TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 357, Number 2, Pages 837–850
S 0002-9947(04)03549-4
Article electronically published on July 22, 2004

HARNACK INEQUALITIES FOR NON-LOCAL OPERATORS OF VARIABLE ORDER

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ABSTRACT. We consider harmonic functions with respect to the operator

$$\mathcal{L}u(x) = \int [u(x+h) - u(x) - 1_{(|h|\le 1)} h \cdot \nabla u(x)] n(x,h)\, dh.$$

Under suitable conditions on $n(x,h)$ we establish a Harnack inequality for functions that are nonnegative and harmonic in a domain. The operator $\mathcal{L}$ is allowed to be anisotropic and of variable order.

1. Introduction

There is a huge literature concerned with Harnack inequalities for functions that are harmonic with respect to second order elliptic operators. Seminal contributions in this field have been made among others by Moser [Mos61], Krylov-Safonov [KS80], and Fabes-Stroock [FS86]. The first and third of these papers deal with differential operators in divergence form, while the second deals with differential operators in non-divergence form. These papers, as well as alternate proofs of their results, all rely heavily on the fact that the operators are local operators, that is, differential operators.

At the same time, in the last few years there has been intense interest in using integral operators (or equivalently, processes with jumps) to model problems in mathematical physics, in finance, and in probability theory. These operators are non-local, in the sense that the behavior of a harmonic function at a point depends on values of the harmonic function at points some distance away rather than just at nearby points.

The purpose of this paper is to consider functions that are harmonic with respect to the integral operator $\mathcal{L}$, where

$$\mathcal{L}u(x) = \int_{\mathbb{R}^d\setminus\{0\}} [u(x+h) - u(x) - 1_{(|h|\le 1)} h \cdot \nabla u(x)] n(x,h) dh \tag{1.1}$$

operates on C^2 functions defined on $\mathbb{R}^d$. This is a reasonably general integro-differential operator, and includes, for example, many of the operators considered by probabilists. In probabilistic terms, $n(x,h)$ represents the relative intensity of the number of jumps of the associated Markov process from a point x to the point $x+h$. We examine what conditions are needed on $n(x,h)$ to guarantee that

Received by the editors May 27, 2003 and, in revised form, October 27, 2003.
2000 *Mathematics Subject Classification.* Primary 45K05; Secondary 60H10.
Key words and phrases. Harnack inequality, non-local operator, stable processes, Lévy processes, jump processes, integral operators.
The first author's research was partially supported by NSF grant DMS-9988496.

a Harnack inequality holds. We start with the assumption that for two positive constants κ_1 and κ_2

$$\frac{\kappa_1}{|h|^{d+\alpha}} \leq n(x,h) \leq \frac{\kappa_2}{|h|^{d+\beta}}, \qquad x \in \mathbb{R}^d, \quad |h| \leq 2, \tag{1.2}$$

where $0 < \alpha < \beta < 2$. This is the analogue of the coercivity and boundedness conditions from the theory of elliptic PDE. Note that the order of the singularity of the kernel with respect to h might depend on x. Moreover, the kernel might exhibit different singularities in different directions. Hence, the corresponding integro-differential operator $\mathcal{L}$ is anisotropic and of variable order. For now let us say that a function u is harmonic with respect to $\mathcal{L}$ in a domain D if $\mathcal{L}u = 0$ in D; a more precise definition is given in Section 2 in terms of martingales.

Our main result is that if $\beta - \alpha < 1$, then a Harnack inequality holds for non-negative functions that are harmonic in a domain; see Theorem 4.1 for a precise statement. We do not know if our condition $\beta - \alpha < 1$ is sharp. The conclusion of Theorem 4.1 says that $u(x) \leq \bar{\kappa}(R)u(y)$ for x, y in a ball of radius $R/2$ when u is harmonic in the concentric ball of radius R. In Proposition 5.1 we give an example to show that the dependence of $\bar{\kappa}$ on R cannot be dispensed with.

At the time of the writing of this paper, there are only a few papers that we know of that consider Harnack inequalities for non-local operators. In [BBG00] a very specific operator was considered; there the interest was not in the Harnack inequality but in a Liouville property for a certain degenerate PDE. In [BL02a] the operator $\mathcal{L}$ given in (1.1) was considered, but in the special case where $\alpha = \beta$, which is sometimes known as the stable-like case. The results of [BL02a] were extended to certain other Markov jump processes in [SV04]. A parabolic Harnack inequality for symmetric jump processes, again with $\alpha = \beta$, together with heat kernel estimates, was proved in [BL02b]. This was extended to more general state spaces in [CK03]. See [BSS02a] and [BSS02b] for related results. A weak Harnack inequality has been obtained in [Kas03] for non-local operators corresponding to jump-diffusions.

The current paper is a major generalization of the results obtained in [BL02a] and [SV04] in that we remove the requirement $\alpha = \beta$. We are able to allow the integro-differential operators to be anisotropic and of variable order.

The method starts with the ideas of [BL02a], but due to the fact that $\alpha \neq \beta$, the techniques are considerably more delicate. Both [BL02a] and the current paper use techniques substantially different from those used in the case of elliptic operators, although the roots of our method come from those of [KS80]. It is interesting that while in [KS80] the hardest part of the proof is obtaining what is essentially an estimate on the probability of hitting sets; here, by contrast, the corresponding estimate is fairly easy. The principal difficulty in this paper is using that estimate to obtain the Harnack inequality.

After a short section on preliminaries, in Section 3 we present some estimates for the Markov process associated with $\mathcal{L}$. These are used in Section 4 to prove the Harnack inequality. Section 5 contains some examples.

2. Preliminaries

We use $B(x, r)$ for the open ball of radius r with center x. The letter c with subscripts will denote positive finite constants whose exact value is unimportant. The Lebesgue measure of a Borel set A will be denoted by $|A|$.

We consider the operator

$$\mathcal{L}u(x) = \int_{h\neq 0} [u(x+h) - u(x) - 1_{(|h|\leq 1)} h \cdot \nabla u(x)] n(x,h)dh. \tag{2.1}$$

Suppose $0 < \alpha < \beta < 2$. We make the following assumptions on $n(x,h)$.

Assumption 2.1. *There exist positive finite constants $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ such that:*

- (a) *For all $|h| \leq 2$ and all x*
$$n(x,h) \geq \frac{\kappa_1}{|h|^{d+\alpha}}.$$
- (b) *For all $|h| \leq 2$ and all x*
$$n(x,h) \leq \frac{\kappa_2}{|h|^{d+\beta}}.$$
- (c) *For all x*
$$\int_{|h|>1} n(x,h)dh \leq \kappa_3.$$
- (d) *For all $x, y,$ and z*
$$n(x, z-x) \leq \kappa_4 n(y, z-y), \qquad |z-x| \geq 1, |z-y| \geq 1, |x-y| \leq 1.$$

Assumptions 2.1 (a)-(c) say that the Lévy kernel $n(x,h)dh$ is bounded between that of a symmetric stable process of index α and that of one of index β for the jumps of size less than 2. Moreover, we have a uniform bound on the number of jumps of size bigger than 1. $n(x,h)$ can be thought of as the intensity of the number of jumps from x to $x+h$; thus $n(x, z-x)$ represents the intensity of the number of jumps from x to z. Assumption 2.1 (d) says that the probability of jumping to a point z is comparable if x, y are more than distance one away from z and within distance one of each other. Note that the constant "one" could be replaced by another appropriate positive constant. In Proposition 5.2 we show that an assumption of this type cannot be avoided.

Our method is probabilistic, and we need to work with the Markov process associated with $\mathcal{L}$. We say a strong Markov process $(\mathbb{P}^x, X_t)$ is associated with $\mathcal{L}$ if for each x we have $\mathbb{P}^x(X_0 = x) = 1$ and for each x and for each $u \in C^2$ that is bounded with bounded first and second partial derivatives, $u(X_t) - u(X_0) - \int_0^t \mathcal{L}u(X_s)ds$ is a martingale under $\mathbb{P}^x$. This is commonly expressed as saying that $\mathbb{P}^x$ solves the martingale problem for $\mathcal{L}$ started at x.

Without some regularity on $n(x,h)$ we do not know that there is a strong Markov process associated with $\mathcal{L}$ or that if there is one, it is unique. One of the major open problems in the area of uniqueness is to formulate simple but not too restrictive sufficient conditions. If $n(x,h)$ depends on x in a Lipschitz fashion, it is known that uniqueness holds, see [Sko65]. Let us assume that $n(x,h)$ satisfies some conditions (see [Kom84, Bas88, Hoh95, Sko65]) which insure that there is one and only one solution to the martingale problem for $\mathcal{L}$ started at x. However, we underline that none of the constants in any of our results depend on the smoothness or regularity of $n(x,h)$.

An equivalent formulation of the connection between the Markov process and the operator $\mathcal{L}$ can be made in terms of a stochastic differential equation driven by a random measure, but this is less direct.

For any Borel set A, let

$$T_A = \inf\{t : X_t \in A\}, \qquad \tau_A = \inf\{t : X_t \notin A\},$$

the first hitting time and first exit time, respectively, of A. We say that a function u is harmonic in a domain D if $u(X_{t\wedge\tau_D})$ is a $\mathbb{P}^x$-martingale for each $x \in D$. It is easy to check that if u satisfies some smoothness conditions (e.g., u and its first and second partials are bounded and continuous in D) and $\mathcal{L}u = 0$ in D, then u is harmonic in D.

Similarly to the diffusion case explained in [SV79], Corollary 6.3.3, uniqueness of solutions to the martingale problem imply that the corresponding process is a Feller process. Therefore our Markov process is a Hunt process (see [BG68], Section 1.9) and in particular left hand limits exist. We write $X_{t-} = \lim_{s\uparrow t} X_s$ and $\Delta X_t = X_t - X_{t-}$. Any harmonic function u is excessive with respect to the semigroup of X_t and therefore $u(X_{t\wedge\tau_D})$ is right-continuous, with the exceptional set having $\mathbb{P}^x$-measure zero for all x; see [BG68], Theorem II.2.12.

3. Some estimates

Throughout this section we assume that Assumption 2.1 holds. Set

$$\overline{\beta} = \max(\beta, 1). \tag{3.1}$$

Proposition 3.1. *There exist constants c_1 and c_2 not depending on x_0 such that if $r < 1$, $\beta \neq 1$ and $t > 0$, then*

$$\mathbb{P}^{x_0}(\tau_{B(x_0,r)} \le c_1 t) \le t r^{-\overline{\beta}},$$

and in particular

$$\mathbb{P}^{x_0}(\tau_{B(x_0,r)} \le c_2 r^{\overline{\beta}}) \le \tfrac{1}{2}.$$

Proof. Let u be a nonnegative C^2 function that is equal to $|x-x_0|^2$ for $|x-x_0| \le r/2$, which equals r^2 for $|x-x_0| \ge r$, and such that u is bounded by $c_3 r^2$, its first partial derivatives are bounded by $c_3 r$, and its second partial derivatives are bounded by c_3. Then, since $\mathbb{P}^{x_0}$ solves the martingale problem,

$$\mathbb{E}^{x_0} u(X_{t\wedge\tau_{B(x_0,r)}}) - u(x_0) = \mathbb{E}^{x_0} \int_0^{t\wedge\tau_{B(x_0,r)}} \mathcal{L}u(X_s)ds. \tag{3.2}$$

We examine $\mathcal{L}u(x)$ for $x \in B(x_0,r)$. We break the integral in (2.1) into two parts, where $|h| \le r$ and where $|h| > r$. For the first part, we have

$$\int_{|h|\le r} [u(x+h) - u(x) - h\cdot\nabla u(x)] n(x,h) dh \le c_4 \int_{|h|\le r} h^2 n(x,h)\, dh,$$

since the expression inside the brackets is bounded by a constant times $h^2\|D^2u\|_\infty$. Since for $|h| \le r$ we have $n(x,h) \le c_5 h^{-d-\beta}$, we bound the above by $c_6 r^{2-\beta}$. For the second part we obtain, using Assumptions 2.1 (b) and (c),

$$\int_{|h|>r} [u(x+h) - u(x)] n(x,h)\, dh \le \|u\|_\infty \int_{|h|>r} n(x,h)\, dh \le c_7 r^{2-\beta},$$

and, using $\|\nabla u\|_\infty \le c_2 r$,

$$\Big|\int_{1>|h|\ge r} h\cdot\nabla u(x) n(x,h) dh\Big| \le c_8 r^{2-\overline{\beta}}.$$

Substituting in (3.2), we obtain

$$\mathbb{E}^{x_0} u(X_{t\wedge\tau_{B(x_0,r)}}) \le c_9 t r^{2-\overline{\beta}}.$$

Note that the left hand side is greater than $r^2\mathbb{P}^{x_0}(\tau_{B(x_0,r)} \le t)$, which yields the first part of the proposition. If we now take $t = c_{10} r^{\overline{\beta}}$, we obtain the second part of the proposition. □

Proposition 3.2. *If A and B are disjoint Borel sets, then for each x*

$$\sum_{s\le t} 1_{(X_{s-}\in A, X_s\in B)} - \int_0^t \int_B 1_A(X_s) n(X_s, u - X_s) du\, ds$$

is a $\mathbb{P}^x$-martingale.

The proof is identical to that of Proposition 2.3 and Remark 2.4 of [BL02a].

The next proposition estimates the hitting probability for certain sets. It is notable that, unavoidably, the conclusion is considerably weaker than that of Theorem 1 in [KS79]; namely, the hitting probability is not bounded away from zero. Despite this difference from the non-degenerate diffusion case, we are able to prove a Harnack inequality.

Proposition 3.3. *Suppose $r < 1$ and $\beta \ne 1$.*

- (a) *There exists c_1 such that if $A \subset B(x_0, r/2)$ and also $y \in B(x_0, r/2)$, then*

$$\mathbb{P}^y(T_A < \tau_{B(x_0,r)}) \ge c_1 r^{\overline{\beta}-\alpha} |A|/|B(x_0, r)|.$$

- (b) *There exists c_1 such that if $A \subset B(x_0, r/2)$ and also $y \in B(x_0, r)$, then*

$$\mathbb{P}^y(T_A < \tau_{B(x_0,r)}) \ge c_1 [\mathrm{dist}\,(y, \partial B(x_0, r))]^{\overline{\beta}} r^{-\alpha} |A|/|B(x_0, r)|.$$

Proof. (a) is an immediate consequence of (b), so we prove (b). Fix y and write τ for $\tau_{B(x_0,r)}$. Let $p = \mathrm{dist}\,(y, \partial B(x_0, r))$. If $\mathbb{P}^y(T_A < \tau) \ge \frac{1}{4}$, we are done, so we assume not. By Proposition 3.1 we can find a constant c_2 such that if $t_0 = c_2 p^{\overline{\beta}}$, then $\mathbb{P}^y(\tau \le t_0) \le \frac{1}{2}$. If $x \in B(x_0, r)$ and $z \in A$, then $|z - x| \le 2r$ and

$$n(x, z - x) \ge c_3 |z - x|^{-d-\alpha} \ge c_4 r^{-d-\alpha}.$$

Then by Proposition 3.2 and optional stopping,

$$\begin{aligned}\mathbb{P}^y(T_A < \tau) &\ge \mathbb{E}^y \sum_{s\le T_A\wedge\tau\wedge t_0} 1_{(X_{s-}\ne X_s, X_s\in A)} \\ &= \mathbb{E}^y \int_0^{T_A\wedge\tau\wedge t_0} \int_A n(X_s, z - X_s) dz\, ds \\ &\ge c_4 |A| r^{-d-\alpha} \mathbb{E}^y (T_A \wedge \tau \wedge t_0).\end{aligned}$$

We also have

$$\begin{aligned}\mathbb{E}^y(T_A \wedge \tau \wedge t_0) &\ge \mathbb{E}^y(t_0; T_A \ge \tau \ge t_0) = t_0 \mathbb{P}^y(T_A \ge \tau \ge t_0) \\ &\ge t_0 \left[1 - \mathbb{P}^y(T_A < \tau) - \mathbb{P}^y(\tau < t_0)\right] \ge t_0/4.\end{aligned}$$

Therefore

$$\mathbb{P}^y(T_A < \tau) \ge \frac{c_4}{4} |A| r^{-d-\alpha} t_0 = c_5 p^{\overline{\beta}} r^{-\alpha} |A|/|B(x_0, r)|.$$

□

Lemma 3.4. *There exist c_1 and c_2 such that if $r < 1/2$ and $\beta \neq 1$, then*

$$\mathbb{E}^x \tau_{B(x,r)} \geq c_1 r^{\overline{\beta}}, \qquad \mathbb{E}^x \tau_{B(x,r)} \leq c_2 r^{\alpha}.$$

Proof. By Proposition 3.1 there exists c_3 such that $\mathbb{P}^x(\tau_{B(x,r)} \leq c_3 r^{\overline{\beta}}) \leq \frac{1}{2}$. So $\tau_{B(x,r)}$ is greater than $c_3 r^{\overline{\beta}}$ with probability at least $\frac{1}{2}$, and the first inequality follows easily from this.

To prove the second inequality, let S be the time of the first jump larger than $2r$. Suppose $\mathbb{P}^z(S \leq r^\alpha) \leq \frac{1}{2}$. Then by Proposition 3.2 and optional stopping,

$$\begin{aligned}
\mathbb{P}^z(S \leq r^\alpha) &= \mathbb{E}^z \sum_{s \leq S \wedge r^\alpha} 1_{(|X_s - X_{s-}| > 2r)} = \mathbb{E}^z \int_0^{S \wedge r^\alpha} \int_{|h| > 2r} n(X_s, h)\, dh\, ds \\
&\geq \mathbb{E}^z \int_0^{S \wedge r^\alpha} \int_{2 \geq |h| > 2r} n(X_s, h)\, dh\, ds \\
&\geq c_4 r^{-\alpha} \mathbb{E}^z (S \wedge r^\alpha) \geq c_4 r^{-\alpha} \mathbb{E}^z (r^\alpha; S > r^\alpha) \\
&\geq c_4 \mathbb{P}^z(S > r^\alpha) \geq c_4/2.
\end{aligned}$$

The other alternative is that $\mathbb{P}^z(S \leq r^\alpha) > \frac{1}{2}$. In either case there exists c_5 such that $\mathbb{P}^z(S \leq r^\alpha) \geq c_5 > 0$.

If θ_t is the shift operator from Markov process theory, then by the Markov property

$$\begin{aligned}
\mathbb{P}^z(S > (m+1) r^\alpha) &\leq \mathbb{P}^z(S > m r^\alpha, S \circ \theta_{m r^\alpha} > r^\alpha) \\
&= \mathbb{E}^z \Big[\mathbb{P}^{X_{m r^\alpha}}(S > r^\alpha); S > m r^\alpha \Big] \\
&\leq (1 - c_5) \mathbb{P}^z(S > m r^\alpha).
\end{aligned}$$

By induction $\mathbb{P}^z(S > m r^\alpha) \leq (1 - c_5)^m$, which proves $\mathbb{E}^x S \leq c_6 r^\alpha$. Our second inequality follows because $\tau_{B(x,r)} \leq S$ when we start the process at x. $\square$

Proposition 3.5. *There exists c_1 such that if $r < 1/2$, $\beta \neq 1$, $z \in B(x, r/4)$, and H is a bounded nonnegative function supported in $B(x,r)^c$, then*

$$\mathbb{E}^x H(X_{\tau_{B(x,r/2)}}) \leq c_1 r^{2(\alpha - \overline{\beta})} \mathbb{E}^z H(X_{\tau_{B(x,r/2)}}).$$

Proof. By linearity and a limit argument, it suffices to consider $H = 1_C$ for a set C contained in $B(x,r)^c$. Note that Assumptions 2.1 (a), (b) and (d) imply that if $v \notin B(x,r)$, then

$$\sup_{y \in B(x,r/2)} n(y, v - y) \leq c_2 r^{\alpha - \overline{\beta}} \inf_{y \in B(x,r/2)} n(y, v - y). \tag{3.3}$$

Write τ for $\tau_{B(x,r/2)}$. For X_τ to be in C, it must get there by a jump of size at least $r/2$. By Proposition 3.2 and optional stopping,

$$\begin{aligned}
\mathbb{E}^z 1_{(X_{t \wedge \tau} \in C)} &= \mathbb{E}^z \sum_{s \leq t \wedge \tau} 1_{(|X_{s-} - X_s| > r/2, X_s \in C)} \\
&= \mathbb{E}^z \int_0^{t \wedge \tau} \int_C n(X_s, v - X_s)\, dv\, ds \\
&\geq (\mathbb{E}^z (t \wedge \tau)) \left(\int_C \inf_{y \in B(x,r/2)} n(y, v - y)\, dv \right).
\end{aligned}$$

Letting $t \to \infty$ and using dominated convergence on the left and monotone convergence on the right, we get

$$\mathbb{P}^z(X_\tau \in C) \geq \mathbb{E}^z\tau \int_C \inf_{y\in B(x,r/2)} n(y, v-y)dv.$$

Since $\mathbb{E}^z\tau \geq \mathbb{E}^z\tau_{B(z,r/4)}$, Lemma 3.4 tells us that

$$\mathbb{P}^z(X_\tau \in C) \geq c_3 r^{\overline{\beta}} \int_C \inf_{y\in B(x,r/2)} n(y, v-y)dv. \tag{3.4}$$

Similarly,

$$\mathbb{P}^x(X_\tau \in C) \leq \mathbb{E}^x\tau \int_C \sup_{y\in B(x,r/2)} n(y, v-y)dv. \tag{3.5}$$

Lemma 3.4, (3.3), (3.4), and (3.5) then imply our result. □

4. Harnack inequality

Theorem 4.1. *Suppose Assumption 2.1 holds. Suppose $\beta-\alpha < 1$. Let $z_0 \in \mathbb{R}^d$ and $R > 0$. Suppose u is nonnegative and bounded on $\mathbb{R}^d$ and harmonic on $B(z_0, R)$. Then there exists a constant $\bar{\kappa}$ depending on $R, \kappa_1, \kappa_2, \kappa_3, \kappa_4$ but not z_0, u, or $\|u\|_\infty$ such that*

$$u(x) \leq \bar{\kappa} u(y), \qquad x, y \in B(z_0, R/2). \tag{4.1}$$

Proof. Since $\beta - \alpha < 1$ and $\alpha > 0$, we can take β bigger if necessary so that $\beta > 1$ and $\beta - \alpha < 1$. So without loss of generality we may assume $\beta > 1$, and hence $\overline{\beta} = \beta$.

Let us first suppose $R \leq 1$. By looking at $u+\varepsilon$ and letting $\varepsilon \downarrow 0$, we may suppose u is bounded below by a positive constant. By looking at au for a suitable constant a, we may suppose $\inf_{B(z_0,R/2)} u \in [\frac{1}{2}, 1]$. (We do not know that u is continuous, so the infimum might not be attained.) We want to bound u above in $B(z_0, R/2)$ by a constant depending only on R and $\kappa_1, \kappa_2, \kappa_3, \kappa_4$. Choose $z_1 \in B(z_0, R/2)$ such that $u(z_1) \leq 1$. Choose ρ such that $1 < \rho < 1/(\beta - \alpha)$.

Let

$$r_i = c_2 R/i^\rho,$$

where c_2 is a constant that will be chosen later. We require first of all that c_2 be small enough so that

$$\sum_{i=1}^\infty r_i \leq R/8\ , \tag{4.2}$$

which, in particular, implies $r_i \leq 1/8$.

Recall that by Proposition 3.3(b) there exists c_3 such that for any $\bar{z}$, $\bar{r}$, $A \subset B(\bar{z}, \bar{r}/2)$ and $\bar{x} \in B(\bar{z}, \bar{r}/2)$ we have

$$\mathbb{P}^{\bar{x}}(T_A < \tau_{B(\bar{z},\bar{r})}) \geq c_3 \bar{r}^{\beta-\alpha}|A|/|B(\bar{z}, \bar{r}/2)|. \tag{4.3}$$

Let c_4 be another constant to be chosen later. Once c_2 and c_4 have been chosen, choose K_1 sufficiently large so that

$$\frac{1}{4}c_3 K_1 \exp(R c_2 c_4 i^{1-\rho(\beta-\alpha)}) c_2^{5(\beta-\alpha)+d} R^{5(\beta-\alpha)} \geq 2 i^{\rho(5(\beta-\alpha)+d)} \tag{4.4}$$

for $i = 1, 2, \ldots$. Such a choice is possible because $1 - \rho(\beta - \alpha) > 0$. K_1 will depend on d, R, ρ, α, and β as well as c_2, c_3 and c_4.

Suppose now that there exists $x_1 \in B(z_0, R/2)$ with $u(x_1) \geq K_1$. We will show that in this case there exists a sequence $\{(x_j, K_j)\}$ with $x_{j+1} \in B(x_j, 2r_j) \subset B(z_0, 3R/4)$, $K_j = u(x_j)$, and

$$K_j \geq K_1 \exp\big(Rc_2c_4 j^{1-\rho(\beta-\alpha)}\big). \tag{4.5}$$

Since $1 - \rho(\beta - \alpha) > 0$, then $K_j \to \infty$, a contradiction to u being bounded. We can then conclude that u must be bounded by K_1, and hence $u(x)/u(y) \leq 2K_1$ if $x, y \in B(z_0, R/2)$.

Suppose $x_1, x_2, \ldots, x_i$ have been selected and that (4.5) holds for $j = 1, \ldots, i$. We will show there exists $x_{i+1} \in B(x_i, 2r_i)$ such that if $K_{i+1} = u(x_{i+1})$, then (4.5) holds for $j = i + 1$; we then use induction to conclude that (4.5) holds for all j.

Let

$$A_i = \{y \in B(x_i, r_i/4) : u(y) \geq K_i r_i^{4(\beta-\alpha)}\}.$$

First, we prove that

$$|A_i|/|B(x_i, r_i/4)| \leq \tfrac{1}{4}. \tag{4.6}$$

To prove this claim, we suppose to the contrary that $|A_i|/|B(x_i, r_i/4)| > \frac{1}{4}$. Let D be a compact subset of A_i with $|D|/|B(x_i, r_i/4)| > \frac{1}{4}$. Recall that $R \geq 8r_i \geq r_i$. By Doob's optional stopping theorem, the facts that u is nonnegative and $u(X_{t\wedge\tau_D})$ is right-continuous, (4.3), (4.4), and (4.5), we can estimate

$$\begin{aligned}
1 \geq u(z_1) &\geq \mathbb{E}^{z_1}[u(X_{T_D\wedge\tau_{B(z_0,R)}}); T_D < \tau_{B(z_0,R)}] \\
&\geq K_i r_i^{4(\beta-\alpha)} \mathbb{P}^{z_1}(T_D < \tau_{B(z_0,R)}) \\
&\geq c_3 K_i r_i^{4(\beta-\alpha)} R^{\beta-\alpha} |D|/|B(z_0, R)| \\
&\geq \frac{1}{4} c_3 K_i r_i^{5(\beta-\alpha)} (r_i/R)^d \geq 2.
\end{aligned}$$

This is a contradiction, and therefore (4.6) is proved.

Write τ_i for $\tau_{B(x_i, r_i/2)}$. Set $M_i = \sup_{B(x_i,r_i)} u(x)$. Let E be a compact subset of $B(x_i, r_i/4) \setminus A_i$ such that $|E|/|B(x_i, r_i/4)| \geq \frac{1}{2}$. In view of (4.6) such a choice is possible. Let

$$p_i = \mathbb{P}^{x_i}(T_E < \tau_i).$$

We have

$$\begin{aligned}
K_i = u(x_i) = &\ \mathbb{E}^{x_i}[u(X_{T_E\wedge\tau_i}); T_E < \tau_i] \\
&+ \mathbb{E}^{x_i}[u(X_{T_E\wedge\tau_i}); T_E \geq \tau_i, X_{\tau_i} \in B(x_i, r_i)] \\
&+ \mathbb{E}^{x_i}[u(X_{T_E\wedge\tau_i}); T_E \geq \tau_i, X_{\tau_i} \notin B(x_i, r_i)].
\end{aligned} \tag{4.7}$$

Since $E \subset B(x_i, r_i/4) \setminus A_i$ is compact, the first term is bounded above by

$$K_i r_i^{4(\beta-\alpha)} \mathbb{P}^{x_i}(T_E < \tau_i) \leq K_i r_i^{4(\beta-\alpha)}.$$

The second term is bounded above by

$$M_i(1 - p_i).$$

We turn to the third term. Inequality (4.6) implies in particular that there exists $y_i \in B(x_i, r_i/4)$ with $u(y_i) \leq K_i r_i^{4(\beta-\alpha)}$. We then have, using Proposition 3.5,

$$\begin{aligned}
K_i r_i^{4(\beta-\alpha)} &\geq u(y_i) \geq \mathbb{E}^{y_i}[u(X_{\tau_i}); X_{\tau_i} \notin B(x_i, r_i)] \\
&\geq c_5 r_i^{2(\beta-\alpha)} \mathbb{E}^{x_i}[u(X_{\tau_i}); X_{\tau_i} \notin B(x_i, r_i)].
\end{aligned} \tag{4.8}$$

Using (4.8), the third term on the right of (4.7) is bounded above by

$$c_5^{-1} r_i^{2(\alpha-\beta)} K_i r_i^{4(\beta-\alpha)} = c_6 K_i r_i^{2(\beta-\alpha)}.$$

Substituting in (4.7), we get

$$K_i \le K_i r_i^{4(\beta-\alpha)} + M_i(1-p_i) + c_6 K_i r_i^{2(\beta-\alpha)}. \tag{4.9}$$

Rearranging,

$$M_i \ge K_i\Big(\frac{1 - r_i^{4(\beta-\alpha)} - c_6 r_i^{2(\beta-\alpha)}}{1-p_i}\Big). \tag{4.10}$$

By (4.3)

$$p_i \ge c_7 c_3 r_i^{\beta-\alpha}. \tag{4.11}$$

Since $r_i \le c_2 R \le c_2$, if we choose c_2 small enough, then

$$r_i^{4(\beta-\alpha)} + c_6 r_i^{2(\beta-\alpha)} \le \tfrac{1}{2} c_7 c_3 r_i^{\beta-\alpha} \tag{4.12}$$

for all i. Therefore

$$M_i \ge K_i\Big(\frac{1-\frac{1}{2}p_i}{1-p_i}\Big) > \Big(1 + \frac{p_i}{2}\Big)K_i.$$

Using the definition of M_i and (4.11), there exists a point $x_{i+1} \in \overline{B(x_i, r_i)} \subset B(x_i, 2r_i)$ such that

$$K_{i+1} = u(x_{i+1}) \ge K_i(1 + c_7 c_3 r_i^{\beta-\alpha}/2).$$

Taking logarithms and writing

$$\log K_{i+1} = \log K_1 + \sum_{j=1}^{i} [\log K_{j+1} - \log K_j],$$

we have

$$\begin{aligned}
\log(K_{i+1}) &\ge \log K_1 + \sum_{j=1}^{i} \log(1 + c_7 c_3 r_j^{\beta-\alpha}/2) \\
&\ge \log K_1 + c_8 \sum_{j=1}^{i} r_j^{\beta-\alpha} \\
&= \log K_1 + R c_2 c_8 \sum_{j=1}^{i} j^{-\rho(\beta-\alpha)} \\
&\ge \log K_1 + R c_2 c_4 (i+1)^{1-\rho(\beta-\alpha)},
\end{aligned}$$

and hence (4.5) holds for $i+1$ provided we choose c_2 small enough so that (4.2) and (4.12) hold. The theorem has thus been proved for $R < 1$.

For $R \ge 1$ we use a standard chain of balls argument. Given any two points $x, y \in B(z_0, R/2)$, we can find N balls $B_1 \ldots, B_N$ of radius $\frac{1}{2}$ such that x is the center of B_1, y is the center of B_N, the centers of B_i and B_{i+1} lie within $\frac{1}{4}$ of each other for each i, and the center of B_i lies in $B(z_0, R/2)$ for each i. Moreover, the number of balls N depends only on R. We then apply the Harnack inequality that we proved above N times to derive $u(x) \le c_9^N u(y)$. □

Remark 4.2. If one keeps careful track of the constants, one sees that $\bar\kappa$ grows at most polynomially in $1/R$ as $R \to 0$. See also Proposition 5.1.

Remark 4.3. We do not know if the condition $\beta-\alpha<1$ can be weakened. It would be very interesting to either weaken this condition or find an example showing it is necessary.

Remark 4.4. Must a function that is bounded in $\mathbb{R}^d$ and is harmonic in a ball be continuous in the ball? This is the case for nondegenerate diffusions and stable-like jump processes (i.e., $\alpha=\beta$), but we do not know the answer to this question in the variable order case. Continuity appears to be a less robust property than the Harnack inequality.

5. Examples

In Theorem 4.1 we allowed the constant $\bar{\kappa}$ to depend on R. This is necessary, as the following proposition shows. To see the idea behind the proof, consider (V_t^1, V_t^2), where V_t^1 is a one-dimensional symmetric stable process of order β and V_t^2 is a one-dimensional symmetric stable process of order α. If $\beta>\alpha$, then over short distances the first component moves much faster than the second. However (V_t^1, V_t^2) does not satisfy Assumption 2.1(a), and so we must use a more complicated example. See [Ber96] for information on Lévy processes.

Proposition 5.1. *Let $0<\alpha<\beta<2$. There exists a function $n(x,h)$ satisfying Assumption 2.1 with the following property.*

- *For $R<1$ there exist functions u_R that are nonnegative and harmonic on $B(0,R)$ and points $x_R, y_R \in B(0,R/2)$ such that $u_R(y_R)/u_R(x_R) \to \infty$ as $R\to 0$.*

Proof. Observe that $\lim_{a\to 1}(a-2+\frac{1}{a})/(1-a)=0$. Choose $a<1$ sufficiently close to 1 so that $\beta > (a-2+\frac{1}{a})/(1-a)$. Take a closer to 1 if necessary so that $a\beta+a-1>\alpha$. Some algebra shows that $\beta+1-\frac{1}{a} > a\beta+a-1$. We now choose γ such that $\beta+1-\frac{1}{a}>\gamma>a\beta+a-1$. Since $a<1$, then $\gamma<\beta$; by our choice of a, we see that $\gamma>\alpha$.

Let

$$A=\{(x_1,x_2): |x_2|>|x_1|^a, |x_2|<1\}.$$

We define a Lévy process $X_t=(X_t^1,X_t^2)$ by specifying that there is no Gaussian component, no drift, and the Lévy measure is given by $n(dh)=n(h)dh$, where

$$n(h)=\frac{1}{|h|^{2+\alpha}}+\frac{1_A(h)}{|h|^{2+\beta}}.$$

If we set $n(x,h)=n(h)$ for all x, clearly Assumption 2.1 holds.

Let $D_R=[-R,R]^2$, $x_R=(-R/4,0)$, $y_R=(R/4,0)$. Define $u_R(x_1,x_2)$ on D_R^c to be 1 if $x_1>0$ and 0 otherwise. Define u_R inside D_R by $u_R(x)=\mathbb{E}^x u_R(X_{\tau_{D_R}})$. Then u_R is harmonic in $B(0,R)$.

We will show that

$$\mathbb{P}(\sup_{s\le R^\gamma}|X_s^2|<R)\to 0 \qquad \text{as } R\to 0 \tag{5.1}$$

and

$$\mathbb{P}(\sup_{s\le R^\gamma}|X_s^1|>R/4)\to 0 \qquad \text{as } R\to 0. \tag{5.2}$$

(5.1) says that for R small, $|X_t^2|$ is very likely to have exceeded R by time R^γ; hence $\tau_{D_R} \le R^\gamma$ with high probability. (5.2) says that by time τ_{D_R} the process X_t^1 is unlikely to have moved as far as $R/4$. Consequently

$$\mathbb{P}^{x_R}(X^1_{\tau_{D_R}} > 0) \to 0 \qquad \text{as } R \to 0,$$

$$\mathbb{P}^{y_R}(X^1_{\tau_{D_R}} > 0) \to 1 \qquad \text{as } R \to 0.$$

This shows that $u_R(x_R) \to 0$ and $u_R(y_R) \to 1$; hence $u_R(y_R)/u_R(x_R) \to \infty$ as $R \to 0$.

We now prove (5.1) and (5.2). Write $h = (h_1, h_2)$. Since $a < 1$, then for $h_2 \in [2R, 3R]$ and $h_1 \in (0, |h_2|^{1/a}]$ we have that $|h|$ is comparable to h_2. We calculate

$$\begin{aligned} I_1(R) = \int_{A\cap(|h_2|>2R)} n(dh) &\ge 4 \int_{2R}^1 \int_0^{|h_2|^{1/a}} \frac{1}{|h|^{2+\beta}} dh_1\, dh_2 \\ &\ge 4 \int_{2R}^{3R} \int_0^{|h_2|^{1/a}} \frac{1}{|h|^{2+\beta}} dh_1\, dh_2 \ge c_1 R^{\frac{1}{a}-1-\beta}. \end{aligned}$$

The number of times that $\Delta X_s \in A \cap (|h_2| > 2R)$ for $s \le t$ is a Poisson random variable with parameter greater than $tI_1(R)$. By our choice of γ, $R^\gamma I_1(R) \to \infty$ as $R \to 0$. Hence the probability that there are no jumps with ΔX_s in $A\cap(|h_2| > 2R)$ by time R^γ tends to 0 as $R \to 0$. But if $\Delta X_s \in A \cap (|h_2| > 2R)$ for some $s \le R^\gamma$, then $|X_s^2|$ will exceed R. This proves (5.1).

We turn to (5.2). We can write $X_t = Y_t + Z_t$, where Y and Z are independent Lévy processes, Y has Lévy measure $n_Y(dh) = |h|^{-(2+\alpha)}dh$, and Z has Lévy measure $n_Z(dh) = 1_A(h)|h|^{-(2+\beta)}dh$.

By scaling and the fact that $\gamma > \alpha$,

$$\mathbb{P}(\sup_{s\le R^\gamma} |Y_s| > R/8) \to 0 \qquad \text{as } R \to 0. \tag{5.3}$$

We calculate

$$\begin{aligned} I_2(R) &= \int_{A\cap(|h_1|>R/16)} n_Z(dh) \\ &\le 4 \int_{R/16}^1 \int_{|h_1|^a}^1 \frac{1}{|h|^{2+\beta}} dh_2\, dh_1 \\ &\le 4 \int_{R/16}^1 \int_{|h_1|^a}^1 \frac{1}{|h_2|^{2+\beta}} dh_2\, dh_1 \\ &\le c_2 R^{1-a-a\beta} \end{aligned}$$

and

$$I_3(R) = 4 \int_0^R \int_{|h_1|^a}^1 \frac{(h_1)^2}{|h|^{2+\beta}} dh_2\, dh_1 \le c_3 R^{3-a-a\beta}.$$

The expected number of times ΔZ_s is in $A \cap (|h_1| > R/16)$ for $s \le R^\gamma$ is $R^\gamma I_2(R)$, which tends to 0 as $R \to 0$ by our choice of γ. Therefore

$$\mathbb{P}(|\Delta Z_s^1| > R/16 \text{ for some } s \le R^\gamma) \to 0 \qquad \text{as } R \to 0. \tag{5.4}$$

Let W_t be the process Z_t with all jumps such that ΔZ_s is in $A \cap (|h_1| > R/16)$ removed, that is,

$$W_t = Z_t - \sum_{s \le t} \Delta Z_s 1_{A\cap(|h_1|>R/16)}(\Delta Z_s).$$

Then W_t is the Lévy process with Lévy measure

$$n_W(dh) = \frac{1_{A\cap(|h_1|\le R/16)}(h)}{|h|^{2+\beta}} dh.$$

Since a Lévy process with bounded jumps has moments of all orders and W has no drift component, then W_t^1 is a martingale. By Doob's inequality,

$$\mathbb{P}(\sup_{s\le R^\gamma} |W_s^1| > R/16) \le 4 \frac{\mathbb{E}\,(W^1_{R^\gamma})^2}{(R/16)^2}.$$

But $\mathbb{E}\,(W_t^1)^2 = tI_3(R)$, and so by our choice of γ we have $\mathbb{E}\,(W^1_{R^\gamma})^2/R^2 \to 0$ as $R \to 0$. Therefore

$$\mathbb{P}(\sup_{s\le R^\gamma} |W_s^1| > R/16) \to 0 \qquad \text{as } R \to 0. \tag{5.5}$$

Putting (5.3), (5.4), and (5.5) together gives (5.2). □

The following example shows that a hypothesis along the lines of Assumption 2.1(d) is necessary for a Harnack inequality to hold.

Proposition 5.2. *There exists a function $n(x,h)$ satisfying Assumptions 2.1*(a)-(c) *(but not (d)) for which the Harnack inequality fails for the corresponding operator.*

Proof. We work in two dimensions. Let $B = B(0,1)$, let $y_0 = (1/8, 0)$ and for $m \ge 4$ let $x_m = (-1/8, 2^{-m})$, $z_m = (16, 2^{-m})$, $C_m = B(x_m, 2^{-m-4})$, and $E_m = B(z_m, 2^{-m-4})$. Define

$$n(x,h) = |h|^{-d-\alpha} 1_{(|h|\le 3)} + \sum_{m=4}^{\infty} 1_{C_m}(x) 1_{E_m}(x+h).$$

It is clear that $n(x,h)$ satisfies Assumptions 2.1 (a)-(c), because the C_m are disjoint and the E_m are disjoint. It is also not hard to see that there is a unique solution to the martingale problem for $\mathcal{L}$, because n differs from the Lévy kernel of a symmetric stable process only in the jumps of size larger than 3, see [Kom84].

Next we show that $\mathbb{P}^{y_0}(T_{C_m} < \tau_B)$ is small when m is large. Note that Lemma 3.4 does not use Assumption 2.1(d), and therefore $\mathbb{E}^{\,y_0}\tau_B \le c_1 < \infty$. Fix m, let $\varepsilon = 2^{-m-4}$, let $g(x) = |x - x_m|^{-d-\alpha}$, let φ be a nonnegative C^∞ function with support in $B(0,1/2)$ whose integral is 1, let $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(x/\varepsilon)$, and let $f_\varepsilon = g * \varphi_\varepsilon$. Let $n_0(x,h) = |h|^{-d-\alpha}$ and let $\mathcal{L}_0$ be the operator corresponding to n_0. Then $f_\varepsilon \ge c_2\varepsilon^{-\alpha-d}$ on C_m and $f_\varepsilon \in C^\infty$. It is well known that $\mathcal{L}_0 f_\varepsilon(x) = -c_3\varphi_\varepsilon(x - x_m)$, and hence $\mathcal{L}_0 f_\varepsilon(x) = 0$ for $x \notin C_m$. It is also easy to check that $f_\varepsilon(y_0) \le c_4$ and $|\mathcal{L} f_\varepsilon(x) - \mathcal{L}_0 f_\varepsilon(x)| \le c_5$ if $x \in B \setminus C_m$. Therefore $\mathcal{L} f_\varepsilon(x) \le c_5$ for $x \in B \setminus C_m$. Since $\mathbb{P}^{y_0}$ is a solution to the martingale problem for $\mathcal{L}$, we have

$$\begin{aligned}\mathbb{E}^{\,y_0} f_\varepsilon(X_{T_{C_m}\wedge\tau_B}) - f_\varepsilon(y_0) &= \mathbb{E}^{\,y_0} \int_0^{T_{C_m}\wedge\tau_B} \mathcal{L} f_\varepsilon(X_s)\,ds \\ &\le c_5 \mathbb{E}^{\,y_0}\tau_B \le c_1 c_5.\end{aligned}$$

Therefore

$$c_2\varepsilon^{-\alpha-d}\,\mathbb{P}^{y_0}(T_{C_m} < \tau_B) \le \mathbb{E}^{\,y_0} f_\varepsilon(X_{T_{C_m}\wedge\tau_B}) \le c_1c_5 + c_4.$$

Thus $\mathbb{P}^{y_0}(T_{C_m} < \tau_B)$ will be small if m is large.

Now suppose that the Harnack inequality did hold for nonnegative functions that are harmonic in B, that is, suppose there exists c_6 such that

$$u(x) \le c_6 u(y), \qquad x, y \in B(0, 1/2),$$

whenever u is bounded in $\mathbb{R}^d$ and nonnegative and harmonic in B. Let

$$u_m(x) = \mathbb{E}^{\,x}[1_{E_m}(X_{\tau_B})].$$

This function is bounded, nonnegative, and harmonic in B. Note the only way that X_{τ_B} can be in E_m is if X_{τ_B-} is in C_m. We then have

$$\begin{aligned} u_m(y_0) &= \mathbb{E}^{\,y_0}[1_{E_m}(X_{\tau_B}); T_{C_m} < \tau_B] \\ &= \mathbb{E}^{\,y_0}\Big[\mathbb{E}^{\,X_{T_{C_m}}}[1_{E_m}(X_{\tau_B})]; T_{C_m} < \tau_B\Big] \\ &= \mathbb{E}^{\,y_0}[u_m(X_{T_{C_m}}); T_{C_m} < \tau_B] \\ &\le c_6 u_m(x_m)\mathbb{P}^{y_0}(T_{C_m} < \tau_B), \end{aligned}$$

where we used the assumption that a Harnack inequality holds to get the last inequality. But then

$$\frac{u_m(x_m)}{u_m(y_0)} \ge \frac{1}{c_6\mathbb{P}^{y_0}(T_{C_m} < \tau_B)},$$

which can be made arbitrarily large if we take m large enough. This is a contradiction, and therefore the Harnack inequality cannot hold. $\square$

Acknowledgments

We would like to thank Martin Barlow for helpful conversations on the subject of this paper.

References

[Bas88] R. F. Bass. Uniqueness in law for pure jump Markov processes. *Probab. Theory Related Fields*, 79(2):271–287, 1988. MR 89h:60118

[BBG00] M.T. Barlow, R.F. Bass, and C. Gui. The Liouville property and a conjecture of De Giorgi. *Comm. Pure Appl. Math.*, 53(8):1007–1038, 2000. MR 2001m:35095

[Ber96] J. Bertoin. *Lévy Processes.* Cambridge University Press, Cambridge, 1996. MR 98e:60117

[BG68] R. M. Blumenthal and R. K. Getoor. *Markov Processes and Potential Theory.* Pure and Applied Mathematics, Vol. 29. Academic Press, New York, 1968. MR 41:9348

[BL02a] R.F. Bass and D.A. Levin. Harnack inequalities for jump processes. *Potential Anal.*, 17(4):375–388, 2002. MR 2003e:60194

[BL02b] R.F. Bass and D.A. Levin. Transition probabilities for symmetric jump processes. *Trans. Amer. Math. Soc.*, 354(7):2933–2953, 2002. MR 2002m:60132

[BSS02a] K. Bogdan, A. Stós, and P. Sztonyk. Harnack inequality for stable processes on d-sets. *Studia Math.* 158(2):163–198, 2003. MR 2013738

[BSS02b] K. Bogdan, A. Stós, and P. Sztonyk. Potential theory for Lévy stable processes. *Bull. Polish Acad. Sci. Math.* 50(3):361–372, 2002. MR 2003m:60125

[CK03] Z.-Q. Chen and T. Kumagai. Heat kernel estimates for stable-like processes on d-sets. *Stochastic Processes Appl.* 108(1):27–62, 2003. MR 2008600

[FS86] E.B. Fabes and D.W Stroock. A new proof of Moser's parabolic Harnack inequality via the old ideas of Nash. *Arch. Mech. Rat. Anal.*, 96:327–338, 1986. MR 88b:35037

[Hoh95] W. Hoh. Pseudodifferential operators with negative definite symbols and the martingale problem. *Stochastics Stochastics Rep.*, 55(3-4):225–252, 1995. MR 97c:35225

[Kas03] M. Kassmann. On regularity for Beurling-Deny type Dirichlet forms. *Potential Analysis*, 19(1):69–87, 2003. MR 2004a:31005

[Kom84] T. Komatsu. On the martingale problem for generators of stable processes with perturbations. *Osaka J. Math.*, 21(1):113–132, 1984. MR 86e:60060

[KS79] N. V. Krylov and M. V. Safonov. An estimate for the probability of a diffusion process hitting a set of positive measure. *Dokl. Akad. Nauk SSSR*, 245(1):18–20, 1979; English transl., *Soviet Math. Dokl.* 20(1):253–255, 1979. MR 80b:60101

[KS80] N. V. Krylov and M. V. Safonov. A property of the solutions of parabolic equations with measurable coefficients. *Izv. Akad. Nauk SSSR Ser. Mat.*, 44(1):161–175, 239, 1980; English transl., *Math. USSR–Izv.* 16(1):151–164, 1981. MR 83c:35059

[Mos61] J. Moser. On Harnack's theorem for elliptic differential equations. *Comm. Pure Appl. Math.*, 14:577–591, 1961. MR 28:2356

[Sko65] A. V. Skorokhod. *Studies in the Theory of Random Processes*. Addison-Wesley Publishing Co., Inc., Reading, Mass., 1965. MR 32:3082b

[SV04] R. Song and Z. Vondracek. Harnack inequality for some classes of Markov processes. *Math. Z.* 246(1-2):177–202, 2004.

[SV79] D. W. Stroock and S. R. S. Varadhan. *Multidimensional Diffusion Processes*, Volume 233 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1979. MR 81f:60108

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