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ON THE BEHAVIOR OF THE ALGEBRAIC TRANSFER

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Dedicated to Professor Huỳnh Mùi on the occasion of his sixtieth birthday

Abstract. Let $Tr_k : \mathbb{F}_2 \underset{GL_k}{\otimes} PH_i(B\mathbb{V}_k) \to Ext_{\mathcal{A}}^{k,k+i}(\mathbb{F}_2, \mathbb{F}_2)$ be the algebraic transfer, which is defined by W. Singer as an algebraic version of the geometrical transfer $tr_k : \pi_*^S((B\mathbb{V}_k)_+) \to \pi_*^S(S^0)$. It has been shown that the algebraic transfer is highly nontrivial and, more precisely, that Tr_k is an isomorphism for $k = 1, 2, 3$. However, Singer showed that Tr_5 is not an epimorphism. In this paper, we prove that Tr_4 does not detect the nonzero element $g_s \in Ext_{\mathcal{A}}^{4,12\cdot 2^s}(\mathbb{F}_2, \mathbb{F}_2)$ for every $s \geq 1$. As a consequence, the localized $(Sq^0)^{-1}Tr_4$ given by inverting the squaring operation Sq^0 is not an epimorphism. This gives a negative answer to a prediction by Minami.

1. Introduction and statement of results

The subject of the present paper is the algebraic transfer

$$Tr_k : \mathbb{F}_2 \underset{GL_k}{\otimes} PH_i(B\mathbb{V}_k) \to Ext_{\mathcal{A}}^{k,k+i}(\mathbb{F}_2, \mathbb{F}_2),$$

which is defined by W. Singer as an algebraic version of the geometrical transfer $tr_k : \pi_*^S((B\mathbb{V}_k)_+) \to \pi_*^S(S^0)$ to the stable homotopy groups of spheres. Here $\mathbb{V}_k$ denotes a k-dimensional $\mathbb{F}_2$-vector space, and $PH_*(B\mathbb{V}_k)$ is the primitive part consisting of all elements in $H_*(B\mathbb{V}_k)$ that are annihilated by every positive-degree operation in the mod 2 Steenrod algebra, $\mathcal{A}$. Throughout the paper, the homology is taken with coefficients in $\mathbb{F}_2$.

It has been proved that Tr_k is an isomorphism for $k = 1, 2$ by Singer [14] and for $k = 3$ by Boardman [1]. These data together with the fact that $Tr = \bigoplus_{k\geq 0} Tr_k$ is an algebra homomorphism (see [14]) show that Tr_k is highly nontrivial. Therefore, the algebraic transfer is considered to be a useful tool for studying the mysterious cohomology of the Steenrod algebra, $Ext_{\mathcal{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$. In [14], Singer also gave computations to show that Tr_4 is an isomorphism up to a range of internal degrees. However, he proved that Tr_5 is not an epimorphism.

Based on these data, we are particularly interested in the behavior of the fourth algebraic transfer. The following theorem is the main result of this paper.

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Theorem 1.1. *For each $s \geq 1$, the nonzero element $g_s \in Ext_{\mathcal{A}}^{4,12\cdot 2^s}(\mathbb{F}_2, \mathbb{F}_2)$ is not in the image of Tr_4.*

The reader is referred to May [11] for the generator g_1 and to Lin [8] or [9] for the generators g_s.

As a consequence, we get a negative answer to a prediction by Minami [13].

Corollary 1.2. *The localization of the fourth algebraic transfer*

$$(Sq^0)^{-1}Tr_4 : (Sq^0)^{-1}\mathbb{F}_2 \underset{GL_4}{\otimes} PH_*(B\mathbb{V}_4) \to (Sq^0)^{-1}Ext_{\mathcal{A}}^{4,4+*}(\mathbb{F}_2, \mathbb{F}_2)$$

given by inverting Sq^0 is not an epimorphism.

It is well known (see [10]) that there are squaring operations Sq^i ($i \geq 0$) acting on the cohomology of the Steenrod algebra, which share most of the properties with Sq^i on the cohomology of spaces. However, Sq^0 is not the identity. We refer to Section 2 for the precise meaning of the operation Sq^0 on the domain of the algebraic transfer.

We next explain the idea of the proof of Theorem 1.1.

Let $P_k := H^*(B\mathbb{V}_k)$ be the polynomial algebra of k variables, each of degree 1. Then, the domain of Tr_k, $\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$, is dual to $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)^{GL_k}$. In order to prove Theorem 1.1, it suffices to show that $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)^{GL_4}_{12\cdot 2^s - 4} = 0$, for every $s \geq 1$.

Direct calculation of $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)_{12\cdot 2^s - 4}$ is difficult, as P_4 in degree $12 \cdot 2^s - 4$ is a huge $\mathbb{F}_2$-vector space, e.g. its dimension is 1771 for $s = 1$. To compute it, we observe that the iterated dual squaring operation

$$(Sq^0_*)^s : (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)_{12\cdot 2^s - 4} \to (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)_8$$

is an isomorphism of GL_4-modules for any $s \geq 1$. This isomorphism is obtained by applying repeatedly the following proposition.

Proposition 1.3. *Let k and r be positive integers. Suppose that each monomial $x_1^{i_1} \cdots x_k^{i_k}$ of P_k in degree $2r + k$ with at least one exponent i_t even is hit. Then*

$$Sq^0_* : (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)_{2r+k} \to (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)_r$$

is an isomorphism of GL_k-modules.

Here, as usual, we say that a polynomial Q in P_k is *hit* if it is $\mathcal{A}$-decomposable.

Further, we show that $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)_8$ is an $\mathbb{F}_2$-vector space of dimension 55. Then, by investigating a specific basis of it, we prove that $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)^{GL_4}_8 = 0$. As a consequence, we get $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)^{GL_4}_{12\cdot 2^s - 4} = 0$ for every $s \geq 1$.

The reader who does not wish to follow the invariant theory computation above may be satisfied by the following weaker theorem, and then would not need to read the paper's last 3 sections.

Theorem 1.4. *Tr_4 is not an isomorphism.*

This theorem is proved by observing that, on the one hand,

$$(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)^{GL_4}_{20} \cong (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)^{GL_4}_8,$$

and on the other hand,

$$Ext_{\mathcal{A}}^{4,4+20}(\mathbb{F}_2,\mathbb{F}_2) = \mathbb{F}_2 \cdot g_1 \ncong Ext_{\mathcal{A}}^{4,4+8}(\mathbb{F}_2,\mathbb{F}_2) = 0.$$

The paper is divided into six sections and organized as follows. Section 2 starts with a recollection of the squaring operation and ends with a proof of the isomorphism $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)_{12\cdot 2^s-4} \cong (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)_8$. Theorem 1.4 is proved in Section 3. We compute $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)_8$ and its GL_4-invariants in Section 4. We prove Theorem 1.1 in Section 5. Finally, in Section 6, we describe the GL_4-module structure of $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)_8$.

2. A sufficient condition for the squaring operation to be an isomorphism

This section starts with a recollection of Kameko's squaring operation

$$Sq^0 : \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k) \to \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k).$$

The most important property of Kameko's Sq^0 is that it commutes with the classical Sq^0 on $Ext_{\mathcal{A}}^*(\mathbb{F}_2,\mathbb{F}_2)$ (defined in [10]) through the algebraic transfer (see [1], [13]).

This squaring operation is constructed as follows.

As is well known, $H^*(B\mathbb{V}_k)$ is the polynomial algebra, $P_k := \mathbb{F}_2[x_1,\dots,x_k]$, on k generators $x_1,\dots,x_k$, each of degree 1. By dualizing,

$$H_*(B\mathbb{V}_k) = \Gamma(a_1,\dots,a_k)$$

is the divided power algebra generated by $a_1,\dots,a_k$, each of degree 1, where a_i is dual to $x_i \in H^1(B\mathbb{V}_k)$. Here the duality is taken with respect to the basis of $H^*(B\mathbb{V}_k)$ consisting of all monomials in $x_1,\dots,x_k$.

In [6] and [7] Kameko defined a homomorphism

$$\begin{array}{rccc} Sq^0 : & H_*(B\mathbb{V}_k) & \to & H_*(B\mathbb{V}_k), \\ & a_1^{(i_1)}\cdots a_k^{(i_k)} & \mapsto & a_1^{(2i_1+1)}\cdots a_k^{(2i_k+1)}, \end{array}$$

where $a_1^{(i_1)}\cdots a_k^{(i_k)}$ is dual to $x_1^{i_1}\cdots x_k^{i_k}$. The following lemma is well known. We give a proof to make the paper self-contained.

Lemma 2.1. *Sq^0 is a GL_k-homomorphism.*

Proof. We use the explanation of Sq^0 by Crabb and Hubbuck [3], which does not depend on the chosen basis of $H_*(B\mathbb{V}_k)$. The element $a(\mathbb{V}_k) = a_1\cdots a_k$ is nothing but the image of the generator of $\Lambda^k(\mathbb{V}_k)$ under the (skew) symmetrization map

$$\Lambda^k(\mathbb{V}_k) \to H_k(B\mathbb{V}_k) = \Gamma_k(\mathbb{V}_k) = (\underbrace{\mathbb{V}_k \otimes \cdots \otimes \mathbb{V}_k}_{k \text{ times}})_{S_k},$$

where the symmetric group S_k acts on $\mathbb{V}_k \otimes\cdots\otimes \mathbb{V}_k$ by permutations of the factors. Let $c : H_*(B\mathbb{V}_k) \to H_*(B\mathbb{V}_k)$ be the degree-halving epimorphism, which is dual to the Frobenius monomorphism $F : H^*(B\mathbb{V}_k) \to H^*(B\mathbb{V}_k)$ defined by $F(x) = x^2$ for any x. We have

$$Sq^0(c(y)) = a(\mathbb{V}_k)y,$$

for $y \in H_*(B\mathbb{V}_k)$. To prove that this is well defined we need to show that if $c(y) = 0$, then $a(\mathbb{V}_k)y = 0$. Indeed, $c(y) = 0$ implies $\langle c(y), x\rangle = \langle y, x^2\rangle = 0$ for every $x \in H^*(B\mathbb{V}_k)$. Here $\langle\cdot,\cdot\rangle$ denotes the dual pairing between $H_*(B\mathbb{V}_k)$ and

$H^*(B\mathbb{V}_k)$. So, if we write $y = \sum a_1^{(i_1)} \cdots a_k^{(i_k)}$, then there is at least one i_t which is odd in each term of the sum. Therefore,

$$a(\mathbb{V}_k)y = a_1 \cdots a_k(\sum a_1^{(i_1)} \cdots a_k^{(i_k)}) = 0,$$

because $a_t a_t^{(i_t)} = 0$ for any odd i_t. So, Sq^0 is well defined.

As c is a GL_k-epimorphism, the map Sq^0 is a GL_k-homomorphism.

The lemma is proved. □

Further, it is easy to see that $cSq_*^{2t+1} = 0$, $cSq_*^{2t} = Sq_*^t c$. So we have

$$Sq_*^{2t+1} Sq^0 = 0, \ \ Sq_*^{2t} Sq^0 = Sq^0 Sq_*^t.$$

(See [4] for an explicit proof.) Therefore, Sq^0 maps $PH_*(B\mathbb{V}_k)$ to itself.

Kameko's Sq^0 is defined by

$$Sq^0 = 1 \underset{GL_k}{\otimes} Sq^0 : \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k) \to \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k).$$

The dual homomorphism $Sq_*^0 : P_k \to P_k$ of Sq^0 is obviously given by

$$Sq_*^0(x_1^{j_1} \cdots x_k^{j_k}) = \begin{cases} x_1^{\frac{j_1-1}{2}} \cdots x_k^{\frac{j_k-1}{2}}, & j_1, \ldots, j_k \text{ odd}, \\ 0, & \text{otherwise}. \end{cases}$$

Hence

$$Ker(Sq_*^0 : P_k \to P_k) = \overline{Even},$$

where $\overline{Even}$ denotes the vector subspace of P_k spanned by all monomials $x_1^{i_1} \cdots x_k^{i_k}$ with at least one exponent i_t even.

Let $s : P_k \to P_k$ be a right inverse of Sq_*^0 defined as follows:

$$s(x_1^{i_1} \cdots x_k^{i_k}) = x_1^{2i_1+1} \cdots x_k^{2i_k+1}.$$

It should be noted that s does not commute with the doubling map on $\mathcal{A}$, that is, in general

$$Sq^{2t} s \neq s Sq^t.$$

However, in one particular circumstance we have the following.

Lemma 2.2. *Under the hypothesis of Proposition 1.3, the map*

$$\begin{array}{rcl} \overline{s} : (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)_r & \to & (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)_{2r+k}, \\ \overline{s}[X] & = & [sX] \end{array}$$

is a well-defined linear map.

Proof. We start with an observation that

$$Im(Sq^{2t} s - s Sq^t) \subset \overline{Even}.$$

We prove this by showing equivalently that

$$Sq_*^0(Sq^{2t} s - s Sq^t) = 0.$$

Indeed,

$$\begin{array}{rcl} Sq_*^0(Sq^{2t} s - s Sq^t) & = & Sq_*^0 Sq^{2t} s - Sq_*^0 s Sq^t \\ & = & Sq^t Sq_*^0 s - Sq_*^0 s Sq^t \\ & = & Sq^t \cdot id - id \cdot Sq^t \\ & = & 0. \end{array}$$

As a consequence, s maps $(\mathcal{A}^+P_k)_r$ to $(\mathcal{A}^+P_k + \overline{Even})_{2r+k}$. Here and in what follows, $\mathcal{A}^+$ denotes the submodule of $\mathcal{A}$ consisting of all positive degree operations. Further, by the hypothesis of Proposition 2.3, we have

$$(\mathcal{A}^+P_k + \overline{Even})_{2r+k} \subset (\mathcal{A}^+P_k)_{2r+k}.$$

Hence, s maps $(\mathcal{A}^+P_k)_r$ to $(\mathcal{A}^+P_k)_{2r+k}$. So the map $\overline{s}$ is well defined. Then it is a linear map, as s is.

The lemma is proved. □

The following proposition is also numbered as Proposition 1.3.

Proposition 2.3. *Let k and r be positive integers. Suppose that each monomial $x_1^{i_1}\cdots x_k^{i_k}$ of P_k in degree $2r+k$ with at least one exponent i_t even is hit. Then*

$$Sq^0_* : (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)_{2r+k} \to (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)_r$$

is an isomorphism of GL_k-modules.

Proof. On the one hand, we have $Sq^0_*\overline{s} = id_{(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)_r}$. Indeed, from $Sq^0_* s = id_{P_k}$, it follows that

$$Sq^0_*\overline{s}[X] = Sq^0_*[sX] = [Sq^0_* sX] = [X],$$

for any X in degree r of P_k.

On the other hand, we have $\overline{s}Sq^0_* = id_{(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)_{2r+k}}$. Indeed, by the hypothesis, any monomial with at least one even exponent represents the 0 class in $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)_{2r+k}$, so we need only to check on the classes of monomials with all exponents odd. We have

$$\begin{aligned} \overline{s}Sq^0_*[x_1^{2i_1+1}\cdots x_k^{2i_k+1}] &= \overline{s}[x_1^{i_1}\cdots x_k^{i_k}] \\ &= [s(x_1^{i_1}\cdots x_k^{i_k})] \\ &= [x_1^{2i_1+1}\cdots x_k^{2i_k+1}], \end{aligned}$$

for any $x_1^{2i_1+1}\cdots x_k^{2i_k+1}$ in degree $2r+k$ of P_k.

Combining the two equalities, $Sq^0_*\overline{s} = id_{(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)_r}$ and $\overline{s}Sq^0_* = id_{(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)_{2r+k}}$, we see that $Sq^0_* : (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)_{2r+k} \to (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)_r$ is an isomorphism with inverse $\overline{s} : (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)_r \to (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)_{2r+k}$.

The proposition is proved. □

The target of this section is the following.

Lemma 2.4. *For every positive integer s,*

$$(Sq^0_*)^s : (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)_{12\cdot 2^s-4} \to (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)_8$$

is an isomorphism of GL_4-modules.

Proof. By using Proposition 2.3 repeatedly, it suffices to show that any monomial of P_4 in degree $m = 12\cdot 2^s - 4$ with at least one even exponent is hit. Since m is even, the number of even exponents in such a monomial must be either 2 or 4. If all exponents of the monomial are even, then it is hit by Sq^1. Hence we need only to consider the case of a monomial R with exactly two even exponents (and so exactly two odd exponents). Wood proves ([15]) that if $\alpha(m+\alpha_0(R)) > \alpha_0(R)$

then R is hit, where $\alpha_0(R)$ is the number of odd exponents in the monomial R, and $\alpha(n)$ is the number of ones in the binary expansion of n. We have $\alpha_0(R) = 2$ and $\alpha(m + \alpha_0(R)) = \alpha(12 \cdot 2^s - 2) = s + 2$, so Wood's criterion is met, and R is hit.

The lemma is proved. $\square$

3. The fourth algebraic transfer is not an isomorphism

The target of this section is to prove the following theorem, which is also numbered as Theorem 1.4.

Theorem 3.1.

$$Tr_4 : \mathbb{F}_2 \underset{GL_4}{\otimes} PH_i(B\mathbb{V}_4) \to Ext_{\mathcal{A}}^{4,4+i}(\mathbb{F}_2, \mathbb{F}_2)$$

is not an isomorphism.

Proof. For any r, we have a commutative diagram

$$\begin{array}{ccc}
(\mathbb{F}_2 \underset{GL_4}{\otimes} PH_i(B\mathbb{V}_4))_r & \xrightarrow{Tr_4} & Ext_{\mathcal{A}}^{4,4+r}(\mathbb{F}_2, \mathbb{F}_2) \\
\downarrow{\scriptstyle Sq^0} & & \downarrow{\scriptstyle Sq^0} \\
(\mathbb{F}_2 \underset{GL_4}{\otimes} PH_i(B\mathbb{V}_4))_{2r+4} & \xrightarrow{Tr_4} & Ext_{\mathcal{A}}^{4,8+2r}(\mathbb{F}_2, \mathbb{F}_2)\,,
\end{array}$$

where the first vertical arrow is the Kameko Sq^0 and the second vertical one is the classical Sq^0.

The dual statement of Lemma 2.4 for $s = 2$ claims that

$$Sq^0 : (\mathbb{F}_2 \underset{GL_4}{\otimes} PH_i(B\mathbb{V}_4))_8 \to (\mathbb{F}_2 \underset{GL_4}{\otimes} PH_i(B\mathbb{V}_4))_{20}$$

is an isomorphism. On the other hand, it is known (May [11]) that

$$Ext_{\mathcal{A}}^{4,4+8}(\mathbb{F}_2, \mathbb{F}_2) = 0 \ncong Ext_{\mathcal{A}}^{4,4+20}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2 \cdot g_1.$$

This implies that Tr_4 is not an isomorphism. The theorem is proved. $\square$

Remark 3.2. This proof does not show whether Tr_4 fails to be a monomorphism or fails to be an epimorphism. We will see that actually Tr_4 is not an epimorphism in Section 5 below.

4. GL_4-invariants of the indecomposables of P_4 in degree 8

From now on, let us write $x = x_1$, $y = x_2$, $z = x_3$ and $t = x_4$ and denote the monomial $x^a y^b z^c t^d$ by (a, b, c, d) for abbreviation.

Proposition 4.1. $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)_8$ *is an* $\mathbb{F}_2$*-vector space of dimension* 55 *with a basis consisting of the classes represented by the following monomials:*

(A) $(7,1,0,0), (7,0,1,0), (7,0,0,1), (1,7,0,0), (1,0,7,0), (1,0,0,7),$
$(0,7,1,0), (0,7,0,1), (0,1,7,0), (0,1,0,7), (0,0,7,1), (0,0,1,7),$

(B) $(3,3,1,1), (3,1,3,1), (3,1,1,3), (1,3,3,1), (1,3,1,3), (1,1,3,3),$

(C) $(6,1,1,0),(6,1,0,1),(6,0,1,1),(1,6,1,0),(1,6,0,1),(1,1,6,0),$
$(1,1,0,6),(1,0,6,1),(1,0,1,6),(0,6,1,1),(0,1,6,1),(0,1,1,6),$

(D) $(5,3,0,0),(5,0,3,0),(5,0,0,3),(0,5,3,0),(0,5,0,3),(0,0,5,3),$

(E) $(5,2,1,0),(5,2,0,1),(5,0,2,1),(2,5,1,0),(2,5,0,1),(2,1,5,0),$
$(2,1,0,5),(2,0,5,1),(2,0,1,5),(0,5,2,1),(0,2,5,1),(0,2,1,5),$

(F) $(5,1,1,1),(1,5,1,1),(1,1,5,1),(1,1,1,5),$

(G) $(4,2,1,1),(4,1,2,1),(1,4,2,1).$

The proposition is proved by combining a couple of lemmas.

Lemma 4.2. $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)_8$ *is generated by the* 55 *elements listed in Proposition 4.1.*

Proof. It is easy to see that every monomial (a,b,c,d) with a,b,c,d all even is hit (more precisely by Sq^1).

The only monomials (a,b,c,d) in degree 8 with at least one of a,b,c,d odd are the following up to permutations of the variables:

$$(7,1,0,0),(3,3,1,1),(6,1,1,0),(5,3,0,0),(5,2,1,0),(5,1,1,1),(4,2,1,1),$$
$$(4,3,1,0),(3,3,2,0),(3,2,2,1).$$

The last 3 monomials and their permutations are expressed in terms of the first 7 monomials and their permutations as follows:

$$\begin{aligned}
(4,3,1,0) &= (2,5,1,0)+Sq^4(1,2,1,0)+Sq^2(2,3,1,0),\\
(3,3,2,0) &= (5,2,1,0)+(2,5,1,0)+Sq^4(2,1,1,0)+Sq^4(1,2,1,0)\\
&\quad +Sq^2(3,2,1,0)+Sq^2(2,3,1,0)+Sq^1(3,3,1,0),\\
(3,2,2,1) &= (5,1,1,1)+(4,2,1,1)+(4,1,2,1)\\
&\quad +Sq^2(3,1,1,1)+Sq^1(4,1,1,1)+Sq^1(3,2,1,1)+Sq^1(3,1,2,1).
\end{aligned}$$

Hence, $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)_8$ is generated by the following 7 monomials and their permutations:

$$(7,1,0,0),(3,3,1,1),(6,1,1,0),(5,3,0,0),(5,2,1,0),(5,1,1,1),(4,2,1,1).$$

By the family of a monomial (a,b,c,d) we mean the set of all monomials which are obtained from (a,b,c,d) by permutations of the variables.

The monomials in the 7 families above which are not in Proposition 4.1 can be expressed in terms of the 55 elements listed there as follows. (We give only one expression from each symmetry class.)

$$\begin{aligned}
(3,5,0,0) &= (5,3,0,0)+Sq^4(2,2,0,0)+Sq^2(3,3,0,0),\\
(5,1,2,0) &= (6,1,1,0)+(5,2,1,0)+Sq^1(5,1,1,0),\\
(4,1,1,2) &= (4,2,1,1)+(4,1,2,1)+Sq^1(4,1,1,1),\\
(2,4,1,1) &= (4,2,1,1)+Sq^4(1,1,1,1)+Sq^2(2,2,1,1),\\
(2,1,1,4) &= (4,2,1,1)+(4,1,2,1)\\
&\quad +Sq^4(1,1,1,1)+Sq^2(2,1,1,2)+Sq^1(4,1,1,1),
\end{aligned}$$

$$\begin{aligned}
(1,4,1,2) &= (4,2,1,1)+(1,4,2,1)\\
&\quad +Sq^4(1,1,1,1)+Sq^2(2,2,1,1)+Sq^1(1,4,1,1),\\
(1,2,1,4) &= (4,2,1,1)+(1,4,2,1)\\
&\quad +Sq^2(2,2,1,1)+Sq^2(1,2,1,2)+Sq^1(1,4,1,1),\\
(1,1,4,2) &= (4,1,2,1)+(1,4,2,1)\\
&\quad +Sq^2(2,1,2,1)+Sq^2(1,2,2,1)+Sq^1(1,1,4,1),\\
(1,1,2,4) &= (4,1,2,1)+(1,4,2,1)+Sq^4(1,1,1,1)\\
&\quad +Sq^2(2,1,2,1)+Sq^2(1,2,2,1)+Sq^2(1,1,2,2)+Sq^1(1,1,4,1).
\end{aligned}$$

The lemma is proved. □

Lemma 4.3. *The* 55 *elements listed in Proposition 4.1 are linearly independent in* $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)_8$.

Proof. We will use an equivalence relation defined by saying that, for two polynomials P and Q, P is equivalent to Q, denoted by $P \sim Q$, if $P - Q$ is hit.

If X is one of the letters from A to G, let X_i be the i-th element in family X according to the order listed in Proposition 4.1. (This is the lexicographical order in each family.)

Suppose there is a linear relation between the 55 elements listed there,

$$\sum_{i=1}^{12} a_i A_i + \sum_{i=1}^{6} b_i B_i + \sum_{i=1}^{12} c_i C_i + \sum_{i=1}^{6} d_i D_i + \sum_{i=1}^{12} e_i E_i + \sum_{i=1}^{4} f_i F_i + \sum_{i=1}^{3} g_i G_i = 0,$$

where $a_i, b_i, c_i, d_i, e_i, f_i, g_i \in \mathbb{F}_2$. We need to show that all these coefficients are zero. The proof is divided into 4 steps.

Step 1. We call a monomial *a spike* if each of its exponents is of the form $2^n - 1$ for some n. It is well known that spikes do not appear in the expression of Sq^iY for any i positive and any monomial Y, since the powers x^{2^n-1} are not hit in the one variable case. Hence, the coefficient of any spike is zero in every linear relation in $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k$.

Among the 55 elements of Proposition 4.1, the classes of families A and B are spikes. So $a_i = b_j = 0$, for every i and j. Then, we get

$$\sum_{i=1}^{12} c_i C_i + \sum_{i=1}^{6} d_i D_i + \sum_{i=1}^{12} e_i E_i + \sum_{i=1}^{4} f_i F_i + \sum_{i=1}^{3} g_i G_i = 0.$$

Step 2. Consider the homomorphism $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4 \to \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_2$ induced by the projection $P_4 \to P_4/(z,t) \cong P_2$. Under this homomorphism, the image of the above linear relation is $d_1(5,3) = 0$.

In order to show that $d_1 = 0$, we need to prove that $(5,3)$ is nonzero in $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_2$. The linear transformation $x \mapsto x, y \mapsto x + y$ sends $(5,3)$ to $(8,0) + (7,1) + (6,2) + (5,3) \sim (7,1) + (5,3)$. As the action of the Steenrod algebra commutes with linear maps, if $(5,3)$ is hit then so is $(7,1) + (5,3)$. But it is impossible, because $(7,1)$ is a spike. Hence, $(5,3) \neq 0$ in $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_2$ and $d_1 = 0$.

Similarly, using all the projections of P_4 to its quotients by the ideals generated by each pair of the four variables, we get $d_i = 0$ for every i. So we get

$$\sum_{i=1}^{12} c_i C_i + \sum_{i=1}^{12} e_i E_i + \sum_{i=1}^{4} f_i F_i + \sum_{i=1}^{3} g_i G_i = 0.$$

Step 3. Consider the homomorphism $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4 \to \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_3$ induced by the projection $P_4 \to P_4/(t) \cong P_3$. Under this homomorphism, the linear relation above is sent to

$$c_1(6,1,1) + c_4(1,6,1) + c_6(1,1,6) + e_1(5,2,1) + e_4(2,5,1) + e_6(2,1,5) = 0.$$

Applying the linear map $x \mapsto x, y \mapsto x, z \mapsto y$ to this relation, we obtain

$$\begin{aligned}&(c_1 + c_4 + e_1 + e_4)(7,1) + c_6(2,6) + e_6(3,5)\\ &= (c_1 + c_4 + e_1 + e_4)(7,1) + e_6(3,5) = 0.\end{aligned}$$

Since $(7,1)$ is a spike, $(c_1 + c_4 + e_1 + e_4) = 0$, hence $e_6(3,5) = 0$. As for $(5,3)$, we can show that $(3,5) \neq 0 \in \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_2$ and get $e_6 = 0$.

By similar arguments, we have $e_1 = e_4 = e_6 = 0$. The equality $(c_1+c_4+e_1+e_4) = 0$ shows that $c_1 + c_4 = 0$ or $c_1 = c_4$. By similar arguments, $c_1 = c_4 = c_6$. We denote this common coefficient by c and get

$$c\{(6,1,1) + (1,6,1) + (1,1,6)\} = 0.$$

We prove that $c = 0$ by showing that $(6,1,1) + (1,6,1) + (1,1,6) \neq 0$. Suppose the contrary, that $(6,1,1) + (1,6,1) + (1,1,6)$ is hit. Then, by the unstable property of the action of $\mathcal{A}$ on the polynomial algebra, we have

$$(6,1,1) + (1,6,1) + (1,1,6) = Sq^1(P) + Sq^2(Q) + Sq^4(R),$$

for some polynomials P, Q, R. By the degree information, $Sq^4(R) = R^2$ and this element is hit by Sq^1. Therefore, it suffices to assume $(6,1,1)+(1,6,1)+(1,1,6) = Sq^1(P) + Sq^2(Q)$.

Let $Sq^2Sq^2Sq^2$ act on the both sides of this equality. The right hand side is sent to zero, as $Sq^2Sq^2Sq^2$ annihilates Sq^1 and Sq^2. On the other hand,

$$Sq^2Sq^2Sq^2\{(6,1,1) + (1,6,1) + (1,1,6)\} = (8,4,2) + \text{symmetries} \neq 0.$$

This is a contradiction. So, it implies $(6,1,1) + (1,6,1) + (1,1,6) \neq 0$ and $c = 0$. We get

$$\sum_{i=1}^{4} f_i F_i + \sum_{i=1}^{3} g_i G_i = 0.$$

Step 4. Apply the linear map $x \mapsto x, y \mapsto y, z \mapsto y, t \mapsto y$ to the above equality, and we have

$$\begin{aligned}&f_1(5,3) + (f_2 + f_3 + f_4 + g_3)(1,7) + (g_1 + g_2)(4,4)\\ &= f_1(5,3) + (f_2 + f_3 + f_4 + g_3)(1,7) = 0.\end{aligned}$$

As $(7,1)$ is a spike, we obtain $(f_2+f_3+f_4+g_3) = 0$ and $f_1(5,3) = 0$. As $(5,3) \neq 0$, it yields $f_1 = 0$.

Next, apply the linear map $x \mapsto x, y \mapsto y, z \mapsto x, t \mapsto x$ to the equality $\sum_{i\neq 1} f_i F_i + \sum_{i=1}^{3} g_i G_i = 0$, and we have

$$\begin{aligned}&f_2(3,5) + (f_3 + f_4 + g_2)(7,1) + g_1(6,2) + g_3(4,4)\\ &= f_2(3,5) + (f_3 + f_4 + g_2)(7,1) = 0.\end{aligned}$$

As $(7,1)$ is a spike, we get $(f_3+f_4+g_2)=0$ and $f_2(3,5)=0$. Since $(3,5)\neq 0$, it implies $f_2=0$.

Similarly, apply the linear map $x\mapsto x, y\mapsto x, z\mapsto y, t\mapsto x$ to the equality $f_3F_3+f_4F_4+\sum_{i=1}^3 g_iG_i=0$, and we have

$$\begin{aligned} &f_3(3,5)+(f_4+g_1)(7,1)+(g_2+g_3)(6,2)\\ &\quad = f_3(3,5)+(f_4+g_1)(7,1)=0. \end{aligned}$$

As $(7,1)$ is a spike, we get $f_4+g_1=0$ and then $f_3=0$.

Finally, apply the linear map $x\mapsto x, y\mapsto x, z\mapsto x, t\mapsto y$ to the equality $f_4F_4+\sum_{i=1}^3 g_iG_i=0$, and we have

$$\begin{aligned} &f_4(3,5)+(g_1+g_2+g_3)(7,1)\\ &\quad = f_4(3,5)+(g_1+g_2+g_3)(7,1)=0. \end{aligned}$$

As $(7,1)$ is a spike, we get $g_1+g_2+g_3=0$ and then $f_4=0$.

Substituting $f_1=f_2=f_3=f_4=0$ into the equations $(f_2+f_3+f_4+g_3)=0$, $(f_3+f_4+g_2)=0$, $f_4+g_1=0$, we get $g_1=g_2=g_3=0$.

We have shown that all coefficients of an arbitrary linear relation between the 55 elements listed in Proposition 4.1 are zero. The lemma follows. $\square$

Combining Lemmas 4.2–4.3, we get Proposition 4.1.

Proposition 4.4. $(\mathbb{F}_2\underset{\mathcal{A}}{\otimes}P_4)_8^{GL_4}=0$.

Proof. If X is one of the letters A,B,C,D,E,F,G, let $\mathcal{L}(X)$ be the vector subspace of $(\mathbb{F}_2\underset{\mathcal{A}}{\otimes}P_4)_8$ spanned by the elements of family X in Proposition 4.1. Let S_k denote the symmetric subgroup of GL_k. According to the relations listed in the proof of Lemma 4.2, $\mathcal{L}(A)$, $\mathcal{L}(B)$, $\mathcal{L}(C)$, $\mathcal{L}(D)$, $\mathcal{L}(F)$, $\mathcal{L}(G)$ are S_4-submodules. The subspace $\mathcal{L}(E)$ is not an S_4-submodule. However, the sum

$$\mathcal{L}(C,E)=\mathcal{L}(C)\oplus\mathcal{L}(E)$$

is. We have a decomposition of S_4-modules

$$(\mathbb{F}_2\underset{\mathcal{A}}{\otimes}P_4)_8=\mathcal{L}(A)\oplus\mathcal{L}(B)\oplus\mathcal{L}(C,E)\oplus\mathcal{L}(D)\oplus\mathcal{L}(F)\oplus\mathcal{L}(G).$$

Let α be an arbitrary GL_4-invariant in $(\mathbb{F}_2\underset{\mathcal{A}}{\otimes}P_4)_8$. It can uniquely be written in the form

$$\alpha=\alpha_A+\alpha_B+\alpha_{C,E}+\alpha_D+\alpha_F+\alpha_G,$$

where $\alpha_X\in\mathcal{L}(X)$ for $X\in\{A,B,D,F,G\}$, and $\alpha_{C,E}\in\mathcal{L}(C,E)$. Each term of this sum is S_4-invariant.

Note that if a linear combination of elements in a family is S_4-invariant, then all of its coefficients are equal, because each element in the family can be obtained from any other by a suitable permutation. Let s_X denote the sum of all the elements in the family X listed in Proposition 4.1. Then, we have $\alpha_A=as_A$, $\alpha_B=bs_B$, $\alpha_D=ds_D$, $\alpha_F=fs_F$, $\alpha_G=gs_G$, and $\alpha_{C,E}=cs_C+es_E$, where $a,b,c,d,e,f,g\in\mathbb{F}_2$.

Let p be the transposition given by $p(x)=y$, $p(y)=x$, $p(z)=z$, $p(t)=t$. It is easy to see that

$$\begin{aligned} p(2,1,0,5) &= (1,2,0,5)=(2,1,0,5)+(1,1,0,6),\\ p(2,1,5,0) &= (1,2,5,0)=(2,1,5,0)+(1,1,6,0). \end{aligned}$$

Further, the 10 elements different from $(2,1,0,5)$ and $(2,1,5,0)$ in family E are divided into 5 pairs with p acting on each pair by twisting. So, $p(s_E) = s_E + (1,1,0,6) + (1,1,6,0)$. On the other hand, as the family C is full, in the sense that it contains all the variable permutations of a monomial, we have $p(s_C) = s_C$. Hence, we get

$$p(\alpha_{C,E}) = p(cs_C + es_E) = cs_C + es_E + e(1,1,0,6) + e(1,1,6,0).$$

As $\alpha_{C,E}$ is S_4-invariant, $e(1,1,0,6) + e(1,1,6,0) = 0$. So $e = 0$, because the two elements are linearly independent by Lemma 4.3. We obtain

$$\alpha = \alpha_A + \alpha_B + \alpha_C + \alpha_D + \alpha_F + \alpha_G,$$

where $\alpha_C = \alpha_{C,E} = cs_C$.

Let us now consider the transvection φ given by $\varphi(x) = x$, $\varphi(y) = y$, $\varphi(z) = z$, $\varphi(t) = x + t$. A routine computation shows

$$\begin{aligned}
\varphi(s_A) &= s_A + (7,1,0,0) + (7,0,1,0) + (7,0,0,1) + (1,7,0,0) + (1,0,7,0) \\
&\quad + (6,1,0,1) + (6,0,1,1) + (1,1,0,6) + (1,0,1,6), \\
\varphi(s_B) &= s_B + (6,1,1,0) + (1,6,1,0) + (1,1,6,0) + (2,5,1,0) + (2,1,5,0) \\
&\quad + (5,1,1,1) + (1,5,1,1) + (1,1,5,1) + (4,2,1,1) + (4,1,2,1) \\
&\quad + (3,3,1,1) + (3,1,3,1), \\
\varphi(s_C) &= s_C + (6,1,1,0) + (1,6,1,0) + (1,1,6,0), \\
\varphi(s_D) &= s_D + (7,0,0,1) + (1,6,0,1) + (1,0,6,1) + (5,3,0,0) + (5,0,3,0), \\
\varphi(s_F) &= s_F + (2,5,1,0) + (2,1,5,0) + (5,1,1,1) + (4,2,1,1) + (4,1,2,1), \\
\varphi(s_G) &= s_G + (6,1,1,0).
\end{aligned}$$

Let $r_X = \varphi(s_X) - s_X$ where X is one of the letters A, B, C, D, F, G. The equality $\varphi(\alpha) = \alpha$ is rewritten as

$$\varphi(as_A + bs_B + cs_C + ds_D + fs_F + gs_G) = as_A + bs_B + cs_C + ds_D + fs_F + gs_G,$$

or equivalently

$$ar_A + br_B + cr_C + dr_D + fr_F + gr_G = 0.$$

In this linear combination, r_B and r_D are the only terms containing $(3,3,1,1)$ in family B and $(5,3,0,0)$ in family D respectively. From Lemma 4.3, we get $b = d = 0$, and therefore $ar_A + cr_C + fr_F + gr_G = 0$.

In the new linear combination, as r_A, r_C and r_F are the only terms containing $(7,1,0,0)$ in family A, $(1,6,1,0)$ in family C and $(4,2,1,1)$ in family F respectively, we have $a = c = f = 0$. As a consequence, $gr_G = 0$, so we finally get $g = 0$.

In summary, we have shown that every GL_4-invariant α in $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)_8$ equals zero. The proposition is proved. $\square$

5. The fourth algebraic transfer is not an epimorphism

The goal of this paper is to prove the following theorem, which is also numbered as Therem 1.1.

Theorem 5.1. *For each $s \geq 1$,*

$$Tr_4 : \mathbb{F}_2 \underset{GL_4}{\otimes} PH_i(B\mathbb{V}_4) \to Ext_{\mathcal{A}}^{4,4+i}(\mathbb{F}_2, \mathbb{F}_2)$$

does not detect the nonzero element $g_s \in Ext_{\mathcal{A}}^{4,12\cdot 2^s}(\mathbb{F}_2, \mathbb{F}_2)$.

Proof. Combining Lemma 2.4 and Proposition 4.4 we get

$$(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)^{GL_4}_{12\cdot 2^s-4} = 0,$$

for every nonnegative integer s.

On the other hand, it is well known that $Ext_{\mathcal{A}}^{4,24}(\mathbb{F}_2, \mathbb{F}_2)$ is spanned by the generator g_1 (see May [11]). Further, $g_s = (Sq^0)^{s-1}(g_1)$ is nonzero in $Ext_{\mathcal{A}}^{4,12\cdot 2^s}(\mathbb{F}_2, \mathbb{F}_2)$ (see Lin [8] and also [9]).

As $\mathbb{F}_2 \underset{GL_4}{\otimes} PH_{12\cdot 2^s-4}(B\mathbb{V}_4)$ is dual to $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)^{GL_4}_{12\cdot 2^s-4}$,

$$Tr_4 : \mathbb{F}_2 \underset{GL_4}{\otimes} PH_{12\cdot 2^s-4}(B\mathbb{V}_4) \to Ext_{\mathcal{A}}^{4,12\cdot 2^s}(\mathbb{F}_2, \mathbb{F}_2)$$

does not detect the generator g_s, for every nonnegative integer s.

The theorem is proved. □

As a consequence, we get a negative answer to a prediction by Minami [13]. (This corollary is also numbered as Corollary 1.2.)

Corollary 5.2. *The localization of the fourth algebraic transfer*

$$(Sq^0)^{-1}Tr_4 : (Sq^0)^{-1}\mathbb{F}_2 \underset{GL_4}{\otimes} PH_*(B\mathbb{V}_4) \to (Sq^0)^{-1}Ext_{\mathcal{A}}^{4,4+*}(\mathbb{F}_2, \mathbb{F}_2)$$

given by inverting Sq^0 is not an epimorphism.

Proof. Indeed, it does not detect the nonzero element g, which is represented by the family $(g_s)_{s>0}$ with $g_s = (Sq^0)^{s-1}(g_1)$. The corollary follows. □

Remark 5.3. Our result does not affect Singer's conjecture that the k-th algebraic transfer is a monomorphism for every k. (See [14].)

6. Final remark: GL_4-module structure

Boardman's study of the 3 variable problem shows that the GL_k module structure of $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k$ may be a useful tool. In this vein we close with a description of the module $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)_8$ as a GL_4-module. From the "Modular Atlas" [5] we find that there are 8 irreducible modules for GL_4 in characteristic 2, of dimensions 1, 4, 4, 6, 14, 20, 20, and 64. With a little calculation we find the following description of them:

1: the trivial module $\mathbb{F}_2$,
N: the natural module $\mathbb{F}_2^4$,
N^*: the dual of the natural module,
Λ: the alternating square of N or N^*,
S: the nontrivial constituent of $N \otimes N^*$, which has composition factors $1, S, 1$,
T: a constituent of $N \otimes \Lambda$, which has composition factors N^* and T,
T^*: a constituent of $N^* \otimes \Lambda$, which has composition factors N and T^*,
St: the Steinberg module.

Using a "meataxe" program written in MAGMA, together with a MAGMA program to compute Brauer characters, we have found that $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)_8$ is an extension

$$0 \longrightarrow N^* \oplus T \longrightarrow (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)_8 \longrightarrow \Lambda \oplus M \longrightarrow 0,$$

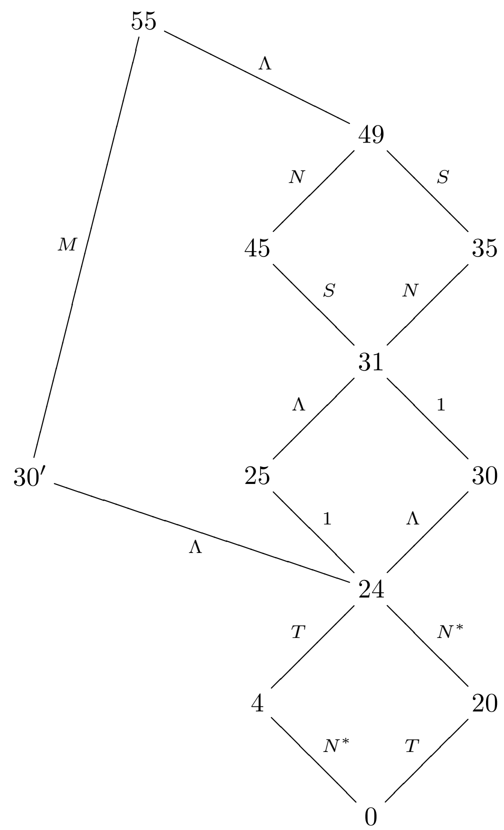

FIGURE 1. Some GL_4-submodules of $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)_8$.

where the 25-dimensional module M is an extension

$$0 \longrightarrow 1 \oplus \Lambda \longrightarrow M \longrightarrow N \oplus S \longrightarrow 0.$$

The corresponding lattice of submodules of $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_4)_8$ is shown in Figure 1. We name the submodules by their dimension, using a prime to distinguish the two submodules of dimension 30. We label the edges by the corresponding quotient module. In it, intersections are shown, but sums are omitted for clarity. That is, the intersection of the submodules $30'$ and 35 is the submodule 24, but the sum of $30'$ and 35 (a submodule of dimension 41) is not shown. The two extensions above can be seen in the lattice, in the sense that, for example, the submodule of dimension 24 is the direct sum of the submodules of dimensions 4 and 20, since their intersection is trivial. Further, the quotient of 55 by 24 is the direct sum of the quotients of $30'$ by 24 and of 49 by 24.

The generators for these submodules are provided by the same computer program used to find this decomposition and are listed below. When all the monomials in one of the seven families listed in Proposition 4.1 appear, we simply write the name of the family, so that, for example, all the monomials in family A are in the submodule of dimension 20. Also, recall the element

$$s_G = (4,2,1,1) + (4,1,2,1) + (1,4,2,1)$$

used in the proof of Proposition 4.4. Finally note that elements which form bases for the subquotients can be read off by comparing these lists of generators. For example, the quotient of the module 30 by the submodule 24 is Λ, and the elements of family D generate it.

4: $(6,1,1,0)+(1,6,1,0)+(1,1,6,0)$, $(6,1,0,1)+(1,6,0,1)+(1,1,0,6)$, $(6,0,1,1)+(1,0,6,1)+(1,0,1,6)$, $(0,6,1,1)+(0,1,6,1)+(0,1,1,6)$.

20: (A), $(6,1,1,0)+(1,1,6,0)$, $(6,1,0,1)+(1,1,0,6)$, $(6,0,1,1)+(1,0,1,6)$, $(1,6,1,0)+(1,1,6,0)$, $(1,6,0,1)+(1,1,0,6)$, $(1,0,6,1)+(1,0,1,6)$, $(0,6,1,1)+(0,1,1,6)$, $(0,1,6,1)+(0,1,1,6)$.

24: (A) and (C).

25: $(A),(C)$, and s_G.

30: $(A),(C)$, and (D).

30′: (A), (C) and $(5,1,1,1)+(1,5,1,1)+s_G+(3,3,1,1)$, $(5,1,1,1)+(1,1,5,1)+s_G+(3,1,3,1)$, $(5,1,1,1)+(1,1,1,5)+s_G+(3,1,1,3)$, $(1,5,1,1)+(1,1,5,1)+s_G+(1,3,3,1)$, $(1,5,1,1)+(1,1,1,5)+s_G+(1,3,1,3)$, $(1,1,5,1)+(1,1,1,5)+s_G+(1,1,3,3)$.

31: $(A),(C),(D)$ and s_G.

35: $(A),(C),(D)$, s_G and
$(5,2,1,0)+(5,2,0,1)+(5,0,2,1)+(5,1,1,1)$,
$(2,5,1,0)+(2,5,0,1)+(0,5,2,1)+(1,5,1,1)$,
$(2,1,5,0)+(2,0,5,1)+(0,2,5,1)+(1,1,5,1)$,
$(2,1,0,5)+(2,0,1,5)+(0,2,1,5)+(1,1,1,5)$.

45: $(A),(C),(D),(E)$ and (G).

49: $(A),(C),(D),(E),(F)$ and (G).

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