ON DEGREES OF IRREDUCIBLE BRAUER CHARACTERS

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Abstract. Based on a large amount of examples, which we have checked so far, we conjecture that

\[ |G| \Phi_1(1) \leq \sum_{\varphi \in \text{IBr}_p(G)} \varphi(1)^2 \]

where \( p \) is a prime and the sum runs through the set of irreducible Brauer characters in characteristic \( p \) of the finite group \( G \). We prove the conjecture simultaneously for \( p \)-solvable groups and groups of Lie type in the defining characteristic. In non-defining characteristics we give asymptotically an affirmative answer in many cases.

1. Introduction

Let \( G \) be a finite group and let \( \text{IBr}_p(G) \), resp. \( \text{IBr}_p(B) \), be the set of irreducible \( p \)-Brauer characters of \( G \), resp. of a \( p \)-block \( B \). For \( \varphi \in \text{IBr}_p(G) \) let \( \Phi_\varphi \) denote the projective indecomposable character corresponding to \( \varphi \). Due to a result of Brauer and Nesbitt [6] the term \( |G| - \frac{|G|}{\Phi_1(1)} \), where \( \Phi_1 \) denotes the projective character corresponding to the trivial character, is an upper bound for the dimension of the Jacobson radical of the \( p \)-modular group algebra of \( G \). An obvious reformulation of this result leads to

\[ \frac{|G|}{\Phi_1(1)} \leq \sum_{\varphi \in \text{IBr}_p(G)} \varphi(1)^2. \]

Moreover, equality holds if and only if \( G \) has a normal Sylow \( p \)-subgroup. This was proved by Wallace in [18] for \( p \)-solvable groups, and in full generality by Brockhaus in [7] using the classification of finite simple groups.

In case \( p = 2 \) the classification can be avoided to show the normality of a Sylow \( p \)-subgroup if equality holds in (1). Since \( \Phi_\varphi \) is a constituent of \( \varphi \Phi_1 \), equality in (1) implies that \( \Phi_\varphi = \varphi \Phi_1 \). Now, if \( p \mid \varphi(1) \), the trivial character has multiplicity at least 2 in \( \varphi \Phi_1 \), and a contradiction follows by considering the scalar product relation

\[ 1 = (\varphi, \Phi_\varphi) = (\varphi, \varphi \Phi_1) = (\varphi \Phi_1, \Phi_1) \geq 2 \]

(see [13], VII, 8.5 d)). Thus \( p \) does not divide any irreducible Brauer character degree, and a nice argument of Okuyama (see [16], Theorem 2.33) implies that for \( p = 2 \) the Sylow \( p \)-subgroup of \( G \) is normal.

The lower bound in (1) has been improved in [14], replacing \( \Phi_1(1) \) by the spectral radius \( \rho(C) \) of the Cartan matrix \( C \), i.e. the maximum value \( |\lambda| \) where \( \lambda \) runs through the set of complex eigenvalues of \( C \).

However, for obvious reasons we are interested in a lower bound which only depends on terms of \( G \). This is a weaker question than the one Brauer asked in

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Problem 15 of his famous list [3]. He actually wanted to have a characterization of
the dimension of the Jacobson radical by group-theoretical properties.

In the case of \( p \)-solvable groups (the existence of a \( p \)-complement suffices) we
have \( \Phi_1(1) = |G|_p \) (see [11], Chap. X, 3.2). Hence the desired bound is

\[
|G|_p' \leq \sum_{\varphi \in \text{IBr}_p(G)} \varphi(1)^2.
\]

But for non-\( p \)-solvable groups, \( \Phi_1(1) \) (and \( \rho(C) \)) may differ extremely from \( |G|_p \).
For instance, let \( G = R(q) \) with \( q = 3^{2m+1} \) (\( m \geq 1 \)) be a Ree group and let \( p = 2 \).
Then \( \Phi_1(1) = 2(q^3 + 1) \gg 8 \) and \( \rho(C) \approx 16.38 \) (see [15] for the Cartan matrix),
but \( |G|_2 = 8 \) and moreover

\[
\frac{|G|}{8} = |G|_2' \leq \sum_{\varphi \in \text{IBr}_p(G)} \varphi(1)^2.
\]

(In order to prove the inequality it is enough to consider the \( q^2 - 1 \) characters of
defect zero of degree \( (q^2 - 1)(q + 1 + 3m) \), see [19].) Nevertheless for the sporadic
groups that we have checked so far, the bound in (2) holds true. The same bound
turns out to be true if the Sylow \( p \)-subgroups of \( G \) are cyclic [14]. So for any finite

group \( G \) we are led to the

Conjecture. We always have

\[
|G|_p' \leq \sum_{\varphi \in \text{IBr}_p(G)} \varphi(1)^2,
\]

and equality holds if and only if \( G \) has a normal Sylow \( p \)-subgroup.

Note that the if part is trivial, since a normal \( p \)-subgroup is always contained in
the kernel of an irreducible representation in characteristic \( p \).

Suppose that the conjecture has an affirmative answer. Thus in the extreme case
\( |G|_p' = \sum_{\varphi \in \text{IBr}_p(G)} \varphi(1)^2 \) the group \( G \) has a normal Sylow \( p \)-subgroup, say \( P \). By
Lemma 4.26 of ([11], Chapter IV), we know that

\[
\Phi_\varphi(x) = |C_P(x)| \varphi(x)
\]

for \( p' \)-elements \( x \in G \). On the other hand, by Lemma 3.8 of ([11], Chapter IV), we have

\[
\sum_{\varphi \in \text{IBr}_p(G)} \Phi_\varphi(x) \overline{\varphi(y)} = \begin{cases} 0 & \text{if } x \not\sim y, \\ |C_G(x)| & \text{if } x \sim y, \end{cases}
\]

for \( p' \)-elements \( x, y \in G \) where \( \sim \) denotes conjugation in \( G \). Thus we get the relation

\[
\sum_{\varphi \in \text{IBr}_p(G)} \varphi(x) \overline{\varphi(y)} = \begin{cases} 0 & \text{if } x \not\sim y, \\ \frac{|C_G(x)|}{|C_P(x)|} & \text{if } x \sim y. \end{cases}
\]

So we may ask the

Question. Is it even true that for \( p' \)-elements \( x \in G \) we always have

\[
\frac{|C_G(x)|}{|G|_p} \leq \sum_{\varphi \in \text{IBr}_p(G)} |\varphi(x)|^2 ?
\]
If so, the bound may be sharp for \( x \neq 1 \) even if the Sylow \( p \)-subgroup is not normal. For instance, if \( p = 5 \) and \( x \) is an element of order 3 in \( G = L_2(16) \), then

\[
3 = \frac{15}{5} = \frac{|C_G(x)|}{|G|_5} = \sum_{\varphi \in \text{IBr}_5(G)} \varphi(x)\varphi(x) = 1^2 + 1^2 + (-1)^2 = 3.
\]

Note that the inequality (4) reduces to (3) for \( x = 1 \).

We are furthermore tempted to ask a \( p \)-local version of (3). In this case we may replace

\[
|G|_p = \frac{|G|}{p^n} = \frac{\dim KG}{p^n}
\]

by \( \frac{\dim B}{p^d} \) where \( K \) is a splitting field of \( G \) of characteristic \( p \) and \( B \) is a \( p \)-block of defect \( d \). Thus we ask whether

\[
\frac{\dim B}{p^d} \leq \sum_{\varphi \in \text{IBr}_p(B)} \varphi(1)^2?
\]

An affirmative answer to (5) was given by Kiyota and Wada in [14] in case \( G \) is \( p \)-solvable or \( B \) is a \( p \)-block with cyclic defect group. In the latter case equality holds if and only if the Brauer tree is a star and all irreducible Brauer characters have the same degree.

Unfortunately the principal 2-block \( B_0 \) of the alternating group \( A_5 \) shows that inequality (5) fails to be true in general, because

\[
11 = \frac{\dim B_0}{2^4} > \sum_{\varphi \in \text{IBr}_{2^4}(B)} \varphi(1)^2 = 1 + 2^2 + 2^2 = 9.
\]

A more advanced counterexample is the non-principal 3-block of maximal defect of \( 6.A_7 \). This example was brought to my attention by Thomas Breuer.

2. On the conjecture

In this section we prove the conjecture for groups which have a projective character with properties similar to those of the Steinberg character for groups of Lie type. We start with a result due to Alperin. Let \( \Phi \) be any projective character of a finite group \( G \) and let \( \text{IBr}(G) = \{ \varphi_1, \varphi_2, \ldots, \varphi_s \} \). Then \( \Phi \) uniquely determines a matrix \( A = (a_{ij}) \) by the equations

\[
\varphi_i \Phi = \sum_{j=1}^s a_{ij} \varphi_j \quad (i = 1, \ldots, s).
\]

If \( x_1, \ldots, x_s \) are representatives of the \( p' \)-conjugacy classes of \( G \) and if \( C \) denotes the Cartan matrix of \( G \), then we have

**Lemma 2.1** (II).

\[
\det A = \frac{\prod_{i=1}^s \Phi(x_i)}{\det C}.
\]

**Proof.** For \( C = (c_{ij}) \), we have

\[
\varphi_i \Phi = \sum_{j,k=1}^s a_{ij} c_{jk} \varphi_k.
\]
Therefore we get the matrix equation
\[
\begin{pmatrix}
\Phi(x_1) & 0 & \cdots & 0 \\
0 & \Phi(x_2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \Phi(x_s)
\end{pmatrix}
= AC(\varphi_i(x_l)).
\]
Since \((\varphi_i(x_l))\) and \(C\) are invertible ([11], Chap. IV, 3.6), the assertion follows.

**Theorem 2.2.** Let \(G\) be a finite group. Suppose that \(G\) has a projective character \(\Phi\) satisfying the following two conditions.

(a) \(\Phi(1) = |G|p\).
(b) \(\Phi(x) \neq 0\) for all \(p'\)-elements \(x \in G\).

Then
\[|G|p' \leq \sum_{i=1}^{s} \varphi_i(1)^2,\]
and equality holds if and only if \(G\) has normal Sylow \(p\)-subgroup.

**Proof.** Condition (b) implies that \(\det A \neq 0\) in the above lemma. Thus there exists a permutation \(\pi\) of \(\{1, \ldots, s\}\) such that
\[\prod_{i=1}^{s} a_{\pi_i, i} \neq 0.\]
This means \(\varphi_{\pi_i} \Phi = \Phi_i + \cdots\). If \(\rho\) denotes the regular character of \(G\), then
\[\sum_{i=1}^{s} \varphi_i(1) \varphi_{\pi_i}(1) \Phi = \sum_{i=1}^{s} \varphi_i(1) \Phi_i + \cdots = \rho + \cdots.\]
In particular we have
\[\sum_{i=1}^{s} \varphi_i(1) \varphi_{\pi_i}(1) \Phi(1) \geq \rho(1) = |G|,\]
and by condition (a)
\[\sum_{i=1}^{s} \varphi_i(1) \varphi_{\pi_i}(1) \geq |G|p'.\]
Applying the Cauchy-Schwarz inequality yields
\[\left(\sum_{i=1}^{s} \varphi_i(1) \varphi_{\pi_i}(1)\right)^2 \leq \sum_{i=1}^{s} \varphi_i(1)^2 \sum_{i=1}^{s} \varphi_{\pi_i}(1)^2,\]
and hence
\[\sum_{i=1}^{s} \varphi_i(1) \varphi_{\pi_i}(1) \leq \sum_{i=1}^{s} \varphi_i(1)^2.\]
Putting (7) and (9) together, we obtain the desired assertion
\[|G|p' \leq \sum_{i=1}^{s} \varphi_i(1)^2.\]
So it remains to prove that \(|G|p' = \sum_{i=1}^{s} \varphi_i(1)^2\) implies that a Sylow \(p\)-subgroup of \(G\) is normal.
By (6) we get \( \varphi_i \Phi = \Phi_i \) for all \( i \). In particular we have \( \varphi_1 \Phi = \Phi_1 \) for the trivial character \( \varphi_1 \). The equality in the Cauchy-Schwarz inequality (see (8)) forces \( \varphi_1(1) = \varphi_1(1) = 1 \). Thus we obtain \( \Phi_1(1) = |G|_p \) and by a result of Brockhaus [7] the group \( G \) has a normal Sylow \( p \)-subgroup.

The properties we required on the projective character in 2.2 seem to be very strong. However there are interesting examples for which the theorem applies.

Examples 2.3. a) Let \( G \) be a finite group with a split \( BN \)-pair of characteristic \( p \) and satisfying the commutator relations (see [9]). For \( \Phi \) we take the Steinberg character. Then conditions a) and b) are satisfied, since

\[
\Phi(x) = |C_G(x)|_p
\]

for all \( p' \)-elements \( x \) of \( G \) (see [9], 6.4.7). Note that for Chevalley groups (twisted or non-twisted) defined in characteristic \( p \), the square of the degree of the Steinberg character already dominates the \( p' \)-part of the order of \( G \).

b) Let \( G \) be a finite group with a \( p \)-complement. Now we may take for \( \Phi \) the projective indecomposable character corresponding to the trivial character. Observe that \( \Phi \) is the trivial character of a \( p \)-complement induced to \( G \) and satisfies the assumptions of 2.2.

c) More subtle is the following. Let \( N \) be a normal \( p \)-solvable subgroup of \( G \), and suppose that \( H = G/N \) possess a projective character satisfying conditions a) and b). We prove that \( G \) has a projective character with the same properties. So let \( M \leq N \) be a minimal normal subgroup of \( G \). By the inductive hypothesis the group \( G/M \) has a projective character, say \( \Psi \), satisfying a) and b). If \( M \) is a \( p' \)-group, we may take for \( \Phi \) the inflation \( \Phi = \text{infl}_G(\Psi) \) of \( \Psi \). Thus suppose that \( M \) is a \( p \)-group. Let \( \lambda \) be the Brauer character of the conjugation action of \( G \) on \( M \). Now we put

\[
\Phi = \lambda \text{infl}_G(\Psi).
\]

That \( \Phi \) is projective follows from a result of Alperin, Collins and Sibley (see [2]). Clearly,

\[
\Phi(1) = \lambda(1) \text{infl}_G(\Psi)(1) = |M| |H| = |G|_p
\]

and

\[
\Phi(x) = \lambda(x) \text{infl}_G(\Psi)(x) = |C_M(x)| \text{infl}_G(\Psi)(x) \neq 0
\]

for all \( p' \)-elements \( x \in G \).

3. Asymptotic results for groups of Lie type

To give more evidence for inequality (3) we will look in this section asymptotically on groups of Lie type in non-defining characteristics. For a detailed introduction to the character theory of groups of Lie type, the reader is referred to [9].

Let \( G \) be a simple linear algebraic group over an algebraically closed field \( K \) of positive characteristic \( r \neq p \), and let \( G^F = G(q) \) (where \( q = r^f \)) denote the group of fixed points under a Frobenius map \( F \). Furthermore let \( T_w = T^F \) be the maximal torus in \( G^F \) corresponding to \( w \in W = W(T) \) where \( W \) denotes the Weyl group of an \( F \)-stable maximal torus \( T \subseteq G \).

Now we look at irreducible Deligne-Lusztig characters \( R_{T,\theta} \) (up to a sign) where \( \theta \in \hat{T}^F = \text{Hom}(T^F, \mathbb{C}^*) \) is an irreducible character of \( T_w = T^F \) and \( \theta \) is in general
position. Let $\mathcal{R}(q)$ denote the set of all such $R_{T,\theta}$’s. Note that $R_{T,\theta} = R_{T,\theta'}$ if and only if $\theta = \theta'$ for some

$$w' \in W(T)^F = C_{w,F} = \{ w' \in W \mid w'^{-1}wF(w') = w \}$$

(see [9], p. 219). Thus the number $|\mathcal{R}(q)|$ of different $R_{T,\theta}$’s is equal to the number of $W(T)^F$-orbits of characters $\theta$ in general position. By a result of Veldkamp (17) this number is equal to the number of $W(T)^F$-conjugacy classes of elements in general position in the fixed point group $T^*F^*$ of the dual torus $T^* \subseteq G^*$.

For simplicity let us suppose that $G^*$ is simply connected. In this case the regular elements in $T^*F^*$ coincide with the elements in general position. (Note that $t \in G$ is regular (resp. in general position) if $C_G(t)^g = T$ (resp. $C_G(t) = T$).) Let $l$ denote the Lie rank of $G$. Applying 3.1 and 3.2 in [10] (see also [12]), we get

$$|\mathcal{R}(q)| = \left| R_{T,\theta} \mid \theta \text{ in general position} \right| = \frac{q^l}{|C_{w,F}|} + O(q^{l-1}).$$

Note that $R_{T,\theta}$ is of degree

$$R_{T,\theta}(1) = \frac{|G(q)|\omega'}{|T_w|}.$$

Thus we obtain

$$\sum_{R_{T,\theta} \in \mathcal{R}(q)} R_{T,\theta}(1)^2 = \left( \frac{q^l}{|C_{w,F}|} + O(q^{l-1}) \right) \frac{|G(q)|^2}{|T_w|^2}.$$  

(10)

Let $N$ denote the number of positive roots. As polynomials in $q$ we have

- $|T_w| = q^l + \text{lower terms in } q$,
- $|G(q)|_{\omega'} = q^{l+N} + \text{lower terms in } q$,
- $|G(q)| = q^{l+2N} + \text{lower terms in } q$.

Inserting these equations in (10), we obtain

$$\sum_{R_{T,\theta} \in \mathcal{R}(q)} R_{T,\theta}(1)^2 = \frac{1}{|C_{w,F}|} q^{l+2N} + O(q^{l+2N-1})$$

$$= |G(q)|_{\omega'} + \left( \frac{1}{|C_{w,F}|} - \frac{1}{|G(q)|} \right) q^{l+2N} + O(q^{l+2N-1}).$$

(11)

We may assume that $p \mid |G(q)|$. If we exclude the Steinberg triality $^3D_4(q^2)$ for a moment, then

$$|G(q)|_{\omega'} = \prod_{i=1}^l (q^{d_i} - \epsilon_i)$$

where $\epsilon_i \in \{1,-1\}$ and the $d_i$ are the exponents of the underlying group (see [9], p. 75). Thus $p \mid q^{2d_i} - 1$. Note that

$$(q^{2d_i} - 1)_p < (q^{2pd_i} - 1)_p$$

and $|W|$ is independent of $q$ (see [3], Planche II). Thus there exists an $s \in \mathbb{N}$ (for instance $s$ a suitable power of $p$) such that

$$|C_{w,F}| \leq |W| < |G(q^s)|_p,$$

and therefore

$$|C_{w,F}| < |G(q^s)|_p.$$
for all $t \in \mathbb{N}$ and all $w \in W$. This is also true for $3D_4(q^3)$, as a similar argument shows. By (11) we obtain
\[
\sum_{R_{T,\theta} \in \mathcal{R}(q^{st})} R_{T,\theta}(1)^2 \geq |G(q^{st})|_p
\]
for all $t \geq t_0$. Now, if there exists a torus $T_w = T_w(q^{st}) \subseteq G(q^{st})$ with $p \nmid |T_w|$ for $t \geq t_0$, then all characters $R_{T,\theta}$ with $\theta$ in general position are irreducible and of $p$-defect zero. In particular they are irreducible Brauer characters for the prime $p$ and we have the desired inequality
\[
\sum_{\varphi \in \text{IBr}_p(G(q^{st}))} \varphi(1)^2 \geq |G(q^{st})|_p
\]
for all $t \geq t_0$. Thus we have proved

**Theorem 3.1.** Let $G(q) = G^F$ be a finite group of Lie type where $G$ is a linear simple algebraic group and $G^*$ is simply connected. Let $p$ be a prime with $p \nmid q$ and $p \mid |G(q)|$.

a) Then there exists an $s \in \mathbb{N}$ such that
\[
\sum_{R_{T,\theta} \in \mathcal{R}(q^{st})} R_{T,\theta}(1)^2 \geq |G(q^{st})|_p
\]
for all $t \geq t_0$.

b) If $T_w = T_w(q^{st})$ is a torus in $G(q^{st})$ for some $w \in W$ and $p \nmid |T_w|$ for all $t \geq t_0$, then
\[
\sum_{\varphi \in \text{IBr}_p(G(q^{st}))} \varphi(1)^2 \geq |G(q^{st})|_p
\]
for all $t \geq t_0$.

The critical point in the whole process is the assumption $p \nmid |T_w| = |T_w(q^{st})|$ for a suitable $w \in W$ and all $t \geq t_0$. The author would like to thank an anonymous referee for pointing out the following argument.

If we replace $s$ by $s(p - 1)$, then we have $q^s \equiv 1 \mod p$. Note that the order of $T_w(q)$ can be written as
\[
|T_w(q)| = \Phi_{n_1}(q) \cdots \Phi_{n_r}(q)
\]
where $\Phi_n$ denotes the $n$-th cyclotomic polynomial. Now we choose $w \in W$ in such a way that $1 < n_j$ and $p \nmid n_j$ for all $j = 1, \ldots, r$. Since
\[
\Phi_{n_j}(q^{st}) \mid \frac{q^{stn_j} - 1}{q^{st} - 1} = 1 + q^{st} + \ldots + q^{st(n_j - 1)} \equiv 1 + 1 + \ldots + 1 \mod p
\]
\[
\equiv n_j \mod p,
\]
we get $p \nmid \Phi_{n_j}(q^{st})$ for all $t$; hence $p \nmid |T_w(q^{st})|$ for all $t$.

The following example may illustrate the above. In particular, in the case where $G(q) = \text{PGL}(2,q)$ we are not able to find a torus as required for the prime $p = 2$. Thus our method fails for $p = 2$.

**Example.** Let $G^F = G(q) = \text{PGL}(l + 1, q)$ be the finite adjoint group of type $A_l$ for $l \geq 2$ with $q$ a power of $r$. Note that $G^*=\text{SL}(l+1,q)$ is the fixed point group of a simply connected group. Let $w \in W$ be a Coxeter element, i.e. $w$ is a cycle of length $l + 1$, and $C_{w,F} = C_{S_{l+1}}(w) = \langle w \rangle$ where $S_n$ is the symmetric group on
n letters. Let p be a prime with $p > l + 1$. Thus we either have $p \mid |G(q)|$ and (3) holds with equality, or $|C_w,F| < |G(q)|_p$ and we have $\sum_{R_r \in R} R_{T_r,\theta}(1)^2 \geq |G(q)|_p$ for $q \geq q_0$ and $p \mid |G(q)|$. Let such a q be given. Now if $p \mid |T_w| = 1 + q + \ldots + q^t$, then

$$\sum_{\varphi \in IBr_{\varphi}(G(q))} \varphi(1)^2 \geq |G(q)|_p.$$

The condition $p \mid |T_w|$ is satisfied for instance if $p \mid q - 1$ and $p \mid q^a - 1$ where $2 < s < l + 1$ and gcd$(s, l + 1) = 1$.

Let us specialize to $G(q) = \text{PGL}(2,q)$ and let $2 < p \neq r$, hence $2 = |W| < p$. Since $|G(q)| = q(q+1)(q-1)$, we have $p \not\mid |T_w| = (q+1)$ or $p \not\mid |T_1| = q-1$ where $T_w$ is the Coxeter and $T_1$ the split torus. Thus for all q large enough and all odd primes inequality (3) holds true for PGL$(2,q)$. However in this small case we get (3) for any q. If $p = r$ we may apply (2.3a). In case $2 \neq p \neq r$ the Sylow p-subgroups of $G(q)$ are cyclic. Thus the inequality (3) holds true for every p-block by [14], which obviously implies (3). For $p = 2$ one easily checks inequality (3) using the results of Section VIII in [3].

**Added in proof**

There is an analogue of the Conjecture for $p'$-class lengths, namely $|G|_{p'} \leq \sum_x |G : C_G(x)|$ where the sum runs through a set of representatives of the $p'$-conjugacy classes, and with equality if and only if $G$ has a normal $p'$-complement. In contrast to characters this can easily be proved using the Frobenius Conjecture which we know to be true. We shall discuss this and similar questions in a forthcoming paper.

**References**


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