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A QUADRATIC APPROXIMATION TO THE SENDOV RADIUS NEAR THE UNIT CIRCLE

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ABSTRACT. Define $S(n, \beta)$ to be the set of complex polynomials of degree $n \geq 2$ with all roots in the unit disk and at least one root at β. For a polynomial P, define $|P|_\beta$ to be the distance between β and the closest root of the derivative P'. Finally, define $r_n(\beta) = \sup\{|P|_\beta : P \in S(n, \beta)\}$. In this notation, a conjecture of Bl. Sendov claims that $r_n(\beta) \leq 1$.

In this paper we investigate Sendov's conjecture near the unit circle, by computing constants C_1 and C_2 (depending only on n) such that $r_n(\beta) \sim 1 + C_1(1 - |\beta|) + C_2(1 - |\beta|)^2$ for $|\beta|$ near 1. We also consider some consequences of this approximation, including a hint of where one might look for a counterexample to Sendov's conjecture.

1. INTRODUCTION

In 1962, Sendov conjectured that if a polynomial (with complex coefficients) has all its roots in the unit disk, then within one unit of each of its roots lies a root of its derivative. More than 50 papers have been published on this conjecture, but it has been verified in general only for polynomials of degree at most 8 [4].

Let $n \geq 2$ be an integer and let β be a complex number of modulus at most 1. Define $S(n, \beta)$ to be the set of polynomials of degree n with complex coefficients, all roots in the unit disk and at least one root at β. For a polynomial P, define $|P|_\beta$ to be the distance between β and the closest root of the derivative P'. Finally, define $r_n(\beta) = \sup\{|P|_\beta : P \in S(n, \beta)\}$, and note that $r_n(\beta) \leq 2$ (since by the Gauss-Lucas Theorem [5, Theorem 6.1] all roots of each P' are also in the unit disk, and so each $|P|_\beta \leq 2$). In this notation, Sendov's conjecture claims that $r_n(\beta) \leq 1$.

In estimating $r_n(\beta)$, we will assume without loss of generality (by rotation) that $0 \leq \beta \leq 1$. It is already known that $r_2(\beta) = (1 + \beta)/2$ and that

$$r_3(\beta) = [3\beta + (12 - 3\beta^2)^{1/2}]/6$$

[9, Theorem 2], that $r_n(0) = (1/n)^{1/(n-1)}$ [2, Lemma 4 and $p(z) = z^n - z$], that $r_n(1) = 1$ [10, Theorem 1], and that $r_n(\beta) \leq \min(1.08332, 1 + 0.72054/n)$ [1, Corollary 1 and equation (3)].

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Since $r_n(1) = 1$, an obvious place to look for counterexamples to Sendov's conjecture is in a neighborhood of $\beta = 1$. This has already been done in [7, Theorem 3] and [12], where a linear upper bound on $r_n(\beta)$ suffices to verify the Sendov conjecture if β is sufficiently close to 1. Unfortunately, having only an upper bound leaves many interesting questions about the conjecture unanswered. In this paper we investigate Sendov's conjecture much more thoroughly near $\beta = 1$, by providing a quadratic approximation to $r_n(\beta)$ with

Theorem 1. *Let $n \geq 3$, let k be the largest integer such that $k \leq (n+1)/3$ and let*

$$\begin{aligned}
&u_1 = \cos\frac{2\pi k}{n+1}, \qquad u_2 = \cos\frac{2\pi(k+1)}{n+1},\\
&D_1 = \frac{-2u_1u_2 - 1}{2(1-u_1)(1-u_2)}, \qquad D_2 = \frac{-1}{2(1-u_1)(1-u_2)},\\
&D_3 = (-1 - 4D_1 - 3D_1^2 + 2D_2^2)/2,\\
&D_4 = (3D_1 - 4D_2 + 3D_1^2 - 2D_1D_2 - 6D_2^2)/2,\\
&D_5 = (2 + 4D_1 + 5D_2 + 2D_1^2 + 4D_1D_2 + 3D_2^2)/2,\\
&D_6 = (2D_2 + 2D_1D_2 + 3D_2^2)/2 \quad \textit{and}\\
&D = D_3 n + D_4 + D_5/n + D_6/n^2.
\end{aligned}$$

If $n = 3$ or $n = 5$, then let $\alpha = 3/2$; otherwise let $\alpha = 2$. If $n = 5$, then let $\Delta = 7/225$; otherwise let $\Delta = 0$. Then for β sufficiently close to 1, we have

$$r_{n+1}(\beta) = 1 + (D_1 + D_2/n)(1-\beta) + (D+\Delta)(1-\beta)^2 + \mathcal{O}(1-\beta)^{\alpha+1}.$$

Before proving this theorem, we will examine some of its consequences. Our first consequence improves on estimates in [7] and [12] (by providing a value for the coefficient of the linear term) with

Corollary 2. *For all $n \geq 2$ we have $r_n(\beta) \leq 1 - (3/10)(1-\beta) + \mathcal{O}(1-\beta)^2$.*

Proof. Recall that for $2 \leq n \leq 3$ we have formulas for $r_n(\beta)$, and so the result for those values of n follows from the Taylor series of these formulas at $\beta = 1$. As we will show in part 6 of Lemma 8, the quantity $D_1 + D_2/n \leq -3/10$ for all $n \geq 3$, and so the rest of Corollary 2 follows from Theorem 1. □

As we will show in part 6 of Lemma 8, at $n = 4$ we have $D_1 + D_2/n = -3/10$, so Corollary 2 provides the smallest possible linear upper bound for $r_n(\beta)$ that is independent of n.

A second consequence of Theorem 1 shows that the result of [7, Theorem 3] is the best possible (in the sense that 1/3 cannot be replaced by a larger number), with

Corollary 3. *There exist constants $K_n > 0$ with $\lim_{n\to\infty} K_n = 1/3$ such that*

$$r_{n+1}(\beta) = 1 - K_n(1-\beta) + \mathcal{O}(1-\beta)^2.$$

Proof. Choose $K_n = -(D_1 + D_2/n)$ and note that by Theorem 1 we have $r_{n+1}(\beta) = 1 - K_n(1-\beta) + \mathcal{O}(1-\beta)^2$. As we shall see in parts 5 and 6 of Lemma 8, the quantity $D_1 + D_2/n$ is negative and tends to $-1/3$. □

Recall that $r_n(0) = (1/n)^{1/(n-1)}$. This quantity is increasing in n, so it is tempting to conjecture that for all fixed β the quantity $r_n(\beta)$ is increasing in n. Indeed, the graphs in [6, figure 4.8] provide some evidence of this for $n = 4$, 6, 8, 10, and 12. Unfortunately, this conjecture is false, as is shown by

Corollary 4. *For β sufficiently close to* 1 *we have $r_6(\beta) < r_4(\beta)$.*

Proof. By Theorem 1 and the constants we will compute at the beginning of section 2 we know that

$$r_4(\beta) = 1 - (1/3)(1-\beta) + \mathcal{O}(1-\beta)^2$$

and that

$$r_6(\beta) = 1 - (11/30)(1-\beta) + \mathcal{O}(1-\beta)^2,$$

and the conclusion follows. □

Corollary 2 hints of the existence of a better-than-Sendov result, for near $\beta = 1$ it appears that $r_n(\beta)$ is bounded above by a function that is independent of n and strictly less than one. Unfortunately, moving up to the quadratic approximation in Theorem 1 casts doubt upon such a result. To see this, note that as n goes to infinity, then $k/(n+1)$ tends to $1/3$, so u_1 and u_2 tend to $-1/2$, so D_3 tends to $4/81$ and D_4 tends to $-1/9$, and so $D+\Delta$ tends to infinity. Indeed, for n sufficiently large one might expect $r_{n+1}(\beta) > 1$ roughly when $D_1(1-\beta) + (D_3 n + D_4)(1-\beta)^2 > 0$, i.e. when $\beta < 1 + D_1/(D_3 n + D_4) \sim 1 - 27/(4n-9)$, provided that this β is "sufficiently close to 1". This is an intriguing possibility that is clearly worthy of further investigation.

We will verify Theorem 1 by proving the following three propositions:

Proposition 5. *Assume the notation of Theorem 1. Then for all polynomials $P \in S(n+1,\beta)$, we have*

$$|P|_\beta \le 1 + (D_1 + D_2/n)(1-\beta) + (D+\Delta)(1-\beta)^2 + \mathcal{O}(1-\beta)^{\alpha+1}.$$

Proposition 6. *There are polynomials $P \in S(6,\beta)$ with*

$$|P|_\beta = 1 - (11/30)(1-\beta) + (29/450)(1-\beta)^2 + \mathcal{O}(1-\beta)^{5/2}.$$

Proposition 7. *Assume the notation of Theorem 1. Then there are real polynomials $P \in S(n+1,\beta)$ with*

$$|P|_\beta = 1 + (D_1 + D_2/n)(1-\beta) + D(1-\beta)^2 + \mathcal{O}(1-\beta)^{\alpha+1}.$$

From the definition of D in Theorem 1 and the constants we will compute at the beginning of section 2 we see that for $n = 5$ we have $D_1 + D_2/n = -11/30$ and $D + \Delta = 29/450$, so Propositions 5 and 6 together imply that Theorem 1 is true for $n = 5$. Note that for $n \neq 5$ we have $\Delta = 0$, so Propositions 5 and 7 taken together imply that Theorem 1 is true for $n \neq 5$.

In [8] it was proved that if $n = 5$ and if β is sufficiently close to 1, then maximal polynomials in $S(n+1,\beta)$ (those for which $|P|_\beta = r_{n+1}(\beta)$) must be nonreal. Taken together, Theorem 1 and Proposition 7 provide strong evidence that this is true only for $n = 5$ (although it is conceivable that this could fail for higher-order approximations).

2. Preliminaries

We begin by computing some values (that we will subsequently need) for the constants that appear in Theorem 1, obtaining:

n	u_1	u_2	D_1	D_2	$D_1 + D_2/n$
3	0	-1	$-1/4$	$-1/4$	$-1/3$
4	$\dfrac{-1+\sqrt{5}}{4}$	$\dfrac{-1-\sqrt{5}}{4}$	$-1/5$	$-2/5$	$-3/10$
5	$-1/2$	-1	$-1/3$	$-1/6$	$-11/30$
6	-0.2225	-0.9010	-0.3014		
7	0	-0.7071	-0.2929	-0.2929	-0.3347
9	-0.3090	-0.8090			
10	-0.1423	-0.6549	-0.3138		

We next establish some relationships between these constants with

Lemma 8. *Assume the notation of Theorem 1. Then*

1. $u_2 < -1/2 \le u_1$, *and* $u_1 \le 0$ *for* $n \ne 4$, *and* $u_2 > -1$ *for* $n \ne 3, 5$,
2. $u_1 + u_2 < 0$ *and* $u_1 u_2 > -1$,
3. $2nu_1 + n + 1 \ge 1$ *and* $2nu_2 + n + 1 < 0$,
4. $D_1 < 0$ *and* $D_2 < 0$,
5. $\lim_{n\to\infty} D_1 + D_2/n = -1/3$,
6. $-1 < D_1 + D_2/n \le -3/10$, *with equality only at* $n = 4$, *and*
7. $1 + (1 + D_1 + D_2)(u_i - 1) - D_2(2u_i^2 - 2) = 0$ *for* $i = 1$ *and* $i = 2$.

Proof. From the definition of k in Theorem 1, the relationship between k and n depends on the residue of n modulo 3. For increasing values of n in each of the three residue classes, the sequence $k/(n+1)$ increases to (or is equal to) $1/3$ and the sequence $(k+1)/(n+1)$ strictly decreases to $1/3$, so the values of u_1 decrease to (or are equal to) $-1/2$ and the values of u_2 strictly increase to $-1/2$. Since the values of u_1 decrease (or remain constant) in each residue class, and since $u_1 \le 0$ for $n = 3$, 5 and 7, then $u_1 \le 0$ for all $n \ne 4$. Since the values of u_2 strictly increase in each residue class, and since $u_2 > -1$ for $n = 4$ and $u_2 = -1$ for $n = 3$ and $n = 5$, then $u_2 > -1$ for $n \ne 3, 5$. This completes the proof of part 1 of the lemma.

For $n = 4$, we have $u_1 + u_2 = -1/2$ and $u_1 u_2 = -1/4$. For $n \ne 4$ we have from part 1 that $u_2 < u_1 \le 0$, and part 2 of the lemma follows trivially.

Since $u_1 \ge -1/2$, then $2nu_1 + n + 1 \ge 1$. For $n = 3$, 4 and 5 we have (k+1)/(n+1) $\le 1/2$. Since in each residue class this quotient strictly decreases to $1/3$, then for all $n \ge 3$ we have $2\pi(k+1)/(n+1) \in (2\pi/3, \pi]$. Now $\cos x \le 1/2 - 3x/(2\pi)$ on this interval, and from the definition of k in Theorem 1 we know that $k \ge (n-1)/3$, so

$$u_2 = \cos\frac{2\pi(k+1)}{n+1} \le \frac{1}{2} - \frac{3(k+1)}{n+1} \le \frac{1}{2} - \frac{n+2}{n+1} < -\frac{n+1}{2n}$$

which completes the proof of part 3 of the lemma.

At $n = 4$, we have $D_1 = -1/5$ and $D_2 = -2/5$. For $n \ne 4$ we know from part 1 of Lemma 8 that $u_2 < u_1 \le 0$ so from the definitions of D_1 and D_2 in Theorem 1 we see that $D_1 < 0$ and $D_2 < 0$. This completes the proof of part 4 of the lemma.

As n tends to infinity, u_1 and u_2 tend to $-1/2$, so D_1 tends to $-1/3$ and D_2 is bounded. This completes the proof of part 5 of the lemma.

By part 2 of Lemma 8 we have $u_1 + u_2 < 0$ and $u_1 u_2 > -1$. Since by part 4 of Lemma 8 we know that $D_2 < 0$, then

$$D_1 + D_2/n > D_1 + D_2 = -\frac{1 + u_1 u_2}{1 + u_1 u_2 - (u_1 + u_2)} > -1.$$

From part 1 of Lemma 8 we know that $u_2 < -1/2 \leq u_1$, so by computing the partial derivatives of D_1 we see that $\partial D_1/\partial u_1 > 0$ and $\partial D_1/\partial u_2 \leq 0$. Since in each residue class u_1 decreases to $-1/2$ and u_2 increases to $-1/2$, then in each residue class D_1 decreases to $-1/3$. At $n = 5$, 6 and 10 we have $D_1 < -3/10$, and hence $D_1 + D_2/n < D_1 < -3/10$ for all $n \geq 3$ except possibly $n = 3$, 4 and 7. Checking the values of $D_1 + D_2/n$ (computed at the beginning of section 2) for these exceptional values completes the proof of part 6 of the lemma.

Expressing D_1 and D_2 in terms of u_1 and u_2 and simplifying the result verifies part 7, and thus completes the proof of Lemma 8. □

We now estimate the size of the coefficients of P' with

Proposition 9. *Suppose that $P \in S(n+1, \beta)$ with P' monic and $|P|_\beta \geq \beta$. Let $P'(z) = \prod_{j=1}^{n}(z - \zeta_j) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$. Then*

1. *each $\Re[\zeta_j] = \mathcal{O}(1-\beta)$ and each $\Im[\zeta_j] = \mathcal{O}(1-\beta)^{1/2}$,*
2. *each $a_{n-k} = \mathcal{O}(1-\beta)^{k/2}$,*
3. *for k odd, each $\Re[a_{n-k}] = \mathcal{O}(1-\beta)^{(k+1)/2}$, and*
4. *for k even, each $\Im[a_{n-k}] = \mathcal{O}(1-\beta)^{(k+1)/2}$.*

Proof. Parts 1–3 were proved in [8, Proposition 4]. Part 4 is proved similarly to part 3, by noting that each term of $\Im[a_{n-k}]$ is a product of k of the $\Re[\zeta_j]$'s and $\Im[\zeta_j]$'s, and that for k even, each term has at least one $\Re[\zeta_j]$, so from part 1 of Proposition 9 we have that $\Im[a_{n-k}] = \mathcal{O}(1-\beta)^{(k+1)/2}$. □

To have $P \in S(n+1, \beta)$ requires that the moduli of the roots of P are all at most 1. We estimate these moduli with

Proposition 10. *Assume the notation of Theorem 1. Let P be a polynomial with $P'(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ and $P(\beta) = 0$. Let $z \neq \beta$ be a root of P, let ω be the $(n+1)$th root of 1 that is closest to z and let $R = (1-\beta) + a_{n-1}(\omega^n - 1)/n + \cdots + a_0(\omega - 1)$.*

1. *For $0 < r \leq 1$, if each $a_k = \mathcal{O}(1-\beta)^r$, then $|z|^2 = 1 - 2\Re[R] + \mathcal{O}(1-\beta)^{2r}$.*
2. *Suppose that*

$$a_{n-1} = n(1 + D_1 + D_2)(1-\beta) + \mathcal{O}(1-\beta)^\alpha,$$
$$a_{n-2} = -(n-1)D_2(1-\beta) + \mathcal{O}(1-\beta)^\alpha, \textit{and}$$
$$a_{n-k} = \mathcal{O}(1-\beta)^\alpha \quad \textit{for } k \geq 3$$

and define

$$\Gamma_2 = 2(1 + D_1 + D_2)(D_1 - 2D_2 + nD_2) \textit{ and}$$
$$\Gamma_1 = -\Gamma_2 + (-2 - 4D_1)n + (1 + 4D_1 - 4D_2).$$

If $\Re[\omega] = u_i$ for $i = 1$ or $i = 2$, then

$$|z|^{2n+2} = 1 - 2(n+1)\Re[R] + (n+1)(\Gamma_1 + \Gamma_2 u_i)(1-\beta)^2 + \mathcal{O}(1-\beta)^{\alpha+1}.$$

Proof. Since $\beta = 1 - (1-\beta)$, then by the binomial theorem $\beta^k = 1 - k(1-\beta) + \mathcal{O}(1-\beta)^2$. Since z is a root of P we have

$$0 = P(z) = \int_\beta^z P'(t)\,dt = \frac{z^{n+1} - \beta^{n+1}}{n+1} + a_{n-1}\frac{z^n - \beta^n}{n} + \cdots + a_0(z-\beta),$$

and solving for z^{n+1} gives us

$$z^{n+1} = \beta^{n+1} - (n+1)\left[a_{n-1}\frac{z^n - \beta^n}{n} + \cdots + a_0(z-\beta)\right]. \tag{2.1}$$

By hypothesis, as β goes to 1 the a_k all tend to 0 so the roots of P tend to the roots of $z^{n+1} - 1$, and so the ω appearing in the hypotheses is well defined.

Now each $\beta^k = 1 + \mathcal{O}(1-\beta)$, and by the hypothesis of part 1 each $a_k = \mathcal{O}(1-\beta)^r$. Putting these estimates into equation (2.1), we see that $z^{n+1} = 1 + \mathcal{O}(1-\beta)^r$. Then $z = \omega + \mathcal{O}(1-\beta)^r$ and so $(z^k - \beta^k)/k = (\omega^k - 1)/k + \mathcal{O}(1-\beta)^r$. Now note that each $a_{n-k} = \mathcal{O}(1-\beta)^r$ and that each $\beta^k = 1 - k(1-\beta) + \mathcal{O}(1-\beta)^2$. Substituting these estimates into equation (2.1) gives

$$\begin{aligned} z^{n+1} &= 1 - (n+1)(1-\beta) - (n+1)\left[a_{n-1}\frac{\omega^n - 1}{n} + \cdots + a_0(\omega - 1)\right] + \mathcal{O}(1-\beta)^{2r} \\ &= 1 - (n+1)R + \mathcal{O}(1-\beta)^{2r}. \end{aligned}$$

Note that $R = \mathcal{O}(1-\beta)^r$ so

$$(1-R)^{n+1} = 1 - (n+1)R + \mathcal{O}(1-\beta)^{2r} = z^{n+1} + \mathcal{O}(1-\beta)^{2r},$$

so $z = \omega(1-R) + \mathcal{O}(1-\beta)^{2r}$ and hence $|z|^2 = z\overline{z} = 1 - 2\Re[R] + \mathcal{O}(1-\beta)^{2r}$. This finishes the proof of part 1.

From the hypotheses of part 2, we know that $\Re[\omega] = u_i$ for $i = 1$ or $i = 2$. Suppose for the moment that $\Re[\omega] = u_1$ and write $\omega = u_1 + iv_1$. Since $\omega^{n+1} = 1$, then $|\omega| = 1$, so $\omega^n = \overline{\omega}$ and $\Re[\omega^2] = 2u_1^2 - 1$. Let $A = \big[-(1+D_1+D_2) + 2D_2u_1\big]v_1$. From part 7 of Lemma 8 we see that

$$\Re[1 + (1 + D_1 + D_2)(\overline{\omega} - 1) - D_2(\overline{\omega}^2 - 1)] = 0$$

and so using the estimates of the a_{n-k}'s given in the hypotheses of part 2, we get

$$\begin{aligned} R &= (1-\beta) + a_{n-1}\frac{\overline{\omega} - 1}{n} + a_{n-2}\frac{\overline{\omega}^2 - 1}{n-1} + \cdots + a_0(\omega - 1) \\ &= \big[1 + (1 + D_1 + D_2)(\overline{\omega} - 1) - D_2(\overline{\omega}^2 - 1)\big](1-\beta) + \mathcal{O}(1-\beta)^\alpha \\ &= iA(1-\beta) + \mathcal{O}(1-\beta)^\alpha. \end{aligned}$$

The hypotheses of part 2 imply that each $a_k = \mathcal{O}(1-\beta)$, so from the proof of part 1 with $r = 1$ we have $z = \omega(1-R) + \mathcal{O}(1-\beta)^2 = \omega\big[1 - iA(1-\beta)\big] + \mathcal{O}(1-\beta)^\alpha$ and so

$$(z^k - \beta^k)/k = (\omega^k - 1)/k + (1 - iA\omega^k)(1-\beta) + \mathcal{O}(1-\beta)^\alpha.$$

Let $G = n/2 - n(1 + D_1 + D_2)(1 - iA\overline{\omega}) + (n-1)D_2(1 - iA\overline{\omega}^2)$. Then from equation (2.1) and the estimates of the a_k's given in the hypotheses of part 2 we get

$$\begin{aligned}z^{n+1} &= 1-(n+1)(1-\beta)+\frac{(n+1)n}{2}(1-\beta)^2\\ &\quad -(n+1)\Big[a_{n-1}\left(\frac{\omega^n-1}{n}+(1-iA\omega^n)(1-\beta)\right)\\ &\qquad +a_{n-2}\left(\frac{\omega^{n-1}-1}{n-1}+(1-iA\omega^{n-1})(1-\beta)\right)\\ &\qquad +a_{n-3}\frac{\omega^{n-2}-1}{n-2}+\cdots+a_0(\omega-1)\Big]+\mathcal{O}(1-\beta)^{\alpha+1}\\ &= 1-(n+1)R+(n+1)G(1-\beta)^2+\mathcal{O}(1-\beta)^{\alpha+1}.\end{aligned}$$

Then since $R = iA(1-\beta)+\mathcal{O}(1-\beta)^{\alpha}$ we have

$$\begin{aligned}|z|^{2n+2} &= z^{n+1}\overline{z}^{n+1}\\ &= 1-2(n+1)\Re[R]+(n+1)\big[2\Re[G]+(n+1)A^2\big](1-\beta)^2+\mathcal{O}(1-\beta)^{\alpha+1}.\end{aligned}$$

Thus to complete the proof of part 2 of Proposition 10 for the case $\Re[\omega]=u_1$ we need only verify that $2\Re[G]+(n+1)A^2=\Gamma_1+\Gamma_2u_1$.

Let $D_0 = 1+D_1+D_2$, so from the definition of A we see that

$$A = (-D_0+2D_2u_1)v_1.$$

Note that $\Re[i\,\overline{\omega}]=\Im[\omega]$. Then from the definition of G we have

$$\begin{aligned}\Re[G] &= n/2-nD_0(1-Av_1)+(n-1)D_2(1-2Au_1v_1)\\ &= n/2-nD_0+(n-1)D_2-A\big[n(-D_0v_1+2D_2u_1v_1)-2D_2u_1v_1\big]\\ &= (-n/2-nD_1-D_2)-nA^2+2AD_2u_1v_1\end{aligned}$$

so

$$2\Re[G]+(n+1)A^2 = (-n-2nD_1-2D_2)+(-n+1)A^2+4AD_2u_1v_1. \tag{2.2}$$

Now

$$2D_2u_1^2 = \frac{-u_1^2}{(1-u_1)(1-u_2)} = D_0u_1+(D_2-D_1),$$

so

$$\begin{aligned}Av_1 &= (-D_0+2D_2u_1)(1-u_1^2)\\ &= -D_0+2D_2u_1-u_1(-D_0u_1+2D_2u_1^2)\\ &= -D_0+(D_1+D_2)u_1.\end{aligned}$$

Using these two equalities, we see that

$$\begin{aligned}A^2 &= (-D_0+2D_2u_1)\big[-D_0+(D_1+D_2)u_1\big]\\ &= D_0^2+(-D_0D_1-3D_0D_2)u_1+(D_1+D_2)(2D_2u_1^2)\\ &= D_0^2-D_1^2+D_2^2-2D_0D_2u_1\end{aligned}$$

and

$$\begin{aligned}2AD_2u_1v_1 &= 2D_2u_1[-D_0+(D_1+D_2)u_1]\\ &= -2D_0D_2u_1+(D_1+D_2)[D_0u_1+(D_2-D_1)]\\ &= D_0(D_1-D_2)u_1+(D_2^2-D_1^2).\end{aligned}$$

Thus from equation (2.2) we have

$$\begin{aligned}2\Re[G]+(n+1)A^2 &= (-n-2nD_1-2D_2)+(-n+1)(D_0^2-D_1^2+D_2^2-2D_0D_2u_1)\\ &\quad +2\big[D_0(D_1-D_2)u_1+(D_2^2-D_1^2)\big]\\ &= (-1-2D_1-D_0^2+D_1^2-D_2^2)n+(-2D_2+D_0^2-3D_1^2+3D_2^2)\\ &\quad +2D_0u_1(D_1-2D_2+nD_2)\\ &= \Gamma_1+\Gamma_2u_1.\end{aligned}$$

This finishes the proof of part 2 of Proposition 10 for the case $\Re[\omega]=u_1$. Since D_1 and D_2 are symmetric in u_1 and u_2, swapping u_1 and u_2 in this proof verifies part 2 of Proposition 10 for the remaining case $\Re[\omega]=u_2$, and thus completes the proof of Proposition 10. □

Finally, consider the linear transformation $\mathcal{T}$ which takes functions to real numbers via

$$\mathcal{T}(f)=\frac{(2nu_1+n+1)f(u_2)-(2nu_2+n+1)f(u_1)}{2(u_1-u_2)}. \tag{2.3}$$

Recall that by Lemma 8 we have $u_1-u_2>0$, $2nu_1+n+1>0$ and $2nu_2+n+1<0$, so $\mathcal{T}/n$ is a weighted average. This implies that $\mathcal{T}$ preserves inequalities, in the sense that if $f(u_1)\le g(u_1)$ and $f(u_2)\le g(u_2)$, then $\mathcal{T}(f)\le\mathcal{T}(g)$.

In the process of analyzing several inequalities, we will need the following values of the transformation $\mathcal{T}$:

$$\begin{aligned}\mathcal{T}(1)&=n,\\ \mathcal{T}(2+2u)&=n-1,\\ \mathcal{T}(1+4u+4u^2)&=-[n+2+2(n+1)(u_1+u_2)+4nu_1u_2]\\ &=-\frac{n+1+D_1+3nD_1+3D_2}{D_2},\\ \mathcal{T}\left(\frac{1}{1-u}\right)&=n+nD_1+D_2,\\ \mathcal{T}\left(\frac{u}{1-u}\right)&=nD_1+D_2.\end{aligned} \tag{2.4}$$

We will also use the results of

Lemma 11. *For the linear transformation $\mathcal{T}$ defined in equation* (2.3) *we have*

1. $\mathcal{T}(1+4u+4u^2)/(n-2)<1/2$ *for* $n\ne 3$, 4 *and* 6, *and*
2. $\mathcal{T}(8u^2+8u^3)\ge 0$ *for all* n.

Proof. From the formula for $\mathcal{T}(1+4u+4u^2)$ in (2.4) and from part 3 of Lemma 8 we have

$$\begin{aligned}\partial\mathcal{T}(1+4u+4u^2)/\partial u_1&=-2(2nu_2+n+1)>0 \text{ and}\\ \partial\mathcal{T}(1+4u+4u^2)/\partial u_2&=-2(2nu_1+n+1)<0.\end{aligned}$$

Recall from the proof of Lemma 8 that for each residue class of n modulo 3 the values of u_1 decrease and the values of u_2 increase, so the signs of the partial derivatives above imply that in each residue class the values of $\mathcal{T}(1+4u+4u^2)$ decrease. Since $1+4u+4u^2=(1+2u)^2\ge 0$ and since $\mathcal{T}$ preserves inequalities, then $\mathcal{T}(1+4u+4u^2)\ge 0$, so the values of $\mathcal{T}(1+4u+4u^2)/(n-2)$ also decrease in each residue class. Using the formula for $\mathcal{T}(1+4u+4u^2)$ in (2.4) and the values of the u_i computed at the beginning of section 2, we calculate the values of $\mathcal{T}(1+4u+4u^2)/(n-2)$ at $n=5$, 7 and 9, getting respectively 1/3, 0.4627 and 0.3372. Since they are all less than 1/2, this proves part 1 of Lemma 11.

Since by definition $u_i \geq -1$, then $8u_i^2+8u_i^3 = 8u_i^2(1+u_i) \geq 0$ for both $i=1$ and $i=2$, and so part 2 of Lemma 11 follows from our observation that $\mathcal{T}$ preserves inequalities. □

Finally, we will deal with polynomials that are "almost" in $S(n,\beta)$ using

Lemma 12. *Suppose that P is a polynomial of degree n with all roots in $\{z : |z| \leq 1+\mathcal{O}(1-\beta)^r\}$, one root at β, and all other roots bounded away from β. Then there is a polynomial $Q \in S(n,\beta)$ such that $|Q|_\beta = |P|_\beta + \mathcal{O}(1-\beta)^r$.*

Proof. If $P \in S(n,\beta)$, then we may take $Q = P$. If not, then at least one root of P has modulus greater than 1. In this case, let

$$c = \max\left\{\frac{|z|^2-1}{|z-\beta|^2} : z \text{ is a root of } P \text{ and } |z|>1\right\}.$$

Since by hypothesis $|z-\beta|$ is bounded away from 0 and $|z| \leq 1+\mathcal{O}(1-\beta)^r$, then $0 < c \leq \mathcal{O}(1-\beta)^r$. In particular, for β sufficiently close to 1 we have $0<c<1$.

Let Q be the polynomial with roots $\{z - c(z-\beta) : z \text{ is a root of } P\}$. Since the mapping $z \mapsto z - c(z-\beta)$ is a contraction of the plane that leaves β fixed and moves all roots of P (and hence P') toward β by at most $\mathcal{O}(1-\beta)^r$, then $Q(\beta)=0$ and $|Q|_\beta = |P|_\beta + \mathcal{O}(1-\beta)^r$. Thus we need only show that all roots of Q are in the unit disk.

Note that for t real the image of the mapping $t \mapsto z - t(z-\beta)$ is a line, with $t=0$ mapping to z, and $t=1$ mapping to β, and $t = (|z|^2-1)/|z-\beta|^2$ mapping to

$$z - \frac{|z|^2-1}{|z-\beta|^2}(z-\beta) = z - \frac{z\overline{z}-1}{\overline{z}-\beta} = \frac{1-\beta z}{\overline{z}-\beta}.$$

If z is in the unit disk, then the images of every t between 0 and 1 lie on the line between z and β, hence in the unit disk. If z is not in the unit disk, then $|(1-\beta z)/(z-\beta)| < 1$ and so the images of every t between $(|z|^2-1)/|z-\beta|^2$ and 1 lie on the line between $(1-\beta z)/(\overline{z}-\beta)$ and β, hence in the unit disk. Thus for every root z of P, the image of c lies in the unit disk, so all roots of Q are in the unit disk and so $Q \in S(n,\beta)$. This completes the proof of Lemma 12. □

3. Proof of Proposition 5

Take any $P \in S(n+1,\beta)$, assume without loss of generality that P' is monic, and write $P'(z) = \prod_{j=1}^n (z-\zeta_j) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$.

If $|P|_\beta \leq 1+(D_1+D_2/n)(1-\beta)+(D+\Delta)(1-\beta)^2$, then Proposition 5 is trivially true. Thus we may assume without loss of generality that

$$|P|_\beta \geq 1 + (D_1+D_2/n)(1-\beta) + (D+\Delta)(1-\beta)^2. \tag{3.1}$$

From part 6 of Lemma 8 we have that $D_1 + D_2/n > -1$, and so inequality (3.1) implies that $|P|_\beta \geq \beta$ as long as β is sufficiently close to 1. Note that P thus satisfies all the hypotheses of Proposition 9.

We begin by estimating some relationships between the coefficients of P' with

Lemma 13. *Suppose that* $\Im[a_{n-1}] = \mathcal{O}(1-\beta)^{3/2}$ *and that each*

$$|\zeta_j - \beta| = 1 + (D_1 + D_2/n)(1-\beta) + \mathcal{O}(1-\beta)^2.$$

Then

1. $\Im[a_{n-2}] = (-3/2)\Im[a_{n-3}] + \mathcal{O}(1-\beta)^{5/2}$ *and*
2. $\Re[a_{n-3}] + 2\Re[a_{n-4}] = (n-2)(1 + D_1 + D_2/n)(1-\beta)\Re[a_{n-2}] + \mathcal{O}(1-\beta)^3$.

Proof. Let each $\zeta_j = x_j + iy_j$ and note that by Proposition 9 we have $x_j = \mathcal{O}(1-\beta)$ and $y_j = \mathcal{O}(1-\beta)^{1/2}$. Note that by hypothesis, $\sum_i y_i = -\Im[a_{n-1}] = \mathcal{O}(1-\beta)^{3/2}$ and that each

$$(\beta - x_j)^2 + y_j^2 = |\beta - \zeta_j|^2 = 1 + 2(D_1 + D_2/n)(1-\beta) + \mathcal{O}(1-\beta)^2,$$

so solving for x_j gives us

$$x_j = y_j^2/2 - (1 + D_1 + D_2/n)(1-\beta) + \mathcal{O}(1-\beta)^2. \tag{3.2}$$

Note that $\Im[a_{n-3}] = -\sum_{i<j<k} \Im[\zeta_i\zeta_j\zeta_k] = \sum_{i<j<k} y_i y_j y_k + \mathcal{O}(1-\beta)^{5/2}$, so

$$\begin{aligned}\mathcal{O}(1-\beta)^{5/2} &= \sum_i y_i \sum_{i<j} y_i y_j = \sum_{i\neq j} y_i^2 y_j + 3\sum_{i<j<k} y_i y_j y_k \\ &= \sum_{i\neq j} y_i^2 y_j + 3\Im[a_{n-3}] + \mathcal{O}(1-\beta)^{5/2}\end{aligned}$$

and so $\sum_{i\neq j} y_i^2 y_j = -3\Im[a_{n-3}] + \mathcal{O}(1-\beta)^{5/2}$. Then using equation (3.2) we have

$$\begin{aligned}\Im[a_{n-2}] &= \sum_{i<j} \Im[\zeta_i\zeta_j] = \sum_{i\neq j} x_i y_j \\ &= (1/2)\sum_{i\neq j} y_i^2 y_j - (1 + D_1 + D_2/n)(1-\beta)\sum_{i\neq j} y_j + \mathcal{O}(1-\beta)^{5/2} \\ &= (-3/2)\Im[a_{n-3}] + \mathcal{O}(1-\beta)^{5/2},\end{aligned}$$

which completes the proof of part 1 of Lemma 13.

Let S be the set of triples (i,j,k) of distinct integers from 1 to n with $j<k$. Note that $\Re[a_{n-2}] = \sum_{i<j} \Re[\zeta_i\zeta_j] = -\sum_{i<j} y_i y_j + \mathcal{O}(1-\beta)^2$ and $\Re[a_{n-3}] = -\sum_{i<j<k} \Re[\zeta_i\zeta_j\zeta_k] = \sum_S x_i y_j y_k + \mathcal{O}(1-\beta)^3$. Furthermore,

$$\mathcal{O}(1-\beta)^3 = \sum_i y_i \sum_{j<k<l} y_j y_k y_l = \sum_S y_i^2 y_j y_k + 4\sum_{i<j<k<l} y_i y_j y_k y_l,$$

so

$$\begin{aligned}\Re[a_{n-4}] &= \sum_{i<j<k<l} \Re[\zeta_i\zeta_j\zeta_k\zeta_l] = \sum_{i<j<k<l} y_i y_j y_k y_l + \mathcal{O}(1-\beta)^3 \\ &= (-1/4)\sum_S y_i^2 y_j y_k + \mathcal{O}(1-\beta)^3.\end{aligned}$$

Then using equation (3.2) we have

$$\begin{aligned}\Re[a_{n-3}] + 2\Re[a_{n-4}] &= \sum_S \left(x_i - y_i^2/2\right) y_j y_k + \mathcal{O}(1-\beta)^3 \\ &= -(1 + D_1 + D_2/n)(1-\beta)(n-2)\sum_{j<k} y_j y_k + \mathcal{O}(1-\beta)^3 \\ &= (n-2)(1 + D_1 + D_2/n)(1-\beta)\Re[a_{n-2}] + \mathcal{O}(1-\beta)^3,\end{aligned}$$

which completes the proof of Lemma 13. □

We now establish a lower bound on $\Re[a_{n-4}]$ with

Lemma 14. *Suppose that*

$$\begin{aligned}&\Im[a_{n-1}] = \mathcal{O}(1-\beta)^\alpha, \\ &\Re[a_{n-2}] = -(n-1)D_2(1-\beta) + \mathcal{O}(1-\beta)^\alpha, \text{ and} \\ &\Im[a_{n-3}] = \mathcal{O}(1-\beta)^\alpha.\end{aligned}$$

If $n = 5$, then define $\delta = -1/15$; otherwise define $\delta = 0$. Then

$$\Re[a_{n-4}] \geq \delta(1-\beta)^2 + \mathcal{O}(1-\beta)^{\alpha+1}.$$

Proof. Let each $\zeta_j = x_j + iy_j$ and recall by Proposition 9 that $x_j = \mathcal{O}(1-\beta)$ and $y_j = \mathcal{O}(1-\beta)^{1/2}$. Let $F(y) = \prod_{i=1}^n (y + y_i) = y^n + b_{n-1}y^{n-1} + \cdots + b_0$. Note that

$$\begin{aligned}\Re[a_{n-4}] &= \sum_{i<j<k<l} \Re[\zeta_i\zeta_j\zeta_k\zeta_l] = \sum_{i<j<k<l} y_i y_j y_k y_l + \mathcal{O}(1-\beta)^3 \\ &= b_{n-4} + \mathcal{O}(1-\beta)^3\end{aligned}$$

and that by hypothesis

$$\begin{aligned}b_{n-1} &= \sum_i y_i = \sum_i \Im[\zeta_i] = -\Im[a_{n-1}] \\ &= \mathcal{O}(1-\beta)^\alpha, \\ b_{n-2} &= \sum_{i<j} y_i y_j = -\sum_{i<j} \Re[\zeta_i\zeta_j] + \mathcal{O}(1-\beta)^2 = -\Re[a_{n-2}] + \mathcal{O}(1-\beta)^2 \\ &= (n-1)D_2(1-\beta) + \mathcal{O}(1-\beta)^\alpha, \text{ and} \\ b_{n-3} &= \sum_{i<j<k} y_i y_j y_k = -\sum_{i<j<k} \Im[\zeta_i\zeta_j\zeta_k] + \mathcal{O}(1-\beta)^{5/2} = \Im[a_{n-3}] + \mathcal{O}(1-\beta)^{5/2} \\ &= \mathcal{O}(1-\beta)^\alpha.\end{aligned}$$

Let

$$\begin{aligned}f(y) &= F^{(n-4)}(y) \\ &= \frac{n!}{24}y^4 + \frac{(n-1)!}{6}b_{n-1}y^3 + \frac{(n-2)!}{2}b_{n-2}y^2 + (n-3)!\,b_{n-3}y + (n-4)!\,b_{n-4}.\end{aligned}$$

Now by definition F has all real roots, hence by Rolle's Theorem (from elementary calculus) so does f. Then the "reverse" of f defined by $y^4 f(1/y) = (n-4)!b_{n-4}y^4 + \cdots + n!/24$ has all real roots, so by Rolle's theorem so does the reverse's second derivative

$$12(n-4)!b_{n-4}y^2 + 6(n-3)!b_{n-3}y + (n-2)!\ b_{n-2}.$$

Since this quadratic has all real roots, then its discriminant is nonnegative, so

$$[6(n-3)!b_{n-3}]^2 - 48(n-2)!(n-4)!b_{n-2}b_{n-4} \geq 0.$$

Using our estimates of the b_{n-k}'s (including $b_{n-4} = \mathcal{O}(1-\beta)^2$), this implies that $-D_2(1-\beta)b_{n-4} \geq \mathcal{O}(1-\beta)^{2\alpha}$ and so $b_{n-4} \geq \mathcal{O}(1-\beta)^{2\alpha-1}$. Now for $n \neq 3, 5$ we have $\alpha = 2$ and so $\Re[a_{n-4}] = b_{n-4} + \mathcal{O}(1-\beta)^3 \geq \mathcal{O}(1-\beta)^3$, which finishes the proof of Lemma 14 for these values of n.

Lemma 14 is trivially true for $n = 3$, since then $\Re[a_{n-4}] \equiv 0 \geq \mathcal{O}(1-\beta)^{5/2}$.

Finally, for $n = 5$ we have that

$$f(y) = 5y^4 + 4b_{n-1}y^3 + 3b_{n-2}y^2 + 2b_{n-3}y + b_{n-4}$$

has all real roots, hence by Rolle's theorem so does its derivative

$$f'(y) = 20y^3 + 12b_{n-1}y^2 + 6b_{n-2}y + 2b_{n-3}.$$

A classical result (see e.g. [11, p. 289]) states that if a cubic polynomial $ax^3 + bx^2 + cx + d$ has all real roots, then its discriminant is nonnegative, so

$$18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2 \geq 0.$$

Applying this to $f'(y)$, we have

$$-4[20][6b_{n-2}]^3 - 27[20]^2[2b_{n-3}]^2 \geq \mathcal{O}(1-\beta)^4,$$

which implies that $2b_{n-2}^3 + 5b_{n-3}^2 \leq \mathcal{O}(1-\beta)^4$. Since for $n = 5$ we have $D_2 = -1/6$, then by hypothesis $b_{n-2} = (-2/3)(1-\beta) + \mathcal{O}(1-\beta)^{3/2}$, and so

$$\begin{aligned} b_{n-3}^2 &\leq (-2/5)b_{n-2}^3 + \mathcal{O}(1-\beta)^4 \\ &= (16/135)(1-\beta)^3 + \mathcal{O}(1-\beta)^{7/2}. \end{aligned}$$

We also have that the first derivative of the reverse of f

$$4b_{n-4}y^3 + 6b_{n-3}y^2 + 6b_{n-2}y + 4b_{n-1}$$

has all real roots, so applying our classical result gives

$$[6b_{n-3}]^2[6b_{n-2}]^2 - 4[4b_{n-4}][6b_{n-2}]^3 \geq \mathcal{O}(1-\beta)^6.$$

Dividing this by $144b_{n-2}^2$ and recalling that $b_{n-2} = (-2/3)(1-\beta) + \mathcal{O}(1-\beta)^{3/2}$ yields

$$9b_{n-3}^2 + 16(1-\beta)b_{n-4} \geq \mathcal{O}(1-\beta)^{7/2}.$$

Combining these two inequalities implies that for $n = 5$ we have

$$\begin{aligned} \Re[a_{n-4}] &= b_{n-4} + \mathcal{O}(1-\beta)^3 \\ &\geq \frac{-9b_{n-3}^2}{16(1-\beta)} + \mathcal{O}(1-\beta)^{5/2} \\ &\geq (-1/15)(1-\beta)^2 + \mathcal{O}(1-\beta)^{5/2}. \end{aligned}$$

This completes the proof of Lemma 14. □

We now begin the proof of Proposition 5. Our first step will be to show that $|P|_\beta \leq 1 + (D_1 + D_2/n)(1-\beta) + \mathcal{O}(1-\beta)^2$. Recall that P satisfies the hypotheses of Proposition 9, so each $a_{n-k} = \mathcal{O}(1-\beta)^{k/2}$. Let $\omega \neq 1$ be any $(n+1)$st root of 1 and let z be the root of P (so $|z| \leq 1$) closest to ω. Then in Proposition 10 we have

$$\begin{aligned} R &= (1-\beta) + a_{n-1}(\omega^n - 1)/n + \cdots + a_0(\omega - 1) \\ &= a_{n-1}(\omega^n - 1)/n + \mathcal{O}(1-\beta) \end{aligned}$$

and so by part 1 of Proposition 10 with $r = 1/2$, we have

$$|z|^2 = 1 - 2\Re[a_{n-1}(\omega^n - 1)/n] + \mathcal{O}(1-\beta).$$

Since $|z| \leq 1$ and $\omega^n = \overline{\omega}$, this implies that $\Re[a_{n-1}(\overline{\omega} - 1)] \geq \mathcal{O}(1-\beta)$. Expanding the product and noting that by Proposition 9 we have $\Re[a_{n-1}] = \mathcal{O}(1-\beta)$, we get that $\Im[a_{n-1}]\Im[\omega] \geq \mathcal{O}(1-\beta)$. Choosing ω non-real and repeating this argument with $\overline{\omega}$ substituted for ω provides that $\Im[a_{n-1}]\Im[\overline{\omega}] \geq \mathcal{O}(1-\beta)$ and so $\Im[a_{n-1}] = \mathcal{O}(1-\beta)$. Thus we have $a_{n-1} = \mathcal{O}(1-\beta)$.

Recall that each $a_{n-k} = \mathcal{O}(1-\beta)^{k/2}$, so we know now that each $a_{n-k} = \mathcal{O}(1-\beta)$. Since $\omega^{n-k} = \overline{\omega}^{k+1}$, by part 1 of Proposition 10 with $r = 1$ we have

$$|z|^2 = 1 - 2\Re\left[(1-\beta) + a_{n-1}\frac{\overline{\omega}-1}{n} + a_{n-2}\frac{\overline{\omega}^2-1}{n-1} + a_{n-3}\frac{\overline{\omega}^3-1}{n-2}\right] + \mathcal{O}(1-\beta)^2.$$

Since $|z| \leq 1$ this implies that

$$-\Re\left[a_{n-1}\frac{\overline{\omega}-1}{n} + a_{n-2}\frac{\overline{\omega}^2-1}{n-1} + a_{n-3}\frac{\overline{\omega}^3-1}{n-2}\right] \leq (1-\beta) + \mathcal{O}(1-\beta)^2. \tag{3.3}$$

Averaging the expressions obtained by substituting ω and $\overline{\omega}$ into inequality (3.3) and noting that by Proposition 9 we have $\Re[a_{n-3}] = \mathcal{O}(1-\beta)^2$ we get

$$\Re[a_{n-1}]\Re\left[\frac{1-\omega}{n}\right] + \Re[a_{n-2}]\Re\left[\frac{1-\omega^2}{n-1}\right] \leq (1-\beta) + \mathcal{O}(1-\beta)^2. \tag{3.4}$$

Let $u = \Re[\omega]$. Note that since $|\omega| = 1$, then $\Re[\omega^2] = 2u^2 - 1$, so dividing inequality (3.4) by $1 - u$, we get

$$\frac{\Re[a_{n-1}]}{n} + \frac{\Re[a_{n-2}]}{n-1}(2+2u) \leq \frac{1-\beta}{1-u} + \mathcal{O}(1-\beta)^2 \tag{3.5}$$

for each $\omega \neq 1$. In particular, inequality (3.5) holds for $u = u_1$ and $u = u_2$ as defined in Theorem 1.

Applying the linear transformation $\mathcal{T}$ defined in equation (2.3) to inequality (3.5), and using the values computed in (2.4), we see that

$$\Re[a_{n-1}] + \Re[a_{n-2}] \leq (n + nD_1 + D_2)(1-\beta) + \mathcal{O}(1-\beta)^2. \tag{3.6}$$

Recall that $P'(z) = \prod_{j=1}^{n}(z - \zeta_j) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$, that each $a_{n-k} = \mathcal{O}(1-\beta)$ and that $\Re[a_{n-3}] = \mathcal{O}(1-\beta)^2$. Then

$$\begin{aligned}
|P|_\beta^{2n} &= (\min_j |\beta - \zeta_j|)^{2n} \leq \prod_{j=1}^{n} |\beta - \zeta_j|^2 = |P'(\beta)|^2 \\
&= P'(\beta)\overline{P}'(\beta) = \beta^{2n} + 2\Re[a_{n-1}]\beta^{2n-1} + 2\Re[a_{n-2}]\beta^{2n-2} + \mathcal{O}(1-\beta)^2 \\
&= 1 - 2n(1-\beta) + 2\Re[a_{n-1}] + 2\Re[a_{n-2}] + \mathcal{O}(1-\beta)^2 \\
&= \left[1 - (1-\beta) + (\Re[a_{n-1}] + \Re[a_{n-2}])/n\right]^{2n} + \mathcal{O}(1-\beta)^2
\end{aligned} \tag{3.7}$$

and so using inequalities (3.7) and then (3.6) we have

$$\begin{aligned}
|P|_\beta &\leq 1 - (1-\beta) + (\Re[a_{n-1}] + \Re[a_{n-2}])/n + \mathcal{O}(1-\beta)^2 \\
&\leq 1 + (D_1 + D_2/n)(1-\beta) + \mathcal{O}(1-\beta)^2.
\end{aligned} \tag{3.8}$$

This completes our first step.

Our second step will be to verify the hypotheses of part 2 of Proposition 10, by showing that

$$\begin{aligned} a_{n-1} &= n(1+D_1+D_2)(1-\beta)+\mathcal{O}(1-\beta)^\alpha, \\ a_{n-2} &= -(n-1)D_2(1-\beta)+\mathcal{O}(1-\beta)^\alpha, \text{ and} \\ a_{n-k} &= \mathcal{O}(1-\beta)^\alpha \text{ for } k \geq 3. \end{aligned}$$

Combining inequalities (3.1) and (3.8), we see that

$$|P|_\beta = 1 + (D_1 + D_2/n)(1-\beta) + \mathcal{O}(1-\beta)^2. \tag{3.9}$$

Since equation (3.8) is thus an equality, then so are equations (3.7) and (3.6), and thus equation (3.5) for $u = u_i$ and equations (3.4) and (3.3) for $\Re[\omega] = u_i$.

Since equation (3.5) is an equality for $u = u_i$, we can solve the resulting linear system in the variables $\Re[a_{n-1}]$ and $\Re[a_{n-2}]$ and get

$$\begin{aligned} \Re[a_{n-1}] &= \frac{-n(u_1+u_2)}{(1-u_1)(1-u_2)}(1-\beta)+\mathcal{O}(1-\beta)^2 \\ &= n(1+D_1+D_2)(1-\beta)+\mathcal{O}(1-\beta)^2 \quad \text{and} \\ \Re[a_{n-2}] &= \frac{n-1}{2(1-u_1)(1-u_2)}(1-\beta)+\mathcal{O}(1-\beta)^2 \\ &= -(n-1)D_2(1-\beta)+\mathcal{O}(1-\beta)^2. \end{aligned}$$

Note that from Proposition 9 we have that $\Re[a_{n-k}] = \mathcal{O}(1-\beta)^2$ for $k \geq 3$, so we now have the correct real parts for our second step. Thus we need only show that each $\Im[a_{n-k}] = \mathcal{O}(1-\beta)^\alpha$.

Recalling the definitions of u_1 and u_2 in Theorem 1, we can choose ω_1 and ω_2 to be $(n+1)$st roots of 1 so that $\Re[\omega_i] = u_i$. For $\omega = \omega_i$, expanding the products in equality (3.3) and cancelling those terms of equality (3.4) gives us

$$\frac{\Im[a_{n-1}]}{n}\Im[\omega_i] + \frac{\Im[a_{n-2}]}{n-1}\Im[\omega_i^2] + \frac{\Im[a_{n-3}]}{n-2}\Im[\omega_i^3] = \mathcal{O}(1-\beta)^2. \tag{3.10}$$

Consider the case $i = 1$. Since $|\omega_1| = 1$ and since by part 1 of Lemma 8 we have $-1/2 \leq u_1 < 1$, then $\Im[\omega_1] \neq 0$. Now by Proposition 9, $\Im[a_{n-k}] = \mathcal{O}(1-\beta)^{3/2}$ for $k \geq 2$, so equation (3.10) implies that $\Im[a_{n-1}] = \mathcal{O}(1-\beta)^{3/2}$. If $n = 3$ or $n = 5$, then by definition $\alpha = 3/2$, so this completes our second step for those two values of n.

Assume then without loss of generality that $n \neq 3, 5$. Again by part 1 of Lemma 8 we have $-1 < u_2 < u_1 < 1$ so $\Im[\omega_i] \neq 0$. Thus we may divide equation (3.10) by $\Im[\omega_i]$ to obtain

$$\frac{\Im[a_{n-1}]}{n} + \frac{\Im[a_{n-2}]}{n-1}(2u_i) + \frac{\Im[a_{n-3}]}{n-2}(4u_i^2-1) = \mathcal{O}(1-\beta)^2. \tag{3.11}$$

Now subtracting equality (3.11) with $i = 2$ from equality (3.11) with $i = 1$ and dividing by $2(u_1 - u_2)$ produces

$$\frac{\Im[a_{n-2}]}{n-1} + \frac{\Im[a_{n-3}]}{n-2}2(u_1+u_2) = \mathcal{O}(1-\beta)^2. \tag{3.12}$$

Since equation (3.7) is an equality, we have each $|\beta - \zeta_j| = |P|_\beta + \mathcal{O}(1-\beta)^2$. Recall that $\Im[a_{n-1}] = \mathcal{O}(1-\beta)^{3/2}$ and that $|P|_\beta = 1 + (D_1 + D_2/n)(1-\beta) +$

$\mathcal{O}(1-\beta)^2$. Then by part 1 of Lemma 13 we have $\Im[a_{n-2}] = (-3/2)\Im[a_{n-3}] + \mathcal{O}(1-\beta)^{5/2}$, so substituting into (3.12) we have

$$\Im[a_{n-3}]\left[\frac{-3/2}{n-1} + \frac{2(u_1+u_2)}{n-2}\right] = \mathcal{O}(1-\beta)^2.$$

Now by part 2 of Lemma 8, we have $u_1 + u_2 < 0$ so the quantity in brackets is non-zero. Then $\Im[a_{n-3}] = \mathcal{O}(1-\beta)^2$, and so solving back in equations (3.12) and (3.11) we find that $\Im[a_{n-k}] = \mathcal{O}(1-\beta)^2$ for all $k \leq 3$. Note that by Proposition 9, we have $a_{n-k} = \mathcal{O}(1-\beta)^2$ for all $k \geq 4$, and so $\Im[a_{n-k}] = \mathcal{O}(1-\beta)^2$ for all k. Since $n \neq 3, 5$, then by definition $\alpha = 2$, and so this finishes the proof of our second step.

We will now finish the proof of Proposition 5. Consider only those roots z of P such that the nearest ω has $\Re[\omega] = u_i$. In our second step, we verified the hypotheses of part 2 of Proposition 10, so we have

$$|z|^{2n+2} = 1 - 2(n+1)\Re[R] + (n+1)(\Gamma_1 + \Gamma_2 u_i)(1-\beta)^2 + \mathcal{O}(1-\beta)^{\alpha+1}.$$

Since $|z| \leq 1$, this implies that

$$-\Re[R] \leq -\frac{\Gamma_1 + \Gamma_2 u_i}{2}(1-\beta)^2 + \mathcal{O}(1-\beta)^{\alpha+1}$$

and so from the definition of R in Proposition 10 we have

$$-\Re\left[a_{n-1}\frac{\overline{\omega}-1}{n} + a_{n-2}\frac{\overline{\omega}^2-1}{n-1} + \cdots + a_0(\omega-1)\right] \leq (1-\beta) - \frac{\Gamma_1+\Gamma_2 u_i}{2}(1-\beta)^2 + \mathcal{O}(1-\beta)^{\alpha+1}.$$

Since $\Re(\overline{\omega}) = u_i$, this inequality is also valid when ω is replaced by $\overline{\omega}$. Note that by Proposition 9 we have $\Re[a_{n-k}] = \mathcal{O}(1-\beta)^3$ for $k \geq 5$, so averaging these two inequalities gives us

$$\frac{\Re[a_{n-1}]}{n}\Re[1-\omega] + \cdots + \frac{\Re[a_{n-4}]}{n-3}\Re[1-\omega^4] \leq (1-\beta) - \frac{\Gamma_1+\Gamma_2 u_i}{2}(1-\beta)^2 + \mathcal{O}(1-\beta)^{\alpha+1}. \tag{3.13}$$

Note that since $|\omega| = 1$, then $\Re[\omega^2] = 2u_i^2 - 1$, $\Re[\omega^3] = 4u_i^3 - 3u_i$ and $\Re[\omega^4] = 8u_i^4 - 8u_i^2 + 1$. Dividing inequality (3.13) by $1 - u_i$, we get

$$\frac{\Re[a_{n-1}]}{n} + \frac{\Re[a_{n-2}]}{n-1}(2+2u_i) + \frac{\Re[a_{n-3}]}{n-2}(1+4u_i+4u_i^2) + \frac{\Re[a_{n-4}]}{n-3}(8u_i^2+8u_i^3) \leq \frac{1-\beta}{1-u_i} - \frac{(\Gamma_1+\Gamma_2 u_i)(1-\beta)^2}{2(1-u_i)} + \mathcal{O}(1-\beta)^{\alpha+1}.$$

Applying to this the linear transformation $\mathcal{T}$ defined in (2.3) and using the values computed in (2.4), we get an inequality of the form

$$\begin{aligned} &\Re[a_{n-1}] + \Re[a_{n-2}] + c_3\Re[a_{n-3}] + c_4\Re[a_{n-4}] \\ &\quad \leq (n + nD_1 + D_2)(1-\beta) \\ &\quad\quad - \left[(\Gamma_1/2)(n+nD_1+D_2) + (\Gamma_2/2)(nD_1+D_2)\right](1-\beta)^2 \\ &\quad\quad + \mathcal{O}(1-\beta)^{\alpha+1}, \end{aligned} \tag{3.14}$$

where $c_3 = \mathcal{T}(1+4u+4u^2)/(n-2)$ and $c_4 = \mathcal{T}(8u^2+8u^3)/(n-3)$.

Define

$$Q = (-\Gamma_1/2)(n + nD_1 + D_2) - (\Gamma_2/2)(nD_1 + D_2) - (n-1)(n-2)(1-c_3)D_2(1 + D_1 + D_2/n). \tag{3.15}$$

Recall from our second step that for all n we have that $\Im[a_{n-1}] = \mathcal{O}(1-\beta)^{3/2}$, and that $\Re[a_{n-2}] = -(n-1)D_2(1-\beta) + \mathcal{O}(1-\beta)^2$, and that each $|\zeta_j - \beta| = 1 + (D_1 + D_2/n)(1-\beta) + \mathcal{O}(1-\beta)^2$. Then by part 2 of Lemma 13, we have

$$\Re[a_{n-3}] + 2\Re[a_{n-4}] = -(n-1)(n-2)D_2(1 + D_1 + D_2/n)(1-\beta)^2 + \mathcal{O}(1-\beta)^3.$$

Adding $1 - c_3$ times this to inequality (3.14) gives us

$$\Re[a_{n-1}] + \Re[a_{n-2}] + \Re[a_{n-3}] + (2 - 2c_3 + c_4)\Re[a_{n-4}] \leq (n + nD_1 + D_2)(1-\beta) + Q(1-\beta)^2 + \mathcal{O}(1-\beta)^{\alpha+1}. \tag{3.16}$$

Note that Lemma 11 implies that $c_3 < 1/2$ for $n \neq 3$, 4, and 6 and that $c_4 \geq 0$ for all n. Using the definition of $\mathcal{T}$ in (2.3), we calculate that for $n = 4$ we have $c_3 = 3/2$ and $c_4 = 4$, and for $n = 6$ we have $c_3 = 0.729$ and $c_4 = 0.972$. Thus for all $n \geq 4$ we have $1 - 2c_3 + c_4 > 0$. Note also that by our second step and Lemma 14 we have $\Re[a_{n-4}] \geq \delta(1-\beta)^2 + \mathcal{O}(1-\beta)^{\alpha+1}$. Since $\delta = 0$ except when $n = 5$, and for $n = 5$ we calculate $c_3 = 1/3$ and $c_4 = 2$, then

$$\begin{aligned} -(1 - 2c_3 + c_4)\Re[a_{n-4}] &\leq -(1 - 2c_3 + c_4)\delta(1-\beta)^2 + \mathcal{O}(1-\beta)^{\alpha+1} \\ &= (-7\delta/3)(1-\beta)^2 + \mathcal{O}(1-\beta)^{\alpha+1}. \end{aligned}$$

Adding this to equation (3.16) gives us

$$\Re[a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}] \leq (n + nD_1 + D_2)(1-\beta) + (Q - 7\delta/3)(1-\beta)^2 + \mathcal{O}(1-\beta)^{\alpha+1}. \tag{3.17}$$

Let

$$\begin{aligned} Q_1 &= -n(1-\beta) + a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4} + a_{n-5} \quad \text{and} \\ Q_2 &= n(n-1)(1-\beta)^2/2 - \big[(n-1)a_{n-1} + (n-2)a_{n-2}\big](1-\beta). \end{aligned}$$

Recall from our first step that each $a_{n-k} = \mathcal{O}(1-\beta)$ so $Q_1 = \mathcal{O}(1-\beta)$ and $Q_2 = \mathcal{O}(1-\beta)^2$.

Now from our second step we know that $a_{n-k} = \mathcal{O}(1-\beta)^\alpha$ for $k \geq 3$, and from Proposition 9 we know that $a_{n-k} = \mathcal{O}(1-\beta)^3$ for $k \geq 6$, so

$$\begin{aligned} P'(\beta) &= \beta^n + a_{n-1}\beta^{n-1} + \cdots + a_0 \\ &= 1 - n(1-\beta) + \frac{n(n-1)}{2}(1-\beta)^2 + a_{n-1}\big[1 - (n-1)(1-\beta)\big] \\ &\qquad + a_{n-2}\big[1 - (n-2)(1-\beta)\big] + a_{n-3} + a_{n-4} + a_{n-5} + \mathcal{O}(1-\beta)^{\alpha+1} \\ &= 1 + Q_1 + Q_2 + \mathcal{O}(1-\beta)^{\alpha+1}. \end{aligned}$$

Then $|P'(\beta)|^2 = P'(\beta)\overline{P'(\beta)} = 1 + 2\Re[Q_1] + 2\Re[Q_2] + |Q_1|^2 + \mathcal{O}(1-\beta)^{\alpha+1}$. Note from our second step that each $\Im[a_{n-k}] = \mathcal{O}(1-\beta)^\alpha$ so $\Im[Q_1] = \mathcal{O}(1-\beta)^\alpha$. Then $(1 + \Re[Q_1] + \Re[Q_2])^2 = |P'(\beta)|^2 + \mathcal{O}(1-\beta)^{\alpha+1}$ and so $|P'(\beta)| = 1 + \Re[Q_1] + \Re[Q_2] +$

$\mathcal{O}(1-\beta)^{\alpha+1}$. Substituting the values of Q_1 and Q_2 and using the results of our second step gives us

$$\begin{aligned}|P'(\beta)| &= 1 - n(1-\beta) + \Re[a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}] \\ &\quad + (n-1)\big[n/2 - n(1+D_1+D_2) + (n-2)D_2\big](1-\beta)^2 \\ &\quad + \mathcal{O}(1-\beta)^{\alpha+1}.\end{aligned} \tag{3.18}$$

Using the first line of inequality (3.7), then inequalities (3.18) and (3.17), we have

$$\begin{aligned}|P|_\beta^n &\le |P'(\beta)| \\ &\le 1 + (nD_1 + D_2)(1-\beta) \\ &\quad + \big[Q - 7\delta/3 - (n-1)(n/2 + nD_1 + 2D_2)\big](1-\beta)^2 \\ &\quad + \mathcal{O}(1-\beta)^{\alpha+1}.\end{aligned} \tag{3.19}$$

We now seek to compute the coefficient of $(1-\beta)^2$ in this inequality. Note first that from the definitions of Γ_1 and Γ_2 in Proposition 10 we have

$$\begin{aligned}-\frac{\Gamma_1}{2}&(n + nD_1 + D_2) - \frac{\Gamma_2}{2}(nD_1 + D_2) \\ &= -\frac{\Gamma_1 + \Gamma_2}{2}(n + nD_1 + D_2) + \frac{n\Gamma_2}{2} \\ &= \left[(1+2D_1)n - \left(\frac{1}{2} + 2D_1 - 2D_2\right)\right]\big[(1+D_1)n + D_2\big] \\ &\qquad + n(1+D_1+D_2)\big[nD_2 + (D_1 - 2D_2)\big].\end{aligned}$$

Now from the definition of c_3 (after inequality (3.14)) combined with equalities (2.4) we have $(n-2)c_3D_2 = -(n+1+D_1+3nD_1+3D_2)$ and so

$$(n-2)(1-c_3)D_2 = (1+3D_1+D_2)n + (1+D_1+D_2).$$

Substituting these values into equation (3.15) and collecting like powers of n, we conclude that

$$\begin{aligned}Q &= \big[-D_1 - D_1^2 + D_2^2\big]n^2 + \left[-\frac{1}{2} + \frac{1}{2}D_1 + D_1^2 - 3D_2^2\right]n \\ &\quad + \left[1 + 2D_1 + \frac{1}{2}D_2 + D_1^2 + D_1D_2 + 2D_2^2\right] + \big[D_2 + D_1D_2 + D_2^2\big]/n\end{aligned} \tag{3.20}$$

and so comparing this with the definition of D in Theorem 1, we see that

$$Q - (n-1)(n/2 + nD_1 + 2D_2) = nD + \frac{n(n-1)}{2}(D_1 + D_2/n)^2. \tag{3.21}$$

Substituting this into inequality (3.19), we have

$$\begin{aligned}|P|_\beta^n &\le 1 + (nD_1 + D_2)(1-\beta) \\ &\quad + \left[nD + \frac{n(n-1)}{2}(D_1 + D_2/n)^2 - 7\delta/3\right](1-\beta)^2 + \mathcal{O}(1-\beta)^{\alpha+1} \\ &= \left[1 + (D_1 + D_2/n)(1-\beta) + \left(D - \frac{7\delta}{3n}\right)(1-\beta)^2\right]^n + \mathcal{O}(1-\beta)^{\alpha+1}.\end{aligned}$$

Note that (from the definitions of δ in Lemma 14 and Δ in Theorem 1) for all n we have $\Delta = -7\delta/(3n)$, and so

$$|P|_\beta \leq 1 + (D_1 + D_2/n)(1-\beta) + (D+\Delta)(1-\beta)^2 + \mathcal{O}(1-\beta)^{\alpha+1}.$$

This completes the proof of Proposition 5.

4. Proof of Proposition 6

This proof parallels the proof of [8, Theorem 2]. We begin by letting

$$u = \frac{-i\sqrt{15}}{15}(1-\beta)^{1/2} - \frac{6}{10}(1-\beta) + \frac{i\sqrt{15}}{300}(1-\beta)^{3/2} - \frac{33}{600}(1-\beta)^2$$

and

$$v = \frac{4i\sqrt{15}}{15}(1-\beta)^{1/2} - \frac{1}{10}(1-\beta) + \frac{46i\sqrt{15}}{300}(1-\beta)^{3/2} + \frac{532}{600}(1-\beta)^2.$$

Let $P'(z) = (z-u)^4(z-v)$ and let $P(z) = \int_\beta^z P'(t)\,dt$. Note that $u - \beta = -1 + u + (1-\beta)$ so

$$\begin{aligned}|u-\beta|^2 &= \big[-1 + (4/10)(1-\beta) - (33/600)(1-\beta)^2\big]^2 \\ &\quad + \big[(-\sqrt{15}/15)(1-\beta)^{1/2} + (\sqrt{15}/300)(1-\beta)^{3/2}\big]^2 \\ &= 1 - (11/15)(1-\beta) + (79/300)(1-\beta)^2 + \mathcal{O}(1-\beta)^3\end{aligned}$$

and $v - \beta = -1 + v + (1-\beta)$ so

$$\begin{aligned}|v-\beta|^2 &= \big[-1 + (9/10)(1-\beta) + (532/600)(1-\beta)^2\big]^2 \\ &\quad + \big[(4\sqrt{15}/15)(1-\beta)^{1/2} + (46\sqrt{15}/300)(1-\beta)^{3/2}\big]^2 \\ &= 1 - (11/15)(1-\beta) + (79/300)(1-\beta)^2 + \mathcal{O}(1-\beta)^3.\end{aligned}$$

Now

$$\begin{aligned}&\big[1 - (11/30)(1-\beta) + (29/450)(1-\beta)^2\big]^2 \\ &\qquad = 1 - (11/15)(1-\beta) + (79/300)(1-\beta)^2 + \mathcal{O}(1-\beta)^3,\end{aligned}$$

and so we have

$$\begin{aligned}|P|_\beta &= \min\{|u-\beta|, |v-\beta|\} \\ &= 1 - (11/30)(1-\beta) + (29/450)(1-\beta)^2 + \mathcal{O}(1-\beta)^3.\end{aligned}$$

By definition P is of degree 6 and $P(\beta) = 0$. Thus to verify that $P \in S(6,\beta)$ we need only show that all the roots of P remain in the closed unit disk when β is sufficiently close to 1. Now

$$\begin{aligned}u^2 &= (-1/15)(1-\beta) + (2i\sqrt{15}/25)(1-\beta)^{3/2} + \mathcal{O}(1-\beta)^2, \\ u^3 &= (i\sqrt{15}/225)(1-\beta)^{3/2} + \mathcal{O}(1-\beta)^2, \quad \text{and} \\ u^4 &= \mathcal{O}(1-\beta)^2,\end{aligned}$$

so writing $P'(z) = z^5 + a_4 z^4 + \cdots + a_0$, we calculate that

$$\begin{aligned} a_4 &= -(4u+v) = (5/2)(1-\beta) - (i\sqrt{15}/6)(1-\beta)^{3/2} - (2/3)(1-\beta)^2 \\ a_3 &= u(6u+4v) \\ &= (2/3)(1-\beta) - (2i\sqrt{15}/15)(1-\beta)^{3/2} + 3(1-\beta)^2 + \mathcal{O}(1-\beta)^{5/2} \\ a_2 &= -u^2(4u+6v) = (4i\sqrt{15}/45)(1-\beta)^{3/2} + (7/5)(1-\beta)^2 + \mathcal{O}(1-\beta)^{5/2} \\ a_1 &= u^3(u+4v) = (-1/15)(1-\beta)^2 + \mathcal{O}(1-\beta)^{5/2} \\ a_0 &= -u^4 v = \mathcal{O}(1-\beta)^{5/2}. \end{aligned}$$

Recall from the values computed at the beginning of section 2 that for $n = 5$ we have $\alpha = 3/2$, $u_1 = -1/2$, $u_2 = -1$, $D_1 = -1/3$ and $D_2 = -1/6$. Note that in part 2 of Proposition 10 the values of the a_k's computed above satisfy the hypotheses, and that $\Gamma_2 = -5/6$ and $\Gamma_1 = -13/6$.

Let us apply part 2 of Proposition 10 to the case $\omega = -1$. Note that $\Re[\omega] = u_2$ and $\Gamma_1 + \Gamma_2 u_2 = -4/3$. Since $\omega = -1$ we have

$$R = (1-\beta) - (2/5)a_4 - (2/3)a_2 - 2a_0,$$

and so

$$\begin{aligned} \Re[R] &= (1-\beta) - (2/5)\big[(5/2)(1-\beta) - (2/3)(1-\beta)^2\big] \\ &\quad - (2/3)(7/5)(1-\beta)^2 + \mathcal{O}(1-\beta)^{5/2} \\ &= (-2/3)(1-\beta)^2 + \mathcal{O}(1-\beta)^{5/2}. \end{aligned}$$

Thus by part 2 of Proposition 10 we have

$$\begin{aligned} |z|^{12} &= 1 - 12(-2/3)(1-\beta)^2 + 6(-4/3)(1-\beta)^2 + \mathcal{O}(1-\beta)^{5/2} \\ &= 1 + \mathcal{O}(1-\beta)^{5/2}, \end{aligned}$$

and so $|z| = 1 + \mathcal{O}(1-\beta)^{5/2}$.

Let us now apply part 2 of Proposition 10 to the case $\omega = (1/2)(-1 \pm i\sqrt{3})$. Note that $\Re[\omega] = u_1$ and $\Gamma_1 + \Gamma_2 u_1 = -7/4$. Now

$$\begin{aligned} R = (1-\beta) &+ (a_4/10)(-3 \mp i\sqrt{3}) + (a_3/8)(-3 \pm i\sqrt{3}) \\ &+ (a_1/4)(-3 \mp i\sqrt{3}) + (a_0/2)(-3 \pm i\sqrt{3}) \end{aligned}$$

so

$$\begin{aligned} \Re[R] &= (1-\beta) - (3/10)\big[(5/2)(1-\beta) - (2/3)(1-\beta)^2\big] \\ &\quad \pm (\sqrt{3}/10)(-\sqrt{15}/6)(1-\beta)^{3/2} - (3/8)\big[(2/3)(1-\beta) + 3(1-\beta)^2\big] \\ &\quad \mp (\sqrt{3}/8)(-2\sqrt{15}/15)(1-\beta)^{3/2} - (3/4)(-1/15)(1-\beta)^2 + \mathcal{O}(1-\beta)^{5/2} \\ &= (-7/8)(1-\beta)^2 + \mathcal{O}(1-\beta)^{5/2}. \end{aligned}$$

Thus by part 2 of Proposition 10 we have

$$\begin{aligned} |z|^{12} &= 1 - 12(-7/8)(1-\beta)^2 + 6(-7/4)(1-\beta)^2 + \mathcal{O}(1-\beta)^{5/2} \\ &= 1 + \mathcal{O}(1-\beta)^{5/2}, \end{aligned}$$

so $|z| = 1 + \mathcal{O}(1-\beta)^{5/2}$.

Finally, let us apply part 1 of Proposition 10 with $r = 1$ to the case $\omega = (1/2)(1 \pm i\sqrt{3})$. Note that

$$R = (1-\beta) + (a_4/10)(-1 \mp i\sqrt{3}) + (a_3/8)(-3 \mp i\sqrt{3}) + \mathcal{O}(1-\beta)^{3/2}$$

so

$$\begin{aligned}\Re[R] &= (1-\beta) + (-1/10)(5/2)(1-\beta) + (-3/8)(2/3)(1-\beta) + \mathcal{O}(1-\beta)^{3/2}\\ &= (1/2)(1-\beta) + \mathcal{O}(1-\beta)^{3/2}.\end{aligned}$$

Thus by part 1 of Proposition 10 we have $|z|^2 = 1 - (1-\beta) + \mathcal{O}(1-\beta)^{3/2}$ and so $|z| = 1 - (1/2)(1-\beta) + \mathcal{O}(1-\beta)^{3/2}$.

At this stage, we know that $|P|_\beta = 1 - (11/30)(1-\beta) + (29/450)(1-\beta)^2 + \mathcal{O}(1-\beta)^3$ and that if β is sufficiently close to 1, then all roots z of P have $|z| \leq 1 + \mathcal{O}(1-\beta)^{5/2}$. Since the roots of P approach the roots of $z^6 - 1$, then the non-β roots of P are bounded away from β. Thus by Lemma 12, there is a polynomial $Q \in S(6,\beta)$ with $|Q|_\beta = 1 - (11/30)(1-\beta) + (29/450)(1-\beta)^2 + \mathcal{O}(1-\beta)^{5/2}$. This completes the proof of Proposition 6.

5. Proof of Proposition 7

Let $b_1 = 1 + D_1 + D_2/n$, let $b_2 = (n-1)D_2$, and let $z_0 = -b_1(1-\beta) - D(1-\beta)^2$. Then $z_0 - \beta = -1 + (1-b_1)(1-\beta) - D(1-\beta)^2$, and (for β near 1) this is real and negative so $|z_0 - \beta| = 1 + (D_1 + D_2/n)(1-\beta) + D(1-\beta)^2$.

Now let x be a real constant, depending only on n (and to be determined later), and let

$$\begin{aligned}q(z) = z^2 &+ \big[(b_2 + 2b_1)(1-\beta) - 2x(1-\beta)^2\big]z\\ &+ \big[-b_2(1-\beta) + (b_1^2 + b_2 + 2D + 2x)(1-\beta)^2\big].\end{aligned}$$

Now by part 4 of Lemma 8 we have $D_2 < 0$ and so $b_2 < 0$. Since the discriminant of $q(z)$ is $4b_2(1-\beta) + \mathcal{O}(1-\beta)^2$, then (for β near 1) the roots of q are complex conjugates. If we denote these roots by z_1 and $\overline{z}_1$, then by writing $\beta = 1 - (1-\beta)$ we have

$$\begin{aligned}|z_1 - \beta|^2 &= (z_1 - \beta)(\overline{z}_1 - \beta) = q(\beta)\\ &= 1 + (2b_1 - 2)(1-\beta) + (1 - 2b_1 + b_1^2 + 2D)(1-\beta)^2 + \mathcal{O}(1-\beta)^3\\ &= \big[1 + (b_1 - 1)(1-\beta) + D(1-\beta)^2\big]^2 + \mathcal{O}(1-\beta)^3,\end{aligned}$$

so $|z_1 - \beta| = 1 + (D_1 + D_2/n)(1-\beta) + D(1-\beta)^2 + \mathcal{O}(1-\beta)^3$.

Let $P'(z) = (z - z_0)^{n-2}q(z)$ and $P(z) = \int_\beta^z P'(t)\,dt$, so

$$|P|_\beta = 1 + (D_1 + D_2/n)(1-\beta) + D(1-\beta)^2 + \mathcal{O}(1-\beta)^3.$$

Now $z_0 = \mathcal{O}(1-\beta)$, so

$$\begin{aligned}(z - z_0)^{n-2} &= z^{n-2} - (n-2)z_0 z^{n-3} + \binom{n-2}{2} z_0^2 z^{n-4} + \mathcal{O}(1-\beta)^3\\ &= z^{n-2} + (n-2)\big[b_1(1-\beta) + D(1-\beta)^2\big]z^{n-3}\\ &\quad + \binom{n-2}{2} b_1^2 (1-\beta)^2 z^{n-4} + \mathcal{O}(1-\beta)^3.\end{aligned}$$

Then letting $t_1 = (n^2 - n)b_1^2/2 + (n-2)b_1 b_2 + b_2$ we have

$$\begin{aligned} P'(z) &= (z - z_0)^{n-2} q(z) \\ &= z^n + \big[(nb_1 + b_2)(1-\beta) + (nD - 2D - 2x)(1-\beta)^2\big] z^{n-1} \\ &\quad + \big[-b_2(1-\beta) + (t_1 + 2D + 2x)(1-\beta)^2\big] z^{n-2} \\ &\quad - (n-2)b_1 b_2 (1-\beta)^2 z^{n-3} + \mathcal{O}(1-\beta)^3. \end{aligned} \tag{5.1}$$

Note that by its definition, P is a polynomial of degree $n+1$ and $P(\beta) = 0$. Thus to show that $P \in S(n+1, \beta)$ it will suffice to show that all roots of P remain in the unit disk when β is sufficiently close to 1.

Let $\omega \neq 1$ be an $(n+1)$th root of 1, let $u = \Re[\omega]$ and note that since $|\omega| = 1$, then $\Re[\omega^2] = 2u^2 - 1$, $\Re[\omega^3] = 4u^3 - 3u$, and $\omega^{n-k} = \overline{\omega}^{k+1}$. Substituting the coefficients of equation (5.1) into the formula for R in Proposition 10, we have

$$R = (1-\beta) + (nb_1 + b_2)(1-\beta)(\overline{\omega} - 1)/n - b_2(1-\beta)(\overline{\omega}^2 - 1)/(n-1) + \mathcal{O}(1-\beta)^2.$$

Substituting the values of b_1 and b_2 into this formula, we see by part 1 of Proposition 10 with $r = 1$ that

$$|z|^2 = 1 - 2(1-\beta)\big[1 + (1 + D_1 + D_2)(u-1) - D_2(2u^2 - 2)\big] + \mathcal{O}(1-\beta)^2.$$

Recall from part 4 of Lemma 8 that $D_2 < 0$, so the quantity in square brackets is quadratic in u with positive leading coefficient. By elementary calculus, its minimum (over all real numbers) occurs when $1 + D_1 + D_2 - 4D_2 u = 0$, which happens when $u = (1 + D_1 + D_2)/(4D_2) = (u_1 + u_2)/2$, which is between u_1 and u_2. Now u_1 and u_2 are (by definition) the real parts of adjacent $(n+1)$th roots of 1, so there are no possible values of u between u_1 and u_2, so the minimum (over all possible values of u) must occur at either u_1 or u_2. From part 7 of Lemma 8 we see that at these values the quantity in square brackets is 0, and so the minimum value of the quantity in square brackets is 0. Thus for $\Re[\omega] \neq u_i$ the quantity in square brackets is positive, so for these values of ω and for β sufficiently close to 1 we have $|z| < 1$, and so these roots remain in the unit disk.

Thus we need only concern ourselves with the case $\Re[\omega] = u_i$. In this case, by part 2 of Proposition 10 we have

$$|z|^{2n+2} = 1 - 2(n+1)\Re[R] + (n+1)(\Gamma_1 + \Gamma_2 u_i)(1-\beta)^2 + \mathcal{O}(1-\beta)^{\alpha+1}.$$

To get $P \in S(n+1, \beta)$ we will seek a value of x so that $|z| = 1 + \mathcal{O}(1-\beta)^{\alpha+1}$, so we will need

$$\Re[R] - (1/2)(\Gamma_1 + \Gamma_2 u_i)(1-\beta)^2 = \mathcal{O}(1-\beta)^{\alpha+1} \tag{5.2}$$

for both $i = 1$ and $i = 2$.

Substituting the coefficients of equation (5.1) into the formula for R in Proposition 10, we have

$$\begin{aligned} R = (1-\beta) &+ \big[(nb_1 + b_2)(1-\beta) + (nD - 2D - 2x)(1-\beta)^2\big](\overline{\omega} - 1)/n \\ &+ \big[-b_2(1-\beta) + (t_1 + 2D + 2x)(1-\beta)^2\big](\overline{\omega}^2 - 1)/(n-1) \\ &- (n-2)b_1 b_2 (1-\beta)^2 (\overline{\omega}^3 - 1)/(n-2) + \mathcal{O}(1-\beta)^3. \end{aligned} \tag{5.3}$$

Taking the real parts of equation (5.3) and collecting like powers of $(1-\beta)$ gives us

$$\begin{aligned}\Re[R] &= \bigl[1 + (nb_1 + b_2)(u_i - 1)/n - b_2(2u_i^2 - 2)/(n-1)\bigr](1-\beta) \\ &\quad + \Bigl[(nD - 2D - 2x)(u_i - 1)/n + (t_1 + 2D + 2x)(2u_i^2 - 2)/(n-1) \\ &\qquad - b_1 b_2(4u_i^3 - 3u_i - 1)\Bigr](1-\beta)^2 + \mathcal{O}(1-\beta)^3.\end{aligned}$$

Substituting the values of b_1 and b_2 into this formula, we see from part 7 of Lemma 8 that the coefficient of $(1-\beta)$ in $\Re[R]$ is zero, so to satisfy equation (5.2) we need only find a value of x such that the coefficient of $(1-\beta)^2$ in equation (5.2) is 0. We divide this coefficient by $u_i - 1$ and denote the result by Z_i, so

$$\begin{aligned}Z_i &= (nD - 2D - 2x)/n + (t_1 + 2D + 2x)(2u_i + 2)/(n-1) \\ &\quad - (n-1)D_2(1 + D_1 + D_2/n)(4u_i^2 + 4u_i + 1) + (1/2)(\Gamma_1 + \Gamma_2 u_i)/(1 - u_i).\end{aligned} \tag{5.4}$$

Note that the coefficient of x in Z_i is $-2/n + (4u_i + 4)/(n-1)$, which is non-zero by part 3 of Lemma 8, so each equation $Z_i = 0$ has a solution for x. To show that these solutions are identical, we will show that Z_1 and Z_2 (considered as linear expressions in the variable x) are scalar multiples of each other.

To see this, we eliminate x by applying the transformation $\mathcal{T}$ defined in equation (2.3). Since in equation (3.14) we defined $c_3 = \mathcal{T}(1 + 4u + 4u^2)/(n-2)$, then from equations (2.4) we see that

$$\begin{aligned}\mathcal{T}(Z_i) &= nD + t_1 - (n-1)(n-2)c_3 D_2(1 + D_1 + D_2/n) \\ &\quad + (\Gamma_1/2)(n + nD_1 + D_2) + (\Gamma_2/2)(nD_1 + D_2).\end{aligned} \tag{5.5}$$

Comparing this to the value of Q defined in equation (3.15), we see that

$$\mathcal{T}(Z_i) = nD + t_1 - Q - (n-1)(n-2)D_2(1 + D_1 + D_2/n). \tag{5.6}$$

Note that by equation (3.21) we have

$$Q = nD + \frac{n(n-1)}{2}(D_1 + D_2/n)^2 + (n-1)(n/2 + nD_1 + 2D_2).$$

Substituting the values of b_1 and b_2 into our definition of t_1 gives us

$$t_1 = (n-1)\bigl[(n/2)(1 + D_1 + D_2/n)^2 + (n-2)D_2(1 + D_1 + D_2/n) + D_2\bigr]$$

and so $Q - t_1 = nD - (n-1)(n-2)D_2(1 + D_1 + D_2/n)$. Substituting this into equation (5.6) gives us $\mathcal{T}(Z_i) = 0$. Since $\mathcal{T}(Z_i)$ is a linear combination of Z_1 and Z_2, this implies that Z_1 and Z_2 (considered as polynomials in x) are scalar multiples of one another, and so there is a single value of x that satisfies equation (5.2) for both $i = 1$ and $i = 2$.

Using this value of x, we have now constructed a real polynomial P with

$$|P|_\beta = 1 + (D_1 + D_2/n)(1-\beta) + D(1-\beta)^2 + \mathcal{O}(1-\beta)^3$$

and such that all roots z of P have $|z| \le 1 + \mathcal{O}(1-\beta)^{\alpha+1}$. Since the roots of P approach the roots of $z^{n+1} - 1$, then the non-β roots of P are bounded away from β. Thus by Lemma 12, there is a real polynomial $Q \in S(n+1, \beta)$ with

$$|Q|_\beta = 1 + (D_1 + D_2/n)(1-\beta) + D(1-\beta)^2 + \mathcal{O}(1-\beta)^{\alpha+1}.$$

This finishes the proof of Proposition 7.

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