A QUADRATIC APPROXIMATION TO THE SENDOV RADIUS NEAR THE UNIT CIRCLE

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Abstract. Define $S(n, \beta)$ to be the set of complex polynomials of degree $n \geq 2$ with all roots in the unit disk and at least one root at $\beta$. For a polynomial $P$, define $|P|_{\beta}$ to be the distance between $\beta$ and the closest root of the derivative $P'$. Finally, define $r_n(\beta) = \sup \{|P|_{\beta} : P \in S(n, \beta)\}$. In this notation, a conjecture of Bl. Sendov claims that $r_n(\beta) \leq 1$. In this paper we investigate Sendov’s conjecture near the unit circle, by computing constants $C_1$ and $C_2$ (depending only on $n$) such that $r_n(\beta) \sim 1 + C_1(1 - |\beta|) + C_2(1 - |\beta|)^2$ for $|\beta|$ near 1. We also consider some consequences of this approximation, including a hint of where one might look for a counterexample to Sendov’s conjecture.

1. Introduction

In 1962, Sendov conjectured that if a polynomial (with complex coefficients) has all its roots in the unit disk, then within one unit of each of its roots lies a root of its derivative. More than 50 papers have been published on this conjecture, but it has been verified in general only for polynomials of degree at most 8 [4].

Let $n \geq 2$ be an integer and let $\beta$ be a complex number of modulus at most 1. Define $S(n, \beta)$ to be the set of polynomials of degree $n$ with complex coefficients, all roots in the unit disk and at least one root at $\beta$. For a polynomial $P$, define $|P|_{\beta}$ to be the distance between $\beta$ and the closest root of the derivative $P'$. Finally, define $r_n(\beta) = \sup \{|P|_{\beta} : P \in S(n, \beta)\}$, and note that $r_n(\beta) \leq 2$ (since by the Gauss-Lucas Theorem [5, Theorem 6.1] all roots of each $P'$ are also in the unit disk, and so each $|P|_{\beta} \leq 2$). In this notation, Sendov’s conjecture claims that $r_n(\beta) \leq 1$.

In estimating $r_n(\beta)$, we will assume without loss of generality (by rotation) that $0 \leq \beta \leq 1$. It is already known that $r_2(\beta) = (1 + \beta)/2$ and that $r_3(\beta) = [3\beta + (12 - 3\beta^2)^{1/2}]/6$ [9, Theorem 2], that $r_n(0) = (1/n)^{(n-1)}$ [2, Lemma 4 and $p(z) = z^n - z$, that $r_n(1) = 1$ [10, Theorem 1], and that $r_n(\beta) \leq \min(1.08332, 1 + 0.72054/n)$ [1, Corollary 1 and equation (3)].
Since \( r_n(1) = 1 \), an obvious place to look for counterexamples to Sendov’s conjecture is in a neighborhood of \( \beta = 1 \). This has already been done in [7, Theorem 3] and [12], where a linear upper bound on \( r_n(\beta) \) suffices to verify the Sendov conjecture if \( \beta \) is sufficiently close to 1. Unfortunately, having only an upper bound leaves many interesting questions about the conjecture unanswered. In this paper we investigate Sendov’s conjecture much more thoroughly near \( \beta = 1 \), by providing a quadratic approximation to \( r_n(\beta) \) with

**Theorem 1.** Let \( n \geq 3 \), let \( k \) be the largest integer such that \( k \leq (n+1)/3 \) and let

\[
\begin{align*}
    u_1 &= \cos \frac{2\pi k}{n+1}, \quad u_2 = \cos \frac{2\pi(k+1)}{n+1}, \\
    D_1 &= \frac{-2u_1u_2 - 1}{2(1-u_1)(1-u_2)}, \quad D_2 = \frac{-1}{2(1-u_1)(1-u_2)}, \\
    D_3 &= (-1 - 4D_1 - 3D_1^2 + 2D_2^2)/2, \\
    D_4 &= (3D_1 - 4D_2 + 3D_1^2 - 2D_1D_2 - 6D_2^2)/2, \\
    D_5 &= (2 + 4D_1 + 5D_2 + 2D_1^2 + 4D_1D_2 + 3D_2^2)/2, \\
    D_6 &= (2D_2 + 2D_1D_2 + 3D_2^2)/2 \quad \text{and} \\
    D &= D_3n + D_4 + D_5/n + D_6/n^2.
\end{align*}
\]

If \( n = 3 \) or \( n = 5 \), then let \( \alpha = 3/2 \); otherwise let \( \alpha = 2 \). If \( n = 5 \), then let \( \Delta = 7/225 \); otherwise let \( \Delta = 0 \). Then for \( \beta \) sufficiently close to 1, we have

\[
r_{n+1}(\beta) = 1 + (D_1 + D_2/n)(1-\beta) + (D + \Delta)(1-\beta)^2 + O(1-\beta)^{\alpha+1}.
\]

Before proving this theorem, we will examine some of its consequences. Our first consequence improves on estimates in [7] and [12] (by providing a value for the coefficient of the linear term) with

**Corollary 2.** For all \( n \geq 2 \) we have \( r_n(\beta) \leq 1 - (3/10)(1-\beta) + O(1-\beta)^2 \).

**Proof.** Recall that for \( 2 \leq n \leq 3 \) we have formulas for \( r_n(\beta) \), and so the result for those values of \( n \) follows from the Taylor series of these formulas at \( \beta = 1 \). As we will show in part 6 of Lemma 8, the quantity \( D_1 + D_2/n \leq -3/10 \) for all \( n \geq 3 \), and so the rest of Corollary 2 follows from Theorem 1. \( \square \)

As we will show in part 6 of Lemma 8 at \( n = 4 \) we have \( D_1 + D_2/n = -3/10 \), so Corollary 2 provides the smallest possible linear upper bound for \( r_n(\beta) \) that is independent of \( n \).

A second consequence of Theorem 1 shows that the result of [7, Theorem 3] is the best possible (in the sense that 1/3 cannot be replaced by a larger number), with

**Corollary 3.** There exist constants \( K_n > 0 \) with \( \lim_{n \to \infty} K_n = 1/3 \) such that

\[
r_{n+1}(\beta) = 1 - K_n(1-\beta) + O(1-\beta)^2.
\]

**Proof.** Choose \( K_n = -(D_1 + D_2/n) \) and note that by Theorem 1 we have \( r_{n+1}(\beta) = 1 - K_n(1-\beta) + O(1-\beta)^2 \). As we shall see in parts 5 and 6 of Lemma 8 the quantity \( D_1 + D_2/n \) is negative and tends to \(-1/3\). \( \square \)
Recall that $r_n(0) = (1/n)^{1/(n-1)}$. This quantity is increasing in $n$, so it is tempting to conjecture that for all fixed $\beta$ the quantity $r_n(\beta)$ is increasing in $n$. Indeed, the graphs in [6, figure 4.8] provide some evidence of this for $n = 4, 6, 8, 10, 12$. Unfortunately, this conjecture is false, as is shown by

**Corollary 4.** For $\beta$ sufficiently close to 1 we have $r_6(\beta) < r_4(\beta)$.

**Proof.** By Theorem 1 and the constants we will compute at the beginning of section 2 we know that

$$r_4(\beta) = 1 - (1/3)(1 - \beta) + O(1 - \beta)^2$$

and that

$$r_6(\beta) = 1 - (11/30)(1 - \beta) + O(1 - \beta)^2,$$

and the conclusion follows. □

Corollary 2 hints of the existence of a better-than-Sendov result, for near $\beta = 1$ it appears that $r_n(\beta)$ is bounded above by a function that is independent of $n$ and strictly less than one. Unfortunately, moving up to the quadratic approximation in Theorem 1 casts doubt upon such a result. To see this, note that as $n$ goes to infinity, then $k/(n+1)$ tends to $1/3$, so $u_1$ and $u_2$ tend to $-1/2$, so $D_3$ tends to $4/30$ and $D_4$ tends to $-1/9$, and so $D + \Delta$ tends to infinity. Indeed, for $n$ sufficiently large one might expect $r_{n+1}(\beta) > 1$ roughly when $D_1(1 - \beta) + (D_3n + D_4)(1 - \beta)^2 > 0$, i.e. when $\beta < 1 + D_1/(D_3n + D_4) \sim 1 - 27/(4n - 9)$, provided that this $\beta$ is “sufficiently close to 1”. This is an intriguing possibility that is clearly worthy of further investigation.

We will verify Theorem 1 by proving the following three propositions:

**Proposition 5.** Assume the notation of Theorem 1. Then for all polynomials $P \in S(n + 1, \beta)$, we have

$$|P|_\beta \leq 1 + (D_1 + D_2/n)(1 - \beta) + (D + \Delta)(1 - \beta)^2 + O(1 - \beta)^{\alpha + 1}.$$  

**Proposition 6.** There are polynomials $P \in S(6, \beta)$ with

$$|P|_\beta = 1 - (11/30)(1 - \beta) + (29/450)(1 - \beta)^2 + O(1 - \beta)^{5/2}.$$  

**Proposition 7.** Assume the notation of Theorem 1. Then there are real polynomials $P \in S(n + 1, \beta)$ with

$$|P|_\beta = 1 + (D_1 + D_2/n)(1 - \beta) + D(1 - \beta)^2 + O(1 - \beta)^{\alpha + 1}.$$  

From the definition of $D$ in Theorem 1 and the constants we will compute at the beginning of section 2 we see that for $n = 5$ we have $D_1 + D_2/n = -11/30$ and $D + \Delta = 29/450$, so Propositions 5 and 6 together imply that Theorem 1 is true for $n = 5$. Note that for $n \neq 5$ we have $\Delta = 0$, so Propositions 5 and 6 taken together imply that Theorem 1 is true for $n \neq 5$.

In [8] it was proved that if $n = 5$ and if $\beta$ is sufficiently close to 1, then maximal polynomials in $S(n + 1, \beta)$ (those for which $|P|_\beta = r_{n+1}(\beta)$) must be nonreal. Taken together, Theorem 1 and Proposition 4 provide strong evidence that this is true only for $n = 5$ (although it is conceivable that this could fail for higher-order approximations).
2. Preliminaries

We begin by computing some values (that we will subsequently need) for the constants that appear in Theorem 1, obtaining:

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<th>( u_2 )</th>
<th>( D_1 )</th>
<th>( D_2 )</th>
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We next establish some relationships between these constants with

**Lemma 8.** Assume the notation of Theorem 1. Then

1. \( u_2 < -1/2 \leq u_1 \), and \( u_1 \leq 0 \) for \( n \neq 4 \), and \( u_2 > -1 \) for \( n \neq 3, 5 \),
2. \( u_1 + u_2 < 0 \) and \( u_1 u_2 > -1 \),
3. \( 2nu_1 + n + 1 \geq 1 \) and \( 2nu_2 + n + 1 < 0 \),
4. \( D_1 < 0 \) and \( D_2 < 0 \),
5. \( \lim_{n \to \infty} D_1 + D_2/n = -1/3 \),
6. \( -1 < D_1 + D_2/n \leq -3/10 \), with equality only at \( n = 4 \), and
7. \( 1 + (1 + D_1 + D_2)(u_1 - 1) - D_2(2u_1^2 - 2) = 0 \) for \( i = 1 \) and \( i = 2 \).

**Proof.** From the definition of \( k \) in Theorem 1 the relationship between \( k \) and \( n \) depends on the residue of \( n \) modulo 3. For increasing values of \( n \) in each of the three residue classes, the sequence \( k/(n+1) \) increases to (or is equal to) \( 1/3 \) and the sequence \( (k+1)/(n+1) \) strictly decreases to \( 1/3 \), so the values of \( u_1 \) decrease to (or are equal to) \(-1/2\) and the values of \( u_2 \) strictly increase to \(-1/2\). Since the values of \( u_1 \) decrease (or remain constant) in each residue class, and since \( u_1 \leq 0 \) for \( n = 3, 5 \) and 7, then \( u_1 \leq 0 \) for all \( n \neq 4 \). Since the values of \( u_2 \) strictly increase in each residue class, and since \( u_2 > -1 \) for \( n = 4 \) and \( u_2 = -1 \) for \( n = 3 \) and \( n = 5 \), then \( u_2 > -1 \) for \( n \neq 3, 5 \). This completes the proof of part 1 of the lemma.

For \( n = 4 \), we have \( u_1 + u_2 = -1/2 \) and \( u_1 u_2 = -1/4 \). For \( n \neq 4 \) we have from part 1 that \( u_2 < u_1 \leq 0 \), and part 2 of the lemma follows trivially.

Since \( u_1 \geq -1/2 \), then \( 2nu_1 + n + 1 \geq 1 \). For \( n = 3, 4 \) and 5 we have \( (k+1)/(n+1) \leq 1/2 \). Since in each residue class this quotient strictly decreases to \( 1/3 \), then for all \( n \geq 3 \) we have \( 2\pi(k+1)/(n+1) \in (2\pi/3, \pi] \). Now \( \cos x \leq 1/2 - 3x/(2\pi) \) on this interval, and from the definition of \( k \) in Theorem 1 we know that \( k \geq (n-1)/3 \), so

\[
u_2 = \cos \left( \frac{2\pi(k+1)}{n+1} \right) \leq \frac{1}{2} - \frac{3(k+1)}{n+1} \leq \frac{1}{2} - \frac{n+2}{n+1} < -\frac{n+1}{2n},\]

which completes the proof of part 3 of the lemma.

At \( n = 4 \), we have \( D_1 = -1/5 \) and \( D_2 = -2/5 \). For \( n \neq 4 \) we know from part 1 of Lemma 8 that \( u_2 < u_1 \leq 0 \) so from the definitions of \( D_1 \) and \( D_2 \) in Theorem 1 we see that \( D_1 < 0 \) and \( D_2 < 0 \). This completes the proof of part 4 of the lemma.

As \( n \) tends to infinity, \( u_1 \) and \( u_2 \) tend to \(-1/2\), so \( D_1 \) tends to \(-1/3 \) and \( D_2 \) is bounded. This completes the proof of part 5 of the lemma.
By part 2 of Lemma 8 we have $u_1 + u_2 < 0$ and $u_1 u_2 > -1$. Since by part 4 of Lemma 8 we know that $D_2 < 0$, then

$$D_1 + D_2/n > D_1 + D_2 = -\frac{1 + u_1 u_2}{1 + u_1 u_2 - (u_1 + u_2)} > -1.$$  

From part 1 of Lemma 8 we know that $u_2 < -1/2 \leq u_1$, so by computing the partial derivatives of $D_1$ we see that $\partial D_1/\partial u_1 > 0$ and $\partial D_1/\partial u_2 \leq 0$. Since in each residue class $u_1$ decreases to $-1/2$ and $u_2$ increases to $-1/2$, then in each residue class $D_1$ decreases to $-1/3$. At $n = 5, 6$ and 10 we have $D_1 < -3/10$, and hence $D_1 + D_2/n < D_1 < -3/10$ for all $n \geq 3$ except possibly $n = 3, 4$ and 7. Checking the values of $D_1 + D_2/n$ (computed at the beginning of section 2) for these exceptional values completes the proof of part 6 of the lemma.

Expressing $D_1$ and $D_2$ in terms of $u_1$ and $u_2$ and simplifying the result verifies part 7, and thus completes the proof of Lemma 8. □

We now estimate the size of the coefficients of $P'$ with

**Proposition 9.** Suppose that $P \in S(n + 1, \beta)$ with $P'$ monic and $|P|_\beta \geq \beta$. Let $P'(z) = \prod_{j=1}^n (z - \zeta_j) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$. Then

1. each $\Re[\zeta_j] = \mathcal{O}(1 - \beta)$ and each $\Im[\zeta_j] = \mathcal{O}(1 - \beta)^{1/2}$,
2. each $a_{n-k} = \mathcal{O}(1 - \beta)^{k/2}$,
3. for $k$ odd, each $\Re[a_{n-k}] = \mathcal{O}(1 - \beta)^{(k+1)/2}$, and
4. for $k$ even, each $\Im[a_{n-k}] = \mathcal{O}(1 - \beta)^{(k+1)/2}$.

**Proof.** Parts 1–3 were proved in [8, Proposition 4]. Part 4 is proved similarly to part 3, by noting that each term of $\Im[a_{n-k}]$ is a product of $k$ of the $\Re[\zeta_j]$’s and $\Im[\zeta_j]$’s, and that for $k$ even, each term has at least one $\Re[\zeta_j]$, so from part 1 of Proposition 9 we have that $\Im[a_{n-k}] = \mathcal{O}(1 - \beta)^{(k+1)/2}$. □

To have $P \in S(n + 1, \beta)$ requires that the moduli of the roots of $P$ are all at most 1. We estimate these moduli with

**Proposition 10.** Assume the notation of Theorem 1. Let $P$ be a polynomial with $P'(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ and $P(\beta) = 0$. Let $z \neq \beta$ be a root of $P$, let $\omega$ be the $(n + 1)$th root of 1 that is closest to $z$ and let $R = (1 - \beta) + a_{n-1}(\omega^n - 1)/n + \cdots + a_0(\omega - 1)$.

1. For $0 < r \leq 1$, if each $a_k = \mathcal{O}(1 - \beta)^r$, then $|z|^2 = 1 - 2\Re[R] + \mathcal{O}(1 - \beta)^{2r}$.
2. Suppose that

$$a_{n-1} = n(1 + D_1 + D_2)(1 - \beta) + \mathcal{O}(1 - \beta)^{\alpha},$$

$$a_{n-2} = -(n - 1)D_2(1 - \beta) + \mathcal{O}(1 - \beta)^{\alpha},$$

$$a_{n-k} = \mathcal{O}(1 - \beta)^{\alpha} \quad \text{for } k \geq 3$$

and define

$$\Gamma_2 = 2(1 + D_1 + D_2)(D_1 - 2D_2 + nD_2) \quad \text{and}$$

$$\Gamma_1 = -\Gamma_2 + (-2 - 4D_1)n + (1 + 4D_1 - 4D_2).$$

If $\Re[\omega] = u_i$ for $i = 1$ or $i = 2$, then

$$|z|^{2n+2} = 1 - 2(n + 1)\Re[R] + (n + 1)(\Gamma_1 + \Gamma_2 u_i)(1 - \beta)^2 + \mathcal{O}(1 - \beta)^{\alpha+1}.$$
Proof. Since \( \beta = 1 - (1 - \beta) \), then by the binomial theorem \( \beta^k = 1 - k(1 - \beta) + \mathcal{O}(1 - \beta)^2 \). Since \( z \) is a root of \( P \) we have
\[
0 = P(z) = \int_0^z P'(t) \, dt = \frac{z^{n+1} - \beta^{n+1}}{n+1} + a_{n-1} \frac{z^n - \beta^n}{n} + \cdots + a_0(z - \beta),
\]
and solving for \( z^{n+1} \) gives us
\[
(2.1) \quad z^{n+1} = \beta^{n+1} - (n+1) \left[ a_{n-1} \frac{z^n - \beta^n}{n} + \cdots + a_0(z - \beta) \right].
\]
By hypothesis, as \( \beta \) goes to 1 the \( a_k \) all tend to 0 so the roots of \( P \) tend to the roots of \( z^{n+1} - 1 \), and so the \( \omega \) appearing in the hypotheses is well defined.

Now each \( \beta^k = 1 + \mathcal{O}(1 - \beta) \), and by the hypothesis of part 1 each \( a_k = \mathcal{O}(1 - \beta)^r \). Putting these estimates into equation (2.1), we see that \( z^{n+1} = 1 + \mathcal{O}(1 - \beta)^r \). Then \( z = \omega + \mathcal{O}(1 - \beta)^r \) and so \( (z^k - \beta^k)/k = (\omega^k - 1)/k + \mathcal{O}(1 - \beta)^r \).

Now note that each \( a_{n-k} = \mathcal{O}(1 - \beta)^r \) and that each \( \beta^k = 1 - k(1 - \beta) + \mathcal{O}(1 - \beta)^2 \). Substituting these estimates into equation (2.1) gives
\[
z^{n+1} = 1 - (n+1)(1 - \beta) - (n+1) \left[ a_{n-1} \frac{\omega^n - 1}{n} + \cdots + a_0(\omega - 1) \right] + \mathcal{O}(1 - \beta)^{2r}
\]
\[
= 1 - (n+1)R + \mathcal{O}(1 - \beta)^{2r}.
\]
Note that \( R = \mathcal{O}(1 - \beta)^r \) so
\[
(1 - R)^{n+1} = 1 - (n+1)R + \mathcal{O}(1 - \beta)^{2r} = z^{n+1} + \mathcal{O}(1 - \beta)^{2r},
\]
so \( z = \omega(1 - R) + \mathcal{O}(1 - \beta)^{2r} \) and hence \( |z|^2 = z\overline{z} = 1 - 2R|R| + \mathcal{O}(1 - \beta)^{2r} \). This finishes the proof of part 1.

From the hypotheses of part 2, we know that \( \Re[\omega] = u_i \) for \( i = 1 \) or \( i = 2 \). Suppose for the moment that \( \Re[\omega] = u_1 \) and write \( \omega = u_1 + iv_1 \). Since \( \omega^{n+1} = 1 \), then \( |\omega| = 1 \), so \( \omega^n = \overline{\omega} \) and \( \Re[\omega^n] = 2u_1^2 - 1 \). Let \( A = \left[ -(1 + D_1 + D_2) + 2D_2u_1 \right]v_1 \). From part 7 of Lemma 8 we see that
\[
\Re[1 + (1 + D_1 + D_2)(\overline{\omega} - 1) - D_2(\overline{\omega}^2 - 1)] = 0
\]
and so using the estimates of the \( a_{n-k} \)'s given in the hypotheses of part 2, we get
\[
R = (1 - \beta) + a_{n-1} \frac{\overline{\omega} - 1}{n} + a_{n-2} \frac{\overline{\omega}^2 - 1}{n-1} + \cdots + a_0(\omega - 1)
\]
\[
= \left[ 1 + (1 + D_1 + D_2)(\overline{\omega} - 1) - D_2(\overline{\omega}^2 - 1) \right](1 - \beta) + \mathcal{O}(1 - \beta)^\alpha
\]
\[
= iA(1 - \beta) + \mathcal{O}(1 - \beta)^\alpha.
\]

The hypotheses of part 2 imply that each \( a_k = \mathcal{O}(1 - \beta) \), so from the proof of part 1 with \( r = 1 \) we have \( z = \omega(1 - R) + \mathcal{O}(1 - \beta)^2 = \omega[1 - iA(1 - \beta)] + \mathcal{O}(1 - \beta)^\alpha \) and so
\[
(z^k - \beta^k)/k = (\omega^k - 1)/k + (1 - iA\omega^k)(1 - \beta) + \mathcal{O}(1 - \beta)^\alpha.
\]

Let \( G = n/2 - n(1 + D_1 + D_2)(1 - iA\overline{\omega}) + (n - 1)D_2(1 - iA\overline{\omega}^2) \). Then from equation (2.1) and the estimates of the \( a_k \)'s given in the hypotheses of part 2 we
get

\[ z^{n+1} = 1 - (n + 1)(1 - \beta) + \frac{(n + 1)n}{2}(1 - \beta)^2 \]

\[ - (n + 1) \left[ a_{n-1} \left( \frac{\omega^n - 1}{n} + (1 - iA\omega^n)(1 - \beta) \right) \right. \]

\[ + \left. a_{n-2} \left( \frac{\omega^{n-1} - 1}{n - 1} + (1 - iA\omega^{n-1})(1 - \beta) \right) \right. \]

\[ + \left. a_{n-3} \frac{\omega^{n-2} - 1}{n - 2} + \cdots + a_0(\omega - 1) \right] + O(1 - \beta)^{n+1} \]

Then since \( R = iA(1 - \beta) + O(1 - \beta)^n \) we have

\[ |z|^{2n+2} = z^{n+1} - \beta \]

\[ = 1 - 2(n + 1)\Re[R] + (n + 1) \left[ 2\Re[G] + (n + 1)A^2 \right] (1 - \beta)^2 + O(1 - \beta)^{n+1}. \]

Thus to complete the proof of part 2 of Proposition 10 for the case \( \Re[\omega] = u_1 \) we need only verify that \( 2\Re[G] + (n + 1)A^2 = \Gamma_1 + \Gamma_2u_1 \).

Let \( D_0 = 1 + D_1 + D_2 \), so from the definition of \( A \) we see that

\[ A = (-D_0 + 2D_2u_1)v_1. \]

Note that \( \Re[i\omega] = \Im[\omega] \). Then from the definition of \( G \) we have

\[ \Re[G] = n/2 - nD_0(1 - Av_1) + (n - 1)D_2(1 - 2Av_1) \]

\[ = n/2 - nD_0 + (n - 1)D_2 - A[n(-D_0v_1 + 2D_2u_1v_1) - 2D_2u_1v_1] \]

\[ = (-n/2 - nD_1 - D_2) - nA^2 + 2AD_2u_1v_1 \]

so

\[ (2.2) \quad 2\Re[G] + (n + 1)A^2 = (-n - 2nD_1 - 2D_2) + (-n + 1)A^2 + 4AD_2u_1v_1. \]

Now

\[ 2D_2u_1^2 \]

\[ = \frac{-u_2^2}{(1 - u_1)(1 - u_2)} = D_0u_1 + (D_2 - D_1), \]

so

\[ Av_1 = (-D_0 + 2D_2u_1)(1 - u_1^2) \]

\[ = -D_0 + 2D_2u_1 - u_1(-D_0u_1 + 2D_2u_1^2) \]

\[ = -D_0 + (D_1 + D_2)u_1. \]

Using these two equalities, we see that

\[ A^2 = (-D_0 + 2D_2u_1)[-D_0 + (D_1 + D_2)u_1] \]

\[ = D_0^2 + (-D_0D_1 - 3D_0D_2)u_1 + (D_1 + D_2)(2D_2u_1^2) \]

\[ = D_0^2 - D_1^2 + D_2^2 - 2D_0D_2u_1 \]

and

\[ 2AD_2u_1v_1 = 2D_2u_1[-D_0 + (D_1 + D_2)u_1] \]

\[ = -2D_0D_2u_1 + (D_1 + D_2)[D_0u_1 + (D_2 - D_1)] \]

\[ = D_0(D_1 - D_2)u_1 + (D_2^2 - D_1^2). \]
Thus from equation (2.2) we have
\[2\Re[G] + (n + 1)A^2 = (-n - 2nD_1 - 2D_2) + (-n + 1)(D_0^2 - D_1^2 + D_2^2 - 2D_0D_2u_1)\]
\[+ 2\left[D_0(D_1 - D_2)u_1 + (D_2^2 - D_1^2)\right]\]
\[= (-1 - 2D_1 - D_0^2 + D_1^2 - D_2^2)n + (-2D_2 + D_0^2 - 3D_1^2 + 3D_2^2)\]
\[+ 2D_0u_1(D_1 - 2D_2 + nD_2)\]
\[= \Gamma_1 + \Gamma_2u_1.\]

This finishes the proof of part 2 of Proposition 10 for the case \(\Re[\omega] = u_1\). Since \(D_1\) and \(D_2\) are symmetric in \(u_1\) and \(u_2\), swapping \(u_1\) and \(u_2\) in this proof verifies part 2 of Proposition 10 for the remaining case \(\Re[\omega] = u_2\), and thus completes the proof of Proposition 10. □

Finally, consider the linear transformation \(T\) which takes functions to real numbers via
\[(2.3) \quad T(f) = \frac{2nu_1 + (n + 1)f(u_2) - (2nu_2 + n + 1)f(u_1)}{2(u_1 - u_2)}.\]
Recall that by Lemma 8 we have \(u_1 - u_2 > 0, 2nu_1 + n + 1 > 0\) and \(2nu_2 + n + 1 < 0\), so \(T/n\) is a weighted average. This implies that \(T\) preserves inequalities, in the sense that if \(f(u_1) \leq g(u_1)\) and \(f(u_2) \leq g(u_2)\), then \(T(f) \leq T(g)\).

In the process of analyzing several inequalities, we will need the following values of the transformation \(T\):
\[T(1) = n,\]
\[T(2 + 2u) = n - 1,\]
\[T(1 + 4u + 4u^2) = \frac{[n + 2 + 2(n + 1)(u_1 + u_2) + 4nu_1u_2]}{D_2},\]
\[(2.4) \quad T\left(\frac{1}{1 - u}\right) = n + nD_1 + D_2,\]
\[T\left(\frac{u}{1 - u}\right) = nD_1 + D_2.\]

We will also use the results of

Lemma 11. For the linear transformation \(T\) defined in equation (2.3) we have
1. \(T(1 + 4u + 4u^2)/(n - 2) < 1/2\) for \(n \neq 3, 4\) and 6, and
2. \(T(8u^2 + 8u^3) \geq 0\) for all \(n\).

Proof. From the formula for \(T(1 + 4u + 4u^2)\) in (2.4) and from part 3 of Lemma 8 we have
\[\partial T(1 + 4u + 4u^2)/\partial u_1 = -2(2nu_2 + n + 1) > 0\] and
\[\partial T(1 + 4u + 4u^2)/\partial u_2 = -2(2nu_1 + n + 1) < 0.\]
Recall from the proof of Lemma 8 that for each residue class of \(n\) modulo 3 the values of \(u_1\) decrease and the values of \(u_2\) increase, so the signs of the partial derivatives above imply that in each residue class the values of \(T(1 + 4u + 4u^2)\) decrease. Since \(1 + 4u + 4u^2 = (1 + 2u)^2 \geq 0\) and since \(T\) preserves inequalities, then \(T(1 + 4u + 4u^2) \geq 0\), so the values of \(T(1 + 4u + 4u^2)/(n - 2)\) also decrease.
in each residue class. Using the formula for $T(1 + 4u + 4u^2)$ in (2.4) and the values of the $u_i$ computed at the beginning of section 2, we calculate the values of $T(1 + 4u + 4u^2)/(n - 2)$ at $n = 5, 7$ and $9$, getting respectively $1/3, 0.4627$ and $0.3372$. Since they are all less than $1/2$, this proves part 1 of Lemma 11.

Since by definition $u_i \geq -1$, then $8u_i^2 + 8u_i^4 = 8u_i^2(1 + u_i) \geq 0$ for both $i = 1$ and $i = 2$, and so part 2 of Lemma 11 follows from our observation that $T$ preserves inequalities.

Finally, we will deal with polynomials that are “almost” in $S(n, \beta)$ using

**Lemma 12.** Suppose that $P$ is a polynomial of degree $n$ with all roots in $\{z : |z| \leq 1 + O(1 - \beta)^{\gamma}\}$, one root at $\beta$, and all other roots bounded away from $\beta$. Then there is a polynomial $Q \in S(n, \beta)$ such that $|Q|_\beta = |P|_\beta + O(1 - \beta)^{\gamma}$.

**Proof.** If $P \in S(n, \beta)$, then we may take $Q = P$. If not, then at least one root of $P$ has modulus greater than $1$. In this case, let

$$c = \max \left\{ \frac{|z|^2 - 1}{|z - \beta|^2} : z \text{ is a root of } P \text{ and } |z| > 1 \right\}.$$ 

Since by hypothesis $|z - \beta|$ is bounded away from 0 and $|z| \leq 1 + O(1 - \beta)^{\gamma}$, then $0 < c \leq O(1 - \beta)^{\gamma}$. In particular, for $\beta$ sufficiently close to 1 we have $0 < c < 1$.

Let $Q$ be the polynomial with roots $\{z - c(z - \beta) : z \text{ is a root of } P\}$. Since the mapping $z \mapsto z - c(z - \beta)$ is a contraction of the plane that leaves $\beta$ fixed and moves all roots of $P$ (and hence $P'$) toward $\beta$ by at most $O(1 - \beta)^{\gamma}$, then $Q(\beta) = 0$ and $|Q|_\beta = |P|_\beta + O(1 - \beta)^{\gamma}$. Thus we need only show that all roots of $Q$ are in the unit disk.

Note that for $t$ real the image of the mapping $t \mapsto z - t(z - \beta)$ is a line, with $t = 0$ mapping to $z$, and $t = 1$ mapping to $\beta$, and $t = (|z|^2 - 1)/|z - \beta|^2$ mapping to

$$z - \frac{|z|^2 - 1}{|z - \beta|^2}(z - \beta) = z - \frac{z\bar{z} - 1}{\bar{z} - \beta} = \frac{1 - \beta z}{\bar{z} - \beta}.$$ 

If $z$ is in the unit disk, then the images of every $t$ between 0 and 1 lie on the line between $z$ and $\beta$, hence in the unit disk. If $z$ is not in the unit disk, then $|(1 - \beta z)/(z - \beta)| < 1$ and so the images of every $t$ between $(|z|^2 - 1)/|z - \beta|^2$ and 1 lie on the line between $(1 - \beta z)/(\bar{z} - \beta)$ and $\beta$, hence in the unit disk. Thus for every root $z$ of $P$, the image of $c$ lies in the unit disk, so all roots of $Q$ are in the unit disk and so $Q \in S(n, \beta)$. This completes the proof of Lemma 12. □

3. **Proof of Proposition 5**

Take any $P \in S(n + 1, \beta)$, assume without loss of generality that $P'$ is monic, and write $P'(z) = \prod_{j=1}^n (z - \zeta_j) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$.

If $|P|_\beta \leq 1 + (D_1 + D_2/n)(1 - \beta) + (D + \Delta)(1 - \beta)^2$, then Proposition 4 is trivially true. Thus we may assume without loss of generality that

$$|P|_\beta \geq 1 + (D_1 + D_2/n)(1 - \beta) + (D + \Delta)(1 - \beta)^2.$$ 

From part 6 of Lemma 8 we have that $D_1 + D_2/n > -1$, and so inequality (3.1) implies that $|P|_\beta \geq \beta$ as long as $\beta$ is sufficiently close to 1. Note that $P$ thus satisfies all the hypotheses of Proposition 8.
We begin by estimating some relationships between the coefficients of $P'$ with

**Lemma 13.** Suppose that $\Re[a_{n-1}] = \mathcal{O}(1 - \beta)^{3/2}$ and that each

$$|\zeta_j - \beta| = 1 + (D_1 + D_2/n)(1 - \beta) + \mathcal{O}(1 - \beta)^2.$$ 

Then

1. $\Im[a_{n-2}] = (-3/2)\Im[a_{n-3}] + \mathcal{O}(1 - \beta)^{5/2}$ and
2. $\Re[a_{n-3}] + 2\Re[a_{n-4}] = (n - 2)(1 + D_1 + D_2/n)(1 - \beta)\Re[a_{n-2}] + \mathcal{O}(1 - \beta)^3.$

**Proof.** Let each $\zeta_j = x_j + iy_j$ and note that by Proposition 11 we have $x_j = \mathcal{O}(1 - \beta)$ and $y_j = \mathcal{O}(1 - \beta)^{1/2}$. Note that by hypothesis, $\sum_i y_i = -\Im[a_{n-1}] = \mathcal{O}(1 - \beta)^{3/2}$ and that each

$$(\beta - x_j)^2 + y_j^2 = |\beta - \zeta_j|^2 = 1 + 2(D_1 + D_2/n)(1 - \beta) + \mathcal{O}(1 - \beta)^2,$$ 

so solving for $x_j$ gives us

$$x_j = y_j^2/2 - (1 + D_1 + D_2/n)(1 - \beta) + \mathcal{O}(1 - \beta)^2. \quad (3.2)$$

Note that $\Im[a_{n-3}] = -\sum_{i < j < k} \Im[\zeta_i \zeta_j \zeta_k] = \sum_{i < j < k} y_i y_j y_k + \mathcal{O}(1 - \beta)^{5/2},$ so

$$\mathcal{O}(1 - \beta)^{5/2} = \sum_i y_i \sum_{i < j} y_i y_j = \sum_{i < j} y_i^2 y_j + \sum_{i < j < k} y_i y_j y_k + \mathcal{O}(1 - \beta)^{5/2}$$

$$= \sum_{i < j} y_i^2 y_j + 3\Im[a_{n-3}] + \mathcal{O}(1 - \beta)^{5/2}$$

and so $\sum_{i < j} y_i^2 y_j = -3\Im[a_{n-3}] + \mathcal{O}(1 - \beta)^{5/2}.$ Then using equation (3.2) we have

$$\Im[a_{n-2}] = \sum_{i < j} \Im[\zeta_i \zeta_j] = \sum_{i < j} x_i y_j$$

$$= (1/2) \sum_{i < j} y_i^2 y_j - (1 + D_1 + D_2/n)(1 - \beta) \sum_{i < j} y_j + \mathcal{O}(1 - \beta)^{5/2}$$

$$= (-3/2)\Im[a_{n-3}] + \mathcal{O}(1 - \beta)^{5/2},$$

which completes the proof of part 1 of Lemma 13.

Let $S$ be the set of triples $(i, j, k)$ of distinct integers from 1 to $n$ with $j < k$. Note that $\Re[a_{n-2}] = \sum_{i < j} \Re[\zeta_i \zeta_j] = -\sum_{i < j} y_i y_j + \mathcal{O}(1 - \beta)^2$ and $\Re[a_{n-3}] = -\sum_{i < j < k} \Re[\zeta_i \zeta_j \zeta_k] = \sum_{S} x_i y_j y_k + \mathcal{O}(1 - \beta)^3.$ Furthermore,

$$\mathcal{O}(1 - \beta)^3 = \sum_i y_i \sum_{j < k < l} y_j y_k y_l = \sum_S y_i^2 y_j y_k + 4 \sum_{i < j < k < l} y_i y_j y_k y_l,$$

so

$$\Re[a_{n-4}] = \sum_{i < j < k < l} \Re[\zeta_i \zeta_j \zeta_k \zeta_l] = \sum_{i < j < k < l} y_i y_j y_k y_l + \mathcal{O}(1 - \beta)^3$$

$$= (-1/4) \sum_S y_i^2 y_j y_k + \mathcal{O}(1 - \beta)^3.$$
Then using equation (3.2) we have
\[ \Re[a_{n-3}] + 2\Re[a_{n-4}] = \sum_S (x_i - y_i^2/2) y_j y_k + \mathcal{O}(1 - \beta)^3 \]
\[ = -(1 + D_1 + D_2/n)(1 - \beta)(n - 2) \sum_{j < k} y_j y_k + \mathcal{O}(1 - \beta)^3 \]
\[ = (n - 2)(1 + D_1 + D_2/n)(1 - \beta)\Re[a_{n-2}] + \mathcal{O}(1 - \beta)^3, \]
which completes the proof of Lemma 13. \( \square \)

We now establish a lower bound on \( \Re[a_{n-4}] \) with

Lemma 14. Suppose that
\[ \Im[a_{n-1}] = \mathcal{O}(1 - \beta)^\alpha, \]
\[ \Re[a_{n-2}] = -(n - 1)D_2(1 - \beta) + \mathcal{O}(1 - \beta)^\alpha, \text{ and} \]
\[ \Im[a_{n-3}] = \mathcal{O}(1 - \beta)^\alpha. \]
If \( n = 5 \), then define \( \delta = -1/15 \); otherwise define \( \delta = 0. \) Then
\[ \Re[a_{n-4}] \geq \delta(1 - \beta)^2 + \mathcal{O}(1 - \beta)^{\alpha + 1}. \]

Proof. Let each \( \zeta_j = x_j + iy_j \) and recall by Proposition 9 that \( x_j = \mathcal{O}(1 - \beta) \) and \( y_j = \mathcal{O}(1 - \beta)^{1/2} \). Let \( F(y) = \prod_{i=1}^n (y + y_i) = y^n + b_{n-1}y^{n-1} + \cdots + b_0 \). Note that
\[ \Re[a_{n-4}] = \sum_{i < j < k < l} \Re[\zeta_i \zeta_j \zeta_k \zeta_l] = \sum_{i < j < k < l} y_i y_j y_k y_l + \mathcal{O}(1 - \beta)^3 \]
and that by hypothesis
\[ b_{n-1} = \sum_i y_i = \sum_i \Im[\zeta_i] = -\Im[a_{n-1}] \]
\[ = \mathcal{O}(1 - \beta)^\alpha, \]
\[ b_{n-2} = \sum_{i < j} y_i y_j = -\sum_{i < j} \Re[\zeta_i \zeta_j] + \mathcal{O}(1 - \beta)^2 = -\Re[a_{n-2}] + \mathcal{O}(1 - \beta)^2 \]
\[ = (n - 1)D_2(1 - \beta) + \mathcal{O}(1 - \beta)^\alpha, \text{ and} \]
\[ b_{n-3} = \sum_{i < j < k} y_i y_j y_k = -\sum_{i < j < k} \Im[\zeta_i \zeta_j \zeta_k] + \mathcal{O}(1 - \beta)^{5/2} = \Im[a_{n-3}] + \mathcal{O}(1 - \beta)^{5/2} \]
\[ = \mathcal{O}(1 - \beta)^\alpha. \]

Let
\[ f(y) = F^{(n-4)}(y) \]
\[ = \frac{n!}{24} y^4 + \frac{(n - 1)!}{6} b_{n-1} y^3 + \frac{(n - 2)!}{2} b_{n-2} y^2 + (n - 3)! b_{n-3} y + (n - 4)! b_{n-4}. \]

Now by definition \( F \) has all real roots, hence by Rolle’s Theorem (from elementary calculus) so does \( f \). Then the “reverse” of \( f \) defined by \( y^4 f(1/y) = (n - 4)!b_{n-4}y^4 + \cdots + n!/24 \) has all real roots, so by Rolle’s theorem so does the reverse’s second derivative
\[ 12(n - 4)!b_{n-4}y^2 + 6(n - 3)!b_{n-3} y + (n - 2)! b_{n-2}. \]
Since this quadratic has all real roots, then its discriminant is nonnegative, so
\[ 6(n - 3)!b_{n-3}^2 - 48(n - 2)!(n - 4)!b_{n-2}b_{n-4} \geq 0. \]

Using our estimates of the \( b_{n-1} \)'s (including \( b_{n-4} = \mathcal{O}(1 - \beta)^2 \)), this implies that \(-D_2(1 - \beta)b_{n-4} \geq \mathcal{O}(1 - \beta)^{2a} \) and so \( b_{n-4} \geq \mathcal{O}(1 - \beta)^{2a-1} \). Now for \( n \neq 3, 5 \) we have \( \alpha = 2 \) and so \( \Re[a_{n-4}] = b_{n-4} + \mathcal{O}(1 - \beta)^3 \geq \mathcal{O}(1 - \beta)^3 \), which finishes the proof of Lemma \[14\] for these values of \( n \).

Lemma \[14\] is trivially true for \( n = 3 \), since then \( \Re[a_{n-4}] = 0 \geq \mathcal{O}(1 - \beta)^{5/2} \).

Finally, for \( n = 5 \) we have that
\[ f(y) = 5y^4 + 4b_{n-1}y^3 + 3b_{n-2}y^2 + 2b_{n-3}y + b_{n-4} \]
has all real roots, hence by Rolle’s theorem so does its derivative
\[ f'(y) = 20y^3 + 12b_{n-1}y^2 + 6b_{n-2}y + 2b_{n-3}. \]

A classical result (see e.g. [11, p. 289]) states that if a cubic polynomial \( ax^3 + bx^2 + cx + d \) has all real roots, then its discriminant is nonnegative, so
\[ 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2 \geq 0. \]

Applying this to \( f'(y) \), we have
\[ -4[20][6b_{n-2}]^3 - 27[20]^2[2b_{n-3}]^2 \geq \mathcal{O}(1 - \beta)^4, \]
which implies that \( 2b_{n-2}^3 + 5b_{n-3}^2 - \mathcal{O}(1 - \beta)^4 \). Since for \( n = 5 \) we have \( D_2 = -1/6 \), then by hypothesis \( b_{n-2} = (-2/3)(1 - \beta) + \mathcal{O}(1 - \beta)^{3/2} \), and so
\[ b_{n-3}^2 \leq (-2/5)b_{n-2}^2 + \mathcal{O}(1 - \beta)^4 \]
\[ = (16/135)(1 - \beta)^{3} + \mathcal{O}(1 - \beta)^{7/2}. \]

We also have that the first derivative of the reverse of \( f \)
\[ 4b_{n-4}y^3 + 6b_{n-3}y^2 + 6b_{n-2}y + 4b_{n-1} \]
has all real roots, so applying our classical result gives
\[ [6b_{n-3}]^2[6b_{n-2}]^2 - 4[4b_{n-4}][6b_{n-2}]^3 \geq \mathcal{O}(1 - \beta)^6. \]

Dividing this by \( 144b_{n-2}^2 \) and recalling that \( b_{n-2} = (-2/3)(1 - \beta) + \mathcal{O}(1 - \beta)^{3/2} \) yields
\[ 9b_{n-3}^2 + 16(1 - \beta)b_{n-4} \geq \mathcal{O}(1 - \beta)^{7/2}. \]

Combining these two inequalities implies that for \( n = 5 \) we have
\[ \Re[a_{n-4}] = b_{n-4} + \mathcal{O}(1 - \beta)^3 \]
\[ \geq \frac{-9b_{n-3}^2}{16(1 - \beta)} + \mathcal{O}(1 - \beta)^{5/2} \]
\[ \geq (1 - \frac{1}{15})(1 - \beta)^2 + \mathcal{O}(1 - \beta)^{5/2}. \]

This completes the proof of Lemma \[14\] \[\square\]

We now begin the proof of Proposition \[5\]. Our first step will be to show that \( |P|_n \leq 1 + (D_1 + D_2/n)(1 - \beta) + \mathcal{O}(1 - \beta)^2 \). Recall that \( P \) satisfies the hypotheses of Proposition \[9\] so each \( a_{n-k} = \mathcal{O}(1 - \beta)^{k/2} \). Let \( \omega \neq 1 \) be any \((n + 1)\)st root of 1 and let \( z \) be the root of \( P \) (so \( |z| \leq 1 \)) closest to \( \omega \). Then in Proposition \[10\] we have
\[ R = (1 - \beta) + a_{n-1}(\omega^n - 1)/n + \cdots + a_0(\omega - 1) \]
\[ = a_{n-1}(\omega^n - 1)/n + \mathcal{O}(1 - \beta) \]
and so by part 1 of Proposition 10 with \( r = 1/2 \), we have
\[
|z|^2 = 1 - 2\Re[\alpha_{n-1}(\omega^n - 1)/n] + O(1 - \beta).
\]

Since \(|z| \leq 1\) and \(\omega^n = \overline{\omega}\), this implies that \(\Re[\alpha_{n-1}(\overline{\omega} - 1)] \geq O(1 - \beta)\). Expanding the product and noting that by Proposition 9 we have \(\Re[\alpha_{n-1}] = O(1 - \beta)\), we get that \(\Im[\alpha_{n-1}]3|\omega| \geq O(1 - \beta)\). Choosing \(\omega\) non-real and repeating this argument with \(\overline{\omega}\) substituted for \(\omega\) provides that \(\Im[\alpha_{n-1}]3|\overline{\omega}| \geq O(1 - \beta)\) and so \(\Im[\alpha_{n-1}] = O(1 - \beta)\). Thus we have \(\alpha_{n-1} = O(1 - \beta)\).

Recall that each \(\alpha_{n-k} = O((1 - \beta)^k/2\), so we now know that each \(\alpha_{n-k} = O(1 - \beta)\). Since \(\omega^{n-k} = \overline{\omega}^{k+1}\), by part 1 of Proposition 10 with \( r = 1 \) we have
\[
|z|^2 = 1 - 2\Re\left[(1 - \beta) + a_{n-1}\frac{\overline{\omega} - 1}{n} + a_{n-2}\frac{\overline{\omega}^2 - 1}{n - 1} + a_{n-3}\frac{\overline{\omega}^3 - 1}{n - 2}\right] + O(1 - \beta)^2.
\]

Since \(|z| \leq 1\) this implies that
\[
|z|^2 = 1 - 2\Re\left[a_{n-1}\frac{\overline{\omega} - 1}{n} + a_{n-2}\frac{\overline{\omega}^2 - 1}{n - 1} + a_{n-3}\frac{\overline{\omega}^3 - 1}{n - 2}\right] \leq (1 - \beta) + O(1 - \beta)^2.
\]

Averaging the expressions obtained by substituting \(\omega\) and \(\overline{\omega}\) into inequality (3.3) and noting that by Proposition 9 we have \(\Re[\alpha_{n-3}] = O(1 - \beta)^2\) we get
\[
\Re[\alpha_{n-1}]\Re\left[\frac{1 - \omega}{n}\right] + \Re[\alpha_{n-2}]\Re\left[\frac{1 - \omega^2}{n - 1}\right] \leq (1 - \beta) + O(1 - \beta)^2.
\]

Let \(u = \Re[\omega]\). Note that since \(|\omega| = 1\), then \(\Re[\omega^2] = 2u^2 - 1\), so dividing inequality (3.4) by \(1 - u\), we get
\[
\frac{\Re[\alpha_{n-1}]}{n} + \frac{\Re[\alpha_{n-2}]}{n - 1}(2 + 2u) \leq \frac{1 - \beta}{1 - u} + O(1 - \beta)^2
\]
for each \(\omega \neq 1\). In particular, inequality (3.5) holds for \(u = u_1\) and \(u = u_2\) as defined in Theorem 1.

Applying the linear transformation \(T\) defined in equation (2.3) to inequality (3.5), and using the values computed in (2.4), we see that
\[
\Re[\alpha_{n-1}] + \Re[\alpha_{n-2}] \leq (n + nD_1 + D_2)(1 - \beta) + O(1 - \beta)^2.
\]

Recall that \(P(z) = \prod_{j=1}^n(z - \zeta_j) = z^n + a_{n-1}z^{n-1} + \cdots + a_0\), that each \(a_{n-k} = O(1 - \beta)\) and that \(\Re[\alpha_{n-3}] = O(1 - \beta)^2\). Then
\[
|P|^{2n} = (\min_j |\beta - \zeta_j|)^{2n} \leq \prod_{j=1}^n |\beta - \zeta_j|^2 = |P'(\beta)|^2
\]

\[
= P'(\beta)\overline{P'(\beta)} = \beta^{2n} + 2\Re[\alpha_{n-1}]\beta^{2n-1} + 2\Re[\alpha_{n-2}]\beta^{2n-2} + O(1 - \beta)^2
\]

\[
= 1 - 2n(1 - \beta) + 2\Re[\alpha_{n-1}] + 2\Re[\alpha_{n-2}] + O(1 - \beta)^2
\]

\[
= [1 - (1 - \beta) + (\Re[\alpha_{n-1}] + \Re[\alpha_{n-2}])/n]^2 + O(1 - \beta)^2
\]

and so using inequalities (3.7) and then (3.6) we have
\[
|P| \leq 1 - (1 - \beta) + (\Re[a_{n-1}] + \Re[a_{n-2}])/(n + O(1 - \beta)^2)
\]

\[
\leq 1 + (D_1 + D_2/n)(1 - \beta) + O(1 - \beta)^2.
\]

This completes our first step.
Our second step will be to verify the hypotheses of part 2 of Proposition 10 by showing that
\[ a_{n-1} = n(1 + D_1 + D_2)(1 - \beta) + \mathcal{O}(1 - \beta)^\alpha, \]
\[ a_{n-2} = -(n - 1)D_2(1 - \beta) + \mathcal{O}(1 - \beta)^\alpha, \]
and
\[ a_{n-k} = \mathcal{O}(1 - \beta)^\alpha \text{ for } k \geq 3. \]

Combining inequalities (3.1) and (3.8), we see that
\[ |P|_\beta = 1 + (D_1 + D_2/n)(1 - \beta) + \mathcal{O}(1 - \beta)^2. \]

Since equation (3.7) is an equality, we have each \( u \in \mathbb{R} \) so equation (3.10) implies that
\[ |P|_\beta = 1 + D_2(1 - \beta) + \mathcal{O}(1 - \beta)^2. \]

Note that from Proposition 10 we have that \( \mathbb{R}[a_{n-k}] = \mathcal{O}(1 - \beta)^2 \) for \( k \geq 3 \), so we now have the correct real parts for our second step. Thus we need only show that each \( \mathbb{R}[a_{n-k}] = \mathcal{O}(1 - \beta)^\alpha \).

Recalling the definitions of \( u_1 \) and \( u_2 \) in Theorem 1, we can choose \( \omega_1 \) and \( \omega_2 \) to be \((n + 1)\text{st roots of }1\) so that \( \mathbb{R}[\omega_i] = u_i \). For \( \omega = \omega_1 \), expanding the products in equality (3.3) and cancelling those terms of equality (3.4) gives us
\[ (3.10) \quad \frac{\mathbb{R}[a_{n-1}]}{\mathbb{R}[\omega_1]} + \frac{\mathbb{R}[a_{n-2}]}{\mathbb{R}[\omega_1]} = \mathbb{R}[a_{n-3}]/2 - \mathbb{R}[\omega_1] = \mathcal{O}(1 - \beta)^2. \]

Consider the case \( i = 1 \). Since \( |\omega_i| = 1 \) and since by part 1 of Lemma 8 we have \( -1/<u_1 < 1 \), then \( \mathbb{R}[\omega_i] \neq 0 \). Now by Proposition 7 \( \mathbb{R}[a_{n-k}] = \mathcal{O}(1 - \beta)^3/2 \) for \( k \geq 2 \), so equation (3.10) implies that \( \mathbb{R}[a_{n-1}] = \mathcal{O}(1 - \beta)^3/2 \). If \( n = 3 \) or \( n = 5 \), then by definition \( \alpha = 3/2 \), so this completes our second step for those two values of \( n \).

Assume then without loss of generality that \( n \neq 3, 5 \). Again by part 1 of Lemma 8 we have \(-1 < u_2 < u_1 < 1 \) so \( \mathbb{R}[\omega_i] = 0 \). Thus we may divide equation (3.10) by \( \mathbb{R}[\omega_i] \) to obtain
\[ (3.11) \quad \frac{3[a_{n-1}]}{n} + \frac{3[a_{n-2}]}{n-1} (2u_1) + \frac{3[a_{n-3}]}{n-2} (4u_1^2 - 1) = \mathcal{O}(1 - \beta)^2. \]

Now subtracting equality (3.11) with \( i = 2 \) from equality (3.11) with \( i = 1 \) and dividing by \( 2(u_1 - u_2) \) produces
\[ (3.12) \quad \frac{3[a_{n-2}]}{n-1} + \frac{3[a_{n-3}]}{n-2} (2u_1 + u_2) = \mathcal{O}(1 - \beta)^2. \]

Since equation (3.7) is an equality, we have each \( |\beta - \zeta_i| = |P|_\beta + \mathcal{O}(1 - \beta)^2 \). Recall that \( \mathbb{R}[a_{n-1}] = \mathcal{O}(1 - \beta)^3/2 \) and that \( |P|_\beta = 1 + (D_1 + D_2/n)(1 - \beta) + \mathcal{O}(1 - \beta)^2 \).
\( \mathcal{O}(1-\beta)^2 \). Then by part 1 of Lemma 13 we have 
\[ \Im[a_{n-3}] = (-3/2)\Im[a_{n-3}] + \mathcal{O}(1-\beta)^{5/2}, \]
so substituting into (3.12) we have
\[ \Im[a_{n-3}] \left[ \frac{-3}{n-1} + \frac{2(u_1 + u_2)}{n-2} \right] = \mathcal{O}(1-\beta)^2. \]

Now by part 2 of Lemma 3 we have \( u_1 + u_2 < 0 \) so the quantity in brackets is non-zero. Then \( \Im[a_{n-3}] = \mathcal{O}(1-\beta)^2 \), and so solving back in equations (3.12) and (3.11) we find that \( \Im[a_{n-k}] = \mathcal{O}(1-\beta)^2 \) for all \( k \leq 3 \). Note that by Proposition 9 we have \( a_{n-k} = \mathcal{O}(1-\beta)^2 \) for all \( k \geq 4 \), and so \( \Im[a_{n-k}] = \mathcal{O}(1-\beta)^2 \) for all \( k \). Since \( n \neq 3, 5 \), then by definition \( a = 2 \), and so this finishes the proof of our second step.

We will now finish the proof of Proposition 5. Consider only those roots \( z \) of \( P \) such that the nearest \( \omega \) has \( \Re[\omega] = u_i \). In our second step, we verified the hypotheses of part 2 of Proposition 10 so we have
\[ |z|^{2^{n+2}} = 1 - 2(n+1)\Re[R] + (n+1)(\Gamma_1 + \Gamma_2 u_i)(1-\beta)^2 + \mathcal{O}(1-\beta)^{n+1}. \]

Since \( |z| \leq 1 \), this implies that
\[ -\Re[R] \leq -\frac{\Gamma_1 + \Gamma_2 u_i}{2}(1-\beta)^2 + \mathcal{O}(1-\beta)^{n+1} \]
and so from the definition of \( R \) in Proposition 10 we have
\[ -\Re \left[ a_{n-1} \frac{\omega - 1}{n} + a_{n-2} \frac{\omega^2 - 1}{n-1} + \cdots + a_0 (\omega - 1) \right] \leq (1-\beta) - \frac{\Gamma_1 + \Gamma_2 u_i}{2}(1-\beta)^2 + \mathcal{O}(1-\beta)^{n+1}. \]

Since \( \Re(\omega) = u_i \), this inequality is also valid when \( \omega \) is replaced by \( \overline{\omega} \). Note that by Proposition 9 we have \( \Re[a_{n-k}] = \mathcal{O}(1-\beta)^3 \) for \( k \geq 5 \), so averaging these two inequalities gives us
\[
\tag{3.13} \frac{\Re[a_{n-1}]}{n} \Re[1 - \omega] + \cdots + \frac{\Re[a_{n-4}]}{n-3} \Re[1 - \omega^4] \leq (1-\beta) - \frac{\Gamma_1 + \Gamma_2 u_i}{2}(1-\beta)^2 + \mathcal{O}(1-\beta)^{n+1}.
\]

Note that since \( |\omega| = 1 \), then \( \Re[\omega^2] = 2u_i^2 - 1 \), \( \Re[\omega^3] = 4u_i^3 - 3u_i \) and \( \Re[\omega^4] = 8u_i^4 - 8u_i^2 + 1 \). Dividing inequality (3.13) by \( 1-u_i \), we get
\[
\frac{\Re[a_{n-1}]}{n} + \frac{\Re[a_{n-2}]}{n-1}(2+2u_i) + \frac{\Re[a_{n-3}]}{n-2}(1+4u_i + 4u_i^2) + \frac{\Re[a_{n-4}]}{n-3}(8u_i^2 + 8u_i^3) \leq \frac{1-\beta}{1-u_i} - \frac{(\Gamma_1 + \Gamma_2 u_i)(1-\beta)^2}{2(1-u_i)} + \mathcal{O}(1-\beta)^{n+1}.
\]

Applying to this the linear transformation \( \mathcal{T} \) defined in (2.6) and using the values computed in (2.11), we get an inequality of the form
\[
\Re[a_{n-1}] + \mathcal{O}(a_{n-2}) + c_3 \Re[a_{n-3}] + c_4 \Re[a_{n-4}] \leq (n + nD_1 + D_2)(1-\beta)
\]
\[
-\left[ (\Gamma_1/2)(n + nD_1 + D_2) + (\Gamma_2/2)(nD_1 + D_2) \right] (1-\beta)^2 + \mathcal{O}(1-\beta)^{n+1},
\]
where \( c_3 = \mathcal{T}(1+4u+4u^2)/(n-2) \) and \( c_4 = \mathcal{T}(8u^2+8u^3)/(n-3) \).
Define

$$Q = (-\Gamma_1 / 2)(n + nD_1 + D_2) - (\Gamma_2 / 2)(nD_1 + D_2) - (n - 1)(n - 2)(1 - c_3)D_2(1 + D_1 + D_2 / n).$$

Recall from our second step that for all $n$ we have that $\Im[a_{n-1}] = \mathcal{O}(1 - \beta)^{3/2}$, and that $\Re[a_{n-2}] = -(n - 1)D_2(1 - \beta) + \mathcal{O}(1 - \beta)^2$, and that each $|\zeta_j - \beta| = 1 + (D_1 + D_2 / n)(1 - \beta) + \mathcal{O}(1 - \beta)^2$. Then by part 2 of Lemma $13$ we have

$$\Re[a_{n-3}] + 2\Re[a_{n-4}] = -(n - 1)(n - 2)D_2(1 + D_1 + D_2 / n)(1 - \beta)^2 + \mathcal{O}(1 - \beta)^3.$$

Adding $1 - c_3$ times this to inequality (3.14) gives us

$$\Re[a_{n-1}] + \Re[a_{n-2}] + \Re[a_{n-3}] + (2 - 2c_3 + c_4)\Re[a_{n-4}] \leq (n + nD_1 + D_2)(1 - \beta) + Q(1 - \beta)^2 + \mathcal{O}(1 - \beta)^{\alpha + 1}.$$

Note that Lemma $11$ implies that $c_3 < 1 / 2$ for $n \neq 3, 4, 6$ and that $c_4 \geq 0$ for all $n$. Using the definition of $T$ in (3.5), we calculate that for $n = 4$ we have $c_3 = 3 / 2$ and $c_4 = 4$, and for $n = 6$ we have $c_3 = 0.729$ and $c_4 = 0.972$. Thus for all $n \geq 4$ we have $1 - 2c_3 + c_4 > 0$. Note also that by our second step and Lemma $13$ we have $\Re[a_{n-4}] \geq \delta(1 - \beta)^2 + \mathcal{O}(1 - \beta)^{\alpha + 1}$. Since $\delta = 0$ except when $n = 5$, and for $n = 5$ we calculate $c_3 = 1 / 3$ and $c_4 = 2$, then

$$-(1 - 2c_3 + c_4)\Re[a_{n-4}] \leq -(1 - 2c_3 + c_4)\delta(1 - \beta)^2 + \mathcal{O}(1 - \beta)^{\alpha + 1} = (-7\delta / 3)(1 - \beta)^2 + \mathcal{O}(1 - \beta)^{\alpha + 1}.$$

Adding this to equation (3.16) gives us

$$\Re[a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}] \leq (n + nD_1 + D_2)(1 - \beta) + (Q - 7\delta / 3)(1 - \beta)^2 + \mathcal{O}(1 - \beta)^{\alpha + 1}.$$

Let

$$Q_1 = -n(1 - \beta) + a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4} + a_{n-5}$$
$$Q_2 = n(n - 1)(1 - \beta)^2 / 2 - [(n - 1)a_{n-1} + (n - 2)a_{n-2}](1 - \beta).$$

Recall from our first step that each $a_{n-k} = \mathcal{O}(1 - \beta)$ so $Q_1 = \mathcal{O}(1 - \beta)$ and $Q_2 = \mathcal{O}(1 - \beta)^2$.

Now from our second step we know that $a_{n-k} = \mathcal{O}(1 - \beta)^\alpha$ for $k \geq 3$, and from Proposition $9$ we know that $a_{n-k} = \mathcal{O}(1 - \beta)^3$ for $k \geq 6$, so

$$P'(\beta) = \beta^n + a_{n-1} \beta^{n-1} + \cdots + a_0$$
$$= 1 - n(1 - \beta) + \frac{n(n - 1)}{2}(1 - \beta)^2 + a_{n-1}[1 - (n - 1)(1 - \beta)]$$
$$+ a_{n-2}[1 - (n - 2)(1 - \beta)] + a_{n-3} + a_{n-4} + a_{n-5} + \mathcal{O}(1 - \beta)^{\alpha + 1}$$
$$= 1 + Q_1 + Q_2 + \mathcal{O}(1 - \beta)^{\alpha + 1}.$$

Then $|P'(\beta)|^2 = P'(\beta)P'(\beta) = 1 + 2\Re(Q_1) + 2\Re(Q_2) + |Q_1|^2 + \mathcal{O}(1 - \beta)^{\alpha + 1}$. Note from our second step that each $\Im[a_{n-k}] = \mathcal{O}(1 - \beta)^\alpha$ so $\Im(Q_1) = \mathcal{O}(1 - \beta)^\alpha$. Then

$$(1 + \Re(Q_1) + \Re(Q_2))^2 = |P'(\beta)|^2 + \mathcal{O}(1 - \beta)^{\alpha + 1}$$
and so $|P'(\beta)| = 1 + \Re(Q_1) + \Re(Q_2) +$
\( O(1 - \beta)^{\alpha + 1} \). Substituting the values of \( Q_1 \) and \( Q_2 \) and using the results of our second step gives us

\[
|P'(\beta)| = 1 - n(1 - \beta) + \Re \{a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4} \} \\
+ (n - 1) [n/2 - n(1 + D_1 + D_2) + (n - 2)D_2] (1 - \beta)^2 \\
+ O(1 - \beta)^{\alpha + 1}.
\] (3.18)

Using the first line of inequality (3.14), then inequalities (3.18) and (3.17), we have

\[
|P|_\beta^n \leq |P'(\beta)| \\
\leq 1 + (nD_1 + D_2)(1 - \beta) \\
+ [Q - 7\delta/3 - (n - 1)(n/2 + nD_1 + 2D_2)] (1 - \beta)^2 \\
+ O(1 - \beta)^{\alpha + 1}.
\] (3.19)

We now seek to compute the coefficient of \((1 - \beta)^2\) in this inequality. Note first that from the definitions of \( \Gamma_1 \) and \( \Gamma_2 \) in Proposition 10, we have

\[
-\frac{\Gamma_1}{2} (n + nD_1 + D_2) - \frac{\Gamma_2}{2} (nD_1 + D_2) \\
= -\frac{\Gamma_1 + \Gamma_2}{2} (n + nD_1 + D_2) + \frac{n\Gamma_2}{2} \\
= \left[ (1 + 2D_1)n - \left( \frac{1}{2} + 2D_1 - 2D_2 \right) \right] (1 + D_1)n + D_2 \\
+ n(1 + D_1 + D_2)[nD_2 + (D_1 - 2D_2)].
\]

Now from the definition of \( c_3 \) (after inequality (3.14)) combined with equalities (2.4) we have \((n - 2)c_3D_2 = -(n + 1 + D_1 + 3nD_1 + 3D_2)\) and so

\((n - 2)(1 - c_3)D_2 = (1 + 3D_1 + D_2)n + (1 + D_1 + D_2).\)

Substituting these values into equation (3.16) and collecting like powers of \( n \), we conclude that

\[
Q = \left[ -D_1 - D_2 \right] n^2 + \left[ -\frac{1}{2} + \frac{1}{2}D_1 + D_1^2 - 3D_2^2 \right] n \\
+ \left[ 1 + 2D_1 + \frac{1}{2}D_2 + D_1D_2 + 2D_2^2 \right] + [D_2 + D_1D_2 + D_2^2]/n
\]

and so comparing this with the definition of \( D \) in Theorem 1, we see that

\[
(3.21) \quad Q - (n - 1)(n/2 + nD_1 + 2D_2) = nD + \frac{n(n - 1)}{2}(D_1 + D_2/n)^2.
\]

Substituting this into inequality (3.19), we have

\[
|P|_\beta^n \leq 1 + (nD_1 + D_2)(1 - \beta) \\
+ \left[ nD + \frac{n(n - 1)}{2}(D_1 + D_2/n)^2 - 7\delta/3 \right] (1 - \beta)^2 + O(1 - \beta)^{\alpha + 1} \\
= \left[ 1 + (D_1 + D_2/n)(1 - \beta) + \left( D - \frac{7\delta}{3n} \right) (1 - \beta)^2 \right] n + O(1 - \beta)^{\alpha + 1}.
\]
Note that (from the definitions of \( \delta \) in Lemma \[4\] and \( \Delta \) in Theorem \[4\]) for all \( n \) we have \( \Delta = -76\delta/(3n) \), and so

\[
|P|_\beta \leq 1 + (D_1 + D_2/n)(1 - \beta) + (D + \Delta)(1 - \beta)^2 + O(1 - \beta)^{\alpha+1}.
\]

This completes the proof of Proposition \[5\].

4. Proof of Proposition \[5\]

This proof parallels the proof of \[8\], Theorem 2. We begin by letting

\[
u = \frac{-i\sqrt{15}}{15}(1 - \beta)^{1/2} - \frac{6}{10}(1 - \beta) + \frac{i\sqrt{15}}{300}(1 - \beta)^{3/2} - \frac{33}{600}(1 - \beta)^2
\]

and

\[
v = \frac{4i\sqrt{15}}{15}(1 - \beta)^{1/2} - \frac{1}{10}(1 - \beta) + \frac{46i\sqrt{15}}{300}(1 - \beta)^{3/2} + \frac{532}{600}(1 - \beta)^2.
\]

Let \( P'(z) = (z - u)^4(z - v) \) and let \( P(z) = \int_{\beta}^z P'(t) \, dt \). Note that \( u - \beta = -1 + u + (1 - \beta) \) so

\[
|u - \beta|^2 = \left[ -1 + (4/10)(1 - \beta) - (33/600)(1 - \beta)^2 \right]^2
\]

\[
+ \left[ (-\sqrt{15}/15)(1 - \beta)^{1/2} + (\sqrt{15}/300)(1 - \beta)^{3/2} \right]^2
\]

\[
= 1 - (11/15)(1 - \beta) + (79/300)(1 - \beta)^2 + O(1 - \beta)^3
\]

and \( v - \beta = -1 + v + (1 - \beta) \) so

\[
|v - \beta|^2 = \left[ -1 + (9/10)(1 - \beta) + (532/600)(1 - \beta)^2 \right]^2
\]

\[
+ \left[ (4\sqrt{15}/15)(1 - \beta)^{1/2} + (46\sqrt{15}/300)(1 - \beta)^{3/2} \right]^2
\]

\[
= 1 - (11/15)(1 - \beta) + (79/300)(1 - \beta)^2 + O(1 - \beta)^3.
\]

Now

\[
\left[ 1 - (11/30)(1 - \beta) + (29/450)(1 - \beta)^2 \right]^2
\]

\[
= 1 - (11/15)(1 - \beta) + (79/300)(1 - \beta)^2 + O(1 - \beta)^3,
\]

and so we have

\[
|P|_\beta = \min\{|u - \beta|, |v - \beta|\}
\]

\[
= 1 - (11/30)(1 - \beta) + (29/450)(1 - \beta)^2 + O(1 - \beta)^3.
\]

By definition \( P \) is of degree 6 and \( P(\beta) = 0 \). Thus to verify that \( P \in S(6, \beta) \) we need only show that all the roots of \( P \) remain in the closed unit disk when \( \beta \) is sufficiently close to 1. Now

\[
u^2 = (-1/15)(1 - \beta) + (2i\sqrt{15}/25)(1 - \beta)^{3/2} + O(1 - \beta)^2,
\]

\[
u^3 = (i\sqrt{15}/225)(1 - \beta)^{3/2} + O(1 - \beta)^2,
\]

and

\[
u^4 = O(1 - \beta)^2,
\]
A QUADRATIC APPROXIMATION TO THE SENDOV RADIUS 869

so writing $P'(z) = z^5 + a_4 z^4 + \cdots + a_0$, we calculate that

$$a_4 = -(4u + v) = (5/2)(1 - \beta) - (i\sqrt{15}/6)(1 - \beta)^{3/2} - (2/3)(1 - \beta)^2$$

$$a_3 = u(6u + 4v)$$

$$= (2/3)(1 - \beta) - (2i\sqrt{15}/15)(1 - \beta)^{3/2} + 3(1 - \beta)^2 + O(1 - \beta)^{5/2}$$

$$a_2 = -u^2(4u + 6v) = (4i\sqrt{15}/45)(1 - \beta)^{3/2} + (7/5)(1 - \beta)^2 + O(1 - \beta)^{5/2}$$

$$a_1 = u^3(u + 4v) = (-1/15)(1 - \beta)^2 + O(1 - \beta)^{5/2}$$

$$a_0 = -u^4v = O(1 - \beta)^{5/2}.$$

Recall from the values computed at the beginning of section 2 that for $n = 5$ we have $\alpha = 3/2, u_1 = -1/2, u_2 = -1, D_1 = -1/3$ and $D_2 = -1/6$. Note that in part 2 of Proposition 10 the values of the $a_k$’s computed above satisfy the hypotheses, and that $\Gamma_2 = -5/6$ and $\Gamma_1 = -13/6$.

Let us apply part 2 of Proposition 10 to the case $\omega = -1$. Note that $\Re[\omega] = u_2$ and $\Gamma_1 + \Gamma_2 u_2 = -4/3$. Since $\omega = -1$ we have

$$R = (1 - \beta) - (2/5)a_4 - (2/3)a_2 - 2a_0,$$

and so

$$\Re[R] = (1 - \beta) - (2/5)\left[(5/2)(1 - \beta) - (2/3)(1 - \beta)^2\right]$$

$$= (2/3)(7/5)(1 - \beta)^2 + O(1 - \beta)^{5/2}$$

$$= (-2/3)(1 - \beta)^2 + O(1 - \beta)^{5/2}.$$ 

Thus by part 2 of Proposition 10 we have

$$|z|^2 = 1 - 12(-2/3)(1 - \beta)^2 + 6(-4/3)(1 - \beta)^2 + O(1 - \beta)^{5/2}$$

$$= 1 + O(1 - \beta)^{5/2},$$

and so $|z| = 1 + O(1 - \beta)^{5/2}$.

Let us now apply part 2 of Proposition 10 to the case $\omega = (1/2)(-1 \pm i\sqrt{3})$. Note that $\Re[\omega] = u_1$ and $\Gamma_1 + \Gamma_2 u_1 = -7/4$. Now

$$R = (1 - \beta) + (a_4/10)(-3 \mp i\sqrt{3}) + (a_3/8)(-3 \mp i\sqrt{3})$$

$$+ (a_1/4)(-3 \mp i\sqrt{3}) + (a_0/2)(-3 \mp i\sqrt{3})$$

so

$$\Re[R] = (1 - \beta) - (3/10)\left[(5/2)(1 - \beta) - (2/3)(1 - \beta)^2\right]$$

$$\pm (\sqrt{3}/10)(-\sqrt{15}/6)(1 - \beta)^{3/2} - (3/8)\left[(2/3)(1 - \beta) + 3(1 - \beta)^2\right]$$

$$\pm (\sqrt{3}/8)(-2\sqrt{15}/15)(1 - \beta)^{3/2} - (3/4)(-1/15)(1 - \beta)^2 + O(1 - \beta)^{5/2}$$

$$= (-7/8)(1 - \beta)^2 + O(1 - \beta)^{5/2}.$$ 

Thus by part 2 of Proposition 10 we have

$$|z|^2 = 1 - 12(-7/8)(1 - \beta)^2 + 6(-7/4)(1 - \beta)^2 + O(1 - \beta)^{5/2}$$

$$= 1 + O(1 - \beta)^{5/2},$$

so $|z| = 1 + O(1 - \beta)^{5/2}$.
Finally, let us apply part 1 of Proposition 11 with \( r = 1 \) to the case \( \omega = (1/2)(1 + i\sqrt{3}) \). Note that

\[
R = (1 - \beta) + (a_4/10)(-1 \mp i\sqrt{3}) + (a_3/8)(-3 \mp i\sqrt{3}) + \mathcal{O}(1 - \beta)^{3/2}
\]

so

\[
\Re[R] = (1 - \beta) + (-1/10)(5/2)(1 - \beta) + (-3/8)(2/3)(1 - \beta) + \mathcal{O}(1 - \beta)^{3/2}
\]

\[
= (1/2)(1 - \beta) + \mathcal{O}(1 - \beta)^{3/2}.
\]

Thus by part 1 of Proposition 11 we have \(|z|^2 = 1 - (1 - \beta) + \mathcal{O}(1 - \beta)^{3/2} \) and so \(|z| = 1 - (1/2)(1 - \beta) + \mathcal{O}(1 - \beta)^{3/2} \).

At this stage, we know that \(|P|_\beta = 1 - (11/30)(1 - \beta) + (29/450)(1 - \beta)^2 + \mathcal{O}(1 - \beta)^3 \) and that if \( \beta \) is sufficiently close to 1, then all roots \( z \) of \( P \) have \(|z| \leq 1 + \mathcal{O}(1 - \beta)^{3/2} \). Since the roots of \( P \) approach the roots of \( z^n - 1 \), then the non-\( \beta \) roots of \( P \) are bounded away from \( \beta \). Thus by Lemma 12 there is a polynomial \( Q \in S(6, \beta) \) with \(|Q|_\beta = 1 - (11/30)(1 - \beta) + (29/450)(1 - \beta)^2 + \mathcal{O}(1 - \beta)^{3/2} \). This completes the proof of Proposition 11.

5. Proof of Proposition 7

Let \( b_1 = 1 + D_1 + D_2/n \), let \( b_2 = (n-1)D_2 \), and let \( z_0 = -b_1(1 - \beta) - D(1 - \beta)^2 \).

Then \( z_0 - \beta = -1 + (1 - b_1)(1 - \beta) - D(1 - \beta)^2 \), and (for \( \beta \) near 1) this is real and negative so \(|z_0 - \beta| = 1 + (D_1 + D_2/n)(1 - \beta) + D(1 - \beta)^2 \).

Now let \( x \) be a real constant, depending only on \( n \) (and to be determined later), and let

\[
q(z) = z^2 + [(b_2 + 2b_1)(1 - \beta) - 2x(1 - \beta)^2]z
+ [-b_2(1 - \beta) + (b_1^2 + b_2 + 2D + 2x)(1 - \beta)^2].
\]

Now by part 4 of Lemma 8 we have \( D_2 < 0 \) and so \( b_2 < 0 \). Since the discriminant of \( q(z) \) is \( 4b_2(1 - \beta) + \mathcal{O}(1 - \beta)^2 \), then (for \( \beta \) near 1) the roots of \( q \) are complex conjugates. If we denote these roots by \( z_1 \) and \( \overline{z}_1 \), then by writing \( \beta = 1 - (1 - \beta) \) we have

\[
|z_1 - \beta|^2 = (z_1 - \beta)\overline{(z_1 - \beta)} = q(\beta)
= 1 + (2b_1 - 1)(1 - \beta) + (1 - 2b_1 + b_1^2 + 2D)(1 - \beta)^2 + \mathcal{O}(1 - \beta)^3
\]

\[
= [1 + (b_1 - 1)(1 - \beta) + D(1 - \beta)^2]^2 + \mathcal{O}(1 - \beta)^3,
\]

so \(|z_1 - \beta| = 1 + (D_1 + D_2/n)(1 - \beta) + D(1 - \beta)^2 + \mathcal{O}(1 - \beta)^3 \).

Let \( P'(z) = (z - z_0)^{n-2}q(z) \) and \( P(z) = \int_P P'(t) \, dt \), so

\[
|P|_\beta = 1 + (D_1 + D_2/n)(1 - \beta) + D(1 - \beta)^2 + \mathcal{O}(1 - \beta)^3.
\]

Now \( z_0 = \mathcal{O}(1 - \beta) \), so

\[
(z - z_0)^{n-2} = z^{n-2} - (n-2)z_0z^{n-3} + \binom{n-2}{2}z_0^2z^{n-4} + \mathcal{O}(1 - \beta)^3
\]

\[
= z^{n-2} + (n-2)[b_1(1 - \beta) + D(1 - \beta)^2]z^{n-3}
+ \binom{n-2}{2}b_1^2(1 - \beta)^2z^{n-4} + \mathcal{O}(1 - \beta)^3.
\]
Then letting $t_1 = (n^2 - n)b_1^2/2 + (n - 2)b_1b_2 + b_2$ we have

$$P'(z) = (z - z_0)^{n-2}q(z)$$

$$= z^n + [(nb_1 + b_2)(1 - \beta) + (nD - 2D - 2x)(1 - \beta^2)]z^{n-1}$$

$$+ [-b_1(1 - \beta) + (t_1 + 2D + 2x)(1 - \beta^2)]z^{n-2}$$

$$- (n - 2)b_1b_2(1 - \beta^2)z^{n-3} + \mathcal{O}(1 - \beta^3).$$

(5.1)

Note that by its definition, $P$ is a polynomial of degree $n + 1$ and $P(\beta) = 0$. Thus to show that $P \in S(n + 1, \beta)$ it will suffice to show that all roots of $P$ remain in the unit disk when $\beta$ is sufficiently close to 1.

Let $\omega \neq 1$ be an $(n + 1)$th root of 1, let $u = \Re[\omega]$ and note that since $|\omega| = 1$, then $\Re[\omega^2] = 2u^2 - 1$, $\Re[\omega^3] = 4u^3 - 3u$, and $\omega^{n-k} = \overline{\omega}^{k+1}$. Substituting the coefficients of equation (5.1) into the formula for $R$ in Proposition 10 we have

$$R = (1 - \beta) + (nb_1 + b_2)(1 - \beta)(\overline{\omega} - 1)/n$$

$$- b_2(1 - \beta)(\overline{\omega}^2 - 1)/(n - 1) + \mathcal{O}(1 - \beta)^2.$$

Substituting the values of $b_1$ and $b_2$ into this formula, we see by part 1 of Proposition 10 with $r = 1$ that

$$|z|^2 = 1 - 2(1 - \beta)[1 + (1 + D_1 + D_2)(u - 1) - D_2(2u^2 - 2)] + \mathcal{O}(1 - \beta)^2.$$

Recall from part 4 of Lemma 8 that $D_2 < 0$, so the quantity in square brackets is quadratic in $u$ with positive leading coefficient. By elementary calculus, its minimum (over all real numbers) occurs when $1 + D_1 + D_2 - 4D_2u = 0$, which happens when $u = (1 + D_1 + D_2)/(4D_2) = (u_1 + u_2)/2$, which is between $u_1$ and $u_2$. Now $u_1$ and $u_2$ are (by definition) the real parts of adjacent $(n + 1)$th roots of 1, so there are no possible values of $u$ between $u_1$ and $u_2$, so the minimum (over all possible values of $u$) must occur at either $u_1$ or $u_2$. From part 7 of Lemma 8 we see that at these values the quantity in square brackets is 0, and so the minimum value of the quantity in square brackets is 0. Thus for $\Re[\omega] \neq u_i$ the quantity in square brackets is positive, so for these values of $\omega$ and for $\beta$ sufficiently close to 1 we have $|z| < 1$, and so these roots remain in the unit disk.

Thus we need only concern ourselves with the case $\Re[\omega] = u_i$. In this case, by part 2 of Proposition 10 we have

$$|z|^{2n+2} = 1 - 2(n + 1)\Re[R] + (n + 1)(\Gamma_1 + \Gamma_2u_i)(1 - \beta)^2 + \mathcal{O}(1 - \beta)^{n+1}.$$

To get $P \in S(n + 1, \beta)$ we will seek a value of $x$ so that $|z| = 1 + \mathcal{O}(1 - \beta)^{n+1}$, so we will need

$$\Re[R] - (1/2)(\Gamma_1 + \Gamma_2u_i)(1 - \beta)^2 = \mathcal{O}(1 - \beta)^{n+1}$$

(5.2)

for both $i = 1$ and $i = 2$.

Substituting the coefficients of equation (5.1) into the formula for $R$ in Proposition 10 we have

$$R = (1 - \beta) + [(nb_1 + b_2)(1 - \beta) + (nD - 2D - 2x)(1 - \beta^2)](\overline{\omega} - 1)/n$$

$$+ [-b_1(1 - \beta) + (t_1 + 2D + 2x)(1 - \beta^2)](\overline{\omega}^2 - 1)/(n - 1)$$

$$- (n - 2)b_1b_2(1 - \beta^2)(\overline{\omega}^3 - 1)/(n - 2) + \mathcal{O}(1 - \beta)^3.$$

(5.3)
Taking the real parts of equation (5.3) and collecting like powers of \((1 - \beta)\) gives us
\[
\Re[R] = \left[1 + (nb_1 + b_2)(u_i - 1)/n - b_2(2u_i^2 - 2)/(n - 1)\right](1 - \beta) + \left[(nD - 2D - 2x)(u_i - 1)/n + (t_1 + 2D + 2x)(2u_i^2 - 2)/(n - 1) - b_1b_2(4u_i^3 - 3u_i - 1)\right](1 - \beta)^2 + O(1 - \beta)^3.
\]

Substituting the values of \(b_1\) and \(b_2\) into this formula, we see from part 7 of Lemma 8 that the coefficient of \((1 - \beta)\) in \(\Re[R]\) is zero, so to satisfy equation (5.2) we need only find a value of \(x\) such that the coefficient of \((1 - \beta)^2\) in equation (5.2) is 0. We divide this coefficient by \(u_i - 1\) and denote the result by \(Z_i\), so
\[
(5.4) \quad Z_i = (nD - 2D - 2x)/n + (t_1 + 2D + 2x)(2u_i + 2)/(n - 1)
\]
\[
- (n - 1)D_2(1 + D_1 + D_2/n)(4u_i^2 + 4u_i + 1) + (1/2)(\Gamma_1 + \Gamma_2u_i)/(1 - u_i).
\]

Note that the coefficient of \(x\) in \(Z_i\) is\(-2/n + (4u_i + 4)/(n - 1)\), which is non-zero by part 3 of Lemma 8, so each equation \(Z_i = 0\) has a solution for \(x\). To show that these solutions are identical, we will show that \(Z_1\) and \(Z_2\) (considered as linear expressions in the variable \(x\)) are scalar multiples of each other.

To see this, we eliminate \(x\) by applying the transformation \(T\) defined in equation (2.3). Since in equation (3.11) we defined \(c_3 = T(1 + 4u + 4u^2)/(n - 2)\), then from equations (2.4), we see that
\[
(5.5) \quad T(Z_i) = nD + t_1 - (n - 1)(n - 2)c_3D_2(1 + D_1 + D_2/n)
\]
\[
+ (\Gamma_1/2)(n + nD_1 + D_2) + (\Gamma_2/2)(nD_1 + D_2).
\]

Comparing this to the value of \(Q\) defined in equation (3.14), we see that
\[
(5.6) \quad T(Z_i) = nD + t_1 - Q - (n - 1(n - 2)D_2(1 + D_1 + D_2/n).
\]

Note that by equation (3.21) we have
\[
Q = nD + \frac{n(n - 1)}{2}(D_1 + D_2/n)^2 + (n - 1)(n/2 + nD_1 + 2D_2).
\]

Substituting the values of \(b_1\) and \(b_2\) into our definition of \(t_1\) gives us
\[
t_1 = (n - 1)\left[(n/2)(1 + D_1 + D_2/n)^2 + (n - 2)D_2(1 + D_1 + D_2/n) + D_2\right]
\]
and so \(Q - t_1 = nD - (n - 1)(n - 2)D_2(1 + D_1 + D_2/n)\). Substituting this into equation (5.6) gives us \(T(Z_i) = 0\). Since \(T(Z_i)\) is a linear combination of \(Z_1\) and \(Z_2\), this implies that \(Z_1\) and \(Z_2\) (considered as polynomials in \(x\)) are scalar multiples of one another, and so there is a single value of \(x\) that satisfies equation (5.2) for both \(i = 1\) and \(i = 2\).

Using this value of \(x\), we have now constructed a real polynomial \(P\) with
\[
|P|_\beta = 1 + (D_1 + D_2/n)(1 - \beta) + D(1 - \beta)^2 + O(1 - \beta)^3
\]
and such that all roots \(z\) of \(P\) have \(|z| \leq 1 + O(1 - \beta)^{n+1}\). Since the roots of \(P\) approach the roots of \(z^{n+1} - 1\), then the non-\(\beta\) roots of \(P\) are bounded away from \(\beta\). Thus by Lemma 12, there is a real polynomial \(Q \in S(n + 1, \beta)\) with
\[
|Q|_\beta = 1 + (D_1 + D_2/n)(1 - \beta) + D(1 - \beta)^2 + O(1 - \beta)^{n+1}.
\]
This finishes the proof of Proposition 7.
A QUADRATIC APPROXIMATION TO THE SENDOV RADIUS

REFERENCES


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