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STABLE AND FINITE MORSE INDEX SOLUTIONS ON $\mathbf{R}^n$ OR ON BOUNDED DOMAINS WITH SMALL DIFFUSION

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ABSTRACT. In this paper, we study bounded solutions of $-\Delta u = f(u)$ on $\mathbf{R}^n$ (where $n = 2$ and sometimes $n = 3$) and show that, for most f's, the weakly stable and finite Morse index solutions are quite simple. We then use this to obtain a very good understanding of the stable and bounded Morse index solutions of $-\epsilon^2\Delta u = f(u)$ on Ω with Dirichlet or Neumann boundary conditions for small ϵ.

The purpose of the present paper is to study the equation

$$-\Delta u = f(u) \tag{1}$$

on $\mathbf{R}^n$ (or a half space T), where we are interested in solutions which are weakly stable or have finite Morse index (that is, have only finitely many negative eigenvalues in some suitable generalized sense). Usually, we study positive solutions (but not always). For $n = 2$ and sometimes for $n = 3$ we prove that these solutions are very simple and easy to understand. (For example, if $n = 2$, the weakly stable solutions usually have to be constant.) This is an interesting contrast to the results in [8] where for certain nonlinearities there are a great many positive bounded solutions which are periodic in some variables and decay in others. In addition, for many f, it is easy to use variational methods (or bifurcation methods) to construct many solutions periodic (and non-constant) in all variables. Thus it seems that the structure of all solutions of (1) may be quite complicated. Our results show that the finite Morse index solutions are usually quite simple. We also prove closely related results on half spaces.

As an application of these ideas, we have a number of results on the solutions (usually but not always positive) of

$$-\epsilon^2\Delta u = f(u) \text{ in } \Omega$$

where Ω is a bounded open set in $\mathbf{R}^2$ (sometimes in $\mathbf{R}^3$) and we have homogeneous Neumann or Dirichlet boundary conditions. For many f's we obtain the exact number of stable positive solutions for small positive ϵ and show it is independent of the shape of the domain Ω (and easy to calculate). This contrasts strongly with the results for all positive solutions in [9], [10] and [11] for small positive ϵ and results for the stable positive solutions when ϵ is not small. We also prove that stable positive solutions are constant in the case of Neumann boundary conditions, and for either boundary condition there are no stable sign-changing positive solutions for small positive ϵ (for many f's). These two results contrast with results in [31] and [16].

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We also show that if we have a sequence of positive solutions of (2) with ϵ tending to 0 for which the Morse index stays bounded, then the solution is nearly constant in the interior of Ω with finitely many sharp peaks (and possibly a boundary layer in the Dirichlet case). We then sketch briefly two applications of this. First, if Ω is strongly convex and the boundary conditions are Dirichlet, we deduce for certain nonlinearities that the only non-trivial positive solution with finite Morse index is a one peak solution. We also remove (for $n = 2$) the convexity conditions in results on the mountain pass solution in Jang [30] and Dancer-Wei ([18], [19]). We feel our results will have many other applications.

Our results are motivated by the recent work on the De Giorgi conjecture as in [1], [2] and [25]. Note that if we could prove Theorem 1 for all weakly stable solutions when $n = 3$, nearly all the later results would always be true if $n = 3$. This seems an interesting open question. Indeed our question appears more natural than the original De Giorgi conjecture. However, by combining our ideas here with some other ideas, we will prove elsewhere that Theorems 6 and 8 extend to the case $n = 3$ (that is, for positive solutions of the Dirichlet problem). Note that the cases of interest in physical applications are $n = 2, 3$.

Note that in a number of cases, we have proved the main results and mentioned rather more technical extensions in the remarks. This is to help keep the paper to a reasonable length.

In §1 we consider weakly stable solutions on $\mathbf{R}^n$ or half spaces, while in §2 we consider the corresponding problem for finite Morse index solutions. In §3 we consider stable solutions on bounded domains when the diffusion is small, and finally, in §4, we briefly consider solutions of bounded Morse index on bounded domains when the diffusion is small.

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1. Weakly stable solutions on $\mathbf{R}^n$ and half spaces

In this section, we prove some results for weakly stable bounded solutions on $\mathbf{R}^n$ for $n = 2, 3$ or on a half space (with homogeneous Dirichlet or Neumann boundary conditions in the case of a half space). Note that Theorem 1 below seems to be partly known, but it plays a very important role for all our later work. Our main techniques are derived from those used to prove the De Giorgi conjecture in low dimensions (cf. [2] or [25]) and the use of subsolutions.

A bounded solution of

$$-\Delta u = f(u)$$

on $\mathbf{R}^n$ is said to be weakly stable if the quadratic form

$$E(\phi) = \int_{R^n} [|\nabla\phi|^2 - f'(u)\phi^2] \geq 0$$

on $C_0^\infty(R^n)$. Thus, by smoothness, it would be equivalent to assume $\phi \in W^{1,2}(\mathbf{R}^n)$ and ϕ has bounded support. Note that we always assume that $f : \mathbf{R} \longrightarrow \mathbf{R}$ is C^1.

Similarly, a bounded solution on a half space T satisfying Neumann or Dirichlet boundary conditions on ∂T is said to be weakly stable if $E(\phi) \geq 0$ for all C^∞ functions ϕ on $\overline{T}$ with bounded support (and such that $\phi = 0$ on ∂T in the Dirichlet case).

Theorem 1. *Assume that u is a weakly stable solution of* (1) *on $\mathbf{R}^n$, and that $n = 2$ or $n = 3$ and $f(u(x)) \geq 0$ on $\mathbf{R}^n$, or that $\int_{B_R} |\nabla u|^2 \leq \widetilde{C} R^2$ for large R. Then either u is a constant C with $f(C) = 0$ and $f'(C) \leq 0$, or after a rotation of coordinates, $u = u(x_1)$ and u' has fixed sign on R.*

Proof. If $n = 2$, standard $W^{2,p}$ local estimates as in [26] imply that a bounded solution of (1) is bounded in C^1 on $\mathbf{R}^n$, and hence the integral condition is always satisfied if $n = 2$. If $n = 3$ and $f(u(x)) \geq 0$ on $\mathbf{R}^n$, then, modifying an idea in [1], we now prove that the integral condition is satisfied. If u is a solution of (1), then by Green's theorem

$$\int_{B_R} f(u) = \int_{\partial B_r} \tfrac{\partial u}{\partial \nu} ds \leq C_2 R^2,$$

since ∇u is bounded on $\mathbf{R}^n$ and the surface area of ∂B_R is $C_1 R^2$ (since $n = 3$). Here ν is the unit normal. Since f is non-negative and u is bounded above, it follows that

$$\int_{B_R} u f(u) \leq C_3 R^2 \ .$$

Now by Green's theorem again

$$\begin{aligned} \int_{B_R} |\nabla u|^2 &= \int_{B_R} (-\Delta u) u + \int_{\partial B_R} u \tfrac{\partial u}{\partial \nu} \\ &= \int_{B_R} u f(u) + \int_{\partial B_R} u \tfrac{\partial u}{\partial \nu} \\ &\leq C_4 R^2 \ , \end{aligned}$$

as required.

Hence we may assume that $\int_{B_R} |\nabla u|^2 \leq \widetilde{C} R^2$. We next use the weak stability. Since $E(\phi) \geq 0$ for all $\phi \in C_0^\infty(B_R)$, we see by the variational characterization of eigenvalues (and density) that there are a $\lambda_R \geq 0$ and a positive eigenfunction Φ_R on B_R satisfying Dirichlet boundary conditions on ∂B_R so that

$$-\Delta \Phi_R - f'(u) \Phi_R = \lambda_R \Phi_R.$$

We can assume $\Phi_R(0) = 1$. As in [22], we can then use the Harnack inequality to show that a subsequence of Φ_R converges uniformly on compact sets in R^n to a positive solution Φ of

$$-\Delta \Phi - f'(u) \Phi = \overline{\lambda} \Phi \tag{2}$$

on $\mathbf{R}^n$ where $\overline{\lambda} \geq 0$. Note that Φ need not be bounded. (In fact $\overline{\lambda} = 0$, but we will not need this.) Since $\frac{\partial u}{\partial x_i}$ solves (2) with $\overline{\lambda} = 0$, a simple computation (cf. [2] or [25]) shows that

$$\text{div } (\Phi^2 \nabla \sigma_i) = g_i(x) \sigma_i$$

where $g_i(x) \geq 0$ in $\mathbf{R}^n, \sigma_i = \frac{\partial u}{\partial x_i}/\Phi$. Since $\Phi \sigma_i = \frac{\partial u}{\partial x_i}$, $\int_{B_R} (\Phi \sigma_i)^2 \leq \widetilde{C} R^2$. Hence we can apply the Liouville theorem in [4] (or [2]) to check that σ_i is constant, that is, $\frac{\partial u}{\partial x_i} = C_i \Phi$ on $\mathbf{R}^n$ where C_i is constant. By an orthogonal change of coordinates in R^n, we can assume $C_i = 0$ for $1 \leq i \leq n-1$. Thus $\frac{\partial u}{\partial x_i} = 0$ for $1 \leq i \leq n-1$, and hence u is a function of x_n only. Moreover $\frac{\partial u}{\partial x_n} = C_n \Phi$, and hence either $C_n = 0$ (and thus u is constant) or $\frac{\partial u}{\partial x_n}$ has fixed sign in $\mathbf{R}^n$. Finally since $\int_{R^n} [|\nabla \phi|^2 - a\phi^2] < 0$ for some ϕ if $a > 0$, we see that a constant solution C can only be weakly stable if $f'(C) \leq 0$. □

Remarks. 1. The key problem is to try to generalize this result to all bounded solutions when $n = 3$. If we could do this, we could generalize nearly all our later theory to the case $n = 3$. Note that, if $n = 3$ and f changes sign, one can frequently construct bounded positive periodic solutions for which our integral condition on ∇u fails (for example, by the ideas in [8]). This was pointed out to me by Shusen Yan.

2. Since a monotone bounded ODE solution of (1) must converge to zeros a, b of f at $\pm\infty$ such that $\int_a^b f = 0$ (by the first integral of the ordinary differential equation), we see that it is rather difficult for there to be monotone ordinary differential equation solutions.

3. Note that all the solutions we construct in the theorem are weakly stable when they exist.

Theorem 2. *Assume that the conditions of Theorem* 1 *hold except that u is a weakly stable bounded solution of* (1) *on a half space satisfying a Neumann boundary condition and we replace the condition on ∇u by $\int_{B_R \cap T} |\nabla u|^2 \leq \widetilde{C} R^2$. Then u is constant or $u = u(x_i)$ where u' has fixed sign and x_i is a direction on ∂T.*

Proof. If we extend u to be an even function in x_n, we easily see that it is a solution on $\mathbf{R}^n$. We prove that the extended function is weakly stable on $\mathbf{R}^n$. To see this, we consider the principal eigenvalue Φ_R for $-\Delta h - f'(u)h$ for Dirichlet boundary conditions on B_R (where we are assuming $0 \in \partial T$). Note that Φ_R is up to scalar multiplication the only eigenfunction corresponding to the principal eigenvalue λ_R and $\Phi_R(x) > 0$ on B_R. Since u is even for reflection in the hyperplane T, so is $f'(u)$. Hence, by the simplicity of λ_R, Φ_R must be even or odd for this reflection $\widetilde{R}$ (because $\Phi_R(\widetilde{R}x)$ is also an eigenfunction corresponding to λ_R). Since Φ_R is positive, it cannot be odd for this reflection and hence must be even. Hence Φ_R has zero normal derivative on ∂T. Hence $\Phi_R|_T$ is a suitable test function for $E(\phi)$ restricted to T (since it satisfies the boundary condition). Thus the weak stability on T implies $\lambda_R \geq 0$. Hence u is stable on $\mathbf{R}^n$. Thus, by Theorem 1, u is constant or u is a monotone ordinary differential equation solution. In the second case, the other requirements on u follow simply from the boundary condition. □

We now consider the Dirichlet case. If $n = 2$ or if $n = 3$ and $f(0) \geq 0$, a theorem of Berestycki, Caffarelli and Nirenberg [4] ensures that a positive bounded solution on a half space is $u(x_n)$, where u increases in x_n (where $T = \{x \in \mathbf{R}^n : x_n > 0\}$). Thus this case is well understood.

Theorem 3. *Assume that u is a bounded weakly stable solution of the half space Dirichlet problem such that $n = 2$ or $\int_{T \cap B_R} |\nabla u|^2 \leq CR^2$ for large R. Then:*

(i) *u is a function of (x_{n-1}, x_n) only (after a rotation in the first $n-1$ coordinates) and u is a monotone function of x_{n-1} for each $x_n > 0$. Moreover $\lim_{x_{n-1} \longrightarrow \pm\infty} u(x_{n-1}, x_n)$ are monotone solutions of $-v'' = f(v), v$ bounded on $x_n \geq 0, v(0) = 0$.*

(ii) *If there exists M such that $u \longrightarrow M$ as $x_n \longrightarrow \infty$ uniformly in x_{n-1}, then u is a function of x_n only (and is monotone in x_n).*

Remarks. Note that we do not assume u is positive. In case (ii) it is easy to check that $f(M) = 0$ and $f'(M) \leq 0$ (by the weak stability). If in addition $f'(M) < 0$, it is possible to deduce that $u - M$ and its first derivatives decay exponentially in x_n, and hence the integral condition always holds in this case for $n = 3$. If $f(y) \geq 0$ on $\mathbf{R}$, then (i) generalizes to $n = 3$, but this result is weaker than the result in [4].

Proof. (i) We use a modification of the proofs of the De Giorgi conjecture to show that u is a function of (x_{n-1}, x_n) only and is monotone in x_{n-1}. We first construct Φ positive on T and zero on ∂T and $\overline{\lambda} \geq 0$ such that

$$-\Delta\Phi - f'(u)\Phi = \overline{\lambda}\Phi\ .$$

This is much the same as before. We construct $\Phi_R > 0$ on $T \cap B_R$ and vanishing on $\partial(B_R \cap T)$ and λ_R. We normalize Φ_R so that $\Phi_R(e_n) = 1$. We then proceed as before except we replace the ordinary Harnack inequality by the Harnack inequality up to the boundary of Berestycki, Caffarelli and Nirenberg (Theorem 1.4 in [5]).

Now we use an argument based on the De Giorgi proof. As before we find that for $1 \leq i \leq n-1$, $\frac{\partial u}{\partial x_i}$ is a solution of the linearized equation satisfying the boundary condition. We let $\sigma = \frac{\partial u}{\partial x_i}/\Phi$, and as before we find that σ solves

$$\operatorname{div}(\Phi^2\nabla\sigma) = g(x)\sigma$$

in $\operatorname{int} T$ where $g(x) > 0$ in $\operatorname{int} T$. We now multiply this equation by $S_R^2\sigma$ and integrate by parts over T. Here $S_R(x) = S\left(\frac{|x|}{R}\right)$ where S is smooth on R^+ and such that $0 \leq S \leq 1$ always, $S(t) = 1$ if $0 \leq t \leq 1$ and $S(t) = 0$ if $t \geq 2$. There are a couple of points here. It is easier to integrate over $\{x \in T : |x| \leq 2R$ and $x_n \geq \frac{1}{m}\}$ and pass to the limit as $m \longrightarrow \infty$. There is a boundary term $\int_{x_n = \frac{1}{m}} \sigma S_R^2 \Phi^2 \frac{\partial\sigma}{\partial x_n}$. Now σ is a quotient of C^1 functions both vanishing on the boundary, and $\frac{\partial\Phi}{\partial x_n} > 0$ on $x_n = 0$ by the maximum principle. Hence we see that σ is bounded on compact subsets of $\overline{T}$. Now $\Phi^2\nabla\sigma = \Phi\nabla\left(\frac{\partial u}{\partial x_i}\right) - \frac{\partial u}{\partial x_i}\nabla\Phi$, which tends to zero as $x_n \longrightarrow 0$ (uniformly on compact subsets of T) since $\Phi, \frac{\partial u}{\partial x_i}$ are C^1 on $\overline{T}$ and $\Phi, \frac{\partial u}{\partial x_i}$ are zero on ∂T. Hence

$$\int_{T \cap B_R} \Phi^2 \nabla\sigma\nabla(\sigma S_R^2) \leq 0.$$

We can then repeat the part of the proof of Proposition 2.1 in Ambrosio and Cabré [2] (see the top of p.734) and deduce that $\int_T \Phi^2(\nabla\sigma)^2 = 0$. Since $\Phi > 0$ in $\operatorname{int} T$, we conclude that σ is constant.

Hence we have proved that $\frac{\partial u}{\partial x_i} = C_i\Phi$ on T for $1 \leq i \leq n-1$ where C_i is a constant. As before, by an orthogonal rotation of axes in R^{n-1}, we can assume $C_i = 0$ for $1 \leq i \leq n-2$. Thus u is a function of (x_{n-1}, x_n) only and either $C_{n-1} = 0$ (whence u is a function of x_n only) or $C_{n-1} \neq 0$. In the latter case, $\frac{\partial u}{\partial x_{n-1}} = C_{n-1}\Phi$ is non-zero on $\operatorname{int} T$.

Hence we see that $u_+(x_n) = \lim_{x_{n-1} \longrightarrow \infty} u(x_{n-1}, x_n)$ exists. Since u is bounded in C^1, it is easy to check that the convergence is locally uniform, u_+ is continuous, $u_+(0) = 0$ and u_+ is a weak (and hence strong) solution of $-y'' = f(y)$ on $[0, \infty)$. Moreover, by an argument similar to that in the proof of Lemma 3.3 in [4], u_+ is weakly stable (since u is). As at the start of the proof, that implies that there is a positive solution Φ on $[0, \infty)$ of $-\Phi'' = f'(u_+)\Phi + \overline{\lambda}\Phi$ (where $\overline{\lambda} \geq 0$) satisfying $\Phi(0) = 0$. This implies that $u'_+(x) \neq 0$ on $(0, \infty)$. To see this, note that by the first integral of $-y'' = f(y), u_+$ is monotone or is periodic for $x_n \geq T > 0$. In the latter case, u'_+ is a solution of $-v'' = f'(u_+)v$ with many positive zeros. By the Sturm comparison principle, it follows that Φ has many positive zeros, which gives a contradiction. (Here we have used $\overline{\lambda} \geq 0$.) Hence u_+ is monotone. Similarly, u_- is a monotone solution. This proves (i).

(ii) By part (i) and our assumptions, u_+ and u_- are both monotone solutions of $v'' = f(v), v(0) = 0$, and $v(x_n) \longrightarrow M$ as $x_n \longrightarrow \infty$. One easily sees that $f(M) = 0$ and $u'_\pm(x) \longrightarrow 0$ as $x \longrightarrow \infty$. By the first integral of the ordinary differential equation, we see that $\frac{1}{2}(u'_-(0))^2 = F(M) = \frac{1}{2}(u'_+(0))^2$. Here $F' = f, F(0) = 0$. If $u'_+(0) = u'_-(0)$, the uniqueness theorem for ordinary differential equations implies that $u_+ = u_-$, and hence by the monotonicity in $x_{n-1}, u = u_+ = u_-$, which proves the claim. The only other possibility is that $u'_+(0) = -u'_-(0) \neq 0$. Since u_+ and u_- are monotone and $u_+(0) = u_-(0)$, it follows that $\lim_{x_n \longrightarrow \infty} u_+(x) \neq \lim_{x_n \longrightarrow \infty} u_-(x)$, which contradicts our assumptions. This completes the proof. □

Remarks. Note that if $f(0) < 0$, this gives new results for non-negative weakly stable solutions. It is not difficult to show that any solution of the type in Theorem 3 (i) is weakly stable. Note that there are sometimes solutions of this type which depend on x_{n-1}. To see this, we consider f smooth, $f(0) = 0, f(M) = 0, f(t) > 0$ on $(0, M)$. Then it is not difficult to use the method of sub- and super-solutions to construct a positive solution $\widetilde{u}$ of $-\Delta u = f(u)$ in $Q = (0,\infty) \times (0,\infty)$ such that $0 < \widetilde{u} < M$ in Q and $\widetilde{u} = 0$ on ∂Q. Note (cf. [12], Remark 4 on p.433) $\widetilde{u}$ is increasing in x_1 and x_2, and $\lim_{x_1 \longrightarrow \infty} \widetilde{u}(x_1, x_2)$ is the unique positive increasing solution of $-u''(x_2) = f(u(x_2))$ on $(0,\infty)$ with $u(0) = 0$. A similar property holds for $\lim_{x_2 \longrightarrow \infty} \widetilde{u}(x_1, x_i)$. If we then extend f to be an odd function and extend $\widetilde{u}$ to be odd across $x_2 = 0$, it is easy to see that $\widetilde{u}$ is a bounded solution of our equation on the half space which is strictly monotone in x_2 (and is in fact weakly stable). Note that, in this case, $u_- = -u_+$. Similar ideas are used in [41] and [44]. By using similar reflection tricks starting from a solution in a 60° cone in R^2, it is possible to construct a solution in a half space such that $u(x_1, x_2) \longrightarrow M$ pointwise as $x_1 \longrightarrow \infty$ but which is not weakly stable. It is also possible to prove with care that in Theorem 3 (i) $\lim_{x_1 \longrightarrow \infty} u(x_1, x_2)$ exists and is a weakly stable solution of $-u''(x_2) = f(u(x_2))$, and hence

$$\int_{u_-(\infty)}^{u_+(\infty)} f(t)dt = 0 \tag{3}$$

is a necessary condition for such a weakly stable solution to exist which does not depend only on x_1. Here, for simplicity, we are assuming that $n = 2$. Note that (3) is more restrictive than is immediately apparent. It and the first integral of the equations for u_+ and u_- imply that $(u'_+(0))^2 = (u'_-(0))^2$. Since $u'_+(0) = u'_-(0)$ implies $u_+ = u_-$ (by uniqueness) and hence $u = u(x_n)$, solutions depending on both x_{n-1} and x_n can only occur if u_+ and u_- have opposite signs and there are heteroclinic solutions joining $u_+(\infty)$ and $u_-(\infty)$.

2. Solutions of finite Morse index on $\mathbf{R}^n$ or half spaces

In this short section, we use the results of §1 to study solutions of finite Morse index.

A solution u on $\mathbf{R}^n$ is said to have Morse index at most k if there is no $(k+1)$-dimensional subspace Y of $C_0^\infty(\mathbf{R}^n)$ such that $E(u) < 0$ on $Y\backslash\{0\}$. It is said to have finite Morse index if it has Morse index at most k for some k. As usual, by smoothing, we can replace $C_0^\infty(\mathbf{R}^n)$ by $W^{1,2}(\mathbf{R}^n)$ functions of bounded support.

We can analogously define finite Morse index for a Dirichlet or Neumann problem on a half space (where we require the functions to have bounded support and satisfy the appropriate boundary condition on ∂T in the Dirichlet case).

In this section we will usually assume the following extra condition.

Condition $f1$. has only isolated zeros, and f does not have 2 zeros a, b such that $\int_a^b f(t)dt = 0$.

(More generally, in the second part of this condition, it suffices to assume $-y'' = f(y(t))$ does not have any monotone solution joining different zeros of f.)

Remark. In many of our applications to bounded domains in §§3 and 4, the condition can be avoided or weakened.

Theorem 4. *Assume that u is a bounded solution of finite Morse index of* (1) *on $\mathbf{R}^n$, that condition $f1$ holds, and that $n = 2$ or $n = 3$ and $f(u(x)) \geq 0$ on $\mathbf{R}^3$. Then there is a constant C such that $f(C) = 0, f'(C) \leq 0$ and $u(x) \longrightarrow C$ uniformly as $|x| \longrightarrow \infty$.*

Remark. This shows that in these cases, the finite Morse index solutions are quite simple. We will make this clearer below. This contrasts with the ideas in [8] where we show that frequently there is a wide variety of bounded solutions on $\mathbf{R}^n$. If $f'(C) < 0$, one can prove conversely that these solutions have finite Morse index. If $f1$ fails, an example in [43] shows that the finite Morse index solutions can be more complicated.

Proof. Assume the Morse index is k. Then there exist orthogonal functions $\{\phi_i\}_{i=1}^k$ in $C_0^\infty(R^n)$ such that $E(\phi) < 0$ if $\phi \in (\text{span } \{\phi_i\})\backslash\{0\}$. Choose a ball B in $\mathbf{R}^n$ containing all the supports of all the ϕ_i. If B_1 is another ball in $\mathbf{R}^n$ with $B \cap B_1$ empty, then $E(\phi) \geq 0$ for all $\phi \in C_0^\infty(B_1)$. (Otherwise, if $E(\widetilde{\phi}) < 0$ where $\widetilde{\phi} \in C_0^\infty(B_1)$, then $\widetilde{\phi}$ is orthogonal to the ϕ_i, since their supports are disjoint, and we easily see that $E(\phi) < 0$ on $W\backslash\{0\}$, where $W = \text{span } \{\widetilde{T}, \widetilde{\phi}\}$, which contradicts that u has Morse index k.

Now suppose $|x^i| \longrightarrow \infty$. Then $u_i(x) = u(x - x^i)$ is a sequence of solutions of (1) which are bounded in C^1 and hence, after choosing a subsequence, converge uniformly on compact subsets of $\mathbf{R}^n$ to a weak (and hence strong) solution $\hat{u}$ of $-\Delta u = f(u)$ on $\mathbf{R}^n$. Moreover $\hat{u}$ is weakly stable. Suppose by way of contradiction that there exists $\psi \in C_0^\infty(\mathbf{R}^n)$ such that $\int_{R^n} [|\nabla\psi|^2 - f'(\hat{u})\psi^2] < 0$. Since ψ has compact support and $u(x - x^i) \longrightarrow \hat{u}$ uniformly on compact sets (at least for a subsequence), we see that

$$\int_{R^n} [|\nabla\psi|^2 - f'(u(x - x^i))\psi^2] < 0$$

for large i, that is, $E(\psi(x + x_i)) < 0$ for large i. Since $\psi(x + x^i)$ has support missing B for large i, this contradicts what we have already proved. Hence $\hat{u}$ is weakly stable. Thus, by Theorem 1, $\hat{u}$ is a constant C, where $f(C) = 0$ and $f'(C) \leq 0$. It remains to prove that the constant C is independent of the choice of x_i and the choice of subsequence. This follows easily from the discreteness of the zero set of f once we note that, since $\{x \in \mathbf{R}^n : \|x\| \geq R\}$ is connected, it is easy to prove that the set of all possible limit points of $u(x)$ as $|x|$ tends to infinity is a connected set. This completes the proof. □

Frequently, one can deduce that u converges to C from one side and that, up to translation, $u = g(|x|)$ where g decreases to C as $|x| \longrightarrow \infty$. (Thus the solution has one peak.) Thus in this case the finite Morse index solutions are very simple.

We explain these. As an example, assume that $f < 0$ on $(0, \alpha), f'(\alpha) > 0, f > 0$ on (α, d), and $f(x) < 0$ if $x > d$ where $f(\alpha) = f(0) = f(d) = 0$ and $\int_0^d f > 0$. Here we are interested in solutions u such that $0 \le u(x)$ on $\mathbf{R}^n$. By the result above, $u \longrightarrow 0$ as $|x| \longrightarrow \infty$ or $u(x) \longrightarrow d$ as $|x| \longrightarrow \infty$. Let us consider the latter case. Since $\int_0^d f > 0$, it is well known that, if $\beta < d$, then, on large balls $\widetilde{B}$, there is a positive solution $\hat{u}$ of $-\Delta u = f(u)$ on $\widetilde{B}, u = 0$ on $\partial \widetilde{B}, \beta < \|\hat{u}\|_\infty < d$. (cf. Clement and Sweers [7]). We can then obtain a subsolution $\hat{u}$ on $\mathbf{R}^n$ by extending $\hat{u}$ to be zero outside $\widetilde{B}$. Since $\hat{u}$ has compact support and $u \longrightarrow d$ uniformly as $|x| \longrightarrow \infty$, we can choose $\widetilde{x}$ so $\hat{u}(x + \widetilde{x}) \le u(x)$ on $\mathbf{R}^n$. Hence by a result of Serrin on sweeping by families of subsolutions (cf. [7] or [9]), we deduce that $\hat{u}(x + \widetilde{x}) \le u(x), \forall x \in R^n, \forall \widetilde{x} \in \mathbf{R}^n$. Since $\hat{u}(0) \ge \beta$, we see that $u(x) \ge \beta$ on $\mathbf{R}^n$. Hence $u(x) \ge d$ on $\mathbf{R}^n$, as claimed. Hence u must achieve its maximum. If the maximum is larger than d, we get an easy contradiction. Thus either $u \equiv d$ or $u \longrightarrow 0$ as $|x| \longrightarrow \infty$. In the latter case, if we assume in addition that $f'(y) \le 0$ for small positive y, then a theorem of Li and Ni [32] ensures that, up to a translation, $u = g(|x|)$ where g is decreasing. If $\int_0^d f < 0$, we could obtain similar results by constructing supersolutions below d. (In this case, the only non-constant solutions are below d and tend to d as $|x| \longrightarrow \infty$.) As another example, assume $0 < \alpha_1 < \alpha_2 < \alpha_3$ are zeros of f such that $f > 0$ on $(0, \alpha_1) \cup (\alpha_2, \alpha_3), f'(\alpha_2) > 0, f < 0$ on $(\alpha_1, \alpha_2) \cup (\alpha_3, \infty)$ and $\int_{\alpha_1}^{\alpha_3} f > 0$. Then one can use families of subsolutions to prove that, if $0 \le u(x)$ for all x and if $u \longrightarrow \alpha_1$ or α_3 as $|x| \longrightarrow \infty$, then $u(x) \ge \alpha_1$ for all x. If $u(x) > \alpha_1$ for all x, one can use a second family of subsolutions much as before to show that $u(x) \longrightarrow \alpha_3$ as $|x| \longrightarrow \infty$. Thus the only non-constant finite Morse index non-negative solutions satisfy $u(x) > \alpha_1$ and $u(x) \longrightarrow \alpha_1$ as $|x| \longrightarrow \infty$, or $u(x) > 0$ and $u(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$. As above we can nearly always prove that, up to translation, the first type of solutions are radial and decreasing. The second type does not occur if $f(y) \ge C y^{n/(n-2)}$ for small y where $C > 0$ (and $n > 2$) (and never occur if $n = 2$). These last results are well known and follow by averages over spheres (cf. [40] or the appendix to [9], or the first part of the proof of Theorem 1 (ii) in [15]). In general, sub- and supersolutions seem able to give a good deal of information in many cases.

Note that if condition $f1$ fails, we can prove a finite Morse index solution looks asymptotically like ordinary differential equation solutions, but it seems somewhat more difficult to deduce which of these solutions have finite Morse index.

As one very simple application of Theorem 4, note that it implies a bounded positive finite Morse index solution of $-\Delta u = u^p$ on $\mathbf{R}^3$ (where $p > 5$) satisfies $u \longrightarrow 0$ uniformly as $|x| \longrightarrow \infty$. This is of interest because these solutions are not well understood.

To conclude this section, we consider the two half space cases.

Proposition 1. *Assume that $n = 2$ and u is a bounded solution of finite Morse index of the homogenous Neumann problem on the half space T. Assume also that condition $f1$ holds. Then there is a constant C so that $f(C) = 0, f'(C) \le 0$ and $u(x) \longrightarrow C$ uniformly as $|x| \longrightarrow \infty, x \in \overline{T}$.*

Proof. This is very similar to the proof of Theorem 4 except we use Theorems 1 and 2 rather than just Theorem 1. Note that the limit problem as $|x| \longrightarrow \infty$ may be either a full space or a half space problem depending on the direction. Note also the set of limit points of u as $|x| \longrightarrow \infty$ is connected, since $\{x \in \overline{T} : |x| \geq R\}$ is connected. □

By the Berestycki-Caffarelli-Nirenberg result [3], the Dirichlet case is uninteresting if $n = 2$ or if $n = 3$ and $f(0) \geq 0$ for positive solutions, since under weak conditions the only positive bounded solutions are one-dimensional and weakly stable. However, if we consider solutions which have the property that $u(x) \longrightarrow M$ uniformly as $x_n \longrightarrow \infty$, it is not difficult to use an argument similar to the above to prove that, for a finite Morse index solution u on T, $u(x) - \widetilde{u}(x_n) \longrightarrow 0$ uniformly as $|x| \longrightarrow \infty, x \in T$. Here $\widetilde{u}$ is the solution of $-u'' = f(u)$ satisfying $u(0) = 0$ and $u(\infty) = M$, and we are assuming $n = 2$ (though as before, we could cover the case $n = 3$ and $f'(M) < 0$). Note that our earlier arguments imply that the integral condition for $x_n \longrightarrow \infty$ automatically holds if condition $f1$ holds and the zeros of f are discrete. Note that mountain pass solutions of this type sometimes exist. (A mountain pass solution sometimes exists by Dancer and Yan [20] if we modify their nonlinearity for large y.) These results are interesting for non-negative solutions if $n = 2$ and $f(0) < 0$. Note also that, if $f(y) \geq 0$ on R, bounded solutions of the Dirichlet problem are automatically non-negative. One way to prove this is to use the ideas in Lemma 2.6 of Clement and Sweers [7] and consider the function $\widetilde{u}(t) = \inf_{x_n = t} u(x', x_n)$ where u is a solution on the half space satisfying Dirichlet boundary conditions.

3. Stable solutions on bounded domains with small diffusion

In this section, we prove that the stable solutions of

$$-\epsilon^2 \Delta u = f(u) \text{ in } \Omega \tag{4}$$

with homogenous Dirichlet or Neumann boundary conditions on a bounded smooth domain Ω are usually quite simple.

We usually assume condition $f1$ and

Condition $f2$. All the zeros of f are simple.

Sometimes we use instead

Condition $f2'$. Either $f2$ holds, or $f(C) = f'(C) = 0$ implies $C = 0$ and $yf'(y) < f(y)$ for small positive y.

Note that this last condition automatically holds if $f(0) = f'(0) = 0, f$ is real analytic and $yf(y) \leq 0$ near 0.

A solution u of (4) is said to be *weakly stable* if the principal eigenvalue of

$$-\Delta h - f'(u)h = \lambda h \text{ in } \Omega$$

(with the appropriate boundary condition) is non-negative. It is said to be *non-degenerate stable* if the principal eigenvalue is strictly positive, and is said to be *stable* if it is stable as a solution of the corresponding parabolic equation (or equivalently by [15] it is a local minimum of the corresponding energy). Note that to discuss stability, we need to truncate our nonlinearity for $|y|$ large to make sure everything makes sense in $\dot{W}^{1,2}(\Omega)$.

Theorem 5. *Suppose that $n = 2$, that conditions $f1$ and $f2$ hold, and that $K > 0$. Then there is an $\widetilde{\epsilon} > 0$ such that the only weakly stable solutions of* (4) *for Neumann boundary conditions with $\|u\|_\infty \leq K$ and $0 < \epsilon < \widetilde{\epsilon}$ are the constant solutions C with $f(C) = 0$ and $f'(C) < 0$.*

Proof. Let $Z = \{y : f(y) = 0, f'(y) \leq 0, |y| \leq K + 1\}$.

Step 1. We use a blow-up argument to show that any weakly stable solution of (4) for ϵ small is uniformly close to Z on $\overline{\Omega}$. (Since Z is finite, connectedness then implies that any weakly stable solution is uniformly close to a single member of Z.) Suppose that this is false, that is, there exist $\epsilon_i \longrightarrow 0$, solutions u_i for $\epsilon = \epsilon_i$ with $\|u_i\|_\infty \leq K$, and $x^i \in \overline{\Omega}$ such that $d(u(x^i), Z) \geq \delta > 0$ for all i. We rescale x by a factor of ϵ_i and obtain solutions $\widetilde{u}_{\epsilon_i}(x) = u_{\epsilon_i}(\epsilon_i^{-1}(x - x^i))$ of $-\Delta u = f(u)$ on $\epsilon_i^{-1}(\Omega - x^i)$ such that $d(\widetilde{u}_{\epsilon_i}(0), Z) \geq \delta$. A rather standard blowing-up argument (using local $W^{2,p}$ estimates for $p > n$) shows that a subsequence of $\widetilde{u}_{\epsilon_i}(x)$ converges uniformly on compact sets to a solution $\hat{u}$ of $-\Delta u = f(u)$ on $\mathbf{R}^n$ (or a half space T with homogenous Neumann boundary condition on ∂T, depending on how close 0 is to the boundary of $\epsilon_i^{-1}(\Omega - x^i)$) such that $d(\hat{u}(0), Z) \geq \delta$ and $\|\hat{u}\|_\infty \leq K$.

We also prove that $\hat{u}$ is weakly stable. Assuming this for the moment, we complete the proof of Step 1. By Theorem 1 and Theorem 2, $\hat{u}$ is a constant C such that $f(C) = 0$ and $f'(C) \leq 0$. By condition $f2, \hat{u}(0) = C \in Z$, and hence we have a contradiction. Hence Step 1 is proved if we prove that $\hat{u}$ is weakly stable. Suppose that this is false, and first consider the case where $\hat{u}$ is defined on $\mathbf{R}^n$. Hence there exists $\phi \in C_0^\infty(\mathbf{R}^n)$ such that $\int_{R^n}[|\nabla\phi|^2 - f(\hat{u})\phi^2] < 0$. Since ϕ has compact support, $\phi \in C_0^\infty(\epsilon_i^{-1}(\Omega - x^i))$ for large C. Since $\widetilde{u}_{\epsilon_i}$ converges uniformly to $\hat{u}$ on compact sets, in particular on $\operatorname{supp}\phi$, it follows that $\int_{\epsilon_i^{-1}(\Omega - x^i)}[|\nabla\phi|^2 - f'(\widetilde{u}_{\epsilon_i}(x))\phi^2] < 0$ for large i. Thus a simple rescaling of ϕ gives a function ϕ_i in $C_0^\infty(\Omega)$ such that $\int_\Omega[\epsilon_i^2|\nabla\phi_i)^2 - f'(u_{\epsilon_i})\phi_i^2] < 0$ for large i. Hence u_{ϵ_i} is not weakly stable. Since this gives a contradiction, $\hat{u}$ must be weakly stable. We need to be slightly more careful in the half space case. If $\hat{u}$ is not weakly stable, there exists a C^∞ function on $\overline{T}$ of bounded support such that $\int_T[|\nabla\phi|^2 - f'(\hat{u})\phi^2] < 0$. By standard theorems, we can extend ϕ to a C^1 function defined in an open neighbourhood W of $\overline{T}$ and of bounded support. Then ϕ is defined on $\epsilon_i^{-1}(\Omega - x^i)$, and it is easy to see that for large i the symmetric difference of $\overline{T} \cap \operatorname{supp}\phi$ and $\epsilon_i^{-1}(\Omega - x^i) \cap \operatorname{supp}\phi$ has small measure for large i. (Remember that $\epsilon_i^{-1}(\Omega - x^i)$ is a C^1 approximation to $\overline{T}$ on compact sets). Hence we see that $\int_{\epsilon_i^{-1}(\Omega - x^i)}[|\nabla\phi|^2 - f'(\widetilde{u}_{\epsilon_i})\phi^2] < 0$ for large i by continuity, and we can obtain a contradiction much as before. This completes the proof of Step 1.

Step 2. Completion of the proof. If $f(C) = 0$ and $f'(C) < 0$, it is easy to see that C is a stable solution of (3). We prove that if $\epsilon_i \longrightarrow 0$ as $i \longrightarrow \infty$, there is no non-constant solution u_{ϵ_i} for large i such that $u_{\epsilon_i} \longrightarrow C$ uniformly on $\overline{\Omega}$ as $i \longrightarrow \infty$ where $f(C) = 0$ and $f'(C) < 0$. This and the result of Step 1 complete the proof of Theorem 5.

Suppose that such u_{ϵ_i} exist. Then, integrating over Ω, $\int_\Omega f(u_{\epsilon_i}) = 0$ and hence, if u_{ϵ_i} is not constant, $f(u_{\epsilon_i})$ must change sign on Ω. Thus $\sup u_{\epsilon_i} > C$ and $\inf u_{\epsilon_i} < C$. However, this is impossible, since by the equation we have $f(u_{\epsilon_i}(x_M)) \geq 0$, where x_M is a point where u_{ϵ_i} has its maximum while since $u_{\epsilon_i}(x_M) > C, f(u_{\epsilon_i}(x_M)) < 0$. (If u_{ϵ_i} attains its maximum on $\partial\Omega$, we need to use the strong form of the maximum principle on the boundary.) This completes the proof. $\square$

Remarks. 1. Note that we can with care modify our argument to prove that stable solutions of (3) with $\|u\|_\infty \leq K$ and ϵ small are constants under the much weaker conditions that the zeros of f are isolated and $|f'(y)|\ |y-C| \geq |f(y)|$ if y is near C where $f(C) = f'(C) = 0$. Note that this last condition always holds if f is real analytic. In this case, one can use centre manifold theory to prove that the stability of a constant solution (as a solution of the parabolic analogue of (4)) is equivalent to the stability of C as a solution of the ordinary differential equation $\dot{u} = f(u)$ on $\mathbf{R}$, and this is easily determined.

2. Our methods can also be used to study weakly stable solutions of $-\epsilon^2\Delta u = f(x, u(x))$ in Ω with homogeneous Neumann boundary conditions. For example, Theorem 5 continues to hold for nonlinearities $g(x)f(u)$ where f is as in Theorem 1 and g is continuous on $\overline{\Omega}$ and positive on $\overline{\Omega}$. More generally, we could prove a good version of Step 1 if we assume $y \longrightarrow f(x, y)$ satisfies condition $f1$ for each $x \in \overline{\Omega}$. (Here we prove that $(x, u_\epsilon(x))$ is uniformly close on $\overline{\Omega} \times \mathbf{R}$ to $\{(x, v) \in \overline{\Omega} \times [-K, K], f(x, v) = 0\}$.) In many cases one can prove converses, especially if $f_2'(x, y) \neq 0$ when $f(x, y) = 0$.

3. Note that condition $f1$ is crucial, as the results in [31] for "double well" potentials show. Note that the problem of which domains allow construction of stable solutions for small ϵ in this case remains unclear, though there are a number of partial results in [31] and [16]. It does not seem an easy problem. Note that it is known ([6] or [42]) that, if Ω is convex, stable solutions are constant without the condition that ϵ is small.

4. In many cases one can handle stable solutions (and especially positive solutions) whose sup norm grows as $\epsilon \longrightarrow 0$ by establishing a priori bounds, in particular by blow-up techniques. We discuss the corresponding problem for the Dirichlet case in more detail later. Note that the results in [3] are sometimes useful here.

5. Our techniques could also be used to study the case of a Robin boundary condition $\frac{\partial u}{\partial n} + au = 0$ on $\partial\Omega$, or some problems with mixed boundary conditions, though here to prove the existence of solutions we need to use techniques closer to those for the Dirichlet problem later (and restrict the sign of a).

6. Note that in general the unstable positive solutions for small ϵ are much more complicated, as there are frequently many types of peak solutions, cf. [27] or [39].

We now look at the Dirichlet problem. Here it is convenient to split into the case of positive solutions and changing sign solutions.

Theorem 6. *Suppose that $f(0) \geq 0$, condition $f1$ holds and that either $n = 2$ or $n = 3$ and $f(y) \geq 0$ on $[0, K]$.*

(i) *Suppose that u_{ϵ_i} are a sequence of non-trivial non-negative weakly stable solutions of (4) for $\epsilon = \epsilon_i$, and $\|u_{\epsilon_i}\|_\epsilon \leq K$ for all i. Then, after choosing a subsequence if necessary, there are a constant C such that $f(C) = 0$ and $f'(C) \leq 0$ and a solution u_0 of $-y'' = f(y), y > 0, y(0) = 0, y(\infty) = C$ (which is unique), such that $u_{\epsilon_i}(x) - \phi_{\epsilon_i}(x)$ converges uniformly to zero on $\overline{\Omega}$ as $\epsilon \longrightarrow 0$. Here, for fixed small $\delta > 0$, $\phi_\epsilon(x) = C$ if $d(x, \partial\Omega) \geq \delta$ and $\phi_\epsilon(x) = u_0(\epsilon^{-1}t)$ if $d(x, \partial\Omega) < \delta$. (Note that points x near $\partial\Omega$ can be uniquely written as $x = x_0 + t\nu(x_0)$, where $x_0 \in \partial\Omega$ and ν is the inward unit normal to $\partial\Omega$ at x_0.)*

(ii) *In particular if condition $f2'$ also holds, and if $f(K) \neq 0$, then the number of non-trivial stable positive solutions u of (4) with $\|u\|_\infty \leq K$ for homogeneous Dirichlet boundary conditions and for small ϵ is the number of elements in*

$$\widetilde{Z} = \{C \in (0, K) : f(C) = 0, f'(C) \leq 0, \int_0^t f < \int_0^C f \;\; \text{if} \;\; 0 \leq t < C\} .$$

Proof. (i) This is a blow-up argument very similar to that in the proof of Theorem 5. As there, we can easily prove that, if $x^i \in \Omega$ and if $\epsilon_i d(x^i, \partial\Omega) \longrightarrow \infty$, then after choosing a subsequence, we can assume $u_{\epsilon_i}(x^i) \longrightarrow C$ where $f(C) = 0$ and $f'(C) \leq 0$. Since $\{x \in \Omega : d(x, \partial\Omega) \geq t\}$ is connected for all small positive t, we can argue as there to deduce that the C is independent of the choice of x^i. Hence we have that, given $\delta > 0$, there is a $\widetilde{K} > 0$ such that $|u_{\epsilon_i}(x) - C| \leq \delta$ if $d(x, \partial\Omega) \geq \widetilde{K}\epsilon_i$ and i is large. This ensures that, if we blow up at a point x_i whose distance from the boundary is of order ϵ_i, we will obtain a non-negative solution $\hat{u}$ on a half space T so that $\hat{u} = 0$ on ∂T and and $\hat{u} \longrightarrow C$ uniformly as $d(x, \partial T) \longrightarrow \infty$. We will also prove that $\hat{u}$ is weakly stable. Assuming this, Theorem 3 implies that $\hat{u}(x) = u_0(x_n)$, and hence (i) will follow. (If the convergence is not uniform near the boundary, we simply choose the blow-up point suitably.) Thus (i) follows if we prove $\hat{u}$ is weakly stable. This requires a little care. If $\hat{u}$ is not weakly stable, there exists $\phi \in C^\infty(T)$ such that ϕ has compact support and $\int_T [|\nabla\phi|^2 - f'(\hat{u})\phi^2] < 0$. As it stands, ϕ does not belong to $C_0^1(\epsilon_i^{-1}(\Omega - x^i))$, so it is not a suitable test function. However, we can define z_i C^1 such that $z_i(x)$ is C^1 close to x on compact sets such that z_i maps $\epsilon_i^{-1}(\Omega - x^i)$ homeomorphically onto T (at least for compact sets). Assuming this, and since $\widetilde{u}_{\epsilon_i}$ converges to $\hat{u}$ on compact sets, we easily see that

$$\int_{\epsilon_i^{-1}(\Omega - x^i)} [|\nabla(\phi \circ z_i)|^2 - f'(\widetilde{u}_{\epsilon_i})(\phi \circ z_i)^2] < 0$$

for large i. Since $\phi \circ z_i$ vanishes on $\epsilon_i^{-1}(\Omega - x^i)$, this shows that $\widetilde{u}_{\epsilon_i}$ is not stable on $\epsilon_i^{-1}(\Omega - x^i)$, which contradicts our assumptions. Hence we will have proved $\hat{u}$ is weakly stable if we prove that z_i exists. Choose $\overline{x}^i$ to be the point of $\epsilon_i^{-1}(\Omega - x^i)$ closest to zero, and let T_i be the tangent plane to $\epsilon_i^{-1}(\partial\Omega - x^i)$ at x^i. By the blowing-up contraction, we see that, for large i, this boundary (at least on bounded sets) can be written as $t + w_i(t)$ where $t \in T_i, w_i(t) \in T_i^\perp$ and w_i is C^1 small. Hence we see that we can simply define $z_i(t, y) = t + y - w_i(t)$ (where $t \in T_i, y \in T_i^\perp$). This gives the required z_i. This completes the proof of (i).

(ii) We need to prove the existence and uniqueness of the solution near $\phi_\epsilon(x)$ (and its stability). This is essentially known. First note that, by a simple comparison argument, it is easy to see that there is a non-trivial uniformly small stable positive solution when $f'(0) = 0$. (Here we use the condition at zero in condition f_2'.) The case $f'(0) < 0$ is much easier.

Next we prove the existence of a weakly stable positive solution near $\phi_\epsilon(x)$. By Theorem 1.6 in Sweers [35] and the remarks after it, for small $\epsilon > 0$ there is a positive solution u_ϵ of (4) with $\|u_\epsilon\|$ close to but less than C. Note that our condition on the existence of u_0 is equivalent to his assumption. This solution is constructed as a solution between a subsolution and a supersolution, and hence can be chosen to be weakly stable (cf. [17]). Now since $f(y) < 0$ for $y > C$ but close to C, the maximum principle implies that no solution can have $\|u\|_\infty > C$ but close to C. Moreover, by applying the Harnack inequality to $u(x) - C$, we see that no solution u can have $\|u\|_\infty = C$. Hence Sweers' Theorem 1.6 implies the uniqueness. (This could also be proved by the blow-up arguments in §2 of [9].) In fact both arguments show the uniqueness without asking about the positivity of the other solution. The non-degeneracy (and hence the stability of the solution close to ϕ_ϵ) follows from the proofs in [9] or [35]. (One shows that the maximum of an eigenfunction corresponding to a non-negative eigenvalue must occur very close to $\partial\Omega$.)

This completes the proof of (ii) if we recall that the last condition on the definition of $\widetilde{Z}$ is necessary and sufficient for the existence of u_0 (by the first integral). Note that since $f(0) \geq 0$, we can easily prove that $u'(0) > 0$. Then if we note that our blow-up argument shows that u_ϵ rescaled converges C^1 to u_0 near the boundary, we see that u_ϵ is necessarily positive on Ω. □

Before making some remarks on this theorem, we consider stable sign changing solutions.

Theorem 7. *Assume that $n = 2$, and that conditions $f1$ and $f2'$ hold (but for all small y, not only $y > 0$). Then, if $K > 0$, for ϵ small enough there is no weakly stable sign-changing solution u with $\|u\|_\infty \leq K$.*

Proof. The first part of the proof of Theorem 6 shows that any weakly stable solution is uniformly close to $\phi_\epsilon(x)$ (for some C with $f(C) = 0$ and $f'(C) \leq 0$), and in fact is C^1 close near the boundary. Thus there are only two possible ways we could have a weakly stable sign-changing solution for ϵ small, namely the possibilities that $u_0'(0) = 0$ or that $C = 0$ (and thus u_ϵ is uniformly close to zero on $\overline{\Omega}$). In the first case we easily see from the first integral of the ordinary differential equation for u_0 that this implies that $\int_0^{u_0(\infty)} f = 0$. However, by Remark 1 after the proof of Theorem 1 in Clement and Sweers [7], there can be no solution u uniformly close to $\phi_\epsilon(x)$ with $\|u\|_\infty < C$ (where for simplicity we are assuming that $C > 0$). Since as before we can use the maximum principle and the Harnack inequality to eliminate the possibility of solutions with $\|u\|_\infty > C$ but close to C and $\|u\|_\infty = C$ respectively, this case cannot occur.

Finally, a non-trivial uniformly small sign-changing solution u cannot be stable because if it were stable, it would also be stable on the subdomain $D_\epsilon = \{x : u(x) > 0\}$ and we could then use a simple comparison argument comparing the principal eigenvalues of $-\Delta - u^{-1}f(u)I$ and $-\Delta - f'(u)I$ on D_ϵ (for Dirichlet boundary conditions). □

Remarks. **1.** The most important remarks are that with care both the condition that $f(0) \geq 0$ (if $n = 2$) and condition $f1$ can be removed from the statement of Theorem 6. To remove the first condition we use some of the ideas in the proof of Theorem 7 and some from Clement and Sweers [7]. To remove the second condition, we note that if we blow up at a maximum of a weakly stable solution u_ϵ, Theorem 3 shows that the maximum C cannot occur within order ϵ of the boundary, and hence the blow-up problem must be a full space problem. By Theorem 1, we know that either $f(C) = 0$ and $f'(C) \leq 0$, or C lies in a heteroclinic ODE solution $w(x_n)$ (after rotation of axes) where we assume without loss of generality that $w' > 0$. But for points close by $u_\epsilon(x)$ will be close to $w(\infty)$ (which is necessarily a point $\widetilde{C}$ where $f(\widetilde{C}) = 0, f'(\widetilde{C}) \leq 0$). Thus in all cases C lies close to a point $\widetilde{C}$ where $f(\widetilde{C}) = 0, f'(\widetilde{C}) < 0$. (Remember that the positive zeros are simple.) As before

$C > \widetilde{C}$, but close is impossible by considering where u_ϵ has its maximum, and $C = \widetilde{C}$ is impossible by the Harnack inequality. Thus $C < \widetilde{C}$ but is close. If there is a heteroclinic orbit joining $\widetilde{C}$ to $\hat{C} < \widetilde{C}$, we can use a sweeping family of supersolutions (effectively $w(x_n)$ translated) as in the proof of Theorem 1 in Clement and Sweers [7] to deduce that if $\|u_\epsilon\|_\infty < \widetilde{C}$, then $\|u_\epsilon\|_\infty \le \hat{C}$. This contradicts our definition of $\widetilde{C}$, and hence such a heteroclinic does not exist. In this case, our blowing-up argument and Theorem 1 ensure that u_ϵ for small ϵ cannot take any value in the interval $(\check{C}, \widetilde{C})$ except close to the two ends or close to $\partial\Omega$. Here $\check{C}$ is the largest zero of f below $\widetilde{C}$. Hence by connectedness, we see that u_ϵ must be uniformly close to $\widetilde{C}$ except for points within order ϵ of $\partial\Omega$. Thus, if we do a boundary blow-up, we obtain a positive solution $\hat{u}$ on a half space T such that $\hat{u} = 0$ on ∂T and $\hat{u} \longrightarrow \widetilde{C}$ uniformly as $d(x, \partial T) \longrightarrow \infty$. Hence Theorem 3 implies that $\hat{u} = \hat{u}(x_n)$ and $\hat{u}$ is monotone. We can then complete the argument as in the proof of Theorem 6. This contrasts with the Neumann problem. We can make one other improvement. We could replace condition $f2'$ by requiring that $\{C : C \neq 0, f(C) = 0, f'(C) \le 0\}$ consist of points $\widetilde{x}$ which are isolated points in the set of zeros of f and satisfy $f'(x) \le 0$ for x near $\widetilde{x}$.

2. The situation with stable sign-changing solutions is quite different. For certain nonlinearities for which condition $f1$ fails (in particular for certain "double well" non-linearities) it is shown in [31] that for certain (but not all) domains (cf. [16]) there are stable sign-changing solutions for all small ϵ. These are obtained by gamma convergence ideas. These still occur if we allow some order-ϵ perturbations of the nonlinearity (as in [16]).

3. In many cases, one can eliminate the condition in Theorem 6 that $\|u_\epsilon\|_\infty$ is bounded. In many cases one can give uniform bounds for positive solutions u either by considering where u has its maximum or by blowing-up arguments. In a number of other cases (for example the case where $yf'(y) \sim y^p$ as $y \longrightarrow \infty$ where $0 < p < 1$, or the case where $f(y) \longrightarrow M > 0$ as $y \longrightarrow \infty$ and $yf'(y) \longrightarrow 0$ as $y \longrightarrow \infty$), one can prove that for small ϵ there is a unique large solution, and this solution is stable (cf. [9]).

4. As in the Neumann problem, our techniques could be used to study cases where f also depends on x.

4. Solutions on bounded domains of bounded Morse index

In this section we discuss solutions of (4) of bounded Morse index on smooth bounded domains Ω for small ϵ. For simplicity, we will restrict ourselves to positive solutions and Dirichlet boundary conditions, though our methods could be used in the other cases (with slightly weaker results). We then sketch very briefly two applications of our results.

Theorem 8. *Assume that $n = 2$ or that $n = 3$ and $f(y) \ge 0$ for $y \ge 0$, that conditions $f1$ and $f2$ hold, and that $f(0) \ge 0$. Suppose that u_{ϵ_i} are uniformly bounded positive solutions of (4) for Dirichlet boundary conditions such that the Morse index of u_{ϵ_i} is at most k for all i where $\epsilon_i \longrightarrow 0$ as $i \longrightarrow \infty$. Then, after choosing a subsequence if necessary, there exist $C \ge 0$ and a finite number of distinct points $\{x_i^j\}_{j=1}^m$ in Ω where $m \le k$ such that $f(C) = 0, f'(C) \le 0$ and u_{ϵ_i} is uniformly close to $\phi_{\epsilon_i}(x)$ except within order ϵ_i of x_i^j, $\epsilon_i^{-1} d(x_i^j, \partial\Omega) \longrightarrow \infty$ as $i \longrightarrow \infty$, and $u_{\epsilon_i}(\epsilon_i^{-1}(x - x_i^j))$ converges uniformly to a non-negative solution*

$\widetilde{u}_j$ of finite Morse index of $-\Delta u = f(u)$ on $\mathbf{R}^n$ such that $\widetilde{u}_j \longrightarrow C$ uniformly as $\|x\| \longrightarrow \infty$. Moreover, $\widetilde{u}_j$ is not stable and not constant, and $\widetilde{u}_j \geq C$ on $\mathbf{R}^n$.

Remark. Hence we see that our solutions look like the stable solution with maximum close to C but with $m \leq k$ peaks superimposed. In many cases we can prove that the $\widetilde{u}$ are radial, so that we have a good understanding of these solutions. Note that at this stage I am not claiming that the peaks cannot be close together or close to the boundary (though one can frequently eliminate this; cf. Proposition 3.4 in [22]). Note that, if $\widetilde{u}$ is unique, all the peaks will have asymptotically the same shape. Note also that peak solutions of this type have been very extensively studied in recent years; see [34], [24], [36], [37], [38], [22], where many further references can be found. Note that in other cases the peaks could be above or below C, and in the Neumann case the peaks may or may not be on the boundary.

Proof. We consider $Z_1 = \{C : f(C) = 0, f'(C) \leq 0, 0 \leq C \leq K\}$, which is finite. Here K is large and fixed. Suppose that $u_{\epsilon_i}(x_i)$ is bounded away from Z_1 and $\epsilon_i d(x_i, \partial\Omega) \longrightarrow \infty$. Thus, much as earlier, it is easy to see that after taking subsequences $u_{\epsilon_i}(\epsilon_i^{-1}(x - x_i))$ converges uniformly on compact sets to a solution u_1 of $-\Delta u = f(u)$ on $\mathbf{R}^n$ of finite Morse index. Moreover $u_1(0)$ is not close to Z_1. Hence, by Theorem 1, u_1 is not stable and hence has Morse index at least 1. In particular there exists $\phi \in C_0^\infty(\mathbf{R}^n)$ such that $\int_{R^n} [|\nabla\phi|^2 - f'(u_1)\phi^2] < 0$. It follows easily that for large i the function $\widetilde{\phi_i}(x) = \phi(\epsilon_i(x - x_i))$ has the property that $T_i(\phi_i) = \int_\Omega [\epsilon_i^2 |\nabla\phi_i|^2 - f'(u_{\epsilon_i})\phi_i^2] < 0$ and ϕ_i has support in a set of order ϵ_i from x_i. If $y_i \in \Omega$ is such that $\epsilon_i^{-1} d(y_i, \partial\Omega) \longrightarrow \infty$ as $i \longrightarrow \infty$ and $\epsilon_i^{-1} d(x_i, y_i) \longrightarrow \infty$ as $i \longrightarrow \infty$, and if $u_{\epsilon_i}(y_i)$ is not close to Z_1, we see that we have two functions of disjoint support for which $T_i(\phi) < 0$ for some ϕ, and hence a two-dimensional subspace on which $T_i(\phi) < 0$. Continuing this process, we see by our condition on the Morse index that there exist $\{x_i^j\}_{j=1}^{m_i}$ where $m_i \leq k$ such that $\epsilon_i^{-1} d(x_i^j, \partial\Omega) \longrightarrow \infty$ as $i \longrightarrow \infty, \epsilon_i^{-1} d(x_i^{j_i}, x_i^{k_i}) \longrightarrow \infty$ as $i \longrightarrow \infty$ if $j_i \neq k_i$, and $u_{\epsilon_i}(x)$ is uniformly, close to Z_1 if $x \in \Omega$, and if $\epsilon_i^{-1} d(x, \partial\Omega)$ is large unless $\epsilon_i^{-1} d(x, x_i^j)$ is bounded for some j with $1 \leq j \leq m_i$. Since Ω minus a finite number of points is connected, we see that, for fixed large $i, u_{\epsilon_i}(x)$ must be uniformly close to a single element C_1 of Z_1 except within order ϵ_i of the boundary or within order ϵ_i of some x_i^j. On the other hand, if we blow up within order ϵ_i of the boundary, we obtain a bounded non-negative solution of $-\Delta u = f(u)$ on a half space. By [4], this must be a function of x_n only, and monotone in x_n. Since by using the first integral (and by our assumptions) these solutions are isolated and uniquely determined by their value at infinity, we see that the blow-up half space solution must be the same on all of $\partial\Omega$, and the convergence is uniform near the boundary. Suppose we blow up at a point x_i^j for fixed j and obtain a solution $\hat{u}$. By what we have proved, $\hat{u} \longrightarrow C_1$ uniformly as $|x| \longrightarrow \infty$. This proves our claim except to show that $\hat{u} \geq C_1$. To prove this, we use subsolutions. Now by the existence of the half space solution, $F(t) < F(C_1)$ for $0 < t < C_1$ (by the first integral). Since $f(0) \geq 0$, it follows that $F(0) < F(C_1)$. (If $f(0) = 0$, note that, by uniqueness, $u_0'(0) \neq 0$.) Fix $R > 0$. Hence, by Hess [28] or Clement and Sweers [7], there is a positive solution u_R of $-\epsilon^2\Delta u = f(u)$ on B_R with Dirichlet boundary condition for small positive ϵ such that $\|u_R\|_\infty < C_1$, but is close to C_1. By rescaling we obtain a positive solution $\widetilde{u}_R$ of $\Delta u = f(u)$ with support a small ball such that $\|\widetilde{u}_R\|_\infty < C_1$, but is close to C_1. We extend this to be zero outside its support. Just as in the comments after the proof of Theorem

4, we can then start from $\widetilde{u}_R(\widetilde{x}+\ \ )$ where $\widetilde{x}$ is large, and use the Serrin sweeping principle to deduce that $\hat{u} \geq \|\widetilde{u}_R\|_\infty$ on $\mathbf{R}^n$. Hence $\hat{u} \geq C_1$ as required. This completes the proof. □

Remarks. 1. Note that solutions obtained by various minimax arguments (saddle point theorems, etc.) can usually be shown to have bounded Morse index. On the other hand, if Ω has non-trivial homology, the results in [21] show that there are always positive solutions of large Morse index. Our techniques can also be used to obtain a good deal of information on solutions with finite Morse index if f depends on x (because many of our arguments are blowing-up arguments). If $f(0) < 0$, our result is still true provided there does not exist a positive zero C of f such that $f'(C) \leq 0, \int_0^C f = 0$ and $\int_0^t f < \int_0^C f$ if $0 < t < C$. As we see below, in a number of the simpler examples, condition $f1$ is unimportant. If $f'(C) < 0$, it can be shown that any solutions with finitely many peaks (as in the theorem) have bounded Morse index.

2. We define another condition for f to satisfy.

Condition $f3$. $0 < a_1 < a_2$ are zeros of $f, f < 0$ on $(0, a_1) \cup (a_2, \infty)$, and $f > 0$ on (a_1, a_2).

In this case, condition $f1$ can only fail if $\int_0^{a_2} f = 0$. (More correctly, it only can only fail in ways which affect solutions with $0 \leq u \leq a_2$ on Ω.) However, for the Dirichlet problem, whenever $\int_0^{a_2} f \leq 0$, there are no non-trivial non-negative solutions on Ω by Clement and Sweers [7] or Dancer and Schmitt [23]. Similarly, define

Condition $f4$. There exist $0 < a_1 < a_2 < a_3$ such that $f > 0$ on $(0, a_1) \cup (a_2, a_3)$ and $f < 0$ on $(a_1, a_2) \cup (a_3, \infty)$.

If $\int_{a_1}^{a_3} f \leq 0$, we can show as above that every non-negative solution on Ω of the Dirichlet problem satisfies $u(x) \leq a_1$ in Ω. Thus, again in this case, condition $f1$ is unnecessary. (Note that, if $\int_0^{a_3} f = 0$, then $\int_{a_1}^{a_3} f < 0$.)

Finally the case of a radial solution of $-\epsilon^2 \Delta u = u^p - u$ on an annulus shows that large Morse index solutions sometimes exist. (It can be shown that such a solution peaks on a surface.)

We now consider rather briefly two cases where we can combine Theorem 8 with the theory of peak solutions to obtain interesting results.

First assume that condition $f3$ holds and $\int_0^{a_2} f > 0$ (which is the only interesting case), $f'(a_1) > 0$ and $f'(x) \leq 0$ for x close to a_2. By Theorem 6, for small ϵ, there is a stable solution $\overline{u}_\epsilon$ of (4) (for Dirichlet boundary conditions) close to a_2 on "most" of Ω, and zero is easily seen to be a local minimum of the energy $\int_\Omega [\frac{1}{2}\epsilon^2 |\nabla u|^2 - F(u)]$ where $F' = f$. Here we have used that $f'(0) \leq 0$. Hence as in [30] or [33], we can find a mountain pass solution u_ϵ^1 between 0 and $\overline{u}_\epsilon$ for small positive ϵ. By well known results (cf. [29]) u_ϵ^1 has Morse index at most 1. By Theorem 6, it cannot be weakly stable. By Theorem 8, it must be a 1-peak solution which is small except close to the peak, and near the peak the solution rescaled converges to a decaying positive solution $\hat{u}$ of $-\Delta u = f(u)$ on $\mathbf{R}^n$. As in [32], if $f'(y) \leq 0$ for positive y near zero, $\hat{u}$ is radial. Provided f is C^2, we can then use the theory of 1-peak solutions in Ni and Wei [34] or Wei [38] to prove that, if $f'(0) < 0$, then the peak stays away from the boundary, and indeed it is close to a point of Ω at maximal distance from the boundary if the least energy decaying solutions of (1) are non-degenerate in the space of radial functions. It seems very likely that this last condition can be removed by adapting the techniques in [24]. This improves considerably a result in [33] if Ω is 2-dimensional. In particular, we remove the conditions there on Ω. We could obtain similar theorems under condition $f4$ if $\int_{a_1}^{a_3} f > 0$ (and some technical conditions weaker than those in [18]) where we look for the mountain pass solution $\widetilde{u}_\epsilon$ between $u_{1\epsilon}$ and $u_{2\epsilon}$. Here $u_{1\epsilon}$ is the stable solution close to a_1 on most of Ω and $u_{2\epsilon}$ is the stable solution close to a_3 on most of Ω. We certainly can prove $\widetilde{u}_\epsilon$ is a peak on $u_{1\epsilon}$, and we can probably remove the strong geometric restrictions on Ω in [18] in the 2-dimensional case. For the last result, we need to use the theory in [18], [19] and [20, §5], and we need to assume the least energy solutions of $-\Delta v = f(a_1+v), v \geq 0, v \longrightarrow 0$ as $\|x\| \longrightarrow \infty$ on $\mathbf{R}^n$ are non-degenerate in the space of radial functions. We suspect, as in [24], that the non-degeneracy hypothesis is not needed.

Finally we consider $-\epsilon^2 \Delta u = u^p - u$ on Ω with Dirichlet boundary conditions. Here Ω is 2-dimensional and strongly convex, $1 < p < \infty$, and we look for positive solutions. Then for small ϵ, the only solution with bounded Morse index is a 1-peak solution (whose peak is also close to the unique point of Ω of maximum distance from $\partial\Omega$). This one-peak solution is probably unique, but this has not been proved. To see this, note that Theorem 6 and the bound in [34] impliy that are no weakly stable solutions, Theorem 8 implies that the only solutions of finite Morse index are multi-peak solutions, and then a result in [21] implies that there are no k-peak solutions for $k \geq 2$. (A technical point here. We need to use that the solutions in Theorem 8 decay exponentially away from the peaks to check that the solutions here are of the form needed for the theory in [21].) We could prove this result for a rather larger class of nonlinearities (though the present proof still seems to require the uniqueness and non-degeneracy of the positive radial decaying solution on $\mathbf{R}^n$). Note that the arguments here and in Proposition 3.4 in [22] imply that, for the equation at the start of this paragraph, but on any smooth bounded domain Ω in $\mathbf{R}^2$, positive solutions of bounded Morse index consist of finitely many peaks not close to each other or to the boundary. This contrasts with the radial positive solution on an annulus where the maximum occurs close to the inner boundary.

In conclusion, I want to point out several differences when we consider the analogue of Theorem 8 for the Neumann problem. As in §3, condition $f1$ is essential for the Neumann problem. Also, finite Morse index solutions of the Neumann problem may have boundary peaks as well as interior peaks, and peaks could be upwards or downwards peaks. Note that multi-peak positive solutions are well known to occur for certain Neumann problems for any domain; cf. [27] or [39]. Note that these solutions may have large Morse index.

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