lambda). \end{aligned}$$

Note that the transition probabilities sum to 1 because the measures $\{M_n\}$ are coherent. Part (2) is similar and also uses the fact that $\{M_n\}$ is coherent. For part (3) reversibility is immediate from the definitions, provided that the transition probabilities for $J$ sum to 1. But $J$ is simply what one gets by moving up one level in the Bratelli diagram according to the transition mechanism of part (1), and then moving down according to the transition mechanism of part (2). Part (4) is similar; one first moves down the Bratelli diagram and then up. $\square$

One can investigate more general Markov chains where one moves up or down by an amount $k$. This is done in Section 5 where it is applied to tensor products. However, in the application of Stein's method, we will only use the chain $J$ in part (3) of the proposition (the chain in part (4) would work as well). This chain simplifies in many cases of interest. We mention two of them.

**Example 1.** The first example is the Plancherel measure of $S_n$. We already indicated how it fits in with harmonic functions on the Young lattice. Let $parents(\lambda, \mu)$ denote the set of partitions above both $\lambda, \mu$ in the Young lattice (this set has size 0 or 1 unless $\lambda = \mu$). Then $J(\lambda, \mu) = \frac{dim(\mu)|parents(\lambda,\mu)|}{(n+1)dim(\lambda)}$. Lemma 3.6 in the next section will use representation theory to derive a more complicated (but useful) expression for $J(\lambda, \mu)$ as a sum over the symmetric group $S_n$.

**Example 2.** The second example concerns cycles of random permutations. As explained in [K2], the partitions of $n$ index the conjugacy classes of the symmetric group and the probability measure on partitions corresponding to the conjugacy class of a random permutation is a coherent family of measures with respect to the Kingman lattice. Here the underlying lattice is the same as the Young lattice, but the multiplicity function $\kappa(\lambda, \Lambda)$ is the number of rows of length $j$ in $\Lambda$, where $\lambda$ is obtained from $\Lambda$ by removing a box from a row of length $j$. Also $dim(\lambda) = \frac{n!}{\lambda_1! \cdots \lambda_l!}$ where $l$ is the number of rows of $\lambda$ and $\lambda_i$ is the length of row $i$ of $\lambda$. So the Markov chain $J$ amounts to the following. Letting $m_r(\lambda)$ denote the number of rows of length $r$ in $\lambda$, first obtain $\Lambda$ by adding a box to a row of length $r$ with chance $\frac{rm_r(\lambda)}{n+1}$, or to an empty row with probability $\frac{1}{n+1}$. Then remove a box from a row of $\Lambda$ of length $s$ with probability $\frac{sm_s(\Lambda)}{n+1}$. The cycle structure of random permutations (and more generally the Ewens sampling formula) is very well understood, but still we revisit them and study the spectral properties of the Markov chains of Proposition 2.1 in future work.

## 3. Irreducible characters of the symmetric group

This section collects properties we will use about characters of irreducible representations of the symmetric group. For more background on this topic, see the book [Sa]. Lemmas 3.1, 3.2, and 3.3 are well known. Lemma 3.4 is known but not well known. Lemmas 3.5, 3.6 are elementary consequences of known facts but perhaps new.

Throughout $\overline{z}$ denotes the complex conjugate of $z$, and $Irr(G)$ denotes the set of characters of irreducible representations of a finite group $G$. The notation $dim(\chi)$ denotes the dimension of the representation with character $\chi$.

**Lemma 3.1.** *Let $C(g)$ be the conjugacy class of $G$ containing the element $g$. Then for $g \in G$,*

$$\sum_{\chi \in Irr(G)} \chi(g)\overline{\chi(h)}$$

*is equal to $\frac{|G|}{|C(g)|}$ if $h, g$ are conjugate and is 0 otherwise.*

**Lemma 3.2.** *Let $\nu$ be an irreducible character of a finite group $G$, and $\chi$ any character of $G$. Then the multiplicity of $\nu$ in $\chi$ is equal to*

$$\frac{1}{|G|} \sum_{g \in G} \nu(g)\overline{\chi(g)}.$$

**Lemma 3.3.** *The irreducible characters of the symmetric group $S_n$ are real valued.*

Note that Lemma 3.4 generalizes Lemma 3.1.

**Lemma 3.4** ([Sta], Exercise 7.67)**.** *Let $G$ be a finite group with conjugacy classes $C_1, \cdots, C_r$. Let $C_k$ be the conjugacy class of an element $w \in G$. Then the number of $m$-tuples $(g_1, \cdots, g_m) \in G^m$ such that $g_j \in C_{i_j}$ and $g_1 \cdots g_m = w$ is*

$$\frac{\prod_{j=1}^m |C_{i_j}|}{|G|} \sum_{\chi \in Irr(G)} \frac{1}{dim(\chi)^{m-1}} \chi(C_{i_1}) \cdots \chi(C_{i_m})\overline{\chi(C_k)}.$$

For the application of Stein's method we shall only need the case $k = 1$ of Lemmas 3.5 and 3.6.

**Lemma 3.5.** *Let $\chi$ be a character of the symmetric group $S_n$. Let $n_i(g)$ denote the number of cycles of $g$ of length $i$. Let $Res, Ind$ denote the operations of restriction and induction. Then for $k \geq 1$,*

$$Res_{S_n}^{S_{n+k}}(Ind_{S_n}^{S_{n+k}}(\chi))[g] = \chi(g)(n_1(g)+1) \cdots (n_1(g)+k).$$
$$Ind_{S_{n-k}}^{S_n}(Res_{S_{n-k}}^{S_n}(\chi))[g] = \chi(g)(n_1(g)) \cdots (n_1(g)-k+1).$$

*Proof.* Let us prove the first assertion, the second being similar. It is well known ([Sa]) that if $H$ is a subgroup of a finite group $G$, and $\chi$ is a character of $H$, then

$$Ind_H^G(\chi)[g] = \frac{1}{|H|} \sum_{\substack{t \in G \\ t^{-1}gt \in H}} \chi(t^{-1}gt).$$

Now apply this to $H = S_n$, $G = S_{n+k}$, using the fact that if two elements of $S_n$ are conjugate by an element of $S_{n+k}$, they are conjugate in $S_n$. Letting $Cent_G(g)$ denote the centralizer size of an element $g$ in a group $G$, it follows that the induced character at $g \in S_n$ is equal to

$$\chi(g)\frac{|Cent_{S_{n+k}}(g)|}{|Cent_{S_n}(g)|} = \chi(g)(n_1(g)+1) \cdots (n_1(g)+k).$$

Here we have used the fact that $\prod_j j^{n_j} n_j!$ is the centralizer size in $S_{\sum_j jn_j}$ of an element with $n_j$ cycles of length $j$. $\square$

In Lemma 3.6 we use the notation that $dim(\mu/\tau)$ is the number of paths in the Young lattice from $\tau$ to $\mu$ (or equivalently the number of ways of adding boxes one at a time to get from $\tau$ to $\mu$). We let $|\lambda|$ denote the size of a partition.

**Lemma 3.6.** *Let $\mu, \lambda$ be partitions of $n$. Let $n_1(g)$ denote the number of fixed points of a permutation $g$. Then*

$$\sum_{\substack{\tau \\ |\tau|=n+k}} dim(\tau/\lambda)dim(\tau/\mu) = \frac{1}{n!} \sum_{g \in S_n} \chi^{\mu}(g)\chi^{\lambda}(g)(n_1(g)+1)\cdots(n_1(g)+k),$$

$$\sum_{\substack{\tau \\ |\tau|=n-k}} dim(\lambda/\tau)dim(\mu/\tau) = \frac{1}{n!} \sum_{g \in S_n} \chi^{\mu}(g)\chi^{\lambda}(g)(n_1(g))\cdots(n_1(g)-k+1).$$

*Proof.* We prove only the first part as the second part is similar. Both sides of this equation enumerate the multiplicity of $\mu$ in the representation $Res_{S_n}^{S_{n+k}}(Ind_{S_n}^{S_{n+k}}(\lambda))$. This multiplicity is equal to the left-hand side by the branching rules for induction and restriction in the symmetric group (Theorem 2.8.3 in [Sa]). That the right-hand side computes the same multiplicity follows from Lemmas 3.2, 3.3, and 3.5. ☐

Note that as a special case of Lemma 3.6,

$$|parents(\mu, \lambda)| = \frac{1}{n!} \sum_{g \in S_n} \chi^{\mu}(g)\chi^{\lambda}(g)(n_1(g)+1).$$

## 4. Stein's method and Kerov's central limit theorem

In this section we prove Theorem 1.2. Recall that $W(\lambda) = \frac{(n-1)}{\sqrt{2}} \frac{\chi^{\lambda}(12)}{dim(\lambda)}$ and that the Plancherel measure chooses a partition $\lambda$ with probability $\frac{dim(\lambda)^2}{n!}$. If $n = 1$ we use the convention that $W = 0$. Then Lemma 3.4 implies the known fact that the mean and variance of $W$ are 0 and $1 - \frac{1}{n}$ respectively. This will also follow from Stein's method.

In fact, there is an explicit formula (due to Frobenius [Fr])

$$\frac{\chi^{\lambda}(12)}{dim(\lambda)} = \frac{1}{\binom{n}{2}} \sum_i \binom{\lambda_i}{2} - \binom{\lambda_i'}{2}$$

where $\lambda_i$ is the length of row $i$ of $\lambda$ and $\lambda_i'$ is the length of column $i$ of $\lambda$. We shall use this formula only once.

Throughout this section and the remainder of the paper, $W^*$ denotes $W(\lambda^*)$, where given $\lambda$, the partition $\lambda^*$ is $\mu$ with probability $J(\lambda, \mu)$ from Example 1 of Section 2. Recall that $(W, W^*)$ is an exchangeable pair. We also use the notation that $\chi^{\lambda}$ denotes the character of the irreducible representation of the symmetric group parameterized by the partition $\lambda$. We let $|\lambda|$ denote the size of a partition.

Proposition 4.1 shows that the hypothesis needed to apply the Stein method bound is satisfied. It also tells us that $W$ is an eigenvector for the Markov chain $J$, with eigenvalue $(1 - \frac{2}{n+1})$. We shall generalize this observation in Section 5.

**Proposition 4.1.** $E^W(W^*) = (1 - \frac{2}{n+1})W$.

*Proof.* For $n = 1$ we have that $W = 0$ by convention. Otherwise,

$$\begin{aligned} E^W(W^*) &= \sum_{|\mu|=n} \frac{n-1}{\sqrt{2}} \frac{\chi^{\mu}(12)}{dim(\mu)} \frac{dim(\mu)}{(n+1)dim(\lambda)} |parents(\mu, \lambda)| \\ &= \frac{n-1}{\sqrt{2}} \frac{1}{(n+1)dim(\lambda)} \sum_{|\mu|=n} |parents(\mu, \lambda)|\chi^{\mu}(12). \end{aligned}$$

From the representation theory of the symmetric group [Sa], $|parents(\mu,\lambda)|$ is equal to the multiplicity of $\chi^\mu$ in $Res^{S_{n+1}}_{S_n}(Ind^{S_{n+1}}_{S_n}(\chi^\lambda))$, since inducing to $S_{n+1}$ corresponds to the possible ways of adding a box to each corner of the partition $\lambda$, and restricting to $S_n$ corresponds to the possible ways of removing a corner box. Hence

$$E^W(W^*) = \frac{n-1}{\sqrt{2}} \frac{1}{(n+1)dim(\lambda)} Res^{S_{n+1}}_{S_n}(Ind^{S_{n+1}}_{S_n}(\lambda))[(12)].$$

Applying Lemma 3.5, this simplifies to

$$\frac{n-1}{\sqrt{2}} \frac{\chi^\lambda(12)}{(n+1)dim(\lambda)}(n-1) = (1-\frac{2}{n+1})W.$$

□

As a consequence of Proposition 4.1, we obtain a Stein's method proof that the mean $E(W)$ is equal to 0.

**Corollary 4.2.** $E(W) = 0$.

*Proof.* Since the pair $(W, W^*)$ is exchangeable, $E(W^* - W) = 0$. Using Proposition 4.1, we see that

$$E(W^* - W) = E(E^W(W^* - W)) = -\frac{2}{n+1}E(W).$$

Hence $E(W) = 0$. □

Next we shall use Stein's method to compute $E^\lambda(W^*)^2$. Recall that this notation means the expected value of $(W^*)^2$ given $\lambda$. This will be useful for analyzing the error term in Theorem 1.2.

**Proposition 4.3.**

$$\begin{aligned} E^\lambda((W^*)^2) &= (1-\frac{1}{n}) + \frac{2(n-1)(n-2)^2}{n(n+1)} \frac{\chi^\lambda((123))}{dim(\lambda)} \\ &\quad + \frac{(n-1)(n-2)(n-3)^2}{2n(n+1)} \frac{\chi^\lambda((12)(34))}{dim(\lambda)}. \end{aligned}$$

*Here we use the convention that if $n \leq 3$, $\chi^\lambda((12)(34)) = 0$, and if $n \leq 2$, $\chi^\lambda(123) =$ 0.*

*Proof.* If $n = 1$ the result is clear. Otherwise,

$$\begin{aligned} E^\lambda(W^*)^2 &= \frac{(n-1)^2}{2} \sum_{|\mu|=n} \frac{1}{n+1} |parents(\mu,\lambda)| \frac{dim(\mu)}{dim(\lambda)} \left(\frac{\chi^\mu(12)}{dim(\mu)}\right)^2 \\ &= \frac{(n-1)^2}{2(n+1)} \frac{1}{dim(\lambda)} \sum_{|\mu|=n} |parents(\mu,\lambda)| \frac{\chi^\mu(12)^2}{dim(\mu)}. \end{aligned}$$

Applying Lemma 3.6, this can be rewritten as

$$\begin{aligned} &\frac{(n-1)^2}{2(n+1)} \frac{1}{dim(\lambda)} \sum_{|\mu|=n} \frac{\chi^\mu(12)^2}{dim(\mu)} \frac{1}{n!} \sum_{g\in S_n} \chi^\mu(g)\chi^\lambda(g)(n_1(g)+1) \\ = &\frac{(n-1)^2}{2(n+1)} \frac{1}{dim(\lambda)} \sum_{g\in S_n} \chi^\lambda(g)(n_1(g)+1) \frac{1}{n!} \sum_{|\mu|=n} \frac{\chi^\mu(12)^2\chi^\mu(g)}{dim(\mu)}. \end{aligned}$$

The next step is to observe that using Lemmas 3.3 and 3.4, one can compute the expression $\frac{1}{n!}\sum_{|\mu|=n}\frac{\chi^{\mu}(12)^2\chi^{\mu}(g)}{dim(\mu)}$ for any permutation $g$. Indeed, it is simply $\frac{1}{\binom{n}{2}^2}$ multiplied by the number of ordered pairs $(\tau_1,\tau_2)$ of transpositions whose product is $g$. Thus when this expression is nonzero, there are 3 possibilities for the cycle type of $g$: the identity, a 3-cycle, or a product of two 2-cycles on disjoint symbols. In all cases it is elementary to enumerate the number of pairs $(\tau_1,\tau_2)$, and these 3 possibilities yield the 3 terms in the statement of the proposition. □

As a consequence of Lemma 4.3, we compute $Var(W)$.

**Corollary 4.4.** $Var(W)=(1-\frac{1}{n})$.

*Proof.* Since $W$ has mean 0, $Var(W)=E(W^2)$. Since $W$ and $W^*$ have the same distribution, it follows that

$$E(W^2)=E(W^*)^2=E(E^{\lambda}(W^*)^2).$$

The quantity $E^{\lambda}(W^*)^2$ was computed in Proposition 4.3 as a sum of three terms. The Plancherel measure average of the first term is $(1-\frac{1}{n})$. The Plancherel measure averages of the other terms both vanish. Indeed, if $g$ is any nonidentity element of the symmetric group, the Plancherel average of the function $\frac{\chi^{\lambda}(g)}{dim(\lambda)}$ is equal to

$$\frac{1}{n!}\sum_{\lambda} dim(\lambda)\chi^{\lambda}(g)=\frac{1}{n!}\sum_{\lambda}\chi^{\lambda}(1)\chi^{\lambda}(g),$$

which vanishes by Lemma 3.1. □

In order to prove Theorem 1.2, we have to analyze the error terms in Theorem 1.3. To begin note that since $W$ is determined by $\lambda$,

$$E\left(-1+\frac{n+1}{4}E^{W}(W^*-W)^2\right)^2$$

is upper bounded by the expression on the left-hand side of Proposition 4.5 (see Lemma 5 of [Fu3] for details).

**Proposition 4.5.**

$$E\left(-1+\frac{n+1}{4}E^{\lambda}(W^*-W)^2\right)^2=\frac{3n^2-5n+6}{4n^3}.$$

*Proof.* First observe (using Proposition 4.1 in the second equality) that

$$E^{\lambda}(W^*-W)^2=W^2-2WE^{\lambda}W^*+E^{\lambda}(W^*)^2=(\frac{4}{n+1}-1)W^2+E^{\lambda}(W^*)^2.$$

Combining this with Proposition 4.3, it follows that $-1+\frac{n+1}{4}E^{\lambda}(W^*-W)^2$ is equal to $A+B+C+D$ where

(1) $A=\left(\frac{n+1}{4}(1-\frac{1}{n})-1\right)$,
(2) $B=\frac{(n-1)(n-2)(n-3)^2}{8n}\frac{\chi^{\lambda}((12)(34))}{dim(\lambda)}$,
(3) $C=\frac{(n-1)(n-2)^2}{2n}\frac{\chi^{\lambda}(123)}{dim(\lambda)}$,
(4) $D=-\frac{n+1}{4}(1-\frac{4}{n+1})\frac{(n-1)^2}{2}\left(\frac{\chi^{\lambda}(12)}{dim(\lambda)}\right)^2$.

We need to compute the Plancherel measure average of $(A+B+C+D)^2$. Since $A^2$ is constant, the average of $A^2$ is $\left(\frac{n+1}{4}(1-\frac{1}{n})-1\right)^2$. The Plancherel averages of $B^2$,$C^2$ can both be computed using Lemma 3.1. One gets $\frac{(n-1)(n-2)(n-3)^3}{8n^3}$ and $\frac{3(n-1)(n-2)^3}{4n^3}$ respectively. To compute the Plancherel average of $D^2$ one uses Lemma 3.4 to reduce the computation to counting the number of ordered triples $(\tau_1, \tau_2, \tau_3)$ of transpositions whose product is a transposition, which is easy to do. The Plancherel averages of $2AB$, $2AC$, $2BC$ are all 0 by Lemma 3.1. The Plancherel average of $2AD$ is computed by Lemma 3.1. The Plancherel average of $2BD$ is reduced to counting the number of ordered pairs of transpositions whose product consists of two disjoint 2 cycles by Lemma 3.4, and similarly the average of $2CD$ is reduced to counting the number of ordered pairs of transpositions whose product is a 3 cycle. Thus all of the enumerations are elementary and adding up the terms yields the proposition. □

The final ingredient needed to prove Theorem 1.2 is an upper bound on $E|W^*-W|^3$. Note that by Jensen's inequality this is at least

$$(E(W^*-W)^2)^{3/2} = \left(\frac{4(1-\frac{1}{n})}{n+1}\right)^{3/2}.$$

Hence the bound of Proposition 4.6 is sharp up to constants.

**Proposition 4.6.**

$$E|W^*-W|^3 \leq \left(\frac{4e\sqrt{2}}{\sqrt{n}}\right)^3 + 2e^{-2e\sqrt{n}}(2\sqrt{2})^3.$$

*Proof.* From Frobenius' formula,

$$W = \frac{\sqrt{2}}{n}\sum_i \binom{\lambda_i}{2} - \binom{\lambda_i'}{2}.$$

Given this and the way that $\lambda^*$ is constructed from $\lambda$, it follows that

$$|W^*-W| \leq \frac{\sqrt{2}}{n} 2max(\lambda_1, \lambda_1').$$

Indeed, suppose that $\lambda^*$ is obtained from $\lambda$ by moving a box from row $a$ and column $b$ to row $c$ and column $d$. Then

$$W^*-W = \frac{\sqrt{2}}{n}\left(\lambda_c + \lambda_b' - \lambda_a - \lambda_d'\right).$$

Suppose that $\lambda_1$ (the size of the first row of $\lambda$) and $\lambda_1'$ (the size of the first column of $\lambda$) are both at most $2e\sqrt{n}$. Then by the previous paragraph

$$|W^*-W| \leq \frac{4e\sqrt{2}}{\sqrt{n}}.$$

Next note by the first paragraph, even if $\lambda_1 > 2e\sqrt{n}$ or $\lambda_1' > 2e\sqrt{n}$ occurs, $|W^*-W| \leq 2\sqrt{2}$. We claim that the chance that at least one of the events $\lambda_1 > 2e\sqrt{n}$ or $\lambda_1' > 2e\sqrt{n}$ occurs is at most $2e^{-2e\sqrt{n}}$. Indeed, it is a simple lemma proved on page 7 of [Ste] that the chance that the longest increasing subsequence of a random permutation is at least $2e\sqrt{n}$ is at most $e^{-2e\sqrt{n}}$. But it follows from the Robinson-Schensted-Knuth correspondence (see for instance [Sa]) that when $\lambda$

is chosen from the Plancherel measure of the symmetric group on $n$ symbols, both $\lambda_1$ and $\lambda_1'$ have the same distribution as the longest increasing subsequence of a random permutation on $n$ symbols. So as claimed, the chance that at least one of the events $\lambda_1 > 2e\sqrt{n}$ or $\lambda_1' > 2e\sqrt{n}$ occurs is at most $2e^{-2e\sqrt{n}}$.

Combining these observations proves the proposition. □

To close this section, we prove Theorem 1.2.

*Proof.* We use Theorem 1.3, which is applicable with $\tau = \frac{2}{n+1}$ by Propositions 2.1 and 4.1. Since $n \geq 2$, Proposition 4.5 implies that the term

$$2\sqrt{E[1 - \frac{1}{2\tau}E^W(W^* - W)^2]^2}$$

in Stein's bound is at most $\sqrt{3}n^{-1/2} \leq \sqrt{3}n^{-1/4}$. So using Proposition 4.6 and the fact that $e^{-2e\sqrt{n}} \leq n^{-3/2}$, the bound of Theorem 1.3 becomes

$$\sqrt{3}n^{-1/4} + (2\pi)^{-1/4}\sqrt{\frac{n+1}{2}\left( (\frac{4e\sqrt{2}}{\sqrt{n}})^3 + \frac{2(2\sqrt{2})^3}{n^{3/2}} \right)} \leq 40.1n^{-1/4}.$$

□

## 5. Asymptotic multiplicities in tensor products of representations

This section gives an intruiging method for understanding some asymptotic aspects of the decomposition of tensor products of certain representations of the symmetric group. A recent paper which investigates this topic from the viewpoint of free probability theory is [Bi]. However, the methods and results are completely different from those presented here. The main new idea of this section is to use spectral theory of Markov chains. This is developed for more general groups and representations in [Fu1], [Fu2].

Throughout this section $X$ is the set of partitions of size $n$, endowed with Plancherel measure $\pi$. We consider the space of real-valued functions $\ell^2(\pi)$ with the norm

$$||f||_2 = \left( \sum_x |f(x)|^2 \pi(x) \right)^{1/2}.$$

If $J(x,y)$ is the transition rule for a Markov chain, the associated Markov operator (also denoted by $J$) on $\ell^2(\pi)$ is given by $Jf(x) = \sum_y J(x,y)f(y)$. Let $J^r(x,y) = J_x^r(y)$ denote the chance that the Markov chain started at $x$ is at $y$ after $r$ steps.

If the Markov chain with transition rule $J(x,y)$ is reversible with respect to $\pi$ (i.e. $\pi(x)J(x,y) = \pi(y)J(y,x)$ for all $x, y$), then the operator $J$ is self adjoint with real eigenvalues

$$-1 \leq \beta_{min} = \beta_{|X|-1} \leq \cdots \leq \beta_1 \leq \beta_0 = 1.$$

Let $\psi_i$ ($i = 0, \cdots, |X|-1$) be an orthonormal basis of eigenfunctions such that $J\psi_i = \beta_i \psi_i$ and $\psi_0 \equiv 1$. Define $\beta = max\{|\beta_{min}|, \beta_1\}$.

The total variation distance between the measures $J_x^r$ and $\pi$ is defined by $||J_x^r - \pi||_{TV} = \frac{1}{2}\sum_y |J_x^r(y) - \pi(y)|$. (The application to representation theory does not require total variation distance but this concept is useful for understanding the Markov chain $J$; see Theorem 5.3).

The following lemma is well known; for a proof see [DSa]. Part (1) is essentially Jensen's inequality.

**Lemma 5.1.** (1) $2||J_x^r - \pi||_{TV} \leq ||\frac{J_x^r}{\pi} - 1||_2$.
(2) $J^r(x,y) = \sum_{i=0}^{|X|-1} \beta_i^r \psi_i(x)\psi_i(y)\pi(y)$.
(3) $||\frac{J_x^r}{\pi} - 1||_2^2 = \sum_{i=1}^{|X|-1} \beta_i^{2r} |\psi_i(x)|^2 \leq \frac{1-\pi(x)}{\pi(x)} \beta^{2r}$.

Next we specify a class of Markov chains to which we will apply Lemma 5.1. Recall from Proposition 2.1 that there is a natural transition mechanism for moving up one step in the Young lattice or down one step in the Young lattice. For $1 \leq k \leq n$, let $J(k)$ be the Markov chain which from a partition of $n$ first moves down $k$ steps in the Young lattice (one at a time according to part (2) of Proposition 2.1), and then moves back up $k$ steps in the Young lattice (one at a time according to part (1) of Proposition 2.1). Proposition 5.2 finds the eigenvalues and eigenfunctions for the corresponding operators $J(k)$. The word Mult. stands for multiplicity.

**Proposition 5.2.** *The eigenvalues and eigenfunctions of the operator $J(k)$ on partitions of size $n$ are indexed by conjugacy classes $C$ of the symmetric group on $n$ symbols.*

(1) *Letting $n_1(C)$ denote the number of fixed points of the class $C$, the eigenvalue parameterized by $C$ is $\frac{(n_1(C))(n_1(C)-1)\cdots(n_1(C)-k+1)}{n(n-1)\cdots(n-k+1)}$.*
(2) *An orthonormal basis of eigenfunctions $\psi_C$ is defined by $\psi_C(\lambda) = |C|^{\frac{1}{2}} \frac{\chi^\lambda(C)}{dim(\lambda)}$.*

*Proof.* From Proposition 2.1, the chance that the Markov chain $J(k)$ moves from $\lambda$ to $\mu$ is easily seen to be

$$\frac{dim(\mu)}{(n)(n-1)\cdots(n-k+1)dim(\lambda)} \sum_{\substack{\tau \\ |\tau|=n-k}} dim(\lambda/\tau)dim(\mu/\tau).$$

From the branching rules of irreducible representations of the symmetric group [Sa], this is

$$\frac{dim(\mu)}{dim(\lambda)} \frac{Mult.\ \mu\ in\ Ind_{S_{n-k}}^{S_n} Res_{S_{n-k}}^{S_n}(\lambda)}{n(n-1)\cdots(n-k+1)}.$$

Now observe that the $\psi_C(\lambda) = |C|^{\frac{1}{2}} \frac{\chi^\lambda(C)}{dim(\lambda)}$ is an eigenfunction with the asserted eigenvalue because

$$\begin{aligned} & \frac{|C|^{\frac{1}{2}}}{(n)\cdots(n-k+1)dim(\lambda)} \sum_\mu \chi^\mu(C) \cdot Mult.\ \mu\ in\ Ind_{S_{n-k}}^{S_n} Res_{S_{n-k}}^{S_n}(\lambda) \\ = & \frac{|C|^{\frac{1}{2}}}{n(n-1)\cdots(n-k+1)dim(\lambda)} Ind_{S_{n-k}}^{S_n} Res_{S_{n-k}}^{S_n}(\lambda)[C] \\ = & \frac{(n_1(C))(n_1(C)-1)\cdots(n_1(C)-k+1)}{n(n-1)\cdots(n-k+1)} \frac{|C|^{\frac{1}{2}}\chi^\lambda(C)}{dim(\lambda)}. \end{aligned}$$

The last equality is Lemma 3.5. The fact that $\psi_C$ are orthonormal follows from Lemma 3.1. Being linearly independent, they are a basis for $\ell^2(\pi)$ since the number of conjugacy classes of $S_n$ is equal to the number of partitions of $n$. □

*Remarks.* (1) A similar argument shows that the chain which moves up $k$ steps (one at a time) and then down $k$ steps (one at a time) has the $\psi_C$ as an orthonormal basis with eigenvalues $\frac{(n_1(C)+1)(n_1(C)+2)\cdots(n_1(C)+k)}{(n+1)(n+2)\cdots(n+k)}$. The results in the remainder of this section could be applied to that Markov chain as well.

(2) It is remarkable that the set of eigenvalues of the Markov chain $J(1)$ is precisely the set of eigenvalues of the top to random shuffle (see [DFP] for background on top to random shuffles). This is not coincidence; the paper [Fu1] explains this and more general facts in terms of the Robinson-Schensted-Knuth correspondence.

Theorem 5.3 studies the convergence properties of the Markov chains $J(k)$. Throughout, all logs are taken base e, as usual.

**Theorem 5.3.** *Let $(n)$ denote the partition which consists of one part of size $n$ (corresponding to the trivial representation of $S_n$). Let $\beta = \frac{(n-k)(n-k-1)}{(n)(n-1)}$. Then for $n \geq 2, 1 \leq k < n$,*

$$2||J(k)^r_{(n)} - \pi||_{TV} \leq \left( \sum_{\substack{\lambda \\ |\lambda|=n}} |\frac{J(k)^r((n),\lambda)}{\pi(\lambda)} - 1|^2 \pi(\lambda) \right)^{1/2} \leq \sqrt{n!}\,(\beta)^r.$$

*Thus for $r > \frac{nlog(n)+2c}{2log(\frac{1}{\beta})}$,*

$$||J(k)^r_{(n)} - \pi||_{TV} \leq \frac{(2\pi)^{\frac{1}{4}}}{2} e^{-c}.$$

*Proof.* From parts (1) and (3) of Lemma 5.1, to prove the first assertion it is enough to show that for the chain $J(k)$, $\beta = \frac{(n-k)(n-k-1)}{(n)(n-1)}$ and that $\frac{1-\pi((n))}{\pi((n))} \leq n!$. The value of $\beta$ follows from Proposition 5.2 (note that a permutation on $n$ symbols cannot have exactly $n-1$ fixed points) and the inequality holds since $\pi((n)) = \frac{1}{n!}$. This proves the first assertion.

The second assertion follows from the Stirling formula bound [Fe]

$$n! \leq \sqrt{2\pi} e^{-n+\frac{1}{12n}+(n+\frac{1}{2})log(n)},$$

for then

$$\begin{aligned} \sqrt{n!}\,(\beta)^r &\leq (2\pi)^{\frac{1}{4}} e^{rlog(\beta)-\frac{n}{2}+\frac{1}{24n}+\frac{1}{2}(n+\frac{1}{2})log(n)} \\ &\leq (2\pi)^{\frac{1}{4}} e^{rlog(\beta)+\frac{nlog(n)}{2}}. \end{aligned}$$

□

Now we give the application to representation theory. We remind the reader that $Ind_{S_{n-1}}^{S_n}(1)$ is the defining representation of the symmetric group (i.e. the $n$-dimensional permutation representation on the symbols $1, \cdots, n$), and hence corresponds to the case $k = 1$ below. As usual, $\pi(\lambda) = \frac{dim(\lambda)^2}{n!}$ denotes the Plancherel measure of the symmetric group, and the word Mult. is short for multiplicity. We let $\otimes^r$ denote the operation of taking the $r$-fold tensor product of a representation of a symmetric group $S_n$ (yielding the sum of various irreducible representations of $S_n$).

**Theorem 5.4.** *Let $\beta = \frac{(n-k)(n-k-1)}{(n)(n-1)}$. Then for $n \geq 2, 1 \leq k < n$,*

$$\sum_{\substack{\lambda \\ |\lambda|=n}} |\frac{n![Mult.\ \lambda\ in\ \otimes^r Ind_{S_{n-k}}^{S_n}(1)]}{dim(\lambda)\left((n)\cdots(n-k+1)\right)^r} - 1|^2 \pi(\lambda) \leq n!\,(\beta)^{2r}.$$

*For $r > \frac{nlog(n)+c}{2log(\frac{1}{\beta})}$, this is at most $\sqrt{2\pi}e^{-c}$.*

*Proof.* Let $(n)$ be the partition which consists of one row of size $n$, and let $n_1(C)$ denote the number of fixed points of a conjugacy class $C$. We now consider the quantity $||\frac{J(k)^r_{(n)}}{\pi} - 1||_2^2$. On one hand, by part (2) of Lemma 5.1, we know that

$$\begin{aligned}
& J(k)^r_{(n)}(\lambda) \\
= \; & dim(\lambda) \sum_C \left( \frac{(n_1(C)) \cdots (n_1(C) - k + 1)}{n \cdots (n-k+1)} \right)^r \frac{|C| \chi^\lambda(C)}{n!} \\
= \; & \frac{dim(\lambda)}{(n \cdots (n-k+1))^r} \frac{1}{n!} \sum_{g \in S_n} (n_1(g) \cdots (n_1(g) - k + 1))^r \chi^\lambda(g).
\end{aligned}$$

By Lemmas 3.2 and 3.5, and the fact that the character of a tensor product is the product of the characters, this is precisely

$$\frac{dim(\lambda)}{(n(n-1)\cdots(n-k+1))^r} [Mult.\ \lambda\ in\ \otimes^r Ind_{S_{n-k}}^{S_n}(1)].$$

Thus

$$||\frac{J(k)^r_{(n)}}{\pi} - 1||_2^2 = \sum_{\substack{\lambda \\ |\lambda| = n}} |\frac{n![Mult.\ \lambda\ in\ \otimes^r Ind_{S_{n-k}}^{S_n}(1)]}{dim(\lambda)\,((n)\cdots(n-k+1))^r} - 1|^2 \pi(\lambda).$$

The result now follows by the upper bound on $||\frac{J(k)^r_{(n)}}{\pi} - 1||_2^2$ in Theorem 5.3. □

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