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A COMPUTER-ASSISTED PROOF OF SAARI'S CONJECTURE FOR THE PLANAR THREE-BODY PROBLEM

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ABSTRACT. The five relative equilibria of the three-body problem give rise to solutions where the bodies rotate rigidly around their center of mass. For these solutions, the moment of inertia of the bodies with respect to the center of mass is clearly constant. Saari conjectured that these rigid motions are the only solutions with constant moment of inertia. This result will be proved here for the planar problem with three nonzero masses with the help of some computational algebra and geometry.

1. INTRODUCTION

It is a well-known property of the Newtonian n-body problem that the center of mass of the bodies moves along a line with constant velocity. Making a change of coordinates, one may assume that the center of mass is actually constant and remains at the origin. Once this is done, the moment of inertia with respect to the origin provides a natural measure of the size of the configuration. The familiar rigidly rotating periodic solutions of Lagrange provide examples of solutions with constant moment of inertia. Saari conjectured that these are in fact the only such solutions [10]. The goal of this paper is to provide a proof for the planar problem with $n = 3$.

The planar three-body problem concerns the motion of three point masses $m_i > 0$, $i = 1, 2, 3$, under the influence of their mutual gravitational attraction. Let $q_i \in \mathbf{R}^2$ denote the positions of the masses and $v_i \in \mathbf{R}^2$ their velocities. Then Newton's equations of motion are

$$\begin{aligned} \dot{q}_i &= v_i, \\ m_i \dot{v}_i &= U_{q_i}, \end{aligned} \tag{1.1}$$

where

$$U = \frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}}$$

is the Newtonian potential energy and U_{q_i} is the two-dimensional vector of partial derivatives with respect to the components of q_i. Here $r_{ij} = |q_i - q_j|$, $i < j$, are the distances between the masses. The kinetic energy is $\frac{1}{2}K$ where

$$K = m_1|v_1|^2 + m_2|v_2|^2 + m_3|v_3|^2.$$

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The total energy

$$H = \frac{1}{2}K - U = h \tag{1.2}$$

is a constant of motion.

Using the translation symmetry of the problem, one may assume that the center of mass satisfies

$$\tfrac{1}{m}\left(m_1 q_1 + m_2 q_2 + m_3 q_3\right) = (0,0)$$

where $m = m_1 + m_2 + m_3$ is the total mass. Let

$$I = m_1|q_1|^2 + m_2|q_2|^2 + m_3|q_3|^2$$

be the moment of inertia with respect to the origin. Saari's conjecture is about solutions such that $I(t)$ is constant, say $I(t) = c$. One way to get such a solution is to make the configuration rotate rigidly around the origin. Suppose that $q_i(t) = R(\theta(t))q_i(0)$ where $R(\theta)$ is the matrix of the rotation by θ in the plane. The angular momentum of such a solution is $\omega = \nu I$ where $\nu = \theta'(t)$ is the angular velocity. Since ω and I are constant, ν must be constant too (the analogous result holds for rigid motions in $\mathbf{R}^d$, $d \geq 3$, but is more difficult to prove; see [14, 1]). It follows that the acceleration vectors are $\dot{v}_i = -\nu^2 q_i$ and then Newton's equations (1.1) show that the position vectors q_i solve the algebraic equations

$$U_{q_i} + m_i \nu^2 q_i = 0, \qquad i = 1,2,3. \tag{1.3}$$

Solutions of (1.3) are called *central configurations* and the corresponding rigid motions are called *relative equilibrium solutions*. According to Euler and Lagrange there are only five central configurations up to rotation and scaling, namely, the two equilateral triangular configurations (with the masses in clockwise or counterclockwise order) and three collinear configurations, one for each of the rotationally distinct orderings of the masses along the line. Given a configuration with one of these five shapes, there is a corresponding relative equilibrium solution with constant angular velocity ν.

A priori, rigid rotation is a much stronger condition than constant moment of inertia, but the goal of this paper is to prove:

Theorem 1.1. *A solution of the planar three body problem has constant moment of inertia if and only if it is a relative equilibrium solution.*

Actually the theorem also holds for masses which are not necessarily positive, provided $m_i \neq 0$, $m \neq 0$ and $c \neq 0$. The idea of the proof is to use the assumption of constant moment of inertia to derive three algebraic equations for the three mutual distances r_{12}, r_{13}, r_{23}. Then it will be shown that these equations have only a finite number of solutions. Since the mutual distances vary continuously along any solution, it follows that they must be constant. This implies that the motion is a rigid rotation around the center of mass and so the configuration must be a central configuration by the argument above.

The three equations for the mutual distances will be obtained by elimination from six equations involving the positions and velocities. To derive these, begin by differentiating the equation $I(t) = c$:

$$\begin{aligned}
\tfrac{1}{2}\dot{I} &= m_1 q_1 \cdot v_1 + m_2 q_2 \cdot v_2 + m_3 q_3 \cdot v_3 = 0,\\
\tfrac{1}{2}\ddot{I} &= m_1|v_1|^2 + m_2|v_2|^2 + m_3|v_3|^2 + q_1 \cdot U_{q_1} + q_2 \cdot U_{q_2} + q_3 \cdot U_{q_3}\\
&= K - U = 0
\end{aligned}$$

where the second equation uses Newton's law of motion and the fact that $U(q_1, q_2, q_3)$ is homogeneous of degree -1. The equation $K = U$ together with the energy equation (1.2) shows that

$$K = U = -2h.$$

Now any bounded solution of the three-body problem has energy $h < 0$ and by rescaling, one may assume, without loss of generality, that $h = -\frac{1}{2}$. If the solution has constant moment of inertia c, then the following six equations must hold:

$$(1.4) \qquad I = c, \qquad U = 1, \qquad \dot{I} = 0, \qquad \dot{U} = 0, \qquad K = 1, \qquad \ddot{U} = 0.$$

In the next section these equations will be expressed in terms of the mutual distances r_{ij} and their time derivatives. The first two equations in (1.4) involve only the distances. The derivatives will be eliminated from the last four equations to obtain a third equation for the r_{ij}. Then in Section 3 it will be shown that these three equations have only finitely many solutions, completing the proof of the theorem.

The proof presented here would not be feasible without the use of computers. Mathematica was used throughout to perform symbolic computations on the very large expressions which arise [15]. Porta was used to compute convex hulls [3].

Two recent papers settled special cases of the problem [8, 7]. The paper of McCord solves the equal mass case. The paper of Llibre and Piña gives a different proof for the equal mass case which can be adapted to handle other pre-specified mass values; however, the method does not seem to allow the masses to be treated as parameters so that the problem can be settled for all masses at once. Both papers rely on computer algebra.

2. Murnaghan's equations of motion

Equations of motion for the distances r_{ij} were first derived by Lagrange [6]. Here a Hamiltonian system due to Murnaghan will be used instead [9, 12, 13]. Let $a_1 = r_{23}, a_2 = r_{13}, a_3 = r_{12}$. If q_i and v_i satisfy Newton's equations, then Murnaghan shows that the a_i and certain conjugate momentum variables p_i obey Hamilton's differential equations for the Hamiltonian function

$$H = \frac{1}{2}K - U$$

where

$$\begin{aligned}
U &= \frac{m_1 m_2}{a_3} + \frac{m_1 m_3}{a_2} + \frac{m_2 m_3}{a_1},\\
K &= \sum_{i=1}^{3} \frac{1}{m_i}\left[p_j^2 + p_k^2 + \left(p_j p_k + \frac{\omega^2}{9 a_j a_k}\right)\left(\frac{a_j^2 + a_k^2 - a_i^2}{a_j a_k}\right)\right.\\
&\qquad\left. + \frac{\omega^2}{9}\left(\frac{1}{a_j^2} + \frac{1}{a_k^2}\right) + \frac{4\omega\Delta}{3 a_j a_k}\left(\frac{p_j}{a_k} - \frac{p_k}{a_j}\right)\right].
\end{aligned} \tag{2.1}$$

Here Δ is the area of the triangle formed by the three bodies and ω is the total angular momentum (a constant of motion). In the summation for K the triple

(i, j, k) is always a cyclic permutation of $(1, 2, 3)$. One should regard Δ as a function of the mutual distances using Heron's formula

$$\Delta = \sqrt{\sigma(\sigma - a_1)(\sigma - a_2)(\sigma - a_2)}, \qquad \sigma = \tfrac{1}{2}(a_1 + a_2 + a_3). \tag{2.2}$$

Hamilton's differential equations are

$$\begin{aligned} \dot{a}_i &= H_{p_i}, \\ \dot{p}_i &= -H_{a_i} \end{aligned} \tag{2.3}$$

for $i = 1, 2, 3$. Because of (2.2) some of these equations will have a factor of Δ in the denominator. Of course the a_i also occur in the denominators, so when working with Murnaghan's equations, it will always be assumed that

$$a_1 a_2 a_3 \Delta \neq 0. \tag{2.4}$$

For some of the calculations below it is convenient to replace the momentum variables by the velocities $b_i = \dot{a}_i$. Hamilton's equations $b_i = H_{p_i}$ are linear in the momenta p_i and the angular momentum ω:

$$b_i = c_{ii} p_i + c_{ij} p_j + c_{ik} p_k + d_i \omega. \tag{2.5}$$

The coefficients in (2.5) are given by

$$\begin{aligned} c_{ii} &= \frac{m_j + m_k}{m_j m_k}, \qquad d_i = \frac{2\Delta}{3a_i}\left(\frac{1}{m_k a_j^2} - \frac{1}{m_j a_k^2}\right), \\ c_{ij} &= \frac{A_k}{2a_i a_j m_k}, \qquad c_{ik} = \frac{A_j}{2a_i a_k m_j} \end{aligned}$$

where

$$A_k = a_i^2 + a_j^2 - a_k^2. \tag{2.6}$$

As above (i, j, k) is a cyclic permutation of $(1, 2, 3)$.

The determinant of the 3×3 matrix C with entries c_{ij} is

$$\left|C\right| = \frac{4m^2 I \Delta^2}{(m_1 m_2 m_3 a_1 a_2 a_3)^2}.$$

Since $\Delta \neq 0$, (2.5) can be inverted to find the momenta as linear functions of the velocities and the angular momentum

$$p_i = \gamma_{ii} b_i + \gamma_{ij} b_j + \gamma_{ik} b_k + \delta_i \omega \tag{2.7}$$

where

$$\begin{aligned} \gamma_{ii} &= \frac{m_j m_k a_i^2}{4Im^2\Delta^2}\left[4m_j m_k \Delta^2 + m\, m_i a_j^2 a_k^2\right], \\ \gamma_{ij} &= \frac{m_i m_j m_k a_i a_j}{16Im^2\Delta^2}\left[m_k A_i A_j - 2(m_i + m_j) a_k^2 A_k\right], \\ \gamma_{ik} &= \frac{m_i m_j m_k a_i a_k}{16Im^2\Delta^2}\left[m_j A_i A_k - 2(m_i + m_k) a_j^2 A_j\right], \end{aligned}$$

$$\delta_i = \frac{3m_i a_i^2(m_k a_j^2 - m_j a_k^2) + (2m_j m_k a_i^2 - m_i m_j a_k^2 - m_i m_k a_j^2)(a_j^2 - a_k^2)}{12Ima_i\Delta}. \tag{2.8}$$

Now the equations (1.4) will be expressed using these coordinates. The moment of inertia is

$$I = \tfrac{1}{m}(m_1 m_2 a_3^2 + m_1 m_3 a_2^2 + m_2 m_3 a_1^2) = c \tag{2.9}$$

and the potential is

$$U = \frac{m_1 m_2}{a_3} + \frac{m_1 m_3}{a_2} + \frac{m_2 m_3}{a_1} = 1. \tag{2.10}$$

Differentiating (2.9) and (2.10) gives

$$\nabla I \cdot b = \nabla U \cdot b = 0. \tag{2.11}$$

Now it is easy to see that ∇U and ∇I are linearly dependent only when all the mutual distances a_i are equal (the equilateral central configuration). The equation $U = 1$ then determines the a_i uniquely. Since the goal is to show that only finitely many values of the mutual distances are possible, one may exclude this case a priori. Then it follows from (2.11) that b is a scalar multiple of the cross product $\nabla U \times \nabla I$. Let

$$\begin{aligned} B &= \frac{\Delta m^2 a_1 a_2 a_3}{m_1 m_2 m_3} \nabla U \times \nabla I \\ &= 2\Delta m \left(\frac{m_1 a_1 (a_2^3 - a_3^3)}{a_2 a_3}, \frac{m_2 a_2 (a_3^3 - a_1^3)}{a_1 a_3}, \frac{m_3 a_3 (a_1^3 - a_2^3)}{a_1 a_2} \right) \end{aligned}$$

and define a scalar variable β by setting

$$b = \beta B. \tag{2.12}$$

With this assumption on the b_i, the equations $\dot{I} = 0$ and $\dot{U} = 0$ will hold. It only remains to consider the last two equations in (1.4).

Substituting (2.12) into (2.7) gives the momenta as functions

$$p_i = \gamma_i \beta + \delta_i \omega \tag{2.13}$$

where δ_i is given by (2.8) and

$$\gamma_i = \frac{m_i m_j m_k a_j a_k}{4I\Delta} \left[2m_i a_i^3 (a_j^3 - a_k^3) + m_j a_j (a_i^3 - a_k^3) A_k - m_k a_k (a_i^3 - a_j^3) A_j \right].$$

The fifth equation of (1.4) is $K = 1$ where K is given by (2.1). Eliminating the p_i using (2.13) gives a quadratic equation in β and ω:

$$C_0 \beta^2 + \omega^2 - c = 0 \tag{2.14}$$

where C_0 is a polynomial in the a_i and m_i displayed in the appendix. Actually, this equation is just one factor of the numerator of a rational function but the denominator and the other factors of the numerator are nonzero under the assumptions that $m_i \neq 0$, $m \neq 0$ and (2.4).

Finally, the last equation of (1.4) is $\ddot{U} = 0$. First compute

$$\dot{U} = U_{a_1} \dot{a}_1 + U_{a_2} \dot{a}_2 + U_{a_3} \dot{a}_3 = U_{a_1} H_{p_1} + U_{a_2} H_{p_2} + U_{a_3} H_{p_2},$$

then differentiate with respect to time again and use (2.3) to obtain $\ddot{U}$ as a rational function of the a_i, p_i. After eliminating the p_i using (2.13), clearing denominators and eliminating some nonzero factors, one obtains another quadratic function of β and ω:

$$C_1 \beta^2 + C_2 \beta\omega + C_3 \omega^2 - 4c^2 C_4 = 0 \tag{2.15}$$

where the coefficients C_i are polynomials in a_i and m_i relegated to the appendix.

To eliminate β and obtain the third equation for the mutual distances, take the resultant of the equations (2.14) and (2.15) with respect to β. First view the coefficients C_i as variables and replace the angular momemtum ω by a new parameter $\lambda = \omega^2/c$. After canceling a factor of c^2 the resultant is:

$$R = (C_1 - 4\,c\,C_0\,C_4)^2 + \left({C_1}^2 + C_0\,{C_2}^2 - 2\,C_0\,C_1\,C_3 + {C_0}^2\,{C_3}^2\right)\lambda^2 \\ + \left(-2\,{C_1}^2 - C_0\,{C_2}^2 + 2\,C_0\,C_1\,C_3 + 8\,c\,C_0\,C_1\,C_4 - 8\,c\,{C_0}^2\,C_3\,C_4\right)\lambda.$$

Then replacing the variables C_i by the polynomials in the appendix and canceling some nonzero factors, one obtains a large polynomial in the distances a_i with coefficients depending on the parameters m_i, c, λ. This polynomial contains 2154 terms, a few of which are displayed below:

$$\begin{aligned} R = {} & 32c(\lambda-1)m_1m_2^4m_3(m_2+m_3)^4m\,a_1a_2^{16}a_3^{22} \\ & + 96c(\lambda-1)m_1m_2^3m_3^2(m_2+m_3)^4m\,a_1a_2^{18}a_3^{20} \\ & + 64c^2m_1m_2^4m_3^2(m_2+m_3)^3m^2\,a_1a_2^{16}a_3^{19} \\ & - 64c^2m_1m_2^4m_3^2(m_2+m_3)^3m^2\,a_1a_2^{13}a_3^{22} \\ & - 64c^2m_1m_2^3m_3^2(m_2-2m_3)(m_2+m_3)^3m^2\,a_1a_2^{18}a_3^{17} \\ & + 64c^2m_1m_2^3m_3^2(m_2-2m_3)(m_2+m_3)^3m^2\,a_1a_2^{15}a_3^{20} \\ & + 64c^2m_1m_2^2m_3^3(2m_2-m_3)(m_2+m_3)^3m^2\,a_1a_2^{17}a_3^{18} \\ & + 16c^2m_2^4m_3^2(m_2+m_3)^4m^2\,a_2^{16}a_3^{20} \\ & + 32c^2m_2^3m_3^3(m_2+m_3)^4m^2\,a_2^{18}a_3^{18} \\ & + 16c^2m_2^2m_3^4(m_2+m_3)^4m^2\,a_2^{20}a_3^{16} \\ & + \dots. \end{aligned} \tag{2.16}$$

The total degrees in a_i of the terms take the values 36, 39, 40, 42, 43, 44.

The three equations

$$I = c, \qquad U = 1, \qquad R = 0 \tag{2.17}$$

involve only the mutual distances a_i and the parameters m_i, c, λ. By construction, a solution of the planar three-body problem with constant moment of inertia has mutual distances which satisfy these equations for all time. In the next section, it will be shown that if the parameters satisfy $m_i \neq 0, m \neq 0, c \neq 0$, then equations (2.17) have only finitely many solutions a_i. This forces the mutual distances to be constant and therefore the solution will be a rigidly rotating relative equilibrium.

Actually, the equation $R = 0$ was derived under the assumptions that the a_i were not all equal and that (2.4) holds. Although it is not logically necessary to do so, one can check that it also holds for the equilateral case. To handle the case where (2.4) does not hold it will also be shown that

$$I = c, \qquad U = 1, \qquad \Delta^2 = 0 \tag{2.18}$$

also have only finitely many solutions. Of course it is not necessary to consider the possibility that (2.4) is violated with $a_i = 0$.

3. BKK theory

In a remarkable paper [2], D.N. Bernstein considers the basic problem of solving a system of n equations in n variables. Related work of Khovanskii and Kushnirekno appeared around the same time and the resulting circle of ideas is often referenced by the initials BKK [4, 5].

For equations with generic coefficients, Bernstein shows a surprising connection between the number of complex solutions with all variables nonzero and the geometry of certain Newton polytopes. He also gives an effective method based on Puiseux series for verifying that a given system, generic or not, has a finite number of solutions.

A polynomial $f(x_1, \ldots, x_n)$ is a sum of terms of the form $c_k x_1^{k_1} \ldots x_n^{k_n}$ where $k = (k_1, \ldots, k_n)$ is an exponent vector of nonnegative integers. The Newton polytope $P(f)$ is just the convex hull in $\mathbf{R}^n$ of the set of all exponent vectors k which occur in f with nonzero coefficient c_k. Given n polynomials $f_i(x)$, there will be n Newton polytopes $P_i = P(f_i)$. Now the Newton polytopes do not depend on the actual coefficients of the polynomials, only on which coefficients are nonzero. So one can consider the set of all polynomial systems which have the same P_i but with different values for the coefficients. Then Bernstein shows that for generic choices of the coefficients, the number of complex solutions of the system with all variables $x_i \neq 0$ is equal to a certain geometric invariant of the Newton polytopes called the mixed volume (the solutions have to be counted with appropriate multiplicities). A solution $x = (x_1, \ldots, x_n)$ with $x_i \in \mathbf{C} \setminus 0$ is said to lie in the "algebraic torus", $\mathbf{T}$.

The definition of mixed volume depends on some geometrical constructions on polytopes. First define the Minkowski sum of polytopes P_i to be

$$P_1 + \ldots + P_n = \{x \in \mathbf{R}^n : x = x_1 + \ldots + x_n, x_i \in P_i\}.$$

It can be shown that this is again a polytope, i.e., it is the convex hull of a finite set of points. More generally, one can take a linear combination of polytopes

$$\lambda_1 P_1 + \ldots + \lambda_n P_n = \{x \in \mathbf{R}^n : x = \lambda_1 x_1 + \ldots + \lambda_n x_n, x_i \in P_i\}.$$

It turns out that the volume of this linear combination is a homogeneous polynomial function of the λ_i of degree n. If the volume $V(\lambda_1 P_1 + \ldots + \lambda_n P_n)$ is expanded as a sum of powers of the λ_i, then the mixed volume is just the coefficient of the product $\lambda_1 \cdots \lambda_n$. However, it is not easy to compute the mixed volume from this definition.

Bernstein also gives criteria for determining whether or not a given system of polynomial has generic coefficients and so has exactly the predicted number of nonzero solutions. As a first step in justifying these criteria, he has to show that the number of solutions is actually finite. Only this rather minor part of Bernstein's paper is relevant here. The following theorem presents the argument, which is sketched in only a few sentences in Bernstein's paper. The proof is given for the more general case of m equations in n unknowns.

Theorem 3.1. *Suppose that a system of m polynomial equations $f_i(x) = 0$ has infinitely many solutions $x \in \mathbf{T}$. Then there is a rational vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i = 1$ for some i such that the reduced equations $f_{i\alpha}(x) = 0$ also have a solution in $\mathbf{T}$, where the reduced polynomial $f_{i\alpha}(x)$ is the sum of all of the terms of $f_i(x)$ whose exponent vectors k satisfy*

$$\alpha \cdot k = \min_{l \in P_i} \alpha \cdot l. \tag{3.1}$$

Geometrically, the last equation defines a face of the polytope P_i (not necessarily of codimension one). The vector α is an inward pointing normal and for all points $l \in P_i$ not on this face, $\alpha \cdot l$ will be strictly larger.

Proof. The proof uses some basic ideas from algebraic geometry which can be found in [11]. Let $S \subset \mathbf{C}^n$ be the affine algebraic variety defined by the equations $f_i(x) = 0$, $i = 1, \ldots, m$. By hypothesis, the quasi-projective variety $V = S \cap \mathbf{T}$ is infinite. It follows that there is at least one coordinate, x_i, such that the projection $\pi_i(V)$ onto the x_i axis is Zariski dense, i.e, there are no polynomials in x_i alone which vanish on V. For otherwise there would be only finitely many possible values for each of the coordinates and V itself would be a finite set. Since V is a variety, it follows that $\pi_i(V)$ omits at most a finite subset of $\mathbf{C}$.

After reindexing, one may assume that $i = n$. The key to the proof is the claim that after setting $x_n = t$, one can find fractional-power Puiseux series with complex coefficients $x_j(t)$, $j = 1, \ldots, n-1$, not identically zero, such that the vector of Puiseux series $x(t) = (x_1(t), \ldots, x_{n-1}(t), t)$ converges in some (possibly punctured) neighborhood of $t = 0$ and solves the system $f_i(x) = 0$. This can be shown as follows. Let $\mathcal{P}$ denote the algebraically closed field of complex Puiseux series converging in some punctured neighborhood of $t = 0$. Also, let $F_j(x_1, \ldots, x_{n-1}) = f_j(x_1, \ldots, x_{n-1}, t)$. Then F_j can be viewed as an element of the ring $\mathcal{R} = \mathcal{P}[x_1, \ldots, x_{n-1}]$ of polynomials in $n-1$ variables with coefficients in $\mathcal{P}$. The equations $F_j = 0$ define an affine variety in the space $\mathcal{P}^{n-1}$ and it must be shown that this variety contains at least one point with all coordinates nonzero. To get the coordinates to be nonzero, introduce another variable x_0 and another equation

$$F_0(x_0, x_1, \ldots, x_{n-1}) = x_0 x_1 \ldots x_{n-1} - 1 = 0.$$

Now it suffices to show that the variety $W \subset \mathcal{P}^n$ defined by $F_0 = \ldots = F_n = 0$ is nonempty; for if $(x_0, \ldots, x_{n-1}) \in W$, then $x_i(t) \in \mathcal{P}$, $i = 1, \ldots, n-1$, are the required nonzero series solutions.

By the weak Nullstellensatz (which applies in any algebraically closed field), W is empty if and only if an equation of the form

$$1 = F_0\, G_0 + F_1\, G_1 + \ldots + F_n\, G_n \tag{3.2}$$

holds in $\mathcal{R}$. However, the fact that $\pi_n(V)$ is Zariski dense in $\mathbf{C}$ means that for almost every $t \in \mathbf{C}$, there exist $x_i \in \mathbf{C}$, $1 \le i \le n-1$, with $x_i \neq 0$ such that $f_j(x_1, \ldots, x_{n-1}, t) = 0$ for $j = 1, \ldots, n$. Choose such a t in a common neighborhood of convergence of all the Puiseux series occurring as coefficients in (3.2). Substituting t and the corresponding values of x_i into (3.2) makes all the $F_j = 0$, $j = 1, \ldots, n$. Setting $x_0 = (x_1 x_2 \cdots x_{n-1})^{-1} \in \mathbf{C}$ also makes $F_0 = 0$ and gives a contradiction. So $W \neq \emptyset$ and the required nonzero series $x_j(t)$ exist.

Now consider the lowest order terms of the Puiseux series. Let $x_j(t) = a_j t^{\alpha_j} + \ldots$ where $a_j \in \mathbf{C} \setminus 0$ and $\alpha_j \in \mathbf{Q}$. Since $x_n(t) = t$, one has $a_n = \alpha_n = 1$. Let $a = (a_1, \ldots, a_n) \in \mathbf{T}$ and $\alpha = (\alpha_1, \ldots, \alpha_{n-1}, 1) \in \mathbf{Q}^n$. Note that α satisfies the conditions of the theorem.

Substituting the series $x_j(t)$ in the polynomial $f_i(x)$ gives the identically zero Puiseux series and, in particular, the lowest order terms in t must vanish. The lowest order term in t of f_i has degree t^{d_i} where d_i is given by the right-hand side of (3.1). The coefficient of this term is just $f_{i\alpha}(a)$. Thus $a \in \mathbf{T}$ is the required solution of the reduced equations. QED

Bernstein remarks that one need only test a finite number of rational vectors α. The proof requires some elementary polytope theory. Note that the reduced equations $f_{i\alpha}$ corresponding to a given vector α are just the terms whose exponent

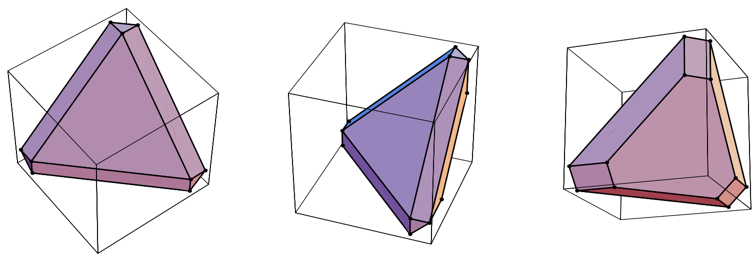

FIGURE 1. The shaded polytope is the Minkowski sum $P_1+P_2+P_3$ of the Newton polytopes for equations (2.18)

vectors lie in the face of P_i determined by the supporting hyperplane with inward normal α. While the facets (faces of codimension 1) of a polytope have a unique inward normal up to scalar multiplication, the lower dimensional faces will have infinitely many. However, they all induce the same reduced system. All the vectors $\alpha \in \mathbf{R}^n$ are partitioned into finitely many sets according to which face of P_i they induce. The sets of the partition turn out to be convex cones and the partition is called the normal fan of P_i. To be sure that every reduced system is examined one needs to construct the common refinement of the normal fans of $P_1, \ldots, P_n$, i.e., the partition obtained by intersecting the sets of the partitions for the individual P_i. It turns out that this refined partition is just the normal fan of the Minkowski sum polytope $P_1 + \ldots + P_n$ [16]. Thus in applying Theorem 3.1, one only needs to consider one inward normal from each face of the Minkowski sum.

As a first application of this result, consider equations (2.18). Clearing denominators, and expanding Heron's formula, the equations become

$$\begin{aligned} f_1 &= m_1m_2a_3^2 + m_1m_3a_2^2 + m_2m_3a_1^2 - mc = 0, \\ f_2 &= m_1m_2a_1a_2 + m_1m_3a_1a_3 + m_2m_3a_2a_3 - a_1a_2a_3 = 0, \\ f_3 &= 2a_1^2a_2^2 + 2a_1^2a_3^2 + 2a_2^2a_3^2 - a_1^4 - a_2^4 - a_3^4 = 0. \end{aligned} \tag{3.3}$$

The exponent vectors appearing in the first equation are

$$(0,0,0), \qquad (2,0,0), \qquad (0,2,0), \qquad (0,0,2),$$

so the first Newton polytope, P_1, is a tetrahedron. Similarly, P_2 is a tetrahedron with vertices

$$(1,1,0), \qquad (0,1,0), \qquad (0,0,1), \qquad (1,1,1).$$

Finally, P_3 is a triangle with vertices

$$(4,0,0), \qquad (0,4,0), \qquad (0,0,4).$$

The other exponent vectors in f_3 are convex combinations of these three. The Minkowski sum $P_1 + P_2 + P_3$ can be found by adding together all possible triples of exponent vectors, taking one from each Newton polytope, and then computing the convex hull. The convex hull was computed using the program Porta 1.3.2 [3]. The result is a polytope with 14 facets, 27 edges and 15 vertices (see Figure 1). To show that equations (3.3) have only finitely many solutions, it suffices to choose one inward normal vector α for each face and to show that the corresponding reduced systems have no solutions in $\mathbf{T}$. Moreover, any face which has no inward normal vectors with at least one $\alpha_i = 1$ can be omitted.

According to the computer program Porta, the facets are determined by the following inequalities:

$$\begin{array}{rrr} k_1 \geq 0 & k_2 \geq 0 & k_3 \geq 0 \\ k_1 + k_2 \geq 1 & k_1 + k_3 \geq 1 & k_2 + k_3 \geq 1 \\ k_1 + k_2 + k_3 \geq 6 & -k_1 - k_2 - k_3 \geq -9 & \\ -k_1 \geq -7 & -k_2 \geq -7 & -k_3 \geq -7 \\ -k_1 - k_2 \geq -8 & -k_1 - k_3 \geq -8 & -k_2 - k_3 \geq -8. \end{array}$$

Comparing with (3.1) one can indentify the inward normal vector α as the vector of coefficients on the left. The inequalities with all coefficients negative can be ignored since even allowing for rescaling the normal vector by a positive factor one cannot get a component $\alpha_i = 1$. For each of the remaining inequalities one must construct the reduced system and show that there are no solutions with all variables nonzero. The virtue of this approach is that, although there are many reduced systems to check, they are all very simple. For example, consider the inequality $k_1 \geq 0$ with normal vector $\alpha = (1, 0, 0)$. Taking the terms from f_i whose exponent vectors minimize $\alpha \cdot k = k_1$ gives the reduced system

$$\begin{aligned} f_{1\alpha} &= m_1 m_2 a_3^2 + m_1 m_3 a_2^2 - mc = 0, \\ f_{2\alpha} &= m_2 m_3 a_2 a_3 = 0, \\ f_{3\alpha} &= 2a_2^2 a_3^2 - a_2^4 - a_3^4 = 0. \end{aligned} \tag{3.4}$$

If the masses are nonzero, the second equation has no solutions with $a_i \neq 0$. Proceeding similarly with the other inequalities (with at least one positive coefficient) one always finds that at least one of the reduced equations is a single term with a nonzero coefficient, assuming that $mm_1m_2m_3 \neq 0$.

Next, one should consider the reduced systems corresponding to the edges of the Minkowski sum. But the following argument shows that no further testing is necessary. Call a facet *trivial* if it leads to a reduced system such as (3.4) where some equation $f_{i\alpha}$ reduces to a single term with a nonzero coefficient. Then any edge or vertex of this facet will lead to reduced equations whose terms are a subset of the terms which appeared for the facet. In particular, an equation with a single term will be unchanged and therefore there will still be no nonzero solutions. Thus one does not need to test edges or vertices contained in trivial facets.

The facet testing above showed that all facets whose inward normals have at least one positive component are all trivial. On the other hand, if an edge or vertex is incident only with facets whose normals α have all components $\alpha_i \leq 0$, then every normal vector for that edge or vertex will have the same property, for these normals are all convex combinations of the normals of the incident facets. This means that no further testing is necessary and so the system (2.18) has a finite number of solutions, as required.

Now the same approach will be used for the more complicated equations (2.17). These are just the equations f_1 and f_2 from (3.3) together with $f_3' = R = 0$ where R is the resultant (2.16). Since f_3' has 2154 terms, there are 2154 exponent vectors which determine the Newton polytope P_3'. Remarkably, most of them are interior points and the polytope itself has only 30 extremal vertices. The Minkowski sum polytope $P_1 + P_2 + P_3'$ which will be used to prove finiteness has 62 facets, 117 edges

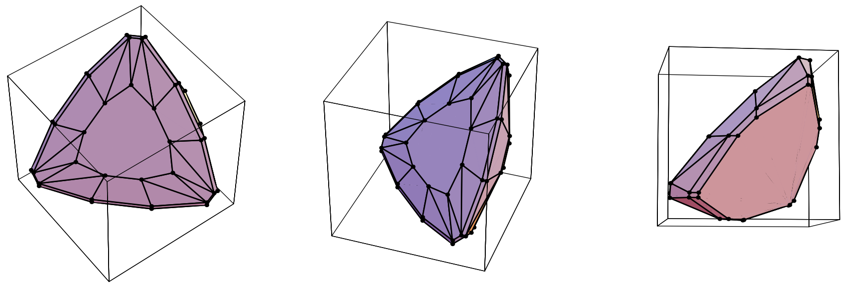

FIGURE 2. Minkowski sum $P_1 + P_2 + P_3'$ of the Newton polytopes for equations (2.17)

and 57 vertices. It is shown in Figure 2. The facet-defining inequalties are

$$\begin{array}{rr}
k_1 \geq 0 & k_1 + k_2 \geq 9 \\
3k_1 + k_2 \geq 17 & 2k_1 + 2k_2 - k_3 \geq -13 \\
k_1 + k_2 + k_3 \geq 38 & 8k_1 + 2k_2 - k_3 \geq 11 \\
2k_1 - k_2 - k_3 \geq -40 & 10k_1 - 2k_2 - 5k_3 \geq -149 \\
-k_1 \geq -35 & -k_1 - k_2 \geq -44 \\
-k_1 - k_2 - k_3 \geq -49 & -2k_1 - 3k_2 - 3k_3 \geq -138 \\
-4k_1 - 5k_2 \geq -203 & -4k_1 - 4k_2 - 5k_3 \geq -223 \\
-8k_1 - 10k_2 - 11k_3 \geq -495 & -10k_1 - 14k_2 - 15k_3 \geq -673
\end{array}$$

where inequalities obtained by permuting the subscripts have been omitted. For example, three facets are given by inequalities like the first one on the list and six facets by an inequality like the last one. The first eight inequalities have at least one positive coefficient. It turns out that all of the corresponding reduced equations are trivial. For example, the inequality $k_1 + k_2 \geq 9$ with normal vector $\alpha = (1, 1, 0)$ determines the reduced system

$$\begin{aligned}
f_{1\alpha} &= m_2 m_3 a_1^2 - mc = 0, \\
f_{2\alpha} &= m_1 m_2 a_1 a_2 + m_1 m_3 a_1 a_3 = 0, \\
f_{3\alpha} &= 16c^2 m_1^2 m_2^4 m_3^4 a_1^{28} a_2^4 a_3^4 = 0.
\end{aligned} \tag{3.5}$$

The last equation, which consists of just one of the 2154 terms of f_3, makes the system trivial. Since the facets whose normals have a positive component are all trivial, the same argument as above shows that it is not necessary to check edges and vertices of the polytope. By Theorem 3.1, equations (2.17) have finitely many solutions.

In showing that (2.17) and (2.18) have finitely many solutions, the only assumptions needed about the parameters m_i, c, λ were that

$$m_i \neq 0, \qquad m = m_1 + m_2 + m_3 \neq 0, \qquad c \neq 0.$$

So Theorem 1.1 is proved.

4. Appendix

Here are the coefficients C_i for the equations (2.14) and (2.15):

$$\begin{aligned}C_0 = m_1m_2m_3m\,\big[&m_1^2a_1^4(a_2^3-a_3^3)^2 + m_2m_3a_2a_3A_1(a_1^3-a_2^3)(a_1^3-a_3^3)\\ &+ m_2^2a_2^4(a_1^3-a_3^3)^2 + m_1m_3a_1a_3A_2(a_2^3-a_1^3)(a_2^3-a_3^3)\\ &+ m_3^2a_3^4(a_1^3-a_2^3)^2 + m_1m_2a_1a_2A_3(a_3^3-a_1^3)(a_3^3-a_2^3)\,\big],\end{aligned}$$

$$\begin{aligned}C_1 = m_1m_2m_3a_1a_2a_3\sum_{i=1}^{3}\big[\,&4m_jm_ka_i^5(m_j^3a_j^7(a_i^3-a_k^3)^2 + m_k^3a_k^7(a_i^3-a_j^3)^2)\\ &+ m_j^2m_k^2a_i^5a_ja_k(m_ja_j(a_i^3-a_k^3)D_{ij} + m_ka_k(a_i^3-a_k^3)D_{ik})\\ &+ 2m_i^3m_jm_ka_i^2(a_j^3-a_k^3)E_i + m_im_j^2m_k^2a_ja_kF_i\,\big]\end{aligned}$$

where

$$\begin{aligned}D_{ij} &= 3(a_i^3-a_k^3)(a_i^2-a_k^2)^2 + 2a_j^2(-5a_i^3 + a_i^3a_k^2 + 3a_i^2a_k^3 + a_k^5)\\ &\quad + a_j^4(7a_i^3-3a_k^3) + 4a_j^5(a_i^2-a_k^2) - 4a_j^7,\\ E_i &= 4a_i^7(a_j^6-a_k^6) + 4a_i^5(a_j^2-a_k^2)(a_j^6-a_j^3a_k^3+a_k^6) - a_i^4a_j^3a_k^3(a_j^3-a_k^3)\\ &\quad - 2a_i^2a_j^3a_k^3(3a_j^5 + a_j^3a_k^2 - a_j^2a_k^3 - 3a_k^5) + 3a_j^3a_k^3(a_j^3-a_k^3)(a_j^2-a_k^2)^2,\\ F_i &= 4{a_i}^4{a_j}^6{a_k}^6 - 8{a_i}^2{a_j}^6{a_k}^6\left({a_j}^2+{a_k}^2\right) + 2{a_i}^{13}\left({a_j}^3+{a_k}^3\right)\\ &\quad - 2{a_i}^7{a_j}^3{a_k}^3\left({a_j}^3+{a_k}^3\right) + 4{a_j}^6{a_k}^6\left({a_j}^4+4{a_j}^2{a_k}^2+{a_k}^4\right)\\ &\quad - 16{a_i}^{11}\left({a_j}^5+{a_k}^5\right) - 4{a_i}^5{a_j}^3{a_k}^3\left({a_j}^5-{a_j}^3{a_k}^2-{a_j}^2{a_k}^3+{a_k}^5\right)\\ &\quad + {a_i}^{10}\left(7{a_j}^6-18{a_j}^3{a_k}^3+7{a_k}^6\right)\\ &\quad + 2{a_i}^3{a_j}^3{a_k}^3\left(3{a_j}^7-14{a_j}^5{a_k}^2-{a_j}^4{a_k}^3-{a_j}^3{a_k}^4-14{a_j}^2{a_k}^5+3{a_k}^7\right)\\ &\quad + 2{a_i}^9\left(7{a_j}^7-6{a_j}^5{a_k}^2-{a_j}^4{a_k}^3-{a_j}^3{a_k}^4-6{a_j}^2{a_k}^5+7{a_k}^7\right)\\ &\quad - 2{a_i}^8\left({a_j}^8+7{a_j}^6{a_k}^2-20{a_j}^5{a_k}^3-20{a_j}^3{a_k}^5+7{a_j}^2{a_k}^6+{a_k}^8\right)\\ &\quad + {a_i}^6\left(-5{a_j}^{10}+2{a_j}^8{a_k}^2-22{a_j}^7{a_k}^3+7{a_j}^6{a_k}^4+60{a_j}^5{a_k}^5\right.\\ &\qquad\qquad \left.+7{a_j}^4{a_k}^6-22{a_j}^3{a_k}^7+2{a_j}^2{a_k}^8-5{a_k}^{10}\right).\end{aligned}$$

As usual the sum is over cyclic permutations of $(1,2,3)$.

The next two are relatively simple:

$$C_2 = -\frac{8}{m}a_1^2a_2^2a_3^2\,C_0, \qquad\qquad C_3 = 4a_1^4a_2^4a_3^4.$$

Finally,

$$\begin{aligned}C_4 = m_1m_2(m_1+m_2)a_1^4a_2^4 &+ m_1m_3(m_1+m_3)a_1^4a_3^4 + m_2m_3(m_2+m_3)a_2^4a_3^4\\ &+ m_1m_2m_3a_1a_2a_3\left(a_1^3A_1 + a_2^3A_2 + a_3^3A_3\right).\end{aligned}$$

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