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A THETA FUNCTION IDENTITY AND ITS IMPLICATIONS

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ABSTRACT. In this paper we prove a general theta function identity with four parameters by employing the complex variable theory of elliptic functions. This identity plays a central role for the cubic theta function identities. We use this identity to re-derive some important identities of Hirschhorn, Garvan and Borwein about cubic theta functions. We also prove some other cubic theta function identities. A new representation for $\prod_{n=1}^{\infty}(1-q^n)^{10}$ is given. The proofs are self-contained and elementary.

1. INTRODUCTION

Throughout this paper we will use q to denote $\exp(2\pi i\tau)$ with $\mathrm{Im}(\tau)>0$. We will use the familiar notation

$$(z;q)_\infty=\prod_{n=0}^{\infty}(1-zq^n), \tag{1.1}$$

and sometimes write

$$(a,b,c,\cdots;q)_\infty=(a;q)_\infty(b;q)_\infty(c;q)_\infty\cdots. \tag{1.2}$$

It is easy to see that for any integer $m>0$,

$$(z,zq,\cdots,zq^{m-1};q^m)_\infty=(z;q)_\infty. \tag{1.3}$$

If we define ω to be the primitive cube root of unity given by $\omega=\exp(\frac{2\pi i}{3})$, then, using the identity $(1-x)(1-x\omega)(1-x\omega^2)=1-x^3$, we find that

$$(z,z\omega,z\omega^2;q)_\infty=(z^3;q^3)_\infty. \tag{1.4}$$

The well-known Jacobi triple product identity is

$$(q,z,q/z;q)_\infty=\sum_{n=-\infty}^{\infty}(-1)^nq^{n(n-1)/2}z^n \tag{1.5}$$

(see [1, pp. 21-22], [2, p. 35], [7], and [10]).

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Jacobi's theta function $\theta_1(z|\tau)$ is defined as

$$\begin{aligned}\theta_1(z|\tau) &= -iq^{1/8}\sum_{n=-\infty}^{\infty}(-1)^n q^{n(n+1)/2}e^{(2n+1)iz}\\ &= 2q^{1/8}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}\sin(2n+1)z \end{aligned}\tag{1.6}$$

(see, for example, [16, p. 463]).

Using the triple product identity, we can find the infinite product representation for $\theta_1(z|\tau)$, namely,

$$\begin{aligned}\theta_1(z|\tau) &= 2q^{1/8}(\sin z)(q, qe^{2iz}, qe^{-2iz};q)_\infty\\ &= iq^{1/8}e^{-iz}(q, e^{2iz}, qe^{-2iz};q)_\infty\end{aligned}\tag{1.7}$$

(see, for example, [16, p. 469]).

In [13], we use the complex variable theory of elliptic functions to establish a general theta function identity. We then derive some remarkable theta function identities related to the modular equations of degree 5; in particular,we give new proofs of the two fundamental identities satisfied by the Rogers-Ramanujan continued fraction. In this paper we set up the following general theta function identity with four parameters by employing the same method. This is a notable identity, which contains many interesting cubic theta function identities as special cases.

Theorem 1. *Suppose $f(z)$ is an entire function satisfying the functional equations*

$$f(z+\pi) = -f(z) \quad \textit{and} \quad f(z+\pi\tau) = -q^{-3/2}e^{-6iz}f(z).\tag{1.8}$$

Then the following identity holds:

$$\begin{aligned}&\{f(z)-f(-z)\}\,\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x+y|\tau)\theta_1(x-y|\tau)\\ =\ &\{f(y)-f(-y)\}\,\theta_1(x|\tau)\theta_1(z|\tau)\theta_1(x+z|\tau)\theta_1(x-z|\tau)\\ -\ &\{f(x)-f(-x)\}\,\theta_1(y|\tau)\theta_1(z|\tau)\theta_1(y+z|\tau)\theta_1(y-z|\tau).\end{aligned}\tag{1.9}$$

The contents are organized as follows. In Section 2, we prove Theorem 1 using the classical theory of elliptic functions. In Section 3, the following theta function identity is first proved using Theorem 1.

Theorem 2. *Let $\theta_1(z|\tau)$ be the Jacobi theta function defined by* (1.6). *Then for any $x, y,$ and z, we have*

$$\begin{aligned}&\theta_1(y|\tfrac{\tau}{3})\theta_1(x|\tau)\theta_1(z|\tau)\theta_1(x-z|\tau)\theta_1(x+z|\tau)\\ &-\theta_1(x|\tfrac{\tau}{3})\theta_1(y|\tau)\theta_1(z|\tau)\theta_1(y-z|\tau)\theta_1(y+z|\tau)\\ =\ &\theta_1(z|\tfrac{\tau}{3})\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau).\end{aligned}\tag{1.10}$$

Then we use Theorem 2 to derive a interesting theta function identity given in Theorem 5 and from which we establish a new representation for $(q;q)_\infty^{10}$. In Section 4, we derive some identities for the Hirschhorn-Garvan-Borwein two-variable cubic theta functions; our method is different from that of Hirschhorn, Garvan, and Borwein. In Section 5, we use Theorem 1 to prove the following remarkable theta function identities.

Theorem 3. *Let $\theta_1(x|\tau)$ be the Jacobi theta function defined as in* (1.6)*, and let $a(q)$ be the Ramanujan function (see* [4]*) defined by*

$$a(q) := 1 + 6\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}}\right). \tag{1.11}$$

Then, we have

$$\theta_1^3(x+\frac{\pi}{3}|\tau) + \theta_1^3(x-\frac{\pi}{3}|\tau) - \theta_1^3(x|\tau) = 3a(q)\theta_1(3x|3\tau) \tag{1.12}$$

and

$$\theta_1^3(x|3\tau) - q^{1/2}e^{2ix}\theta_1^3(x+\pi\tau|3\tau) - q^{1/2}e^{-2ix}\theta_1^3(x-\pi\tau|3\tau) = a(q)\theta_1(x|\tau). \tag{1.13}$$

Equality (1.13) is [3, p.142, Entry 3]. In Section 6, we set up the following identities using Theorem 1.

Theorem 4. *We have*

$$\begin{aligned} &(q;q)_\infty^3\theta_1^3(x|\tau)\theta_1(3y|3\tau) - (q;q)_\infty^3\theta_1^3(y|\tau)\theta_1(3x|3\tau) \\ = \;& 3q^{1/4}(q^3;q^3)_\infty^3\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau) \end{aligned} \tag{1.14}$$

and

$$\begin{aligned} &(q;q)_\infty^3\theta_1^3(x|\tau)\theta_1(y|\frac{\tau}{3}) - (q;q)_\infty^3\theta_1^3(y|\tau)\theta_1(x|\frac{\tau}{3}) \\ = \;& q^{-1/(12)}(q^{1/3};q^{1/3})_\infty^3\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau). \end{aligned} \tag{1.15}$$

2. The proof of Theorem 1

To prove the theorem, we require the following Lemma 1. Lemma 1 is a fundamental theorem of elliptic functions and can be found in [6, p. 22]. Recently, in [9, 11, 12, 13, 14, 15], we have used Lemma 1 to set up many important theta function identities.

Lemma 1. *The sum of all the residues of an elliptic function vanishes in the period parallelogram.*

Proof. Suppose that $f(u)$ is the given function satisfying the functional equations (1.8). Then we consider the function

$$g(u) = \frac{\theta_1(2u|\tau)f(u)}{\theta_1(u|\tau)\theta_1(u-x|\tau)\theta_1(u+x|\tau)\theta_1(u-y|\tau)\theta_1(u+y|\tau)\theta_1(u-z|\tau)\theta_1(u+z|\tau)}. \tag{2.1}$$

Here we temporarily assume that $0 < x, y, z < \pi$ are three distinct parameters different from the zero points of $\theta_1(2u|\tau)f(u)$. Using the functional equations

$$\theta_1(z+\pi|\tau) = -\theta_1(z|\tau) \quad \text{and} \quad \theta_1(z+\pi\tau|\tau) = -q^{-1/2}e^{-2iz}\theta_1(z|\tau), \tag{2.2}$$

we can verify that $g(u+\pi) = g(u)$ and $g(u+\pi\tau) = g(u)$. Hence $g(u)$ is an elliptic function with periods π and $\pi\tau$. We readily find that $x, \pi-x, y, \pi-y, z, \pi-z$ are its only poles and all its poles are simple poles. In this paper we will use $\operatorname{res}(g;\alpha)$ to denote the residue of g at α. Then Lemma 1 gives

$$\operatorname{res}(g;x)+\operatorname{res}(g;\pi-x)+\operatorname{res}(g;y)+\operatorname{res}(g;\pi-y)+\operatorname{res}(g;z)+\operatorname{res}(g;\pi-z) = 0. \tag{2.3}$$

In this paper, we will also use the prime to denote the partial derivative with respect to u. Now, we have the following elementary calculation:

$$\begin{aligned}
\operatorname{res}(g;x) &= \lim_{u\to x}(u-x)g(u)\\
&= \lim_{u\to x}\frac{\theta_1(2u|\tau)f(u)}{\theta_1(u|\tau)\theta_1(u+x|\tau)\theta_1(u-y|\tau)\theta_1(u+y|\tau)\theta_1(u-z|\tau)\theta_1(u+z|\tau)}\\
&\quad\times\lim_{u\to x}\frac{u-x}{\theta_1(u-x|\tau)}\\
&= \frac{f(x)}{\theta_1'(0|\tau)\theta_1(x|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau)\theta_1(x-z|\tau)\theta_1(x+z|\tau)}. \qquad (2.4)
\end{aligned}$$

In the same way we can show that

$$\operatorname{res}(g;\pi-x)=\frac{-f(-x)}{\theta_1'(0|\tau)\theta_1(x|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau)\theta_1(x-z|\tau)\theta_1(x+z|\tau)}. \qquad (2.5)$$

Noting that $g(u)$ is symmetric in x, y, and z, we interchange x and y in (2.4) and (2.5) to obtain

$$\operatorname{res}(g;y)=\frac{-f(y)}{\theta_1'(0|\tau)\theta_1(y|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau)\theta_1(y-z|\tau)\theta_1(y+z|\tau)} \qquad (2.6)$$

and

$$\operatorname{res}(g;\pi-y)=\frac{f(-y)}{\theta_1'(0|\tau)\theta_1(y|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau)\theta_1(y-z|\tau)\theta_1(y+z|\tau)}. \qquad (2.7)$$

Similarly we find that

$$\operatorname{res}(g;z)=\frac{f(z)}{\theta_1'(0|\tau)\theta_1(z|\tau)\theta_1(x-z|\tau)\theta_1(x+z|\tau)\theta_1(y-z|\tau)\theta_1(y+z|\tau)} \qquad (2.8)$$

and

$$\operatorname{res}(g;\pi-z)=\frac{-f(-z)}{\theta_1'(0|\tau)\theta_1(z|\tau)\theta_1(x-z|\tau)\theta_1(x+z|\tau)\theta_1(y-z|\tau)\theta_1(y+z|\tau)}. \qquad (2.9)$$

Substituting (2.4)-(2.9) into (2.3), after some simplification, we obtain (2.1) . By analytic continuation, we know (2.1) holds for all x, y, and z, and so this completes the proof of Theorem 1. ☐

3. A NEW REPRESENTATION FOR $(q;q)_\infty^{10}$

In this section we first give a proof of Theorem 2 using Theorem 1.

Proof. It is easy to verify that $\theta_1(x|\frac{\tau}{3})$ satifies all the conditions of Theorem 1. By taking $f(x)=\theta_1(x|\frac{\tau}{3})$ in Theorem 1, we immediately obtain Theorem 2; and thus we complete the proof of Theorem 2. ☐

Using the identity $(z, z\omega, z\omega^2; q)_\infty = (z^3; q^3)_\infty$, where ω=exp$(\frac{2\pi i}{3})$, and the infinite product representation for $\theta_1(z|\tau)$, we readily find that

$$\theta_1(3z|3\tau)=-\frac{(q^3;q^3)_\infty}{(q;q)_\infty^3}\theta_1(z|\tau)\theta_1(z+\frac{\pi}{3}|\tau)\theta_1(z-\frac{\pi}{3}|\tau). \qquad (3.1)$$

Appealing to the infinite product representation for $\theta_1(z|\tau)$, we can also find that

$$\theta_1(\frac{\pi}{3}|\tau)=\theta_1(\frac{2\pi}{3}|\tau)=\sqrt{3}q^{1/8}(q^3;q^3)_\infty. \qquad (3.2)$$

Taking $z = \frac{\pi}{3}$ in Theorem 2 and then using (3.1) and (3.2) in the resulting equation, we obtain the following identity.

Theorem 5. *For any x, y we have the identity*

$$\begin{aligned} & q^{1/(12)}(q;q)_\infty^2\theta_1(x|\frac{\tau}{3})\theta_1(3y|3\tau) - q^{1/(12)}(q;q)_\infty^2\theta_1(y|\frac{\tau}{3})\theta_1(3x|3\tau) \\ (3.3)\quad = & \ \theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau). \end{aligned}$$

Differentiating both sides of (1.6) with respect to z and then setting $z = 0$, we readily find that

$$\theta_1'(0|\tau) = q^{1/8}\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}, \tag{3.4}$$

and by successive differentiations, we obtain

$$\theta_1'''(0|\tau) = -q^{1/8}\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)^3q^{n(n+1)/2}. \tag{3.5}$$

Differentiation of (1.7) gives

$$\theta_1'(0|\tau) = 2q^{1/8}(q;q)_\infty^3. \tag{3.6}$$

Comparing (3.4) and (3.6), we obtain Jacobi's identity

$$(q;q)_\infty^3 = \frac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}. \tag{3.7}$$

In the next theorem we will give a new representation for $(q;q)_\infty^{10}$. An entirely different proof of this new representation has been given in the paper [5]. Using this identity, the authors have also given a short proof of Ramanujan's famous congruence $p(11n+6) \equiv 0 \pmod{11}$, where $p(n)$ denotes the number of unrestricted partitions of the positive integer n.

Theorem 6. *We have*

$$\begin{aligned} (3.8)\quad & 32(q;q)_\infty^{10} \\ = & \ 9\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)^3q^{3n(n+1)/2}\right)\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{n(n+1)/6}\right) \\ - & \ \left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{3n(n+1)/2}\right)\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)^3q^{n(n+1)/6}\right). \end{aligned}$$

(3.9)

Proof. Dividing both sides of (3.3) by y and then allowing $y \to 0$, we obtain

(3.10)

$$3q^{1/(12)}(q;q)_\infty^2\theta_1'(0|3\tau)\theta_1(x|\frac{\tau}{3}) - q^{1/(12)}(q;q)_\infty^2\theta_1'(0|\frac{\tau}{3})\theta_1(3x|3\tau) = \theta_1'(0|\tau)\theta_1^3(x|\tau).$$

Using (3.6) in this equation, we obtain the following interesting identity:

$$(q;q)_\infty\theta_1^3(x|\tau) = 3q^{1/3}(q^3;q^3)_\infty^3\theta_1(x|\frac{\tau}{3}) - (q^{1/3};q^{1/3})_\infty^3\theta_1(3x|3\tau). \tag{3.11}$$

Differentiating both sides of the identity with respect to x three times, and then setting $x = 0$, we obtain

$$6(q;q)_\infty\theta_1'(0|\tau)^3 = 3q^{1/3}(q^3;q^3)_\infty^3\theta_1'''(0|\frac{\tau}{3}) - 27(q^{1/3};q^{1/3})_\infty^3\theta_1'''(0|3\tau). \tag{3.12}$$

Using (3.6) in the left side of this equation, we find that

$$48q^{3/8}(q;q)_\infty^{10} = 3q^{1/3}(q^3;q^3)_\infty^3\theta_1'''(0|\frac{\tau}{3}) - 27(q^{1/3};q^{1/3})_\infty^3\theta_1'''(0|3\tau). \tag{3.13}$$

Using (3.5) and (3.7) in the right side of the equation, we arrive at (3.9). This completes the proof of Theorem 6. □

4. Some identities for the Hirschhorn-Garvan-Borwein two-variable cubic theta functions

Noting the fact that $\theta_1(x-y|\tau) = -\theta_1(y-x|\tau)$, we can rewrite (3.3) as

$$\begin{aligned} &q^{1/(12)}(q;q)_\infty^2\theta_1(y|\frac{\tau}{3})\theta_1(3x|3\tau) - q^{1/(12)}(q;q)_\infty^2\theta_1(x|\frac{\tau}{3})\theta_1(3y|3\tau) \\ = \;&\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(y-x|\tau)\theta_1(x+y|\tau). \end{aligned} \tag{4.1}$$

Using the second equation of (1.7) in the above equation and then replacing e^{2ix} by x and e^{2iy} by y in the resulting equation, we obtain the following infinite product identity:

Theorem 7. *We have*

$$\begin{aligned} &(q;q)_\infty^2(q^{1/3},x,q^{1/3}/x;q^{1/3})_\infty(q^3,y^3,q^3/y^3;q^3)_\infty \\ - \;&yx^{-1}(q;q)_\infty^2(q^{1/3},y,q^{1/3}/y;q^{1/3})_\infty(q^3,x^3,q^3/x^3;q^3)_\infty \\ = \;&(q,x,q/x;q)_\infty(q,y,q/y;q)_\infty(q,y/x,qx/y;q)_\infty(q,xy,q/xy;q)_\infty. \end{aligned} \tag{4.2}$$

Appealing to the Jacobi triple product identity, we have

$$\begin{aligned} &(q;q)_\infty^2(q^{1/3},x,q^{1/3}/x;q^{1/3})_\infty\sum_{n=-\infty}^{\infty}(-1)^nq^{3n(n-1)/2}y^{3n} \\ - \;&yx^{-1}(q;q)_\infty^2(q^3,x^3,q^3/x^3;q^3)_\infty\sum_{n=-\infty}^{\infty}(-1)^nq^{n(n-1)/6}y^n \\ = \;&(q,x,q/x;q)_\infty\left(\sum_{n=-\infty}^{\infty}(-1)^nq^{n(n-1)/2}y^n\right) \\ \times \;&\left(\sum_{n=-\infty}^{\infty}(-1)^nq^{n(n-1)/2}y^nx^{-n}\right)\left(\sum_{n=-\infty}^{\infty}(-1)^nq^{n(n-1)/2}y^nx^n\right). \end{aligned} \tag{4.3}$$

It is easy to see that the coefficient of y on the right side of the above equation is

$$-(q,x,q/x;q)_\infty\sum_{m,n=-\infty}^{\infty}q^{m^2+mn+n^2-m-n}x^{n-m} \tag{4.4}$$

and the coefficient of y on the left side is

$$-x^{-1}(q;q)_\infty^2(q^3,x^3,q^3/x^3;q^3)_\infty. \tag{4.5}$$

By equating the above two quantities, we arrive at

$$\begin{aligned} &\sum_{m,n=-\infty}^{\infty}q^{m^2+mn+n^2-m-n}x^{n-m} \\ = \;&(q;q)_\infty(q^3;q^3)_\infty(1+x+x^{-1})\frac{(q^3x^3,q^3/x^3;q^3)_\infty}{(qx,q/x;q)_\infty}. \end{aligned} \tag{4.6}$$

Making the changes of indices $m \to -m$ and $n \to -n$, we obtain

$$\begin{aligned}&\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} x^{m-n}\\ (4.7)\qquad &= (q;q)_\infty (q^3;q^3)_\infty (1+x+x^{-1}) \frac{(q^3x^3, q^3/x^3; q^3)_\infty}{(qx, q/x; q)_\infty}.\end{aligned}$$

This is the same as in [8, p.675, Equation (1.23)].

Equating the terms independent of y in (4.3), we find that

$$\begin{aligned}&\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} x^{m-n}\\ &= q^{1/3}(q;q)_\infty (q^3;q^3)_\infty (1+x+x^{-1}) \frac{(q^3x^3, q^3/x^3; q^3)_\infty}{(qx, q/x; q)_\infty}\\ (4.8)\qquad &\quad + (q;q)_\infty (q^{1/3};q^{1/3})_\infty \frac{(q^{1/3}x, q^{1/3}x^{-1}; q^{1/3})_\infty}{(qx, q/x; q)_\infty}.\end{aligned}$$

Combining the above two equations, we find that

$$\begin{aligned}&(q;q)_\infty (q^{1/3};q^{1/3})_\infty \frac{(q^{1/3}x, q^{1/3}x^{-1}; q^{1/3})_\infty}{(qx, q/x; q)_\infty}\\ (4.9)\qquad &= \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} x^{m-n} - q^{1/3} \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} x^{m-n}.\end{aligned}$$

Let ω be the primitive cube root of unity given by $\omega = \exp(\frac{2\pi i}{3})$. Then we readily find that the right side of (4.9) is equal to

$$\sum_{m,n=-\infty}^{\infty} q^{(m^2+mn+n^2)/3} x^n \omega^{m-n} \tag{4.10}$$

(see [8, p. 678] for the details). Therefore, we have

$$\begin{aligned}&\sum_{m,n=-\infty}^{\infty} q^{(m^2+mn+n^2)/3} x^n \omega^{m-n}\\ (4.11)\qquad &= (q;q)_\infty (q^{1/3};q^{1/3})_\infty \frac{(q^{1/3}x, q^{1/3}x^{-1}; q^{1/3})_\infty}{(qx, q/x; q)_\infty}.\end{aligned}$$

Writing q as q^3, we obtain

$$\begin{aligned}&\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} x^n \omega^{m-n}.\\ (4.12)\qquad &= (q;q)_\infty (q^3;q^3)_\infty \frac{(qx, qx^{-1}; q)_\infty}{(q^3x, q^3/x; q^3)_\infty}.\end{aligned}$$

This is the same as in [8, p. 675, equation (1.22)].

If we denote

$$a(q,x) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} x^{m-n}, \tag{4.13}$$

$$b(q,x) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} x^{n}\omega^{m-n}, \tag{4.14}$$

$$c(q,x) = q^{1/3} \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} x^{m-n}, \tag{4.15}$$

then (4.8) can be written as

$$a(q,x) = c(q,x) + b(q^{1/3},x), \tag{4.16}$$

from which one can easily derive the following important identity of Hirschhorn, Garvan, and Borwein (see [8, p.682] for the details).

Theorem 8. *We have*

$$a^3(q,x) = c^3(q,x) + b^2(q,1)b(q,x^3). \tag{4.17}$$

5. The proof of Theorem 3

Using the identity $(z, zq, zq^2; q^3)_\infty = (z;q)_\infty$ and the infinite product representation for $\theta_1(z|\tau)$, we readily find that

$$\theta_1(z|\frac{\tau}{3}) = \frac{(q^{1/3};q^{1/3})_\infty}{(q;q)_\infty^3}\theta_1(z|\tau)\theta_1(z+\frac{\pi\tau}{3}|\tau)\theta_1(z-\frac{\pi\tau}{3}|\tau), \tag{5.1}$$

and we can also find that

$$\theta_1(\frac{\pi\tau}{3}|\tau) = iq^{-1/(24)}(q^{1/3};q^{1/3})_\infty \quad \text{and} \quad \theta_1(\frac{2\pi\tau}{3}|\tau) = iq^{-5/(24)}(q^{1/3};q^{1/3})_\infty. \tag{5.2}$$

We recall the definition of the Ramanujan function

$$a(q) := 1 + 6\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}}\right). \tag{5.3}$$

Using logarithmic differentiation on (1.7), we obtain

$$\frac{\theta_1'}{\theta_1}(z|\tau) = -i - 2i\sum_{n=0}^{\infty}\frac{q^n e^{2iz}}{1-q^n e^{2iz}} + 2i\sum_{n=1}^{\infty}\frac{q^n e^{-2iz}}{1-q^n e^{-2iz}}. \tag{5.4}$$

Comparing the above two equations, we infer that

$$a(q) = -2 + 3i\frac{\theta_1'}{\theta_1}(\pi\tau|3\tau). \tag{5.5}$$

It is well-known that the trigonometric series expansion for the logarithmic derivative of $\theta_1(z|\tau)$ is

$$\frac{\theta_1'}{\theta_1}(z|\tau) = \cot z + 4\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin 2nz \tag{5.6}$$

(see [16, p.489]).

Thus, we have

$$\frac{\theta_1'}{\theta_1}(\frac{\pi}{3}|\tau) = \frac{1}{\sqrt{3}}a(q). \tag{5.7}$$

Now we are ready to prove Theorem 3.

Proof. Differentiating both sides of (1.9) with respect to z and then setting $z = 0$, we obtain

$$\begin{aligned} &\theta_1'(0|\tau)\{f(y) - f(-y)\}\theta_1^3(x|\tau) - \theta_1'(0|\tau)\{f(x) - f(-x)\}\theta_1^3(y|\tau) \\ = \ &2f'(0)\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau). \end{aligned} \tag{5.8}$$

Inserting $\theta_1'(0|\tau) = 2q^{1/8}(q;q)_\infty^3$ in the above equation, we have

$$\begin{aligned} &q^{1/8}(q;q)_\infty^3\{f(y) - f(-y)\}\theta_1^3(x|\tau) - q^{1/8}(q;q)_\infty^3\{f(x) - f(-x)\}\theta_1^3(y|\tau) \\ &= f'(0)\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau). \end{aligned} \tag{5.9}$$

It is easy to verify that $\theta_1^3(x+\frac{\pi}{3}|\tau)$ satisfies (1.8), so we can take $f(x) = \theta_1^3(x+\frac{\pi}{3}|\tau)$ in (5.9). By (3.2) and (5.7) and a direct computation, we have

$$f'(0) = 3\theta_1^3(\frac{\pi}{3}|\tau)\left(\frac{\theta_1'}{\theta_1}\right)(\frac{\pi}{3}|\tau) = 9q^{3/8}(q^3;q^3)_\infty^3 a(q); \tag{5.10}$$

and thus we have

$$\begin{aligned} &q^{1/8}(q;q)_\infty^3\left\{\theta_1^3(y+\frac{\pi}{3}|\tau) + \theta_1^3(y-\frac{\pi}{3}|\tau)\right\}\theta_1^3(x|\tau) \\ - \ &q^{1/8}(q;q)_\infty^3\left\{\theta_1^3(x+\frac{\pi}{3}|\tau) + \theta_1^3(x-\frac{\pi}{3}|\tau)\right\}\theta_1^3(y|\tau) \\ = \ &9q^{3/8}(q^3;q^3)_\infty^3 a(q)\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau). \end{aligned} \tag{5.11}$$

Taking $y = \frac{\pi}{3}$ in the above equation and noting that $\theta_1(\frac{\pi}{3}|\tau) = \theta_1(\frac{2\pi}{3}|\tau) = \sqrt{3}q^{1/8}(q^3;q^3)_\infty$, we find that

$$\begin{aligned} &\theta_1^3(x|\tau) - \theta_1^3(x+\frac{\pi}{3}|\tau) - \theta_1^3(x-\frac{\pi}{3}|\tau) \\ = \ &3a(q)\frac{(q^3;q^3)_\infty}{(q;q)_\infty^3}\theta_1(x|\tau)\theta_1(x+\frac{\pi}{3}|\tau)\theta_1(x-\frac{\pi}{3}|\tau). \end{aligned} \tag{5.12}$$

From (3.1), we have

$$\theta_1(3x|3\tau) = -\frac{(q^3;q^3)_\infty}{(q;q)_\infty^3}\theta_1(x|\tau)\theta_1(x+\frac{\pi}{3}|\tau)\theta_1(x-\frac{\pi}{3}|\tau). \tag{5.13}$$

We substitute the above equation into the right side of (5.12) to obtain (1.12).

Proceeding through the same steps as before, by taking $f(x) = e^{2ix}\theta_1^3(x+\frac{\pi\tau}{3}|\tau)$ in (5.9) and then setting $y = \frac{\pi\tau}{3}$ and appealing to (5.1), (5.2), and (5.5), we can find that

$$\theta_1^3(x|\tau) - q^{1/6}e^{2ix}\theta_1^3(x+\frac{\pi\tau}{3}|\tau) - q^{1/6}e^{-2ix}\theta_1^3(x-\frac{\pi\tau}{3}|\tau) = a(q^{1/3})\theta_1(x|\frac{\tau}{3}). \tag{5.14}$$

Replacing q by q^3, we obtain (1.13). Thus, we complete the proof of Theorem 3. $\square$

6. The proof of Theorem 4

Proof. Taking $z = \frac{\pi}{3}$ in Theorem 1 and appealing to (3.1) and (3.2), we obtain

$$\begin{aligned} & \sqrt{3}q^{1/8}(q;q)_\infty^3 \left\{f(x) - f(-x)\right\} \theta_1(3y|3\tau) \\ = & \sqrt{3}q^{1/8}(q;q)_\infty^3 \left\{f(y) - f(-y)\right\} \theta_1(3x|3\tau) \\ + & \left\{f(\frac{\pi}{3}) - f(-\frac{\pi}{3})\right\} \theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x+y|\tau)\theta_1(x-y|\tau). \end{aligned} \tag{6.1}$$

It is easy to see that $\theta_1^3(x|\tau)$ satisfies all the conditions of Theorem 1. We choose $f(x) = \theta_1^3(x|\tau)$ and then use (3.2) in the resulting equation to obtain (1.14).

Taking $z = \frac{\pi\tau}{3}$ in Theorem 1 and appealing to (5.1) and (5.2), we obtain

$$\begin{aligned} & iq^{-1/(24)}(q;q)_\infty^3 \left\{f(x) - f(-x)\right\} \theta_1(y|\frac{\tau}{3}) \\ = & iq^{-1/(24)}(q;q)_\infty^3 \left\{f(y) - f(-y)\right\} \theta_1(x|\frac{\tau}{3}) \\ - & \left\{f(\frac{\pi\tau}{3}) - f(-\frac{\pi\tau}{3})\right\} \theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x+y|\tau)\theta_1(x-y|\tau). \end{aligned} \tag{6.2}$$

Choosing $f(x) = \theta_1^3(x|\tau)$ and then using (5.2) in the resulting equation, we obtain (1.15). This completes the proof of Theorem 4. □

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References

[1] G. E. Andrews, "The Theory of Partitions," Addison-Wesly, Reading, MA, 1976. MR 58:27738

[2] B. C. Berndt, "Ramanujan's Notebooks, Part III," Springer-Verlag, New York, 1991. MR 92j:01069

[3] B. C. Berndt, "Ramanujan's notebooks. Part IV," Springer-Verlag, New York, 1994. MR 95e:11028

[4] B. C. Berndt, On a certain theta-function in a letter of Ramanujan from Fitzroy House, Ganita 43 (1992), 33–43. MR 94b:11105

[5] B. C. Berndt, S. H. Chan, Z.-G. Liu, and H. Yesilyurt, A new identity for $(q;q)_\infty^{10}$ with an application to Ramanujan's partition congruence modulo 11, Quart. J. Math. **55** (2004), 13-30. MR 2043004

[6] K. Chandrasekharan, "Elliptic Functions," Springer-Verlag, Berlin Heideberg, 1985. MR 87e:11058

[7] M. D. Hirschhorn, "A simple proof of an identity of Ramanujan," J. Austral. Math. Soc. Ser. A, **34** (1983), 31-35. MR 84h:10067

[8] M. Hirschhorn, F. Garvan, and J. Borwein, Cubic analogues of the Jacobian theta function $\theta(z,q)$. Canad. J. Math. 45 (1993), 673–694. MR 94m:33011

[9] Z.-G. Liu, The Borweins' cubic theta function identity and some cubic modular identities of Ramanujan, Ramanujan J. 4 (2000), 43-50. MR 2001f:33026

[10] S. Kongsiriwong and Z.-G. Liu, Uniform proofs of q-series-product identities, Results Math. **44** (2003), 312-339. MR 2028683

[11] Z.-G. Liu, On certain identities of Ramanujan, J. Number Theory 83 (2000), 59-75. MR 2001f:11066

[12] Z.-G. Liu, Some Eisenstein series identities, J. Number Theory 85 (2000), 231–252. MR 2001k:11075

[13] Z.-G. Liu, Some theta function identities associated with the modular equations of degree 5, Integers: Electronic Journal of Combinatorial Number theory 1 (2001), A#03, 14pp. MR 2002c:33017

[14] Z.-G. Liu, Residue theorem and theta function identities, Ramanujan J. 5 (2001), 129-151. MR 2002g:11056

[15] Z.-G. Liu, Some Eisenstein series identities related to modular equations of the seventh order. Pacific J. Math. 209 (2003), 103–130. MR 2004c:11052

[16] E. T. Whittaker and G. N. Watson, A course of modern analysis, 4th ed, Cambridge Univ. Press, Cambridge, 1966. MR 97k:01072

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