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BROWNIAN MOTION IN TWISTED DOMAINS

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ABSTRACT. The tail behavior of a Brownian motion's exit time from an unbounded domain depends upon the growth of the "inner radius" of the domain. In this article we quantify this idea by introducing the notion of a twisted domain in the plane. Roughly speaking, such a domain is generated by a planar curve as follows. As a traveler proceeds out along the curve, the boundary curves of the domain are obtained by moving out $\pm g(r)$ units along the unit normal to the curve when the traveler is r units away from the origin. The function g is called the growth radius. Such domains can be highly nonconvex and asymmetric. We give a detailed account of the case $g(r) = \gamma r^p$, $0 < p \leq 1$. When $p = 1$, a twisted domain can reasonably be interpreted as a "twisted cone."

1. INTRODUCTION

Let G be the planar domain lying above the parabola $y = Ax^2$, $A > 0$. In Bañuelos, DeBlassie and Smits (2001) it was shown that the exit time τ_G of Brownian motion from G has unusual behavior: for some positive constants A_1 and A_2,

$$(1.0) \qquad -A_1 \leq \liminf_{t\to\infty} t^{-1/3} \log P(\tau_G > t) \leq \limsup_{t\to\infty} t^{-1/3} \log P(\tau_G > t) \leq -A_2.$$

Though not stated in that article, the method extends to more general domains $\{(x, y)\colon\ y > Ax^q\}$, $A > 0$, $q > 1$ yielding an analog of (1.0) where $t^{-1/3}$ is replaced by $t^{-(q-1)/(q+1)}$.

Using Gaussian techniques, Li (2002) extended (1.0) to regions $G = \{(x, y) \in \mathbb{R}^{d+1}\colon\ y > A\|x\|^q, x \in \mathbb{R}^d\}$, $A > 0$, $q > 1$, $d \geq 2$, giving the constants A_1 and A_2 explicitly in terms of zeros of Bessel functions. He also obtained estimates for more general unbounded regions. Lifshits and Shi (2002) extended Li's results, explicitly identifying

$$\lim_{t\to\infty} t^{-(q-1)/(q+1)} \log P(\tau_G > t)$$

in terms of zeros of Bessel functions. M. van den Berg (2003) has obtained analogs of these results for the heat kernel.

All these articles exploit the symmetry and convexity of the unbounded region G. Since the power of t multiplying $\log P(\tau_G > t)$ is independent of the dimension, it seems the asymptotic behavior of $\log P(\tau_G > t)$ is essentially determined by the rate at which the domain "expands" along the axis of symmetry as a traveler on the axis moves toward infinity. That is, at distance y from the origin along the axis of symmetry, the "width" of G is $2y^{1/q}$. The key here is that "width" is measured

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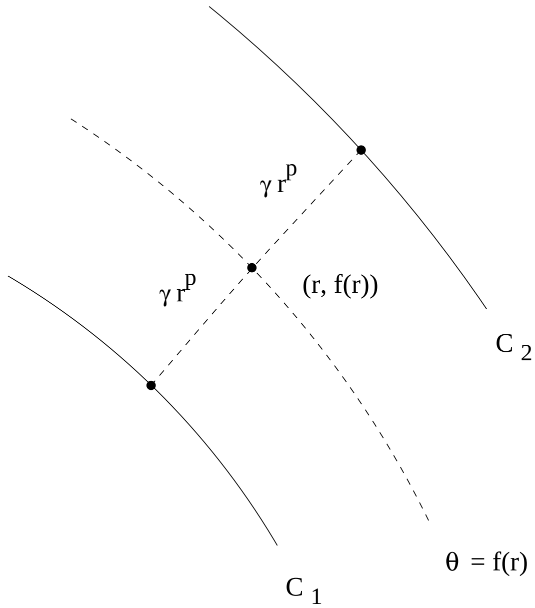


FIGURE 1.

perpendicular to some curve lying within the region. The purpose of this article is to describe the connection between the growth rate of the "inner radius" of the domain and the rate of decay of the tail distribution of the exit time of Brownian motion from the domain.

These considerations motivate the following definition of a *twisted domain*. For simplicity, we restrict attention to planar domains.

Let $D \subseteq \mathbb{R}^2$ be a domain whose boundary consists of three curves (in polar coordinates)

$$\begin{aligned} &C_1\colon\ \theta = f_1(r), \quad r \geq r_1,\\ &C_2\colon\ \theta = f_2(r), \quad r \geq r_1,\\ &C_3\colon\ r = r_1, \quad f_2(r_1) \leq \theta \leq f_1(r_1), \end{aligned}$$

where f_1 and f_2 are smooth and the curves C_1 and C_2 do not cross:

$$0 < f_1(r) - f_2(r) < 2\pi, \quad r \geq r_1.$$

We say D is a *twisted domain* if there are constants $r_0 > 0$, $\gamma > 0$ and $p \in (0, 1]$ and a smooth function $f(r)$ such that the curves $f_1(r)$ and $f_2(r)$, $r \geq r_0$, are obtained from $f(r)$ by moving out $\pm\gamma r^p$ units along the normal to the curve $\theta = f(r)$ at the point whose polar coordinates are $(r, f(r))$. We call γr^p the *growth radius* and $\theta = f(r)$ the *generating curve*. See Figure 1.

There is a question of consistency: the generating curve must yield nonintersecting boundary curves that can be parameterized by distance to the origin. To ensure consistency, we make the following hypotheses on the generating curve:

$$rf'(r) \to 0 \quad \text{as} \quad r \to \infty, \tag{1.1}$$

$$r(rf'(r))' \to 0 \quad \text{as} \quad r \to \infty. \tag{1.2}$$

These hypotheses arise naturally to permit parameterization of the boundary curves by distance to the origin. The condition (1.1) will also ensure that the curves generated by $f(r)$ do not cross. In fact, when $f(r) \uparrow \infty$ as $r \uparrow \infty$ (a spiral), it is possible to show that $rf'(r) \to \infty$ as $r \to \infty$ would force the generated curves to cross. Intuitively, the spiral $\theta = f(r)$ does not have "enough spacing" between successive journeys about the origin.

Note that by (1.1) and (1.2) the generating curve really comes into play only for parts of the domain far away from the origin. Stated another way, the asymptotics in Theorems 1.1 and 1.2 below are unchanged if a compact set is adjoined to the twisted domain D. Also observe that twisted domains can be nonsymmetric and/or nonconvex. Now we state our main theorems. In what follows, for $0 < p < 1$ and $b > 0$ set

$$C_{p,b} = (1+p)\left[\frac{\pi^{2+p}}{8^p p^{2p} b^2 (1-p)^{1-p}} \frac{\Gamma^{2p}\left(\frac{1-p}{2p}\right)}{\Gamma^{2p}\left(\frac{1}{2p}\right)}\right]^{\frac{1}{p+1}}. \tag{1.3}$$

Theorem 1.1. *Let $D \subseteq \mathbb{R}^2$ be a twisted domain with growth radius γr^p, $\gamma > 0$, $0 < p < 1$. Then*

$$\lim_{t\to\infty} t^{-\frac{1-p}{1+p}} \log P_z(\tau_D > t) = \left[\frac{\pi^{2p-1}}{\gamma 2^{2p}(1-p)^{2p}}\right]^{\frac{2}{1+p}} C_{p,1}.$$

Theorem 1.2. *Let $D \subseteq \mathbb{R}^2$ be a twisted domain with growth radius γr, $\gamma > 0$. Then*

$$\lim_{t\to\infty} [\log t]^{-1} \log P_z(\tau_D > t) = \pi \left[4 \arccos \frac{1}{\sqrt{1+\gamma^2}}\right]^{-1}.$$

Remark. Using Theorem 1 from van den Berg (2003), our results imply that the heat kernel $p_D(t,x,y)$ for D in our Theorem 1.1 satisfies

$$\lim_{t\to\infty} t^{-\frac{1-p}{1+p}} \log p_D(t,x,y) = -\left[\frac{\pi^{2p-1}}{\gamma 2^p(1-p)^{2p}}\right]^{\frac{2}{1+p}} C_{p,1}.$$

The article is organized as follows. In sections 2 and 3 we show that our hypotheses (1.1) and (1.2) prevent inconsistencies: distance to the origin r can be used as a parameter and the boundary curves do not cross. In section 4 we collect some results on conformal maps. Functionals of Brownian motion in a strip comprise the object of study in section 5. In section 6 we derive properties of conformal maps from D into strips. Section 7 is the place we prove our main results, and section 8 gives the proof of a technical result from section 6.

We close this introduction with several examples and remarks.

Example 1. If the generating curve is $\theta \equiv$ constant and $p = 1$, then D is a cone and Theorem 1.2 reduces to a weak version of an old result of Spitzer (1958). In light of this, it is reasonable to call any twisted domain with $p = 1$ a *twisted cone*.

Example 2. Let $g(x)$ satisfy

$$g(x) \to L \in [0,\infty], \quad g'(x) \to 0, \quad xg''(x) \to 0,$$

all as $x \to \infty$. Then $y = g(x)$ can be represented in polar coordinates as $\theta = f(r)$, and we get a generating curve satisfying (1.1) and (1.2). Our theorems apply for growth radius γr^p, $0 < p \leq 1$.

For the special case $g \equiv 0$ and $0 < p < 1$, the corresponding twisted domain is

$$\{(x, y) : \ |y| < \gamma |x|^p\},$$

which is a rotation of the domain

$$\widetilde{D} = \{(x, y) : \ y > \gamma^{-1/p} |x|^{1/p}\}$$

considered by Li (2002) and Lifshits and Shi (2002).

Example 3. Spirals. If the generating curve is given by

$$\theta = (\ln r)^q,$$

where $0 < q < 1$, then the hypotheses (1.1) and (1.2) hold and our theorems apply.

Example 4. Oscillation at infinity. Let the generating curve be given by

$$\theta = \sin((\ln r)^q),$$

where $0 < q < 1$. Hypotheses (1.1) and (1.2) hold and our theorems apply. Notice that the domain "oscillates at ∞", in contrast to the previous example, where the domain spins round and round.

Example 5. Consider the domain

$$D = \{(x, y) : \ y > |x|^q \text{ for } x \leq 0 \text{ and } y > |x|^p \text{ for } x > 0\},$$

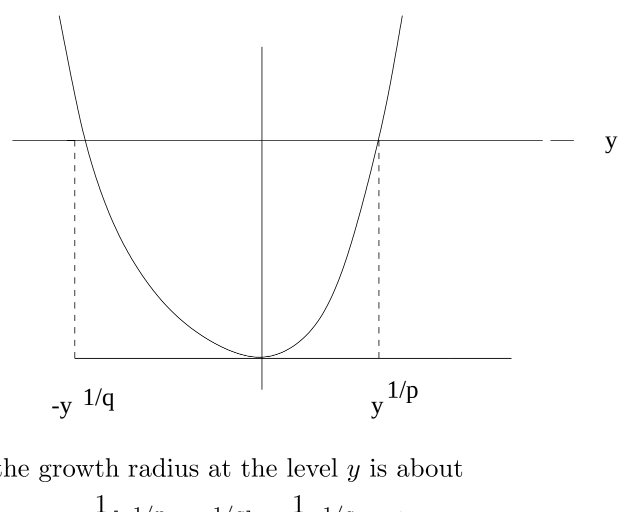
where $1 < q < p$. Then our theorems show that the $|x|^q$ part of the boundary determines the asymptotic behavior: for some positive $A = A(q)$,

$$\lim_{t \to \infty} t^{-(q-1)/(q+1)} \log P(\tau_D > t) = A.$$

Rather than give the rigorous details, we present a plausibility argument. In the rest of this example, we use the notation $g \approx h$ to mean "g is approximately h" in an intuitive sense. Since the normal to the boundary of D converges to a vertical line at ∞, the "halfway curve"

$$x = \frac{1}{2}[y^{1/p} - y^{1/q}]$$

is more or less a generating curve for D.



Moreover, the growth radius at the level y is about

$$\frac{1}{2}[y^{1/p} + y^{1/q}] \approx \frac{1}{2} y^{1/q}, \quad \text{for large} \quad y,$$

since $1 < q < p$. The angular coordinate of a point on the growth curve is

$$\begin{aligned}\theta &= \pi + \tan^{-1}\frac{2y}{y^{1/p} - y^{1/q}} \\ &\approx \pi - \tan^{-1} 2y^{1-\frac{1}{q}}, \quad \text{for large} \quad y,\end{aligned}$$

and the corresponding distance to the origin is

$$\begin{aligned}r &= \sqrt{y^2 + \frac{1}{4}[y^{1/p} - y^{1/q}]^2} \\ &\approx y, \quad \text{for large} \quad y.\end{aligned}$$

Thus our generating curve has the polar form

$$\theta = \pi - \tan^{-1} 2r^{1-1/q},$$

and it is easy to verify that (1.1) and (1.2) hold. Since the growth radius is about $\frac{1}{2}y^{1/q} \approx \frac{1}{2}r^{1/q}$, we get the asserted asymptotic behavior. One can make this argument rigorous by showing that the twisted domains

$$\begin{aligned}&D_1: \text{ growth curve } \theta = \pi - \tan^{-1} 2r^{1-1/q}, \text{ growth radius } \left(\tfrac{1}{2} - \varepsilon\right)r^{1/q}, \\ &D_2: \text{ growth curve } \theta = \pi - \tan^{-1} 2r^{1-1/q}, \text{ growth radius } \left(\tfrac{1}{2} + \varepsilon\right)r^{1/q},\end{aligned}$$

satisfy

$$D_1 \subseteq D \subseteq D_2.$$

2. Consistency: Parameterization by r

Our setup involving the growth radius γr^p leads to a natural parameterization of the boundary curves in terms of the distance t from the origin to the point corresponding to $\theta = f(t)$ on the generating curve. Here are the details.

Let $P = (t\cos f(t), t\sin f(t))$ be the Cartesian coordinates of a point on the generating curve, and suppose Q_1 and Q_2 are the points on C_1 and C_2, respectively, that are $\mp\gamma t^p$ units along the normal at P. The unit tangent vector at P is

$$\frac{(\cos f - tf'\sin f, \sin f + tf'\cos f)}{\sqrt{1 + (tf')^2}},$$

and the unit normal at P (obtained by rotating the unit tangent 90° clockwise) is

$$\vec{N} = \vec{N}(t) = \frac{(\sin f + tf'\cos f, -\cos f + tf'\sin f)}{\sqrt{1 + (tf')^2}}.$$

Thus

$$\begin{aligned}\vec{OQ_1} &= \vec{OP} + \vec{PQ_1} \\ &= \vec{OP} - \gamma t^p \vec{N}.\end{aligned}$$

This yields the following parametric representation of C_1:

$$\begin{aligned}x_1(t) &= t\cos f(t) - \frac{\gamma t^p[\sin f(t) + tf'(t)\cos f(t)]}{\sqrt{1 + [tf'(t)]^2}}, \\ y_1(t) &= t\sin f(t) - \frac{\gamma t^p[-\cos f(t) + tf'(t)\sin f(t)]}{\sqrt{1 + [tf'(t)]^2}}.\end{aligned}$$

Similarly, C_2 is given by

$$x_2(t) = t\cos f(t) + \frac{\gamma t^p[\sin f(t) + tf'(t)\cos f(t)]}{\sqrt{1+[tf'(t)]^2}},$$
$$y_2(t) = t\sin f(t) + \frac{\gamma t^p[-\cos f(t) + tf'(t)\sin f(t)]}{\sqrt{1+[tf'(t)]^2}}.$$

Denote by $r_i(t)$ and $\theta_i(t)$ the polar coordinates of Q_i. Then for

$$h(t) = \frac{t^p f'(t)}{\sqrt{1+[tf'(t)]^2}}, \tag{2.1}$$

we have

$$r_1^2(t) = t^2[1+\gamma^2 t^{2p-2} - 2\gamma h(t)], \tag{2.2}$$
$$r_2^2(t) = t^2[1+\gamma^2 t^{2p-2} + 2\gamma h(t)]. \tag{2.3}$$

Moreover, using that the angle between the position vector $\overrightarrow{OP}$ and the tangent vector at P is no more than $90°$ (their dot product is nonnegative), we have

$$\theta_1(t) = f(t) + \angle(\overrightarrow{OP}, \overrightarrow{OQ}_1),$$
$$\theta_2(t) = f(t) - \angle(\overrightarrow{OP}, \overrightarrow{OQ}_2),$$

where

$$\angle(\overrightarrow{OP}, \overrightarrow{OQ}_i) = \arccos\frac{\overrightarrow{OP}\cdot\overrightarrow{OQ}_i}{|\overrightarrow{OP}||\overrightarrow{OQ}_i|}, \qquad i=1,2.$$

Upon computing the latter and substituting into the equations for θ_i, we get

$$\theta_1(t) = f(t) + \arccos\frac{1-\gamma h(t)}{\sqrt{1+\gamma^2 t^{2p-2} - 2\gamma h(t)}}, \tag{2.4}$$

$$\theta_2(t) = f(t) - \arccos\frac{1+\gamma h(t)}{\sqrt{1+\gamma^2 t^{2p-2} + 2\gamma h(t)}}. \tag{2.5}$$

The main result of this section is the next theorem.

Theorem 2.1. *For $i=1,2$, $r_i(t)$ is increasing for large t.*

Proof. Differentiation of (2.2) yields

$$2r_1 r_1' = 2t[1 + p\gamma^2 t^{2p-2} - 2\gamma h - \gamma t h']. \tag{2.6}$$

Since $(tf')' = f' + tf''$, differentiation of (2.1) gives

$$h' = \frac{t^{p-1}[pf' + (p-1)t^2(f')^3 + tf'']}{(\sqrt{1+(tf')^2})^3}$$
$$= \frac{t^{p-1}[(p-1)f' + (p-1)t^2(f')^3 + (tf')']}{(\sqrt{1+(tf')^2})^3}.$$

Hypothesis (1.1) forces $t^p f'(t) \to 0$ as $t\to\infty$, and so

$$h(t) \to 0 \quad \text{as} \quad t\to\infty. \tag{2.7}$$

By hypotheses (1.1) and (1.2),

$$th'(t) \to 0 \quad \text{as} \quad t\to\infty. \tag{2.8}$$

In fact,

$$t^{1-p}h(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{2.9}$$

By (2.7) and (2.8)

$$1 + p\gamma^2 t^{2p-2} - 2\gamma h(t) - \gamma t h'(t) \to \begin{cases} 1+\gamma^2, & p = 1, \\ 1, & p < 1. \end{cases} \tag{2.10}$$

In any event, we see from (2.6) that $r_1'(t) > 0$ for large t. The argument for r_2 is similar. □

Corollary 2.2. *For* $i = 1, 2$,

$$\lim_{t\to\infty} \frac{r_i(t)}{t r_i'(t)} = 1.$$

Proof. For $i = 1$, we have

$$\begin{aligned} \frac{r_1(t)}{t r_1'(t)} &= \frac{r_1^2(t)}{t r_1(t) r_1'(t)} \\ &= \frac{1 + \gamma^2 t^{2p-2} - 2\gamma h}{1 + p\gamma^2 t^{2p-2} - 2\gamma h - \gamma t h'} \\ &\to 1 \quad \text{as} \quad t \to \infty \end{aligned}$$

by (2.7) and (2.10). The case $i = 2$ is similar. □

An immediate consequence of this theorem is that the boundary curves arising from the generating function f are indeed parameterizable by distance to the origin. In particular, we have (making r_0 large enough)

$$C_i\colon\ f_i(r) = \theta_i(r_i^{-1}(r)), \quad r \ge r_0, \quad i = 1, 2. \tag{2.11}$$

3. Consistency: The boundary curves do not cross

The main result of this section is the next theorem.

Theorem 3.1. *As* $r \to \infty$,

$$\theta_1(r_1^{-1}(r)) - \theta_2(r_2^{-1}(r)) \sim \begin{cases} 2\gamma r^{p-1}, & p < 1, \\ 2 \arccos \dfrac{1}{\sqrt{1+\gamma^2}}, & p = 1. \end{cases}$$

Corollary 3.2. *For large* r,

$$\theta_1(r_1^{-1}(r)) - \theta_2(r_2^{-1}(r)) \in (0, 2\pi).$$ □

By the corollary, the boundary curves do not cross and our setup is consistent, as claimed. To prove Theorem 3.1 we need the following four lemmas.

Lemma 3.3. *For* $i = 1, 2$, $t\theta_i'(t) \to 0$ *as* $t \to \infty$.

Proof. With $h(t)$ from (2.1), write

$$g(t) = \arccos \frac{1 - \gamma h(t)}{\sqrt{1 + \gamma^2 t^{2p-2} - 2\gamma h(t)}}.$$

Then for $i = 1$, by (2.4)

$$\theta_i(t) = f(t) + g(t). \tag{3.1}$$

Now

$$tg'(t) = \frac{\gamma[\gamma t^{2p-2} - h(t)]th'(t) - (1-p)\gamma t^{2p-2}[1-\gamma h(t)]}{\sqrt{t^{2p-2} - h(t)^2}\,[1+\gamma^2 t^{2p-2} - 2\gamma h(t)]}.$$

For $p = 1$, $tg'(t) \to 0$ as $t \to \infty$, by (2.7)–(2.8). By hypotheses, $tf'(t) \to 0$ as $t \to \infty$, so the desired conclusion follows from (3.1).

If $p < 1$, then rearrangement of tg' yields

$$tg'(t) = \frac{\gamma t^{p-1}}{\sqrt{t^{2p-2} - h(t)^2}} \frac{[\gamma t^{p-1} - t^{1-p}h(t)]th'(t) - (1-p)t^{p-1}[1-\gamma h(t)]}{[1+\gamma^2 t^{2p-2} - 2\gamma h(t)]}. \tag{3.2}$$

We have from (2.1)

$$\begin{aligned} t^{2p-2} - h(t)^2 &= t^{2p-2} - \frac{t^{2p}[f'(t)]^2}{1+[tf'(t)]^2} \\ &= \frac{t^{2p-2}}{1+[tf'(t)]^2}. \end{aligned}$$

Then, using (2.7) and (2.8),

$$tg'(t) \to 0 \quad \text{as} \quad t \to \infty.$$

Again we get the desired conclusion.

The case $i = 2$ is similar. □

Lemma 3.4. *For $i = 1, 2$,*

$$\frac{\theta_i(r_i^{-1}(r_2)) - \theta_i(r_i^{-1}(r_1))}{r_2 - r_1} \to 0 \quad \textit{as} \quad r_1, r_2 \to \infty.$$

Proof. For $i = 1$, there is $\tilde{r}$ between r_1 and r_2 such that

$$\begin{aligned} \frac{\theta_1(r_1^{-1}(r_2)) - \theta_1(r_1^{-1}(r_1))}{r_2 - r_1} &= \theta_1'(r_1^{-1}(\tilde{r}))(r_1^{-1})'(\tilde{r}) \\ &= r_1^{-1}(\tilde{r})\theta_1'(r_1^{-1}(\tilde{r})) \cdot \frac{1}{r_1^{-1}(\tilde{r})r_1'(r_1^{-1}(\tilde{r}))}. \end{aligned} \tag{3.3}$$

Since $\tilde{r} \to \infty$ as r_1, $r_2 \to \infty$, it follows that $r_1^{-1}(\tilde{r}) \to \infty$, and by (2.2) and (2.6)

$$r_1'(r_1^{-1}(\tilde{r})) \to \begin{cases} \sqrt{1+\gamma^2}, & p = 1, \\ 1, & p < 1. \end{cases}$$

Hence by Lemma 3.3 we can let $r_1, r_2 \to \infty$ in (3.3) to get the desired conclusion. □

Lemma 3.5. *As $t \to \infty$,*

$$\arccos \frac{1 \pm \gamma h(t)}{\sqrt{1+\gamma^2 t^{2p-2} \pm 2\gamma h(t)}} \sim \begin{cases} \gamma t^{p-1}, & p < 1, \\ \arccos \dfrac{1}{\sqrt{1+\gamma^2}}, & p = 1. \end{cases}$$

Proof. For $p = 1$, this expression follows immediately from the fact that $h(t) \to 0$ (see (2.7)). By the same token, for $p < 1$ the argument of the arccosine converges to 1. Since $\arccos x \sim \sqrt{2}\,\sqrt{1-x}$ as $x \to 1$,

$$\begin{aligned}\left[\arccos \frac{1 \pm \gamma h(t)}{\sqrt{1+\gamma^2 t^{2p-2} \pm 2\gamma h(t)}}\right]^2 &\sim 2\left[1 - \frac{1 \pm \gamma h(t)}{\sqrt{1+\gamma^2 t^{2p-2} \pm 2\gamma h(t)}}\right]\\
&= 2\frac{\gamma^2[t^{2p-2} - h(t)^2]}{\sqrt{1+\gamma^2 t^{2p-2} \pm 2\gamma h(t)}[\sqrt{1+\gamma^2 t^{2p-2} \pm 2\gamma h(t)} + (1 \pm \gamma h(t))]}\\
&\sim \gamma^2[t^{2p-2} - h(t)^2] \quad (\text{since } h(t) \to 0)\\
&= \gamma^2\left[t^{2p-2} - \frac{t^{2p}[f'(t)]^2}{1+[tf'(t)]^2}\right] \quad (\text{by } (2.1))\\
&= \frac{\gamma^2 t^{2p-2}}{1+[tf'(t)]^2}\\
&\sim \gamma^2 t^{2p-2},\end{aligned}$$

by our hypothesis that $tf'(t) \to 0$. □

Lemma 3.6. *Let $t_1 = t_1(r)$ and $t_2 = t_2(r)$ satisfy $t_1 \to \infty$, $t_2 \to \infty$, $\frac{t_1}{t_2} \to L$, and $t_1^2 - t_2^2 = O(t_2^{p-1} t_1^2)$ as $r \to \infty$. Then*

$$\frac{f(t_1) - f(t_2)}{t_2^{p-1}} \to 0 \quad \textit{as} \quad r \to \infty.$$

Proof. For some $\tilde{t}$ between t_1 and t_2,

$$\begin{aligned}\frac{f(t_1) - f(t_2)}{t_2^{p-1}} &= \frac{f'(\tilde{t})(t_1 - t_2)}{t_2^{p-1}}\\
&= \tilde{t} f'(\tilde{t}) \cdot \frac{t_1^2 - t_2^2}{t_2^{p-1}} \cdot \frac{1}{\tilde{t}(t_1+t_2)}.\end{aligned}$$

Since $\tilde{t} \to \infty$ as $r \to \infty$, $\tilde{t} f'(\tilde{t}) \to 0$ as $r \to \infty$. Hence it suffices to show that

$$\frac{t_1^2 - t_2^2}{t_2^{p-1}} \cdot \frac{1}{\tilde{t}(t_1+t_2)} \quad \text{is bounded as} \quad r \to \infty.$$

Indeed, this quantity can be rewritten as

$$\frac{t_1^2 - t_2^2}{t_1^2 t_2^{p-1}} \cdot \frac{t_1}{\tilde{t}} \cdot \frac{t_1}{t_1+t_2},$$

and it is easy to see our hypotheses imply this is bounded as $r \to \infty$. □

Proof of Theorem 3.1. Write $t_i = r_i^{-1}(r)$, so that we have

$$r = r_1(t_1) = r_2(t_2). \tag{3.4}$$

By (2.2)–(2.3),

$$t_1 = r_1(t_1)[1 + \gamma^2 t_1^{2p-2} - 2\gamma h(t_1)]^{-1/2}, \tag{3.5}$$

$$t_2 = r_2(t_2)[1 + \gamma^2 t_2^{2p-2} + 2\gamma h(t_2)]^{-1/2}. \tag{3.6}$$

Then (3.4) forces

$$\frac{t_1}{t_2} \to 1 \quad \text{as} \quad r \to \infty. \tag{3.7}$$

By (3.4)–(3.6),

$$t_1^2 - t_2^2 = r_1^2(t_1)\frac{\gamma^2[t_2^{2p-2} - t_1^{2p-2}] + 2\gamma[h(t_2) + h(t_1)]}{[1 + \gamma^2 t_2^{2p-2} + 2\gamma h(t_2)][1 + \gamma^2 t_1^{2p-2} - 2\gamma h(t_1)]}.$$

Since $r_1^2(t_1) \sim \begin{cases} t_1^2(1+\gamma^2), & p = 1, \\ t_1^2, & p < 1, \end{cases}$ as $r \to \infty$, and since $h(t) \to 0$ as $t \to \infty$,

$$t_1^2 - t_2^2 \sim \begin{cases} \dfrac{2t_1^2\gamma}{1+\gamma^2}[h(t_2) + h(t_1)], & p = 1, \\ t_1^2[\gamma^2[t_2^{2p-2} - t_1^{2p-2}] + 2\gamma[h(t_2) + h(t_1)], & p < 1, \end{cases}$$

as $r \to \infty$.

If $p = 1$, we get

$$\frac{t_1^2 - t_2^2}{t_2^{p-1}t_1^2} \to 0 \quad \text{as} \quad r \to \infty.$$

If $p < 1$,

$$\frac{t_1^2 - t_2^2}{t_2^{p-1}t_1^2} \sim t_2^{p-1}\left[\gamma^2\left[1 - \left(\frac{t_1}{t_2}\right)^{2p-2}\right] + 2\gamma t_2^{2-2p}h(t_2) + 2\gamma\left(\frac{t_2}{t_1}\right)^{2-2p} t_1^{2-2p}h(t_1)\right] \to 0$$

as $r \to \infty$, using (3.7) and (2.9).

In any case, the hypotheses of Lemma 3.6 are fulfilled and the lemma yields

$$\frac{f(t_1) - f(t_2)}{t_2^{p-1}} \to 0 \quad \text{as} \quad r \to \infty. \tag{3.8}$$

For $p = 1$, (2.4)–(2.5) imply

$$\begin{aligned}\theta_1(r_1^{-1}(r)) - \theta_2(r_2^{-1}(r)) &= \theta_1(t_1) - \theta_2(t_2) \\ &= f(t_1) - f(t_2) + \arccos\frac{1 - \gamma h(t_1)}{\sqrt{1 + \gamma^2 - 2\gamma h(t_1)}} + \arccos\frac{1 + \gamma h(t_2)}{\sqrt{1 + \gamma^2 + 2\gamma h(t_2)}} \\ &\to 2\arccos\frac{1}{\sqrt{1+\gamma^2}}\end{aligned}$$

(since $h(t) \to 0$ as $t \to \infty$).

For $p < 1$, as $r \to \infty$

$$\begin{aligned}\frac{\theta_1(r_1^{-1}(r)) - \theta_2(r_2^{-1}(r))}{r^{p-1}} &= \frac{\theta(t_1) - \theta(t_2)}{[r_2(t_2)]^{p-1}} \\ &\sim \frac{\theta(t_1) - \theta(t_2)}{t_2^{p-1}} \quad \text{by (3.6)} \\ &= \frac{f(t_1) - f(t_2)}{t_2^{p-1}} + \left(\frac{t_1}{t_2}\right)^{p-1}\frac{1}{t_1^{p-1}}\arccos\frac{1 - \gamma h(t_1)}{\sqrt{1 + \gamma^2 t_1^{2p-2} - 2\gamma h(t_1)}} \\ &\qquad + \frac{1}{t_2^{p-1}}\arccos\frac{1 + \gamma h(t_2)}{\sqrt{1 + \gamma^2 t_2^{2p-2} + 2\gamma h(t_2)}} \\ &\to 0 + \gamma + \gamma = 2\gamma\end{aligned}$$

by (3.8), (3.7) and Lemma 3.5. □

4. On conformal mappings

In this section we summarize some results on conformal mappings. The first theorem is taken from Theorem X on p. 315 of Warschawski (1942). Before stating the result, we establish some terminology.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two curves in the w-plane ($w = u + iv$) given by continuous functions $v = \phi_i(u)$, $u \geq u_0$, where $\phi_1(u) > \phi_2(u)$. Let $\mathcal{C}_3$ be a Jordan curve lying in the half-plane $u \leq u_0$, connecting the finite endpoints of $\mathcal{C}_1$ and $\mathcal{C}_2$. The curve $\mathcal{C}$ consisting of $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ divides the plane into 2 regions; denote by E that part containing $\phi_2(u) < v < \phi_1(u)$, $u > u_0$. We say E is an *L-strip* with boundary inclination γ at $u = \infty$, $|\gamma| < \frac{\pi}{2}$, if

$$\lim_{\substack{u_1, u_2 \to \infty \\ u_2 > u_1}} \frac{\phi_i(u_2) - \phi_i(u_1)}{u_2 - u_1} = \tan\gamma, \quad i = 1, 2. \tag{4.1}$$

Define

$$\psi(u) = \frac{\phi_1(u) + \phi_2(u)}{2}, \qquad \theta(u) = \phi_1(u) - \phi_2(u).$$

Let

$$W_\beta = \{\tilde{z} = \tilde{x} + i\tilde{y} \colon\ |\tilde{y}| < \beta\}, \tag{4.2}$$

and suppose $J\colon\ E \to W_{\pi/2}$ is a surjective conformal mapping such that $\lim_{u\to\infty} J(w) = \infty$.

Theorem 4.1 (Warschawski)**.** (i) *For $w = u + iv \in E$, uniformly in v,*

$$\lim_{u\to\infty} \frac{J(w)}{\int_{u_0}^{u} \frac{dt}{\theta(t)}} = \pi.$$

(ii) *For $\beta \in \left(0, \frac{\pi}{2}\right)$, set*

$$T_\beta = J^{-1}(W_\beta),$$
$$S_\beta = \left\{ w = u + iv\colon\ u > u_0, \left| \frac{v - \psi(u)}{\theta(u)} \right| < \frac{\beta}{\pi} \right\}.$$

If $\varepsilon > 0$ and $0 < \beta \pm \varepsilon < \frac{\pi}{2}$, then there exists $N = N(\varepsilon, \beta)$ such that

$$S_{\beta-\varepsilon} \cap \{u \geq N\} \subseteq T_\beta \cap \{u \geq N\} \subseteq S_{\beta+\varepsilon} \cap \{u \geq N\}.$$

(iii) *For any $S_\beta, 0 < \beta < \frac{\pi}{2}$,*

$$\lim_{u\to\infty} |J'(w)|\theta(u) = \pi, \quad w = u + iv \in S_\beta,\ \textit{uniformly on } v. \qquad \square$$

We will make use of the following setup. Let R be a domain in the z-plane and suppose G is a conformal map of R into the $\tilde{z}$-plane. If Z_t is Brownian motion in the z-plane, then by conformal invariance, we can represent $G(Z_t)$ as a time-changed Brownian motion in the $\tilde{z}$-plane. Indeed, let $\widetilde{Z}(t) = \widetilde{Z}_t$ be Brownian motion in the $\tilde{z}$-plane started at $G(Z_0)$, and define

$$\eta_t^{-1} = \int_0^t |G'(G^{-1}(\widetilde{Z}_s))|^{-2} ds. \tag{4.3}$$

Then

$$G(Z_t) =_{\mathcal{L}} \widetilde{Z}(\eta_t) \tag{4.4}$$

($=_{\mathcal{L}}$ means "have the same law").

For domains R and $\widetilde{R}$ in the z-plane and the $\tilde{z}$-plane respectively, let

$$\tau_R = \inf\{t > 0\colon\ Z_t \notin R\},$$
$$\tilde{\tau}_{\widetilde{R}} = \inf\{t > 0\colon\ \widetilde{Z}_t \notin \widetilde{R}\}.$$

Then for $\tilde{z} = G(z)$,

$$\begin{aligned} P_z(\tau_R > t) &= P_z(Z_s \in R\ \forall s \le t) \\ &= P_z(G(Z_s) \in G(R)\ \forall s \le t) \\ &= P_{\tilde{z}}(\widetilde{Z}(\eta_s) \in G(R)\ \forall s \le t) \\ &= P_{\tilde{z}}(\tilde{\tau}_{G(R)} > \eta_t) \\ &= P_{\tilde{z}}(\eta^{-1}(\tilde{\tau}_{G(R)}) > t) \\ &= P_{\tilde{z}}\left(\int_0^{\tilde{\tau}_{G(R)}} |G'(G^{-1}(\widetilde{Z}_s))|^{-2} ds > t\right). \end{aligned} \tag{4.5}$$ □

5. Functionals of Brownian motion in a strip

Below we will show that our main theorems on exit times from twisted domains reduce to the following results. Recall $W_\beta = \{\tilde{z} = \tilde{x} + i\tilde{y}\colon\ |\tilde{y}| < \beta\}$.

Theorem 5.1. a) *Suppose $\widetilde{D}$ is a bounded open subset of $\{\tilde{z} = \tilde{x} + i\tilde{y}\colon\ |\tilde{y}| < \frac{\pi}{2}\}$ with smooth boundary. Let $\beta \in \left(0, \frac{\pi}{2}\right)$ and suppose $\tilde{\tau}$ is the exit time of Brownian motion $\widetilde{Z}_t$ from*

$$\widetilde{W}_\beta = \widetilde{D} \cup (W_\beta \cap \{\tilde{x} > M\}).$$

Then for $p < 1$ and $\tilde{z} \in \widetilde{W}_\beta$, with $\widetilde{Z}_1$ denoting the real part of $\widetilde{Z}$,

$$-A_1 \le \liminf_{t\to\infty} t^{-\frac{1-p}{1+p}} \log P_{\tilde{z}}\left(\int_0^{\tilde{\tau}} [1 + |\widetilde{Z}_1(s)|]^{\frac{2p}{1-p}} ds > t\right),$$

where

$$A_1 = [\pi^{1-p} 2^{2p-1}(1-p)^p]^{-\frac{2}{1+p}} C_{p,2\beta/\pi}$$

and $C_{p,\beta/2\pi}$ is from (1.3).

b) *Moreover, for any $a > 0$,*

$$-\frac{\pi}{2a\beta} \le \liminf_{t\to\infty} [\log t]^{-1} \log P_{\tilde{z}}\left(\int_0^{\tilde{\tau}} \exp(a\widetilde{Z}_1(s)) ds > t\right).$$

Theorem 5.2. a) *Let $\tilde{\tau}$ be the exit time of Brownian motion $\widetilde{Z}_t$ from $W_\beta \cap \{\tilde{x} > M\}$, $0 < \beta < \frac{\pi}{2}$. Then, for $p < 1$, $\tilde{z} \in W_\beta \cap \{\tilde{x} > M\}$ and A_1 from Theorem 5.1 a),*

$$\limsup_{t\to\infty} t^{-\frac{1-p}{1+p}} \log P_{\tilde{z}}\left(\int_0^{\tilde{\tau}} [1 + |\widetilde{Z}_1(s)|]^{\frac{2p}{1-p}} ds > t\right) \le -A_1.$$

b) *Moreover, for any $a > 0$,*

$$\limsup_{t\to\infty} [\log t]^{-1} \log P_{\tilde{z}}\left(\int_0^{\tilde{\tau}} \exp(a\widetilde{Z}_1(s)) ds > t\right) \le -\frac{\pi}{2a\beta}.$$ □

Before giving the proof, we describe the idea. Warschawski's theorem gives a conformal mapping J of

$$R = \{z = x + iy\colon\ x > 0, |y| < x^p\}$$

such that for $\tilde{z} = \tilde{x} + i\tilde{y}$,

$$|J'(J^{-1}(\tilde{z}))|^{-2} \approx [1 + |\tilde{x}|]^{\frac{2p}{1-p}}.$$

Moreover, $J(R)$ is not too different from $W_\beta \cap \{\tilde{x} > M\}$, and so $\tilde{\tau}_{J(R)} \approx \tilde{\tau}$. Then by (4.5)

$$P_z(\tau_R > t) \approx P_{\tilde{z}}\left(\int_0^{\tilde{\tau}} [1 + |\widetilde{Z}_1(s)|]^{\frac{2p}{1-p}} ds > t\right).$$

The asymptotics of the left hand side are known (Li, 2002), and the desired conclusion follows. The second part is a bit easier. The mapping $J(z) = \frac{2}{a}\operatorname{Log} z$ takes

$$R = \left\{z = re^{i\theta}\colon\ r > 0, -\frac{a\beta}{2} < \theta < \frac{a\beta}{2}\right\}$$

onto the strip W_β,

$$|J'(J^{-1}(\tilde{z}))|^{-2} \approx e^{a\tilde{x}},$$

and $\tau_{J(R)} \approx \tilde{\tau}$. By (4.5) we have

$$P_z(\tau_R > t) \approx P_{\tilde{z}}\left(\int_0^{\tilde{\tau}} \exp(a\widetilde{Z}_1(s)) ds > t\right).$$

Again the asymptotics of the left hand side are known, and the desired conclusion follows.

First, we describe the asymptotics of $P_z(\tau_R > t)$ for R as in the two cases above.

Lemma 5.3. *For $R = R_b = \{x + iy\colon\ |y| < bx^p,\ x > 0\}$, $b > 0$, $p < 1$, for any compact set $K \subseteq R$,*

$$\lim_{t\to\infty} t^{-\frac{1-p}{1+p}} \log P_z(\tau_R > t) = -C_{p,b}$$

uniformly for $z \in K$, where $C_{p,b}$ is from (1.3).

Proof. Lifshits and Shi (2002) consider the domain

$$\{(x,y)\colon\ y > a|x|^q\}, \qquad q > 1,$$

and compute the limit for $K = \{z\}$. In our setting their results translate into

$$\lim_{t\to\infty} t^{-\frac{1-p}{1+p}} \log P_z(\tau_R > t) = -C_{p,b}$$

for each $z \in R$. To extend this to the version we need, note that for each compact set $K \subseteq R$, there are a translation $\mathcal{R}_1$ of R and $z_1 \in \mathcal{R}_1$ with the following property: for any $z \in K$, there exists a translation $\widetilde{\mathcal{R}}_1$ of $\mathcal{R}_1$ where $z_1 \in \mathcal{R}_1$ is shifted to $z \in \widetilde{\mathcal{R}}_1$ and $\widetilde{\mathcal{R}}_1 \subseteq R$. Hence

$$\begin{aligned} P_z(\tau_R > t) &\geq P_z(\tau_{\widetilde{\mathcal{R}}_1} > t) \\ &= P_{z_1}(\tau_{\mathcal{R}_1} > t). \end{aligned}$$

The point is that z_1 and $\mathcal{R}_1$ are independent of $z \in K$. Similarly, there are a translation $\mathcal{R}_2$ of R and $z_2 \in \mathcal{R}_2$ such that

$$P_z(\tau_R > t) \leq P_{z_2}(\tau_{\mathcal{R}_2} > t), \qquad z \in K.$$

Combined with the previous inequality and the result of Lifshits and Shi, we get the desired uniform limiting behavior. □

Lemma 5.4. *For* $R = R_{a,\beta} = \left\{re^{i\theta}\colon\ r > 0, -\frac{a\beta}{2} < \theta < \frac{a\beta}{2}\right\}$, $a > 0$, $0 < \beta < \pi/2$, *for any compact set* $K \subseteq R$,

$$\lim_{t\to\infty} (\log t)^{-1} \log P_z(\tau_R > t) = -\frac{\pi}{2a\beta},\ \textit{uniformly for } z \in K,$$

Proof. The region R is a wedge with angle $a\beta$, and it is known that

$$P_z(\tau_R > t) \sim C(z)t^{-\pi/2a\beta} \quad \text{as} \quad t \to \infty$$

(Spitzer, 1958). Thus

$$\lim_{t\to\infty} [\log t]^{-1} \log P_z(\tau_R > t) = -\frac{\pi}{2a\beta}.$$

The rest of the proof is similar to that of Theorem 5.2. □

We need the following extension of Lemmas 5.3 and 5.4. Its proof is deferred to the end of this section.

Lemma 5.5. *Let* R *be* R_b *or* $R_{a,\beta}$ *from Lemmas 5.3 or 5.4, and suppose* U *is a bounded open set with smooth boundary such that* $R \cap U \neq \emptyset$. *Then for* $z \in R \cup U$ *and* $C_{p,b}$ *from* (1.3),

$$-C_{p,b} \le \liminf_{t\to\infty} t^{-\frac{1-p}{1+p}} \log P_z(\tau_{R\cup U} > t), \quad R = R_b,$$
$$-\frac{\pi}{2a\beta} \le \liminf_{t\to\infty} [\log t]^{-1} \log P_z(\tau_{R\cup U} > t), \quad R = R_{a,\beta}.$$ □

Proof of Theorem 5.1 a). We are going to use the Warschawski mapping from the z-plane to the $\tilde{z}$-plane. Thus the variables from section four will be $x + iy$ instead of $u + iv$. Let

$$R = \{x + iy\colon\ |y| < x^p, x > 0\}.$$

In Warschawski's theorem (Theorem 4.1) we use

$$\psi(x) = 0,$$
$$\theta(x) = 2x^p,$$
$$x_0 = 0.$$

Thus, for the mapping $J\colon\ R \to W_{\pi/2}$ described there,

$$T_\beta = J^{-1}(W_\beta) = J^{-1}(\{\tilde{x} + i\tilde{y}\colon\ |\tilde{y}| < \beta\}),$$
$$S_\beta = \left\{x + iy\colon\ |y| < \frac{2\beta}{\pi}x^p, x > 0\right\}.$$

By Theorem 4.1, for $0 < \varepsilon < \beta$ with $\beta + \varepsilon < \frac{\pi}{2}$ there exists $N = N(\varepsilon)$ such that

$$S_{\beta-\varepsilon} \cap \{x \ge N\} \subseteq T_\beta \cap \{x \ge N\} \subseteq S_{\beta+\varepsilon} \cap \{x \ge N\}. \tag{5.1}$$

Moreover,

$$\lim_{x\to\infty} |J'(z)|x^p = \frac{\pi}{2}, \quad z = x + iy \in S_{\beta+\varepsilon},\ \text{uniformly in } y,$$

and

$$\lim_{x\to\infty} \frac{J(z)}{x^{1-p}} = 2(1-p)\pi,\ \text{uniformly in } y.$$

Together these give (recall that $\tilde{z} = J(z)$)

$$|J'(J^{-1}(\tilde{z}))| \sim \frac{\pi}{2}[2(1-p)]^{\frac{p}{1-p}}\tilde{x}^{-\frac{p}{1-p}} \quad \text{as} \quad \tilde{x} \to \infty,$$

and $\tilde{z} = \tilde{x} + i\tilde{y} \in J(S_{\beta+\varepsilon})$, uniformly in $\tilde{y}$.

Hence for each $\delta > 0$, for M_1 sufficiently large and $\tilde{z} \in J(S_{\beta+\varepsilon})$,

$$\begin{aligned}(A-\delta)\left[[1+|\tilde{x}|]^{-\frac{p}{1-p}} I_{|\tilde{x}|>M_1} + K_1 I_{|\tilde{x}|\le M_1}\right] &\le |J'(J^{-1}(\tilde{z}))| \\ \le (A+\delta)\left[[1+|\tilde{x}|]^{-\frac{p}{1+p}} I_{|\tilde{x}|>M_1} + K_2 I_{|\tilde{x}|\le M_1}\right], & \end{aligned} \tag{5.2}$$

where

$$A = \frac{\pi}{2}[2(1-p)]^{\frac{p}{1-p}} = \pi[2^{2p-1}(1-p)^p]^{\frac{1}{1-p}}$$

and K_1, K_2 are unimportant constants depending on M_1, δ, p.

Let M be as in the hypothesis of the theorem. By (5.1), making M_1 larger if necessary, we can assume $M_1 > M$ and

$$W_\beta \cap \{\tilde{x} \ge M_1\} = J(T_\beta) \cap \{\tilde{x} \ge M_1\} \subseteq J(S_{\beta+\varepsilon} \cap \{x \ge N\}). \tag{5.3}$$

Then use (5.1) to choose $N_1 > N$ such that

$$J(S_{\beta-\varepsilon} \cap \{x \ge N_1\}) \subseteq J(T_\beta) \cap \{\tilde{x} \ge M_1\} = W_\beta \cap \{\tilde{x} \ge M_1\}. \tag{5.4}$$

The translation G_1 of $S_{\beta-\varepsilon}$ defined by

$$G_1 = \left\{x + iy\colon\ |y| < \frac{2(\beta-\varepsilon)}{\pi}(x - N_1)^p, x > N_1\right\}$$

satisfies

$$G_1 \subseteq S_{\beta-\varepsilon} \cap \{x \ge N_1\}. \tag{5.5}$$

Let $\tilde{z} \in \widetilde{W}_\beta = \widetilde{D} \cup (W_\beta \cap \{\tilde{x} > M\})$ (from our hypothesis) and suppose $U \subseteq R$ is a bounded open set, with smooth boundary, such that $G_1 \cap U \ne \emptyset$ and

$$\tilde{z} \in J(U) \subseteq \widetilde{W}_\beta.$$

Then by (5.4)–(5.5)

$$\begin{aligned}\tilde{z} \in J(U \cup G_1) &\subseteq \widetilde{W}_\beta \cup [W_\beta \cap \{\tilde{x} \ge M_1\}) \\ &= \widetilde{W}_\beta.\end{aligned}$$

In particular, for $\tilde{\tau}_1 = \tilde{\tau}(J(U \cup G_1)) = \inf\{t > 0\colon\ \widetilde{Z}_t \notin J(U \cup G_1)\}$, we get $\tilde{\tau}_1 \le \tilde{\tau}$. For typographical simplicity, write

$$F(z) = [1 + |z_1|]^{\frac{2p}{1-p}}.$$

Then for $z = J^{-1}(\tilde{z})$,

$$
\begin{aligned}
P_{\tilde{z}}&\left(\int_0^{\tilde{\tau}} [1+|\widetilde{Z}_1(s)|]^{\frac{2p}{1-p}} ds > t\right) = P_{\tilde{z}}\left(\int_0^{\tilde{\tau}} F(\widetilde{Z}(s))ds > t\right)\\
&\geq P_{\tilde{z}}\left(\int_0^{\tilde{\tau}_1} F(\widetilde{Z}(s))ds > t\right)\\
&= P_{\tilde{z}}\left(\int_0^{\tilde{\tau}_1} [F(\widetilde{Z}(s))I(|\widetilde{Z}_1(s)| > M_1) + K_1^{-2} I(|\widetilde{Z}_1(s)| \leq M_1)]ds\right.\\
&\quad \left. + \int_0^{\tilde{\tau}_1} F(\widetilde{Z}(s))I(|\widetilde{Z}_1(s)| \leq M_1)ds - \int_0^{\tilde{\tau}_1} K_1^{-2} I(|\widetilde{Z}_1(s)| \leq M_1)ds > t\right)\\
&\geq P_{\tilde{z}}\left(\int_0^{\tilde{\tau}_1} (A-\delta)^2 |J'(J^{-1}(\widetilde{Z}(s)))|^{-2} ds - \int_0^{\tilde{\tau}_1} K_3 I(|\widetilde{Z}_1(s)| \leq M_1)ds > t\right)
\end{aligned}
$$

(by (5.2), where K_3 is independent of t)

$$
\begin{aligned}
&\geq P_{\tilde{z}}\left(\int_0^{\tilde{\tau}_1} |J'(J^{-1}(\widetilde{Z}(s)))|^{-2} ds > [(A-\delta)^{-2}+\delta]t\right)\\
&\quad - P_{\tilde{z}}\left(\int_0^{\tilde{\tau}_1} I(|\widetilde{Z}_1(s)| \leq M_1)ds > K_4 t\right)
\end{aligned}
$$

(where K_4 is independent of t)

(5.6)

$$
= P_z\left(\tau(U \cup G_1) > [(A-\delta)^{-2}+\delta]t\right) - P_{\tilde{z}}\left(\int_0^{\tilde{\tau}_1} I(|\widetilde{Z}_1(s)| \leq M_1)ds > K_4 t\right)
$$

(by (4.5)). By Lemma 5.5 and translation invariance of Brownian motion,

(5.7)

$$
\liminf_{t\to\infty} t^{-\frac{1-p}{1+p}} \log P_z(\tau(U\cup G_1) > [(A-\delta)^{-2}+\delta]t) \geq -[(A-\delta)^{-2}+\delta]^{\frac{1-p}{1+p}} C_{p,2(\beta-\varepsilon)/\pi}.
$$

Since $J(U \cup G_1) \subseteq W_{\frac{\pi}{2}}$, $\tilde{\tau}_1$ is bounded by the first exit time η of $\widetilde{Z}$ from $W_{\frac{\pi}{2}}$. It is well-known that for some $K_5 > 0$,

$$
\log P_{\tilde{z}}(\eta > t) \leq -K_5 t, \quad t \text{ large}.
$$

Thus, for t large,

$$
\begin{aligned}
\log P_{\tilde{z}}\left(\int_0^{\tilde{\tau}_1} I(|\widetilde{Z}_1(s)| \leq M_1) > K_4 t\right) &\leq \log P_{\tilde{z}}(\eta > K_4 t)\\
&\leq -K_5 t.
\end{aligned}
$$

Using this and (5.7), after taking logs on both sides of (5.6) we get

$$
\liminf_{t\to\infty} t^{-\frac{1-p}{1+p}} \log P_{\tilde{z}}\left(\int_0^{\tilde{\tau}} [1+|\widetilde{Z}_1(s)|]^{\frac{2p}{1-p}} ds > t\right) \geq -[(A-\delta)^{-2}+\delta]^{\frac{1-p}{1+p}} C_{p,2(\beta-\varepsilon)/\pi}.
$$

Let $\delta \to 0$ and then $\varepsilon \to 0$, to end up with

$$\liminf_{t\to\infty} t^{-\frac{1-p}{1+p}} \log P_{\tilde z}\left(\int_0^{\tilde\tau} [1+|\widetilde Z_1(s)|]^{\frac{2p}{1+p}} ds > t\right) \ge -A^{-\frac{2(1-p)}{1+p}} C_{p,2\beta/\pi} = -A_1,$$

as desired. □

Proof of Theorem 5.2 a). We continue to use the notation from the proof of Theorem 5.1 a). Choose $\varepsilon > 0$ so close to $\frac{\pi}{2} - \beta$ that

$$W_\beta \cap \{\tilde x > M\} \subseteq J(S_{\beta+\varepsilon}).$$

This is possible because $S_r \uparrow R$ as $r \uparrow \pi/2$. Then for $\tilde z \in W_\beta \cap \{\tilde x > M\}$ and $\tilde\tau_2 = \inf\{t > 0\colon\ \widetilde Z_t \notin J(S_{\beta+\varepsilon})\}$, by (5.2),

$$\begin{aligned}
P_{\tilde z}&\left(\int_0^{\tilde\tau} [1+|\widetilde Z_1(s)|]^{\frac{2p}{1-p}} ds > t\right) \le P_{\tilde z}\left(\int_0^{\tilde\tau_2} [1+|\widetilde Z_1(s)|]^{\frac{2p}{1-p}} ds > t\right)\\
&\le P_{\tilde z}\left(\int_0^{\tilde\tau_2} (A+\delta)^2 |J'(J^{-1}(\widetilde Z(s)))|^{-2} ds + K_6 \int_0^{\tilde\tau_2} I(|\widetilde Z_1(s)| \le M_1) ds > t\right)\\
&\le P_{\tilde z}\left(\int_0^{\tilde\tau_2} |J'(J^{-1}(\widetilde Z(s)))|^{-2} ds > (A+\delta)^{-2}(1-\delta)t\right)\\
&\quad + P_{\tilde z}\left(\int_0^{\tilde\tau_2} I(|\widetilde Z_1(s)| \le M_1) ds > K_7\delta t\right)\\
&= P_z(\tau(S_{\beta+\varepsilon}) > (A+\delta)^{-2}(1-\delta)t)\\
&\quad + P_{\tilde z}\left(\int_0^{\tilde\tau_2} I(|\widetilde Z_1(s)| \le M_1) ds > K_7\delta t\right).
\end{aligned}$$

Since $S_{\beta+\varepsilon} = \{x+iy\colon\ |y| < \frac{2(\beta+\varepsilon)}{\pi} x^p,\ x > 0\}$, we can use Lemma 5.3 on the first term. Now $J(S_{\beta+\varepsilon}) \subseteq W_{\pi/2}$, and so $\tilde\tau_2 \le \eta$. Hence the log of the second term is bounded by $-K_\delta t$ for large t. Take the lim sup as $t \to \infty$, then let $\delta \to 0$, $\varepsilon \to 0$, to end up with the upper bound

$$\limsup_{n\to\infty} P_{\tilde z}\left(\int_0^{\tilde\tau} [1+|\widetilde Z_1(s)|]^{\frac{2p}{1+p}} ds > t\right) \le -A_1. \qquad \square$$

Proof of Theorem 5.1 b). Let

$$J(z) = \frac{2}{a} \operatorname{Log} z \quad \text{(principal branch)}.$$

Given $\tilde z \in \widetilde{W}_\beta$, there exist a bounded open set U with smooth boundary, and a translation $\mathcal R$ of

$$R_{a,\beta} = \left\{ re^{i\theta}\colon\ r > 0, -\frac{a\beta}{2} < \theta < \frac{a\beta}{2} \right\}$$

such that

$$\tilde z \in J(\mathcal R \cup U) \subseteq \widetilde{W}_\beta.$$

This is possible because $\widetilde{W}_\beta \subseteq \left\{\tilde{z} = \tilde{x} + i\tilde{y}\colon\ |\tilde{y}| < \frac{\pi}{2}\right\}$. In particular, $\tilde{\tau} \geq \tilde{\tau}_1 := \tilde{\tau}(J(\mathcal{R} \cup U))$. Since

$$|J'(J^{-1}(\tilde{z}))| = \frac{2}{a} e^{-a\tilde{x}/2}, \quad \tilde{z} = \tilde{x} + i\tilde{y}, \tag{5.8}$$

we have

$$\begin{aligned}
P_{\tilde{z}}\left(\int_0^{\tilde{\tau}} \exp(a\widetilde{Z}_1(s))ds > t\right) &= P_{\tilde{z}}\left(\int_0^{\tilde{\tau}} |J'(J^{-1}(\widetilde{Z}(s)))|^{-2} ds > \frac{a^2 t}{4}\right) \\
&\geq P_{\tilde{z}}\left(\int_0^{\tilde{\tau}_1} |J'(J^{-1}(\widetilde{Z}(s)))|^{-2} ds > \frac{a^2 t}{4}\right) \\
&= P_z\left(\tau_{\mathcal{R}\cup U} > \frac{a^2 t}{4}\right), z = J^{-1}(\tilde{z}) \quad \text{(by (4.5))}.
\end{aligned}$$

Together with translation invariance of Brownian motion and Lemma 5.5, this gives

$$\liminf_{t\to\infty} [\log t]^{-1} \log P_{\tilde{z}}\left(\int_0^{\tilde{\tau}} \exp(a\widetilde{Z}_1(s))ds > t\right) \geq -\frac{\pi}{2a\beta},$$

as desired. □

Proof of Theorem 5.2 b). We continue to use the notation from the proof of Theorem 5.1 b). Notice that J maps the wedge

$$R = R_{a,\beta} = \left\{re^{i\theta}\colon\ r > 0, -\frac{a\beta}{2} < \theta < \frac{a\beta}{2}\right\}$$

onto W_β. Since $\tilde{\tau} = \tilde{\tau}(W_\beta \cap \{\tilde{x} > M\}) \leq \tilde{\tau}(W_\beta)$, we have

$$\begin{aligned}
P_{\tilde{z}}\left(\int_0^{\tilde{\tau}} \exp(a\widetilde{Z}_1(s))ds > t\right) &\leq P_{\tilde{z}}\left(\int_0^{\tilde{\tau}(W_\beta)} \exp(a\widetilde{Z}_1(s))ds > t\right) \\
&= P_{\tilde{z}}\left(\int_0^{\tilde{\tau}(W_\beta)} |J'(J^{-1}(\widetilde{Z}(s)))|^{-2} ds > \frac{a^2 t}{4}\right) \quad \text{(by (5.8))} \\
&= P_z\left(\tau_R > \frac{a^2 t}{4}\right), z = J^{-1}(\tilde{z}) \quad \text{(by (4.5))}.
\end{aligned}$$

Using Lemma 5.4 yields the desired conclusion. □

We end this section with the proof of Lemma 5.5. As we will see, it comes down to the next lemma.

Lemma 5.6. *Let $E \subseteq \mathbb{R}^2$ be a bounded open set with piecewise smooth boundary. Assume $I \subseteq \partial E$ is nonpolar for Brownian motion Z_t and $z \in E$, $T > 0$. Suppose $g(t)$ is twice differentiable with*

$$g(t) \uparrow \infty \quad \textit{and} \quad g'(t) \to 0 \quad \textit{as} \quad t \to \infty$$

and

$$\int^\infty g'(u) e^{\lambda u} du = \infty \quad \textit{for all} \quad \lambda > 0. \tag{5.9}$$

Then there exists a positive constant T_1 such that

$$\liminf_{t\to\infty}[g(t-T_1)]^{-1}\log\int_{T_1}^{t-T} e^{-g(t-s)}d_sP_z(Z(\tau_E)\in I,\tau_E\le s)\ge -1.$$

Here $\tau_E=\inf\{t>0:\ Z(t)\notin E\}$.

Proof. For the harmonic function

$$h(z)=P_z(Z(\tau_E)\in I)$$

let P_z^h denote the corresponding conditioned Brownian motion. Then

$$\begin{aligned}P_z(Z(\tau_E)\in I,\tau_E\le s)&=P_z(Z(\tau_E)\in I)-P_z(Z(\tau_E)\in I,\tau_E>s)\\&=P_z(Z(\tau_E)\in I)[1-P_z^h(\tau_E>s)]\\&=h(z)[1-P_z^h(\tau_E>s)].\end{aligned}$$

Consequently,

$$d_sP_z(Z(\tau_E)\in I,\tau_E\le s)=-h(z)\left[\frac{d}{ds}P_z^h(\tau_E>s)\right]ds.$$

It is known (Bañuelos and Davis (1989)) that for some $C(z)$ and $\lambda>0$,

$$P_z^h(\tau_E>t)\sim C(z)e^{-\lambda t}\quad\text{as}\quad t\to\infty.$$

Given $\varepsilon>0$, choose $T_1>0$ such that

$$(1-\delta)C(z)e^{-\lambda t}\le P_z^h(\tau_E>t)\le(1+\delta)C(z)e^{-\lambda t},\quad t\ge T_1. \tag{5.10}$$

Now for $t-T>T_1$,

$$\begin{aligned}&[h(z)]^{-1}\int_{T_1}^{t-T}e^{-g(t-s)}d_sP_z(Z(\tau_E)\in I,\tau_E\le s)\\&\quad=-\int_{T_1}^{t-T}e^{-g(t-s)}\left[\frac{d}{ds}P_z^h(\tau_E>s)\right]ds\\&\quad=e^{-g(t-T_1)}P_z^h(\tau_E>T_1)-e^{-g(T)}P_z^h(\tau_E>t-T)\\&\qquad+\int_{T_1}^{t-T}P_z^h(\tau_E>s)e^{-g(t-s)}g'(t-s)ds.\end{aligned}\tag{5.11}$$

The boundary term can be written as

$$e^{-g(t-T_1)}[P_z^h(\tau_E>T_1)-e^{-g(T)}P_z^h(\tau_E>t-T)e^{g(t-T_1)}].$$

Now

$$\begin{aligned}&\int_{T_1}^{t-T}P_z^h(\tau_E>s)e^{-g(t-s)}g'(t-s)ds\\&\quad\ge(1-\delta)C(z)\int_{T_1}^{t-T}e^{-\lambda s-g(t-s)}g'(t-s)ds\ \text{(by (5.10))}\\&\quad=(1-\delta)C(z)e^{-\lambda t}\int_{T}^{t-T_1}e^{\lambda u-g(u)}g'(u)du\end{aligned}\tag{5.12}$$

and

$$\begin{aligned}\lim_{t\to\infty}\frac{e^{-\lambda t}\int_T^{t-T_1}e^{\lambda u-g(u)}g'(u)du}{e^{-g(t-T_1)}} &= \lim_{t\to\infty}\frac{\int_T^{t-T_1}e^{\lambda u-g(u)}g'(u)du}{e^{\lambda t-g(t-T_1)}}\\ &= \lim_{t\to\infty}\frac{e^{\lambda(t-T_1)-g(t-T_1)}g'(t-T_1)}{e^{\lambda t-g(t-T_1)}[\lambda-g'(t-T_1)]}\\ &= 0,\ \text{since } g'(t)\to 0 \text{ as } t\to\infty. \qquad (5.13)\end{aligned}$$

Next observe that as $t\to\infty$

$$\begin{aligned}P_z^h(\tau_E>t-T)e^{g(t-T_1)} &\sim C(z)e^{-\lambda(t-T)+g(t-T_1)} \quad (\text{by } (5.8))\\ &= C(z)\exp\left(-(t-T)\left[\lambda-\frac{g(t-T_1)}{t-T}\right]\right)\\ &\to 0, \qquad (5.14)\end{aligned}$$

since $\frac{g(t-T_1)}{t-T}\to 0$ by L'Hôpital's rule and our hypotheses.

Thus, collecting all these relations,

$$\begin{aligned}&\liminf_{t\to\infty}[g(t-T_1)]^{-1}\log\int_{T_1}^{t-T}e^{-g(t-s)}d_s\ P_z(Z(\tau_E)\in I,\tau_E\le s)\\ &=\liminf_{t\to\infty}[g(t-T_1)]^{-1}\Bigg\{-g(t-T_1)+\log\Bigg[P_z^h(\tau_E>T_1)\\ &\qquad -e^{-g(T)}P_z^h(\tau_E>t-T)e^{g(t-T_1)}\\ &\qquad +\frac{\int_{T_1}^{t-T}P_z^h(\tau_E>s)e^{-g(t-s)}g'(t-s)ds}{e^{-g(t-T_1)}}\Bigg]\Bigg\} \quad (\text{by } (5.9))\\ &\ge\liminf_{t\to\infty}[g(t-T_1)]^{-1}\{-g(t-T_1)+\log P_z^h(\tau_E>T_1)\} \quad (\text{by } (5.12)\text{–}(5.14))\\ &=-1,\end{aligned}$$

as desired. □

Proof of Lemma 5.5. Let $z\in R\cup U$ and assume without loss of generality that $z\notin R$; otherwise we would have $P_z(\tau_{R\cup U}>t)>P_z(\tau_R>t)$, and the desired conclusions would follow from Lemmas 5.3 and 5.4. Let $z_1\in R\backslash\overline{U}$ and suppose E is a bounded open set with smooth boundary such that $E\subseteq R\cup U$, $z\in E$ and $z_1\in\partial E$. Let $I\subseteq\partial E$ be a nonpolar set containing z_1.

Let $\delta>0$ and use Lemmas 5.3 and 5.4 to choose $T>0$ such that for $t>T$, uniformly for $w\in I$,

$$-(C_{p,b}+\delta)t^{\frac{1-p}{1+p}}\le\log P_w(\tau_R>t),\quad R=R_b, \qquad (5.15)$$

$$-\left(\frac{\pi}{2a\beta}+\delta\right)\log t\le\log P_w(\tau_R>t),\quad R=R_{a,\beta}. \qquad (5.16)$$

For typographical simplicity, write

$$C=\begin{cases}C_{p,b}+\delta, & R=R_b,\\ \frac{\pi}{2a\beta}+\delta, & R=R_{a,\beta},\end{cases} \qquad (5.17)$$

and

$$g(t)=\begin{cases}Ct^{\frac{1-p}{1+p}}, & R=R_b,\\ C\log t, & R=R_{a,\beta}.\end{cases} \qquad (5.18)$$

By the strong Markov property and (5.15)–(5.18),

$$\begin{aligned}P_z(\tau_{R\cup U} > t) &\geq P_z(t-\tau_E > T, Z(\tau_E) \in I, \tau_{R\cup U} > t)\\ &= \int_0^{t-T} \int P_w(\tau_{R\cup U} > t-s) P_z(\tau_E \in ds, Z(\tau_E) \in dw)\\ &\geq \int_0^{t-T} \int_I e^{-g(t-s)} P_z(\tau_E \in ds, Z(\tau_E) \in dw)\\ &= \int_0^{t-T} e^{-g(t-s)} d_s P_z(Z(\tau_E) \in I, \tau_E \leq s).\end{aligned}$$

The function g satisfies the hypothesis of Lemma 5.6, so for T_1 as there we have

$$\begin{aligned}&\liminf_{t\to\infty} [g(t-T_1)]^{-1} \log P_z(\tau_{R\cup U} > t)\\ &\quad \geq \liminf_{t\to\infty} [g(t-T_1)]^{-1} \int_{T_1}^{t-T} e^{-g(t-s)} d_s P_z(Z(\tau_E) \in I, \tau_E \leq s)\\ &\quad \geq -1.\end{aligned}$$

The desired conclusions follow using (5.17)–(5.18) and then letting $\delta \to 0$. □

6. Conformal transformation of a neighborhood of D to a strip

The estimates for J' in Warschawski's theorem are not uniform for unrestricted v, and this causes some technical problems. We slightly enlarge the original domain D. Let $D_\varepsilon \supseteq \overline{D}$ be a twisted domain with growth radius $(\gamma+\varepsilon)r^p$. We assume D and D_ε have the same generating curve $f(r)$. Abusing the notation, let the boundary curves of D_ε be given by

$$\begin{aligned}&C_1\colon\ \theta = f_1(r), \quad r \geq r_1,\\ &C_2\colon\ \theta = f_2(r), \quad r \geq r_1,\\ &C_3\colon\ r = r_1, \quad f_2(r_1) \leq \theta \leq f_1(r_1).\end{aligned}$$

By (2.11) we have (for r_0 large enough)

$$f_i(r) = \theta_i(r_i^{-1}(r)), \quad r \geq r_0, \tag{6.1}$$

where θ_i and r_i are from (2.2)–(2.5) with γ there replaced by $\gamma+\varepsilon$. Define

$$E_\varepsilon = \{w = u + iv\colon\ u > \ln r_1, f_2(e^u) < v < f_1(e^u)\}. \tag{6.2}$$

Lemma 6.1. *The analytic mapping $F\colon\ E_\varepsilon \to D_\varepsilon$ given by $F(w) = e^w$ is one-to-one and onto.*

Proof. Assume $w_j = u_j + iv_j \in E_\varepsilon$, $j = 1, 2$, with $e^{w_1} = e^{w_2}$. Then $e^{u_1} = e^{u_2}$, yielding $u_1 = u_2$. Thus for some integer n, $v_1 = v_2 + 2\pi n$. It is no loss to assume $n \geq 0$. Now

$$\begin{aligned}2\pi n = v_1 - v_2 &< f_1(e^{u_1}) - f_2(e^{u_2})\\ &= f_1(e^{u_1}) - f_2(e^{u_1})\\ &< 2\pi\end{aligned}$$

by our hypothesis that the boundary curves given by f_1 and f_2 do not cross. Hence $n = 0$, and consequently $v_1 = v_2$.

For surjectivity, let $x+iy \in D_\varepsilon$. With (r,θ) being the corresponding polar coordinates, let $w = \ln r + i\theta$. We know $r > r_1$ and $f_2(r) < \theta < f_2(r)$. Thus $w \in E_\varepsilon$ and $e^w = x+iy$. □

The next step is to apply Warschawski's theorem to E_ε. First we verify that E_ε is an L-strip with boundary inclination 0 at $u=\infty$; that is, we check (4.1) in the present context:

$$\lim_{\substack{u_1,u_2\to\infty\\ u_2>u_1}} \frac{f_i(e^{u_2}) - f_i(e^{u_1})}{u_2-u_1} = 0, \quad i=1,2.$$

Indeed, for some $\tilde{u} \in (u_1,u_2)$,

$$\begin{aligned}\frac{f_i(e^{u_2}) - f_i(e^{u_1})}{u_2-u_1} &= \frac{\theta_i(r_i^{-1}(e^{u_2})) - \theta_i(r_i^{-1}(e^{u_1}))}{u_2-u_1} \quad \text{(by (6.1))}\\ &= \frac{\theta_i'(r_i^{-1}(e^{\tilde{u}}))e^{\tilde{u}}}{r_i'(r_i^{-1}(e^{\tilde{u}}))}\\ &= \theta_i'(r_i^{-1}(e^{\tilde{u}}))r_i^{-1}(e^{\tilde{u}})\frac{e^{\tilde{u}}}{r_i^{-1}(e^{\tilde{u}})r_i'(r_i^{-1}(e^{\tilde{u}}))}. \end{aligned} \tag{6.3}$$

Since $\tilde{u}\to\infty$ as $u_1,u_2\to\infty$, by Lemma 3.3 and Corollary 2.2 the right hand side of (6.3) converges to 0. Thus E_ε is an L-strip with boundary inclination 0 at $u=\infty$, as claimed.

Let $J\colon\ E_\varepsilon \to W_{\pi/2} = \left\{\tilde{z} = \tilde{x}+i\tilde{y}\colon\ |\tilde{y}| < \frac{\pi}{2}\right\}$ be the conformal mapping in Warschawski's theorem. Then

$$\begin{aligned}\psi(u) &= \frac{1}{2}[f_2(e^u)+f_1(e^u)],\\ \theta(u) &= f_1(e^u) - f_2(e^u).\end{aligned} \tag{6.4}$$

Lemma 6.2. *As* $u\to\infty$,

$$\theta(u) \sim \begin{cases} 2(\gamma+\varepsilon)e^{(p-1)u}, & p<1,\\ 2\arccos\dfrac{1}{\sqrt{1+(\gamma+\varepsilon)^2}}, & p=1.\end{cases}$$

Proof. By (6.1),

$$\theta(u) = \theta_1(r_1^{-1}(e^u)) - \theta_2(r_2^{-1}(e^u)) \sim \begin{cases} 2(\gamma+\varepsilon)e^{(p-1)u}, & p<1,\\ 2\arccos\dfrac{1}{\sqrt{1+(\gamma+\varepsilon)^2}}, & p=1,\end{cases}$$

by Theorem 3.1 (recall that our growth radius is $(\gamma+\varepsilon)r^p$, NOT γr^p). □

Define

$$G = J\circ F^{-1}\colon\ D_\varepsilon \to W_{\pi/2}. \tag{6.5}$$

This is a conformal mapping from the enlarged twisted domain onto the strip $W_{\pi/2}$. Recall that $S_\beta = \left\{w = u+iv\colon\ u>\ln r_1, \left|\frac{v-\psi(u)}{\theta(u)}\right| < \frac{\beta}{\pi}\right\}$.

Lemma 6.3. a) *Given* $0<\beta<\frac{\pi}{2}$, $p<1$ *and small* $\delta>0$, *for large* M,

$$\begin{aligned}(A_{p,\gamma+\varepsilon}-\delta)([1+|\tilde{x}|]^{-\frac{p}{1-p}}I_{|\tilde{x}|>M} + K_1 I_{|\tilde{x}\le M}) &\le |G'(G^{-1}(\tilde{z}))|\\ \le (A_{p,\gamma+\varepsilon}+\delta)([1+|\tilde{x}|]^{-\frac{p}{1-p}}I_{|\tilde{x}|>M} + K_2 I_{|\tilde{x}|\le M}),\end{aligned}$$

$\tilde{z} = \tilde{x} + i\tilde{y} \in J(S_\beta)$, where

$$A_{p,\gamma} = \left[\frac{\pi^p}{2\gamma(1-p)^p}\right]^{\frac{1}{1-p}}$$

and K_1, K_2 are independent of M.

b) *For $0 < \beta < \frac{\pi}{2}$, $p = 1$, $M \in \mathbb{R}$ and $\delta > 0$ small, there are positive constants $C_3 = C_3(\delta)$ and $C_4 = C_4(\delta)$ such that*

$$C_3 \exp(-C_5(1+\delta)\tilde{x}) \le |G'(G^{-1}(\tilde{z}))| \le C_4 \exp(-C_5(1-\delta)\tilde{x}), \quad \tilde{z} \in J(S_\beta), \tilde{x} \ge M,$$

where $C_5 = \frac{2}{\pi} \arccos\left(\frac{1}{\sqrt{1+(\gamma+\varepsilon)^2}}\right)$.

Proof. By (6.5), writing $\tilde{z} = G(z)$ and $F^{-1}(z) = w = u + iv$,

$$\begin{aligned} |G'(G^{-1}(\tilde{z}))| &= |G'(z)| \\ &= \left|\frac{J'(F^{-1}(z))}{F'(F^{-1}(z))}\right| \\ &= \left|\frac{J'(w)}{F'(w)}\right| \\ &= \frac{|J'(w)|}{e^u} \\ &\sim \frac{\pi}{\theta(u)} e^{-u} \quad \text{as } u \to \infty, w \in S_\beta, \text{ uniformly in } v, \end{aligned} \tag{6.6}$$

by Theorem 4.1 (ii).

a) If $p < 1$, then by Lemma 6.2, (6.6) yields

$$|G'(G^{-1}(\tilde{z}))| \sim \frac{1}{2(\gamma+\varepsilon)} e^{-pu} \text{ as } u \to \infty, w \in S_\beta, \text{ uniformly in } v. \tag{6.7}$$

By Theorem 4.1 i) and Lemma 6.2, as $u \to \infty$, uniformly in v,

$$\begin{aligned} J(w) &\sim \pi \int_{\ln r_1}^{u} \frac{dt}{\theta(t)} \\ &\sim \frac{\pi}{2(\gamma+\varepsilon)(1-p)} e^{(1-p)u}. \end{aligned}$$

Since $\tilde{z} = G(z) = J(F^{-1}(z)) = J(w)$, this yields

$$\tilde{x} \sim \frac{\pi}{2(\gamma+\varepsilon)(1-p)} e^{(1-p)u} \text{ as } u \to \infty, \text{ uniformly in } v.$$

Combining this with (6.7), we get

$$|G'(G^{-1}(\tilde{z}))| \sim A_{p,\gamma+\varepsilon} \tilde{x}^{-\frac{p}{1-p}} \text{ as } \tilde{x} \to \infty, \tilde{z} = \tilde{x} + i\tilde{y} \in J(S_\beta),$$

uniformly in $\tilde{y}$. Part a) follows from this.

b) If $p = 1$, then by Lemma 6.2, (6.6) implies

$$|G'(G^{-1}(\tilde{z}))| \sim C_5^{-1} e^{-u} \text{ as } u \to \infty, w \in S_\beta, \text{ uniformly in } v. \tag{6.8}$$

By Theorem 4.1 i) and Lemma 6.2, as $u \to \infty$, uniformly in v,

$$\tilde{z} = J(w) \sim C_5^{-1} u$$

yielding

$$\tilde{x} \sim C_5^{-1} u \text{ as } u \to \infty \text{ uniformly in } v. \tag{6.9}$$

Let $\delta > 0$ be small. For large u, by (6.9)

$$(1-\delta)\tilde{x} \leq C_5^{-1} u \leq (1+\delta)\tilde{x},$$

and so

$$\exp(-C_5(1+\delta)\tilde{x}) \leq e^{-u} \leq \exp(-C_5(1-\delta)\tilde{x}).$$

Combined with (6.8), as $u \to \infty$, $w \in S_\beta$, uniformly in v, this yields

$$C_6 \exp(-C_5(1+\delta)\tilde{x}) \leq |G'(G^{-1}(\tilde{z}))| \leq C_7 \exp(-C_5(1+\delta)\tilde{x}).$$

The desired conclusion follows from this. □

Lemma 6.4. *For $\varepsilon > 0$ small, there exist positive $\varepsilon_1, \varepsilon_2, N$ with $1 > \varepsilon_1 > \varepsilon_2$ such that for $\alpha_i = \frac{\pi}{2}(1-\varepsilon_i)$,*

$$S_{\alpha_1} \cap \{u \geq N\} \subseteq F^{-1}(D) \cap \{u \geq N\} \subseteq S_{\alpha_2} \cap \{u \geq N\}. \quad \square$$

We defer the proof to section 8.

Let α_1, α_2, N be as in Lemma 6.4. By Theorem 4.1, for δ_1, δ_2 small and positive, there exists $\widetilde{N} \geq N$ such that

$$\begin{aligned} T_{\alpha_1-\delta_1} \cap \{u \geq \widetilde{N}\} &\subseteq S_{\alpha_1} \cap \{u \geq \widetilde{N}\} \quad \text{(by Theorem 4.1)} \\ &\subseteq F^{-1}(D) \cap \{u \geq \widetilde{N}\} \quad \text{(by Lemma 6.4)} \\ &\subseteq S_{\alpha_2} \cap \{u \geq \widetilde{N}\} \quad \text{(by Lemma 6.4)} \\ &\subseteq T_{\alpha_2+\delta_2} \cap \{u \geq \widetilde{N}\} \quad \text{(by Theorem 4.1)} \end{aligned} \tag{6.10}$$

This implies we can choose M_1 and M_2 such that

$$W_{\alpha_1-\delta_1} \cap \{\tilde{x} \geq M_2\} \subseteq G(D) \cap \{\tilde{x} \geq M_2\} \subseteq W_{\alpha_2+\delta_2} \cap \{\tilde{x} \geq M_1\}. \tag{6.11}$$

By making δ_2 a bit larger and M_1 smaller, if necessary, we can take

$$G(D) \subseteq W_{\alpha_2+\delta_2} \cap \{\tilde{x} \geq M_1\}. \tag{6.12}$$

This is possible because $\overline{D} \subseteq D_\varepsilon$ and (6.11) holds. Similarly, by (6.10) and Theorem 4.1 we can choose $\beta \in \left(0, \frac{\pi}{2}\right)$ such that

$$W_{\alpha_2+\delta_2} \cap \{\tilde{x} \geq M_1\} \subseteq J(S_\beta). \tag{6.13}$$

Remark 6.5. We need β only for the case $p = 1$. Notice too that (6.13) is true if β is made larger.

7. Proof of the main results

We use the notation of §6.

Proof of Theorem 1.1. Let $z \in D$ and suppose $p < 1$. Given $\delta > 0$, making M_2 from (6.11) larger if necessary, by Lemma 6.3 a) we get

$$\begin{aligned} &(A_{p,\gamma+\varepsilon} - \delta)\left[[1+|\tilde{x}|]^{-\frac{p}{1-p}} I_{|\tilde{x}|>M_2} + K_1 I_{|\tilde{x}|\leq M_2}\right] \leq |G'(G^{-1}(\tilde{z}))| \\ &\leq (A_{p,\gamma+\varepsilon} + \delta)\left[[1+|\tilde{x}|]^{-\frac{p}{1-p}} I_{|\tilde{x}|>M_2} + K_2 I_{|\tilde{x}|\leq M_2}\right]. \end{aligned} \tag{7.1}$$

Here and in what follows, K_i will denote a constant independent of t. By (6.11) there is a bounded open subset $\widetilde{D}$ of $W_{\pi/2}$ such that $z \in \widetilde{D}$ and

$$G(D) \supseteq [W_{\alpha_1-\delta_1} \cap \{\tilde{x} > M_2\}] \cup \widetilde{D}. \tag{7.2}$$

Then, with $\eta = \tilde{\tau}(\widetilde{D} \cup [W_{\alpha_1-\delta_1} \cap \{\tilde{x} > M_2\}])$,

$$\begin{aligned}
P_z(\tau_D > t) &= P_{\tilde{z}}\left(\int_0^{\tilde{\tau}(G(D))} |G'(G^{-1}(\widetilde{Z}(s)))|^{-2} ds > t\right) \text{(by (4.5))}\\
&\geq P_{\tilde{z}}\left(\int_0^{\eta} |G'(G^{-1}(\widetilde{Z}(s)))|^{-2} ds > t\right)\\
&\geq P_{\tilde{z}}\left(\int_0^{\eta} [1+|\widetilde{Z}_1(s)|]^{\frac{2p}{1-p}} ds - K_3 \int_0^{\eta} I(|\widetilde{Z}_1(s)| \leq M_2) ds > (A_{p,\gamma+\varepsilon}+\delta)^2 t\right)
\end{aligned}$$

(by (7.1))

$$\begin{aligned}
&\geq P_{\tilde{z}}\left(\int_0^{\eta} [1+|\widetilde{Z}_1(s)|]^{\frac{2p}{1-p}} ds > (A_{p,\gamma+\varepsilon}+2\delta)^2 t\right)\\
&\quad - P_{\tilde{z}}\left(\int_0^{\eta} I(|\widetilde{Z}_1(s)| \leq M_2) ds > K_4 t\right).
\end{aligned} \tag{7.3}$$

Now by (7.2) and (6.12), $\eta \leq \tilde{\tau}(W_{\pi/2})$, and it is well-known that

$$\log P_{\tilde{z}}(\tilde{\tau}(W_{\pi/2}) > t) \leq -K_5 t, \quad t \text{ large}.$$

In particular, for t large,

$$\begin{aligned}
&\log P_{\tilde{z}}\left(\int_0^{\eta} I(|\widetilde{Z}_1(s)| \leq M_2) ds > K_4 t\right)\\
&\quad \leq \log P_{\tilde{z}}(\tilde{\tau}(W_{\pi/2}) > K_4 t)\\
&\quad \leq -K_6 t.
\end{aligned}$$

Then by Theorem 5.1 a), (7.3) yields

$$\liminf_{t\to\infty} t^{-\frac{1-p}{1+p}} \log P_{\tilde{z}}(\tau_D > t) \geq -B_p C_{p,2(\alpha_1-\delta_1)/\pi} [A_{p,\gamma+\varepsilon}+2\delta]^{\frac{2(1-p)}{1+p}}, \tag{7.4}$$

where

$$B_p = [\pi^{1-p} 2^{2p-1} (1-p)^p]^{-\frac{2}{1+p}}. \tag{7.5}$$

As for the lim sup behavior, observe by (6.12) that

$$\tilde{\tau}(G(D)) \leq \tau_1 := \tilde{\tau}(W_{\alpha_2+\delta_2} \cap \{\tilde{x} > M_1\}),$$

and so

$$
\begin{aligned}
P_z(\tau_D > t) &= P_{\tilde z}\left(\int_0^{\tilde\tau(G(D))} |G'(G^{-1}(\widetilde Z(s)))|^{-2} dt > t\right) \quad \text{(by (4.5))}\\
&\le P_{\tilde z}\left(\int_0^{\tilde\tau_1} |G'(G^{-1}(\widetilde Z(s)))|^{-2} dt > t\right)\\
&\le P_{\tilde z}\Bigg(\int_0^{\tau_1} [1+|\widetilde Z_1(s)|]^{\frac{2p}{1-p}} ds \qquad (7.6)\\
&\qquad + K_7 \int_0^{\tau_1} I(|\widetilde Z_1(s)| \le M_2) ds > (A_{p,\gamma+\varepsilon}-\delta)^2 t\Bigg) \quad \text{(by (7.1))}\\
&\le P_{\tilde z}\left(\int_0^{\tau_1} [1+|\widetilde Z_1(s)|]^{\frac{2p}{1-p}} ds > (1-\delta)(A_{p,\gamma+\varepsilon}-\delta)^2 t\right)\\
&\quad + P_{\tilde z}\left(K_7 \int_0^{\tau_1} I(|\widetilde Z_1(s)| \le M_2) ds > \delta(A_{p,\gamma+\varepsilon}-\delta)^2 t\right). \qquad (7.7)
\end{aligned}
$$

Since $\tau_1 \le \tilde\tau(W_{\pi/2})$, the log of the second term on the left is bounded by

$$
\begin{aligned}
\log P_{\tilde z}(\tau_1 > K_8 t) &\le \log P_{\tilde z}(\tilde\tau(W_{\pi/2}) > K_8 t)\\
&\le -K_9 t, \quad t \text{ large}.
\end{aligned}
$$

Then, by Theorem 5.2 a), (7.7) yields

$$
\limsup_{t\to\infty} t^{-\frac{1-p}{1+p}} \log P_z(\tau_D > t) \ge -B_p C_{p,2(\alpha_2+\delta_2)/\pi}[(1-\delta)(A_{p,\gamma+\varepsilon}-\delta)^2]^{\frac{1-p}{1+p}}, \tag{7.8}
$$

where B_p is from (7.5).

To finish, in the following order, let $\delta \to 0$, $\delta_1 \to 0$, $\delta_2 \to 0$, $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$ (which imply $\alpha_i \to \frac{\pi}{2}$) and $\varepsilon \to 0$ in (7.4) and (7.8) to get

$$
\lim_{t\to\infty} t^{-\frac{1-p}{1+p}} \log P_z(\tau_D > t) = B_p C_{p,1}[A_{p,\gamma}]^{\frac{2(1-p)}{1+p}},
$$

as desired. □

Proof of Theorem 1.2. Let $z \in D$. Suppose $p = 1$, β is from (6.13) and M_1 is from (6.12). Let $\delta > 0$ be small and use Lemma 6.3 b) to choose positive $C_3(\delta), C_4(\delta)$ such that

$$
C_3(\delta)\exp(-C_5(1+\delta)\tilde x) \le |G'(G^{-1}(\tilde z))| \le C_4(\delta)\exp(-C_5(1-\delta)\tilde x), \tag{7.9}
$$

for $\tilde z = \tilde x + i\tilde y \in J(S_\beta), \tilde x \ge M_1$, where

$$
C_5 = \frac{2}{\pi}\arccos\frac{1}{\sqrt{1+(\gamma+\varepsilon)^2}}.
$$

Exactly as in the preceding proof, there is a bounded open subset $\widetilde D$ of $W_{\pi/2}$ such that $z \in \widetilde D$ and (7.2) holds. Then for $\eta = \tilde\tau(\widetilde D \cup [W_{\alpha_1-\delta_1} \cap \{\tilde x > M_2\}])$ and $\tau_1 = \tilde\tau(W_{\alpha_2+\delta_2} \cap \{\tilde x > M_1\})$, by (4.5)

$$
\begin{aligned}
&P_{\tilde z}\left(\int_0^{\eta} |G'(G^{-1}(\widetilde Z(s)))|^{-2} ds > t\right)\\
&\qquad \le P_z(\tau_D > t) \le P_{\tilde z}\left(\int_0^{\tau_1} |G'(G^{-1}(\widetilde Z(s)))|^{-2} ds > t\right).
\end{aligned}
$$

Using (7.9),

$$P_{\tilde{z}}\left(\int_0^{\eta}\exp(2C_5(1-\delta)\widetilde{Z}_1(s))ds > C_4^2(\delta)t\right) \le P_z(\tau_D > t)$$
$$\le P_{\tilde{z}}\left(\int_0^{\tilde{\tau}}\exp(2C_5(1+\delta)\widetilde{Z}_1(s))ds > C_3^2(\delta)t\right).$$

By Theorems 5.1 b) and 5.2 b),

$$-\frac{\pi}{4C_5(1-\delta)\beta} \le \liminf_{t\to\infty}[\log t]^{-1}\log P_z(\tau_D > t)$$
$$\le \limsup_{t\to\infty}[\log t]^{-1}\log P_z(\tau_D > t)$$
$$\le -\frac{\pi}{4C_5(1+\delta)\beta}.$$

To finish, let $\delta \to 0$, then $\beta \to \frac{\pi}{2}$ (recall Remark 6.5) and then $\varepsilon \to 0$ to get

$$\lim_{t\to\infty}[\log t]^{-1}\log P_z(\tau_D > t) = \pi\left[4\arccos\frac{1}{\sqrt{1+\gamma^2}}\right]^{-1},$$

as desired. □

8. Proof of Lemma 6.4

We need to refer to the boundary curves of D_ε and D. To distinguish them, we use tildes over the quantities associated with D_ε. In particular, the boundary curves of D_ε will be

$$\widetilde{C}_i\colon\ \theta = \tilde{f}_i(r), \quad r \ge \tilde{r}_1, \quad i = 1, 2,$$
$$\widetilde{C}_3\colon\ r = \tilde{r}_1, \quad \tilde{f}_2(\tilde{r}_1) \le \theta \le \tilde{f}_2(\tilde{r}_1).$$

By (2.11), for r_0 large and for $i = 1, 2$,

$$f_i(r) = \theta_i(r_i^{-1}(r)), \quad r \ge r_0,$$
$$\tilde{f}_i(r) = \tilde{\theta}_i(\tilde{r}_i^{-1}(r)), \quad r \ge r_0,$$

where θ_i and r_i are from (2.2)–(2.5) and $\tilde{\theta}_i$ and $\tilde{r}_i$ are their analogs when γ there is replaced by $\gamma + \varepsilon$. Moreover, since $\overline{D} \subseteq D_\varepsilon$,

$$r_1 > \tilde{r}_1. \tag{8.1}$$

We carefully spell out the S_β notation in the statement of the lemma. From the lines after (6.5),

$$S_\beta = \left\{w = u + iv\colon\ u > \ln\tilde{r}_1, \left|\frac{v - \psi(u)}{\theta(u)}\right| < \frac{\beta}{\pi}\right\}$$

(we wrote r_1 there, but that is really $\tilde{r}_1$ in our current notation), where

$$\psi(u) = \frac{1}{2}[\tilde{f}_2(e^u) + \tilde{f}_1(e^u)],$$
$$\theta(u) = \tilde{f}_1(e^u) - \tilde{f}_2(e^u).$$

Thus for $\alpha_i = \frac{\pi}{2}(1-\varepsilon_i), 0 < \varepsilon_i < 1$,

$$
\begin{aligned}
u+iv \in S_{\alpha_i} &\Leftrightarrow u > \ln \tilde{r}_1 \text{ and } -\frac{1-\varepsilon_i}{2}\theta(u) < v - \psi(u) < \frac{1-\varepsilon_i}{2}\theta(u) \\
&\Leftrightarrow u > \ln \tilde{r}_1 \text{ and } \tilde{f}_2(e^u) + \frac{\varepsilon_i}{2}\theta(u) < v < \tilde{f}_1(e^u) - \frac{\varepsilon_i}{2}\theta(u). \qquad (8.2)
\end{aligned}
$$

Also, since $F(w) = e^w$ (see Lemma 6.1),

$$F^{-1}(D) = \{w = u+iv\colon\ u > \ln r_1, f_2(e^u) < v < f_1(e^u)\}.$$

Hence the proof of the lemma comes down to showing there exist positive $\varepsilon_1, \varepsilon_2$ and N, with $\varepsilon_2 < \varepsilon_1 < 1$, such that for $u \geq N$

$$
\begin{aligned}
f_2(e^u) &< \tilde{f}_2(e^u) + \frac{\varepsilon_1}{2}\theta(u), \\
\tilde{f}_1(e^u) - \frac{\varepsilon_1}{2}\theta(u) &< f_1(e^u), \\
\tilde{f}_2(e^u) + \frac{\varepsilon_2}{2}\theta(u) &< f_2(e^u), \\
f_1(e^u) &< \tilde{f}_1(e^u) - \frac{\varepsilon_2}{2}\theta(u).
\end{aligned}
$$

Equivalently, we must find $0 < \varepsilon_2 < \varepsilon_1 < 1$ such that

$$\frac{\varepsilon_2}{2} < \frac{f_2(e^u) - \tilde{f}_2(e^u)}{\theta(u)} < \frac{\varepsilon_1}{2}, \qquad (8.3)$$

$$\frac{\varepsilon_2}{2} < \frac{\tilde{f}_1(e^u) - f_1(e^u)}{\theta(u)} < \frac{\varepsilon_1}{2} \qquad (8.4)$$

for large u.

Set $t_1 = r_1^{-1}(e^u)$ and $\tilde{t}_1 = \tilde{r}_1^{-1}(e^u)$; then

$$r_1(t_1) = \tilde{r}_1(\tilde{t}_1).$$

By (2.2) and its tilde analog, as $u \to \infty$

$$\frac{t_1^2}{\tilde{t}_1^2} = \frac{1 + (\gamma+\varepsilon)^2 \tilde{t}_1^{2p-2} - 2(\gamma+\varepsilon)h(\tilde{t}_1)}{1 + \gamma^2 t_1^{2p-2} - 2\gamma h(t_1)} \to \begin{cases} \dfrac{1+(\gamma+\varepsilon)^2}{1+\gamma^2}, & p = 1, \\ 1, & p < 1. \end{cases} \qquad (8.5)$$

Moreover,

$$t_1^2 - \tilde{t}_1^2 = \frac{r(t_1)^2[(\gamma+\varepsilon)^2\tilde{t}_1^{2p-2} - \gamma^2 t_1^{2p-2} - 2[(\gamma+\varepsilon)h(\tilde{t}_1) - \gamma h(t_1)]]}{[1 + (\gamma+\varepsilon)^2\tilde{t}_1^{2p-2} - 2(\gamma+\varepsilon)h(\tilde{t}_1)][1 + \gamma^2 t_1^{2p-2} - 2\gamma h(t_1)]}.$$

Since

$$r_1^2(t) \sim \begin{cases} t^2(1+\gamma^2), & p = 1, \\ t^2, & p < 1, \end{cases}$$

as $t \to \infty$ (using (2.2) and that $h(t) \to 0$), we get

$$t_1^2 - \tilde{t}_1^2 \sim \begin{cases} \dfrac{t_1^2[2\varepsilon\gamma + \varepsilon^2 - 2[(\gamma+\varepsilon)h(\tilde{t}_1) - \gamma h(t_1)]]}{[1+(\gamma+\varepsilon)^2]}, \\ t_1^2[(\gamma+\varepsilon)^2\tilde{t}_1^{2p-2} - \gamma^2 t_1^{2p-2} - 2[(\gamma+\varepsilon)h(\tilde{t}_1) - \gamma h(t_1)]] \end{cases} \qquad (8.6)$$

as $u \to \infty$.

If $p = 1$, then (8.5)–(8.6) give

$$\frac{t_1^2 - \tilde{t}_1^2}{\tilde{t}_1^{p-1} t_1^2} \to \frac{2\varepsilon\gamma + \varepsilon^2}{1 + (\gamma+\varepsilon)^2} \quad \text{as} \quad u \to \infty,$$

whereas if $p < 1$, we get

$$\begin{aligned}
&\frac{t_1^2 - \tilde{t}_2^2}{\tilde{t}_1^{p-1} t_1^2} \\
&\quad \sim \tilde{t}_1^{p-1}\left[(\gamma+\varepsilon)^2 - \gamma^2\left(\frac{t_1}{\tilde{t}_1}\right)^{2p-2} - 2(\gamma+\varepsilon)\tilde{t}_1^{2-2p} h(\tilde{t}_1) - 2\gamma\left(\frac{\tilde{t}_1}{t_1}\right)^{2-2p} t_1^{2-2p} h(t_1)\right] \\
&\quad \to 0 \text{ as } u \to \infty, \text{ by (2.9).}
\end{aligned}$$

In any case, the hypothesis of Lemma 3.6 holds, and the lemma yields

$$\frac{f(t_1) - f(\tilde{t}_1)}{\tilde{t}_1^{p-1}} \to 0 \quad \text{as} \quad u \to \infty.$$

Thus by (2.4) and its tilde analog, using Lemma 3.5 and (8.5),

$$\begin{aligned}
\frac{\tilde{f}_1(e^u) - f_1(e^u)}{\tilde{t}_1^{p-1}} &= \frac{\tilde{\theta}_1(\tilde{t}_1) - \theta_1(t_1)}{\tilde{t}_1^{p-1}} \\
&= \frac{f(\tilde{t}_1) - f(t_1)}{\tilde{t}_1^{p-1}} + \frac{1}{\tilde{t}_1^{p-1}} \arccos \frac{1 - (\gamma+\varepsilon)h(\tilde{t}_1)}{\sqrt{1 + (\gamma+\varepsilon)^2 \tilde{t}_1^{2p-2} - 2(\gamma+\varepsilon)h(\tilde{t}_1)}} \\
&\quad - \left(\frac{t_1}{\tilde{t}_1}\right)^{p-1} \frac{1}{t_1^{p-1}} \arccos \frac{1 - \gamma h(t_1)}{\sqrt{1 + \gamma^2 t_1^{2p-2} - 2\gamma h(t_1)}} \\
&\to \begin{cases} 0 + (\gamma+\varepsilon) - \gamma, & p < 1, \\ 0 + \arccos \dfrac{1}{\sqrt{1+(\gamma+\varepsilon)^2}} - \arccos \dfrac{1}{\sqrt{1+\gamma^2}}, & p = 1 \end{cases} \text{ as } u \to \infty.
\end{aligned}$$

On the other hand, since $\tilde{r}_1(\tilde{t}_1) = e^u$, the tilde analog of (2.2) forces $\tilde{t}_1 \sim e^u$ as $u \to \infty$, for $p < 1$. Consequently

$$\frac{\tilde{f}_1(e^u) - f_1(e^u)}{e^{u(p-1)}} \to \begin{cases} \varepsilon, & p < 1, \\ \arccos \dfrac{1}{\sqrt{1+(\gamma+\varepsilon)^2}} - \arccos \dfrac{1}{\sqrt{1+\gamma^2}}, & p = 1, \end{cases}$$

as $u \to \infty$. By Lemma 6.2,

$$\frac{\tilde{f}_1(e^u) - f_1(e^u)}{\theta(u)} \to \begin{cases} \dfrac{\varepsilon}{2(\gamma+\varepsilon)}, & p < 1, \\ \dfrac{\arccos \dfrac{1}{\sqrt{1+(\gamma+\varepsilon)^2}} - \arccos \dfrac{1}{\sqrt{1+\gamma^2}}}{2 \arccos \dfrac{1}{\sqrt{1+(\gamma+\varepsilon)^2}}}, & p = 1, \end{cases}$$

as $u \to \infty$. In a similar way we can show that

$$\frac{f_2(e^u) - \tilde{f}_2(e^u)}{\theta(u)} \text{ converges to the same limit.}$$

For $\varepsilon > 0$ small enough, the limits are positive and small. In particular, we can choose $0 < \varepsilon_2 < \varepsilon_1 < 1$ such that (8.3)–(8.4) hold, as desired. □

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References

[1] Bañuelos, R. and Davis, B. (1989). Heat kernel, eigenfunctions, and conditioned Brownian motion in planar domains, Journal of Functional Analysis **84** 188–200. MR91a:60203

[2] Bañuelos, R., DeBlassie, R.D. and Smits, R. (2001). The first exit time of planar Brownian motion from the interior of a parabola, Annals of Probability **29** 882–901. MR2002h:60165

[3] van den Berg, M. (2003). Subexponential behavior of the Dirichlet heat kernel, Journal of Functional Analysis **198** 28–42. MR2003m:60211

[4] Li, W. (2003). The first exit time of Brownian motion from an unbounded convex domain, Annals of Probability **31** 1078–1096. MR2004c:60126

[5] Lifshits, M. and Shi, Z. (2002), The first exit time of Brownian motion from a parabolic domain, Bernoulli **8** 745–765. MR2004d:60213

[6] Spitzer, F. (1958). Some theorems concerning two-dimensional Brownian motion, Transactions of the American Mathematical Society **87** 187–197. MR21:3051

[7] Warschawski, S.E. (1942). On conformal mapping of infinite strips, Transactions of the American Mathematical Society **51** 280–335. MR4:9b

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