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A CONVERSE TO DYE'S THEOREM

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Abstract. Every non-amenable countable group induces orbit inequivalent ergodic equivalence relations on standard Borel probability spaces. Not every free, ergodic, measure preserving action of $\mathbb{F}_2$ on a standard Borel probability space is orbit equivalent to an action of a countable group on an inverse limit of finite spaces. There is a treeable non-hyperfinite Borel equivalence relation which is not universal for treeable in the $\leq_B$ ordering.

1. Introduction

We show a converse to a consequence of the final strengthening of Dye's theorem proved by Ornstein and Weiss.

Definition. An equivalence relation E is said to be *standard* if it is defined on a standard Borel probability space $(X, \mathcal{B}, \mu)$, it is Borel as a subset of $X \times X$, and all its equivalence classes are countable. It is said to be *measure preserving* if for any measurable $A, B \subset X$ and for any measurable bijection $\varphi : A \to B$ included in the graph of E ($xE\varphi(x)$ all x) we have $\mu(A) = \mu(B)$. It is *ergodic* if any E-invariant set is either null or conull.

Two such equivalence relations, E on $(X, \mathcal{B}, \mu)$, F on $(Y, \mathcal{C}, \nu)$, are said to be *orbit equivalent* if there is a measure preserving bijection

$$\psi : X \to Y$$

such that almost everywhere we have

$$x_1 E x_2 \Leftrightarrow \psi(x_1) F \psi(x_2).$$

An equivalence relation E on a standard Borel probability space $(X, \mathcal{B}, \mu)$ is said to be *induced* by a countable group G if there is a Borel, measure preserving, ergodic, and almost everywhere free action of G on X such that E equals the corresponding equivalence relation E_G. Any such E_G will necessarily be standard, measure preserving, and ergodic.

Theorem 1.1 (Dye; see [5], [6]). *Any two ergodic, standard, measure preserving equivalence relations induced by $\mathbb{Z}$ are orbit equivalent.*

Theorem 1.2 (Ornstein, Weiss; see [14], [3]). *If G is a countably infinite amenable group, then any two ergodic, standard, measure preserving equivalence relations induced by G are orbit equivalent.*

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Theorem 1.3 (Connes, Weiss; Schmidt; see [4], [17])**.** *If G is countable, non-amenable and without property T, then there are at least two orbit inequivalent ergodic, standard, measure preserving equivalence relations induced by actions of G.*

Theorem 1.4. *If G is countably infinite with property T, then there are continuum many orbit inequivalent ergodic, standard, measure preserving equivalence relations induced by actions of G.*

Corollary 1.5. *A countable group is amenable if and only if it induces only one orbit equivalence relation considered up to orbit equivalence.*

The argument, which answers a question raised by Schmidt in 3.10 of [18], was inspired by certain constructions from [15].

We also consider another structural consequence of Dye's work, who in the course of the proof of his theorem provided a normal form for the orbit equivalence relations induced by groups such as $\mathbb{Z}$. Any measurable equivalence relation induced by $\mathbb{Z}$ can be represented as an orbit equivalence relation arising by a kind of inverse limit of actions of $\mathbb{Z}$ on finite spaces. While it is unreasonable to hope that the specifics of Dye's construction, with $\mathbb{Z}$ acting by the *odometer map* as in [5], could provide a canonical model for non-amenable equivalence relations, it does seem to have been open whether arbitrary measure preserving standard ergodic E may allow themselves to be presented as arising from this kind of inverse limit of actions on finite spaces.

We formalize the notion of *modular* in Section 3 to capture this idea, and go onto show that in general $\mathbb{F}_2$ can induce measurable equivalence relations which are not modular in this sense. It turns out that the mixing properties of the Bernoulli shift of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$, in direct contrast to the Bernoulli shifts of amenable groups, are somehow "remembered" at the level of orbit equivalence, and do not allow themselves to be modeled by a modular equivalence relation.

This argument turns out to give information in the context of Borel reducibility. In answer to a question of Jackson, Kechris, and Louveau:

Theorem 1.6. *There is a countable treeable Borel equivalence relation which is not hyperfinite and does not Borel reduce every other countable treeable Borel equivalence relation.*

Here we say that E *Borel reduces* F if there is a Borel function between their respective fields with $x_1 E x_2$ if and only if $\theta(x_1) F \theta(x_2)$; roughly speaking, the quotient space from E Borel injects into the quotient space from F. We say that an equivalence relation is *hyperfinite* if it can be written as the increasing union of Borel equivalence relations with finite classes; equivalently (see [13]) that it is induced by a Borel action of $\mathbb{Z}$. We say that E is *treeable* if there is an acyclic Borel graph on its field whose connected components form the E-equivalence classes; equivalently, there is a collection of partial Borel bijections which generate E and allow no non-trivial loops.

2. Orbit equivalence relations induced by Kazhdan groups

Notation. We generally write $U(\mathcal{H})$, or $U_\infty(\mathcal{H})$ when we know $\mathcal{H}$ to be infinite dimensional, for the group of unitary transformations of a Hilbert space $\mathcal{H}$.

Definition. Let

$$\pi : G \to U(\mathcal{H})$$
$$g \mapsto \pi_g$$

be a unitary representation of a group G. This representation is said to have *almost invariant unit vectors* if for all $\epsilon > 0$ and $F \subset G$ finite there is some $\zeta \in \mathcal{H}$ with $\|h\| = 1$ and $\|\zeta - \pi_g(\zeta)\| < \epsilon$ all $g \in F$.

Definition. A discrete group G is said to be *Kazhdan* or to *have property T* if any unitary representation $\pi : G \to U(\mathcal{H})$ with almost invariant unit vectors has an outright invariant unit vector – that is to say, some $\zeta \in \mathcal{H}$ with $\|\zeta\| = 1$ and

$$\pi_g(\zeta) = \zeta$$

for all $g \in G$.

Possible references for the subject of Kazhdan groups are given by [9] and [22]. We mention two apparent strengthenings of the definition which in fact turn out to be equivalent. In both cases the proofs are routine and can be left as exercises for the reader.

Proposition 2.1. *G has property T if and only if there is some finite $F \subset G$ and $\epsilon > 0$ such that whenever $\pi : G \to U(\mathcal{H})$ is a unitary representation with some $\zeta \in \mathcal{H}$ having $\|\pi_g(\zeta) - \zeta\| < \epsilon$ for all $g \in F$, then π has an invariant unit vector.*

Proposition 2.2. *G has property T if and only if for all $\delta > 0$ there is some finite $F \subset G$ and $\epsilon > 0$ such that whenever $\pi : G \to U(\mathcal{H})$ is a unitary representation with some $\zeta \in \mathcal{H}$ having $\|\pi_g(\zeta) - \zeta\| < \epsilon$ for all $g \in F$, then π has an invariant unit vector η with $\|\eta - \zeta\| < \delta$.*

One natural example of a Kazhdan group is the collection of three by three integer coefficient matrices with determinant one. More generally, at every $n \geq 3$ the group $SL_n(\mathbb{Z})$ is Kazhdan.

In say Chapter 7 of [22] one can find an extended discussion of theorems to the effect that certain kinds of discrete subgroups of certain kinds of Lie groups will, under the appropriate conditions, be Kazhdan. As with Kazhdan's original proof for $SL_3(\mathbb{Z})$, the proofs are analytical in flavor, turning on the topological properties of the ambient Lie group. More recently Andre Zuk in [21] has obtained purely combinatorial proofs that certain discrete groups have property T. Indeed he even shows that in some suitably statistical sense most finitely generated groups in an indicated class are Kazhdan.

In this section we prove that all countably infinite Kazhdan groups have continuum many free, ergodic, measure preserving actions on standard Borel spaces up to orbit equivalence. This theorem was previously known from [8] for certain special classes of property T groups; for instance it was known for $SL_3(\mathbb{Z})$. The first proof that there is at least *some* countable group with continuum many actions can be found in [2], and builds on work by McDuff in the theory of operator algebras. Our proof is more elementary than these previous arguments.

Definition. Let E be a standard, measure preserving equivalence relation on $(X, \mathcal{B}, \mu)$ and let G be a countable group.

We then set $\mathcal{C}(E, G)$ to be the collection of all measurable

$$\alpha : E \to G$$

such that almost everywhere

(i) if $y \in [x]_E$, then $\alpha(y,x) = 1_G$ if and only if $y = x$;

(ii) if $y, z \in [x]_E$, then $\alpha(z,y)\alpha(y,x) = \alpha(z,x)$;

(iii) for all $g \in G$ there exists $y \in [x]_E$ with $\alpha(y,x) = g$.

We identify $\alpha_1, \alpha_2 \in \mathcal{C}(E,G)$ if they agree almost everywhere.

Remark. We can think of $\mathcal{C}(E,G)$ as the space of possible ways to arrange a free action and measurable action of G on X with $E_G = E$.

In the case that Γ induces E, we can identify $\mathcal{C}(E,G)$ with the cocycles from $X \times \Gamma \to G$ which are appropriately "one-to-one" and "onto".

Definition. From now on fix an enumeration $(g_n)_{n\in\mathbb{N}}$ of the group G. For E as above, and for $\alpha, \beta \in \mathcal{C}(E,G)$ we let

$$d_E(\alpha,\beta) = \sum_{n\in\mathbb{N}} 2^{-n}\mu(\{x : \exists y(\alpha(y,x) = g_n, \beta(y,x) \neq g_n)\} \cup \{x : \exists y(\beta(y,x) = g_n, \alpha(y,x) \neq g_n)\}).$$

$\mathcal{C}(E,G)$ equipped with this metric becomes a separable, complete metric space. We will not need the completeness of the metric, but the separability plays a starring role.

Fact 2.3. *For E and G as above, $\mathcal{C}(E,G)$ has a countable dense subset.*

Proof. Let us consider $\mathcal{C}_0(E,G)$, the set of measurable

$$\alpha : E \to G$$

with $|\{y : \alpha(y,x) = g\}| < 2$ for all $g \in G$, a.e. $x \in X$. Since $\mathcal{C}(E,G)$ is included in $\mathcal{C}_0(E,G)$ and the metric for the first extends to a metric for the second, it will suffice to find a countable dense subset of $\mathcal{C}_0(E,G)$.

For this purpose, let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable injections with $(f_n(x))_{n\in\mathbb{N}}$ enumerating $[x]_E$ almost everywhere. Let $\mathcal{B}_0 \subset \mathcal{B}$ be a countable Boolean subalgebra which is dense with respect to the measure algebra, in the sense that for all $B \in \mathcal{B}$, $\epsilon > 0$, there is some $B_0 \in \mathcal{B}_0$ with $\mu(B\Delta B_0) < \epsilon$. Then consider the collection of all $\alpha \in \mathcal{C}_0(E,G)$ satisfying:

(i) for each g, the set of n with $\mu(\{x : \alpha(f_n(x),x) = g\}) \neq 0$ is finite;

(ii) for each g, n, the set $\{x : \alpha(f_n(x),x) = g\}$ is in $\mathcal{B}_0$;

(iii) for all sufficiently large n, $\{x : \alpha(f_n(x),x) = g_n\} = X$.

The countability and density of this collection are routinely verified. □

Note that the metric d_E depends on the choice of the enumeration $(g_n)_{n\in\mathbb{N}}$ of G. In the observations which follow we will want to consider a single countable group G in relation to various choices of E. We will think of a countable group as coming with some fixed choice of an enumeration $(g_n)_{n\in\mathbb{N}}$, and in each case use d_E to refer to the metric which arises on $\mathcal{C}(E,G)$ for that predetermined choice; we will not specifically mention the enumeration.

Definition (compare [3]). For E a standard, measure preserving equivalence relation on $(X,\mathcal{B},\mu)$, let $p_1, p_2 : E \to X$ be the projections onto the first and second coordinates. We then define a measure m on E by

$$m(B) = \int_X |p_1^{-1}[\{x\}] \cap B| d\mu(x).$$

Since E is measure preserving, we could as well have used the projection p_2 in place of p_1 and obtained the same measure.

We then let $\mathcal{U}(E)$ be the collection of all unitary operators on the Hilbert space $L^2(E, m)$ of square integrable $f : E \to \mathbb{C}$. Given $\alpha, \beta \in \mathcal{C}(E, G)$ we define a unitary representation

$$\pi^{\alpha,\beta} : G \to \mathcal{U}(E),$$
$$g \mapsto \pi_g^{\alpha,\beta}$$

by

$$(\pi_g^{\alpha,\beta}(f))(x', y') = f(x, y)$$

where x', y' are defined by the specification that

$$\alpha(x', x) = g,$$
$$\beta(y', y) = g.$$

In other words, if we use α and β in the obvious way to obtain actions

$$a_\alpha : G \times X \to X,$$
$$a_\beta : G \times X \to X,$$

and we take the induced measure preserving transformation

$$a_\alpha \times a_\beta : G \times E \to E$$

given by

$$(a_\alpha \times a_\beta)(g, (x, y)) = (a_\alpha(g, x), a_\beta(g, y)),$$

then in the usual manner we produce a representation of G in $\mathcal{U}(E)$.

Notation. In what follows, $\chi_\Delta : E \to \{0, 1\}$ is the characteristic function of the diagonal; so that $\chi_\Delta(x, y) = 1$ if $x = y$ and 0 if $x \neq y$.

Fact 2.4. *Let G be a countably infinite group. For all $\epsilon > 0$ and finite $F \subset G$, there exists a $\delta > 0$ such that for all standard, measure preserving E, all $\alpha, \beta \in \mathcal{C}(E, G)$, if $d_E(\alpha, \beta) < \delta$, then*

$$\forall g \in F(\|\pi_g^{\alpha,\beta}(\chi_\Delta) - \chi_\Delta\| < \epsilon),$$

where $\|\cdot\|$ refers to the Hilbert space norm on $L^2(E, m)$. □

Lemma 2.5. *Let G be a countably infinite group with property T.*

Then there is a $\delta > 0$ such that for all ergodic, standard, measure preserving E on $(X, \mathcal{B}, \mu)$ and all $\alpha, \beta \in \mathcal{C}(E, G)$ with

$$d_E(\alpha, \beta) < \delta,$$

there is a measurable bijection $\varphi : X \to X$, $\varphi \subset E$, such that for the induced actions

$$a_\alpha : G \times X \to X,$$
$$a_\beta : G \times X \to X,$$

we have

$$a_\alpha(g, x) = \varphi^{-1}(a_\beta(g, \varphi(x)))$$

almost everywhere.

Proof. Since G has T, we may find $\epsilon > 0$ and finite $F \subset G$ such that whenever $\pi : G \to U_\infty(\mathcal{H})$ is a unitary representation and $\eta \in \mathcal{H}$ is a unit vector with

$$\forall g \in F(\|\pi_g(\eta) - \eta\| < \epsilon),$$

then there is a unit vector $\eta_0 \in \mathcal{H}$ which is G-invariant and has

$$\|\eta - \eta_0\| < 10^{-2}.$$

For this $F \subset G$ and $\epsilon > 0$ we choose $\delta > 0$ as in Fact 2.4.

We then consider E as above, $\alpha, \beta \in \mathcal{C}(E, G)$ with $d_E(\alpha, \beta) < \delta$. By the assumption on δ, ϵ, F, we may find G-invariant $f \in L^2(E, m)$ with

$$\|\chi_\Delta - f\| < 10^{-2}.$$

Consider then the set

$$A_f = \{x \in X | \exists! y \in [x]_E(|f(x, y) - 1| < \frac{1}{4})\}.$$

Since f is close to χ_Δ, A_f is non-null. But then by ergodicity of action $a_\alpha : G \times X \to X$ we must have A_f co-null.

G-invariance of f amounts to the assertion that for almost all $(x, y) \in E$ and all $g \in G$

$$f(x, y) = f(a_\alpha(g, x), a_\beta(g, x)).$$

Thus if we define $\varphi : A_f \to X$ by $\varphi(x) = y$ if and only if $|f(x, y) - 1| < \frac{1}{4}$, then this equation implies

$$\varphi(x) = y \Leftrightarrow \varphi(a_\alpha(g, x)) = a_\beta(g, y),$$

which in turn gives

$$a_\alpha(g, x) = \varphi^{-1}(a_\beta(g, \varphi(x)))$$

almost everywhere. □

Corollary 2.6. *If E is an ergodic, standard, measure preserving, equivalence relation on $(X, \mathcal{B}, \mu)$ and G is a property T group, then up to isomorphism*[1] *there are at most $\aleph_0$ many free actions of G by measurable transformations on X which induce E.*

Proof. We may identify such actions with elements of $\mathcal{C}(E, G)$. The last lemma asserts that the equivalence relation of isomorphism of action is open in $\mathcal{C}(E, G)$, and so the corollary follows from the separability of this space. □

Corollary 2.7. *If G is a countably infinite property T group, then it induces $2^{\aleph_0}$ many orbit inequivalent ergodic, standard, measure preserving, equivalence relations.*

Proof. Since the group G is not abelian-by-finite, and hence not type I (see [20]), we may find

$$\pi_\gamma : G \to U_\infty(\mathcal{H}_\gamma),$$

$\gamma \in 2^{\aleph_0}$, non-conjugate infinite dimensional irreducible unitary representations of G. Following [22], 5.2.13, we may find corresponding ergodic actions of G on standard Borel probability spaces $(X_\gamma, \mathcal{B}_\gamma, \mu_\gamma)$, with each induced representation $\sigma_\gamma : G \to U_\infty(L^2(X_\gamma, \mu_\gamma))$ having $\mathcal{H}_\gamma$ as a direct summand. The product of an

[1]Here we say that two actions of G are *isomorphic* if they are simultaneously conjugate: That is to say, $a, b : G \times X \to X$ are *isomorphic* actions if there is measure preserving $\varphi : X \cong X$ with $\varphi(a(g, \varphi^{-1}(x))) = b(g, x)$ almost everywhere.

ergodic action of an infinite group with a mixing action is still ergodic, so after possibly taking the product of this action with the Bernoulli shift, of G acting on $\{0,1\}^G$, we may assume that each action is free.

Then each induced representation $\sigma_\gamma : G \to U_\infty(L^2(X_\gamma, \mu_\gamma))$ has only countably many irreducible representations as direct summands, and is therefore only isomorphic as a unitary representation to countably many other σ_κ's; in particular the original measure preserving action from which it derives is only isomorphic to countably many others. Thus by 2.6 each such E_{σ_γ} is orbit equivalent to at most countably many other such E_{σ_α}'s. □

3. Non-modular equivalence relations induced by $\mathbb{F}_2$

In this section we formulate the notion of a modular equivalence relation, and show that in general the free actions of $\mathbb{F}_2$ may induce non-modular actions.

The argument gives information in the context of Borel reducibility, and thus in answer to question 6.4(A) from [13], we obtain in the following section a treeable countable Borel equivalence relation which is neither universal treeable nor hyperfinite.

Very recently Damien Gaboriau and Sorin Popa have shown that any non-abelian free group gives rise to continuum many non-orbit equivalent free actions. As far as can be determined, their argument does not appear to provide insight at the level of $\leq_B$-reducibility, and it remains open whether $\mathbb{F}_2$, or any $\mathbb{F}_n$ $(n = 2, 3, ..., \aleph_0)$ has more than two non-hyperfinite free Borel actions up to Borel reducibility.

There is a fair amount in the way of prefatory lemmas and definitions. We need to distinguish the highly mixing actions of say the Bernoulli shift from those actions which have generating sets for the Borel algebra with finite orbit. We make that distinction by assigning to each measurable set in $2^{\mathbb{F}_2} =_{\text{df}} \{0,1\}^{\mathbb{F}_2}$ a *center*, cradled between those elements of $\mathbb{F}_2$ which are most important for its definition. The various prefatory lemmas are required for that definition and to formulize the manner in which this center is rapidly mixed by the equivalence relation.

We think of $\mathbb{F}_2 = \langle a, b\rangle$ as having a and b as generators.

Definition. A *weight* for $\mathbb{F}_2$ is a function

$$w : \mathbb{F}_2 \to \mathbb{R}$$

such that $w(\sigma) \geq 0$ all $\sigma \in \mathbb{F}_2$, w is non-zero at some point, and $w \in \ell^1(\mathbb{F}_2)$. An element σ_0 is a *center* for some such weight w if each

$$\sum\{w(\tau x \sigma_0) : \tau \in \mathbb{F}_2, \tau x \text{ is a reduced word}\},$$

as x ranges over $\{a, a^{-1}, b, b^{-1}\}$, is less than

$$\frac{1}{2}\sum\{w(\tau) : \tau \in \mathbb{F}_2\}.$$

σ_0 is a *weak center* if each such

$$\sum\{w(\tau x \sigma_0) : \tau x \text{ is a reduced word}\} < \frac{2}{3}\sum\{w(\tau) : \tau \in \mathbb{F}_2\}.$$

Lemma 3.1. *Every weight has at least one center.*

Proof. We begin with some $e \in \mathbb{F}_2$, and if that is a center we are content.

Otherwise, there will be some $x_0 \in \{a, a^{-1}, b, b^{-1}\}$ with most of the weight lying on reduced words of the form τx_0, and so we shift our attention to x_0. Again, if x_0

is a center, we are done; otherwise, we pass onto some x_1 with most of the weight lying on words of the form $\tau x_1 x_0$. Since the majority of the weight lies within some definite distance from the identity, this process must terminate after considering finitely many x_i's. □

Notation. We let $\mathbb{F}_2$ act on $\ell^1(\mathbb{F}_2)$ by right multiplication:

$$\tau \cdot f(\sigma) = f(\sigma\tau).$$

Lemma 3.2. *σ_0 is a center for a weight w if and only if $\sigma_0\tau^{-1}$ is a center for $\tau \cdot w$; σ_0 is a weak center for a weight w if and only if $\sigma_0\tau^{-1}$ is a weak center for $\tau \cdot w$.*

Proof. Immediate from the structure of the definitions. □

Note then that the centers of a weight and the weak centers of a weight form a finite subset of $\mathbb{F}_2$ which is convex when viewed as being included in the Cayley graph. In fact they are not just convex but linear; this is the key combinatorial fact.

Lemma 3.3. *If w is a weight, then its collection of weak centers is linearly ordered; that is to say, there are $x_0, x_1, ..., x_n$, with $x_0 \in \mathbb{F}_2$, each $x_{i+1} \in \{a, a^{-1}, b, b^{-1}\}$, such that each $x_i x_{i-1}...x_0$ forms a reduced word and*

$$\{x_i x_{i-1}...x_0 : i \leq n\}$$

enumerates the weak centers.

Proof. Assume instead that there is some $\sigma_0 \in \mathbb{F}_2$ such that $a\sigma_0, a^{-1}\sigma_0, b\sigma_0$ are all weak centers. (Without any real loss of generality we can assume that this is the case; the other possibilities are entirely similar.)

Then for $c \in \{a, a^{-1}\}$ we have for $x = c^{-1}$ that

$$\sum\{w(\tau x c\sigma_0) : \tau x \text{ is a reduced word}\} < \frac{2}{3}\|w\|_{\ell^1}$$
$$\therefore \sum\{w(\tau c\sigma_0) : \tau c \text{ is a reduced word}\} > \frac{1}{3}\|w\|_{\ell^1}.$$

For $y = b^{-1}$ we have

$$\{\tau_0 y b\sigma_0 : \tau_0 y \text{ reduced}\} \supset \{\tau_1 a\sigma_0 : \tau_1 a \text{ reduced}\} \cup \{\tau_2 a^{-1}\sigma_0 : \tau_2 a^{-1} \text{ reduced}\}$$
$$\therefore \sum\{w(\tau y b\sigma_0) : \tau y \text{ is a reduced word}\} > \frac{2}{3}\|w\|_{\ell^1},$$

with a slur on $b\sigma_0$'s claim of weak centerhood. □

Notation. We let $2^{\mathbb{F}_2} =_{\text{df}} \{0,1\}^{\mathbb{F}_2}$ be the collection of functions from $\mathbb{F}_2$ to $\{0,1\}$; we equip this with the product measure and let $\mathbb{F}_2$ act by multiplication on the right: For $f \in 2^{\mathbb{F}_2}$, $\sigma, \tau \in \mathbb{F}_2$,

$$\tau \cdot f(\sigma) = f(\sigma\tau).$$

This in turn induces an action on $L^2(2^{\mathbb{F}_2})$ in the usual way: Given

$$\psi : 2^{\mathbb{F}_2} \to \mathbb{C}$$

we define $\tau \cdot \psi$ by the formulation

$$\tau \cdot \psi(f) = \psi(\tau^{-1} \cdot f).$$

For $S \subset \mathbb{F}_2$ finite we define a corresponding function

$$\psi_S : 2^{\mathbb{F}_2} \to \{-1, 1\},$$
$$f \mapsto (-1)^{|\{\sigma \in S : f(\sigma)=1\}|}.$$

Thus if f assumes the value 1 on an odd number of elements of S, then $\psi_S(f) = -1$; otherwise it takes the value 1.

It is well known that $\{\psi_S : S \subset \mathbb{F}_2 \text{ is finite}\}$ forms an orthonormal basis. One can for instance determine by hand that they are orthogonal and then by Stone-Weierstrass check that their linear combinations are dense, and hence that they span.

Note that $\psi_\emptyset : f \mapsto 1$ is the constant function with value 1. $\psi \in L^2(2^{\mathbb{F}_2})$ will be orthogonal to the constant functions if and only if it is in the subspace given by the closed span of $\{\psi_S : S \neq \emptyset\}$.

Notation. Let $\mathcal{F}(\mathbb{F}_2) = \{S \subset \mathbb{F}_2 : S \text{ is finite}\}$.

$\mathcal{F}(\mathbb{F}_2)$ is a countable set, and we can thus form $\ell^2(\mathcal{F}(\mathbb{F}_2))$ in the usual way. This Hilbert space is of course naturally isomorphic to $L^2(2^{\mathbb{F}_2})$ and we introduce some notation to describe that isomorphism.

Notation. Let

$$\Delta : L^2(2^{\mathbb{F}_2}) \to \ell^2(\mathcal{F}(\mathbb{F}_2))$$

be the linear isometry induced by the association

$$\psi_S \mapsto \delta_S,$$

where $\delta_S(T) = 1$ if $T = S$, $= 0$ if $T \neq S$.

From here we wish to convert the elements of $\ell^2(\mathcal{F}(\mathbb{F}_2))$ into weights. Of course now there is no question of a linear isomorphism; instead we ask only for a process of conversion which is Lipschitz on the unit sphere. We begin by mapping $\ell^2(\mathcal{F}(\mathbb{F}_2))$ into the positive part of $\ell^1(\mathcal{F}(\mathbb{F}_2))$.

Notation. We define

$$E : \ell^2(\mathcal{F}(\mathbb{F}_2)) \to \ell^1(\mathcal{F}(\mathbb{F}_2))$$

by

$$E(\phi)(s) = |\phi(s)|^2.$$

Note that $\|E(\phi)\|_{\ell^1} = \|\phi\|^2_{L^2}$.

Lemma 3.4. *For $\phi_1, \phi_2 \in \ell^2(\mathcal{F}(\mathbb{F}_2))$,*

$$\|E(\phi_1) - E(\phi_2)\|_{\ell^1} \leq (\|\phi_1\|_{\ell^2} + \|\phi_2\|_{\ell^2})\|\phi_1 - \phi_2\|_{\ell^2}.$$

Proof.

$$\|E(\phi_1) - E(\phi_2)\|_{\ell^1} = \sum_{s \in \mathcal{F}(\mathbb{F}_2)} |E(\phi_1)(s) - E(\phi_2)(s)|$$

$$= \sum_{s \in \mathcal{F}(\mathbb{F}_2)} |\phi_1(s)^2 - \phi_2(s)^2| = \sum_{s \in \mathcal{F}(\mathbb{F}_2)} |\phi_1(s)[\phi_1(s) - \phi_2(s)] + \phi_2(s)[\phi_1(s) - \phi_2(s)]|$$

$$\leq \sum_{s \in \mathcal{F}(\mathbb{F}_2)} |\phi_1(s)| \cdot |\phi_1(s) - \phi_2(s)| + \sum_{s \in \mathcal{F}(\mathbb{F}_2)} |\phi_2(s)| \cdot |\phi_1(s) - \phi_2(s)|$$

$$= \langle |\phi_1|, |\phi_1 - \phi_2| \rangle + \langle |\phi_2|, |\phi_1 - \phi_2| \rangle \leq \|\phi_1\|_{\ell^2} \|\phi_1 - \phi_2\|_{\ell^2} + \|\phi_2\|_{\ell^2} \|\phi_1 - \phi_2\|_{\ell^2},$$

by Cauchy-Schwarz. □

Notation. We define

$$\Sigma : \ell^1(\mathcal{F}(\mathbb{F}_2)) \to \ell^1(\mathbb{F}_2)$$

by

$$\Sigma(\phi)(\sigma) = \sum_{S \in \mathcal{F}(\mathbb{F}_2), \sigma \in S} |S|^{-1} \phi(S).$$

Σ is a linear contraction, and for ϕ positive, say in the range of E, of unit length, and with $\phi(\emptyset) = 0$, we have that $\Sigma(\phi)$ is a weight with $\|\Sigma(\phi)\|_{\ell^1} = \|\phi\|_{\ell^1}$.

We then let

$$\pi = \Sigma E \Delta : L^2(2^{\mathbb{F}_2}) \to \ell^1(\mathbb{F}_2),$$
$$\varphi \mapsto \Sigma(E(\Delta(\varphi))).$$

Notation. For $S \subset \mathbb{F}_2$ finite and $\tau \in \mathbb{F}_2$ we let

$$\tau \cdot S = \{\sigma\tau^{-1} : \sigma \in S\};$$

thus

$$\sigma \in S \Leftrightarrow \sigma\tau^{-1} \in \tau \cdot S,$$
$$\therefore \sigma\tau \in S \Leftrightarrow \sigma \in \tau \cdot S.$$

Then for $f \in \ell^2(\mathcal{F}(\mathbb{F}_2)) \cup \ell^1(\mathcal{F}(\mathbb{F}_2))$ and $\tau \in \mathbb{F}_2$ we define $\tau \cdot f \in \ell^2(\mathcal{F}(\mathbb{F}_2)) \cup \ell^1(\mathcal{F}(\mathbb{F}_2))$ by

$$\tau \cdot f(S) = f(\tau^{-1} \cdot S);$$

thus if $f = \delta_{S_0}$, then

$$\tau \cdot f(S) = 1 \text{ iff } \tau^{-1} \cdot S = S_0$$
$$\text{iff } S = \tau \cdot S_0;$$

and thus $\tau \cdot \delta_{S_0} = \delta_{\tau \cdot S_0}$.

Lemma 3.5. *(o) For $\varphi \in L^2(2^{\mathbb{F}_2})$ orthogonal to the constant functions and of norm one, $\|\pi(\varphi)\|_{\ell^1} = 1$.*

(i) $\|\pi(\varphi_1) - \pi(\varphi_2)\|_{\ell^1} \leq (\|\varphi_1\|_{L^2} + \|\varphi_2\|_{L^2})\|\varphi_1 - \varphi_2\|_{L^2}$.

(ii) π is an $\mathbb{F}_2$-map, in the sense that for $\sigma \in \mathbb{F}_2$ and $\varphi \in L^2(2^{\mathbb{F}_2})$,

$$\pi(\sigma \cdot \varphi) = \sigma \cdot \pi(\varphi).$$

(iii) For φ not a.e. constant, $\pi(\varphi)$ is a weight.

(iv) For such φ and $\sigma \in \mathbb{F}_2$, σ_0 is a center of $\pi(\varphi)$ if and only if $\sigma_0\sigma^{-1}$ is a center of $\pi(\sigma \cdot \varphi)$; σ_0 is a weak center of $\pi(\varphi)$ if and only if $\sigma_0\sigma^{-1}$ is a weak center of $\pi(\sigma \cdot \varphi)$.

(v) For φ_1, φ_2 unit vectors in $L^2(2^{\mathbb{F}_2})$ orthogonal to the constant functions with $\|\varphi_1 - \varphi_2\|_{L^2} < \frac{1}{12}$, and σ_0 a center of $\pi(\varphi_1)$, we will necessarily have σ_0 a weak center of $\pi(\varphi_2)$.

Proof. (o) and (i) follow from the various facts recorded above regarding E, Δ, and Σ.

(ii): We first observe:

Claim: $\Delta : L^2(2^{\mathbb{F}_2}) \to \ell^2(\mathcal{F}(\mathbb{F}_2))$ is an $\mathbb{F}_2$-map.

Proof of Claim. It suffices to check this for the basis elements, of the form ψ_S. So let $\tau \in \mathbb{F}_2$, $S \subset \mathbb{F}_2$ finite, $f \in 2^{\mathbb{F}_2}$.

$$\tau \cdot \psi_S(f) = \psi_S(\tau^{-1} \cdot f) = (-1)^{|\{\sigma \in S : \tau^{-1} \cdot f(\sigma) = 1\}|}$$
$$= (-1)^{|\{\sigma \in S : f(\sigma\tau^{-1}) = 1\}|} = (-1)^{|\{\sigma\tau \in S : f(\sigma) = 1\}|} = (-1)^{|\{\sigma \in \tau \cdot S : f(\sigma) = 1\}|}$$
$$= \psi_{\tau \cdot S}(f).$$

Thus $\Delta(\tau \cdot \psi_S) = \Delta(\psi_{\tau \cdot S}) = \delta_{\tau \cdot S} = \tau \cdot \delta_S = \tau \cdot \Delta(\psi_S)$. □

It is immediate that E is an $\mathbb{F}_2$-map, and as for Σ we have

$$(\Sigma(\tau \cdot f))(\sigma) = \sum_{\sigma \in S} |S|^{-1} \tau \cdot f(S) = \sum_{\sigma \in S} |S|^{-1} \cdot f(\tau^{-1} \cdot S)$$
$$= \sum_{\sigma \in \tau \cdot S} |S|^{-1} f(S) = \sum_{\sigma\tau \in S} |S|^{-1} f(S) = (\Sigma(f))(\sigma\tau) = (\tau \cdot \Sigma(f))(\sigma).$$

(iii) This is obvious from the definitions and Lemma 3.4.

(iv) We have for any $x \in \{a, a^{-1}, b, b^{-1}\}$ that

$$\sum \{\pi(\sigma \cdot \varphi)(\tau x \sigma_0 \sigma^{-1}) : \tau x \text{ is a reduced word}\}$$
$$= \sum \{\sigma \cdot \pi(\varphi)(\tau x \sigma_0 \sigma^{-1}) : \tau x \text{ is a reduced word}\}$$

by (ii) above,

$$= \sum \{\pi(\varphi)(\tau x \sigma_0) : \tau x \text{ is a reduced word}\},$$

by the definition of the action.

(v) If $\|\varphi_1\|_{L^2}, \|\varphi_2\|_{L^2} = 1$, $\varphi_1, \varphi_2 \perp \mathbb{C}1$, and

$$\|\varphi_1 - \varphi_2\|_{L^2} < \frac{1}{12},$$

then using (0) and (i) we have

$$\|\pi(\varphi_1) - \pi(\varphi_2)\|_{\ell^1} < \frac{1}{6} \min\{\|\pi(\varphi_1)\|_{\ell^1}, \|\pi(\varphi_2)\|_{\ell^1}\} = \frac{1}{6}.$$

Thus for each $\sigma_0 \in \mathbb{F}_2$, $x \in \{a, a^{-1}, b, b^{-1}\}$,

$$|\sum \{\pi(\varphi_1)(\tau x \sigma_0) : \tau x \text{ reduced}\} - \sum \{\pi(\varphi_2)(\tau x \sigma_0) : \tau x \text{ reduced}\}|$$

is bounded by $\frac{1}{6}\|\pi(\varphi_2)\|_{\ell^1}$, so that if

$$\sum \{\pi(\varphi_1)(\tau x \sigma_0) : \tau x \text{ reduced}\} < \frac{1}{2}\|\pi(\varphi_1)\|_{\ell^1} = \frac{1}{2}\|\pi(\varphi_2)\|_{\ell^1},$$

then

$$\sum \{\pi(\varphi_2)(\tau x \sigma_0) : \tau x \text{ reduced}\}$$
$$< \frac{1}{2}\|\pi(\varphi_2)\|_{\ell^1} + \frac{1}{6}\|\pi(\varphi_2)\|_{\ell^1} = \frac{4}{6}\|\pi(\varphi_2)\|_{\ell^1} = \frac{2}{3}\|\pi(\varphi_2)\|_{\ell^1}.$$

Definition. An equivalence relation E on a standard Borel space $(X, \mathcal{B})$ has *modular type* if there is a countable group G acting by Borel automorphisms on X with $E = E_G$ and there is an increasing sequence of finite Boolean algebras

$$\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset \ldots \subset \mathcal{B}_n \subset \mathcal{B}_{n+1} \subset \ldots\mathcal{B},$$

such that G's action permutes each $\mathcal{B}_n$ and $\mathcal{B}$ is generated as a σ-algebra by

$$\bigcup_{n\in\mathbb{N}} \mathcal{B}_n.$$

In particular then we have $X, \emptyset \in \mathcal{B}_0$ and if A is an atom in some $\mathcal{B}_n$ and $g \in G$, then $g \cdot A$ is still an atom in $\mathcal{B}_n$.

Example. Consider the space $2^{\mathbb{N}} =_{\rm df} \{f | f : \mathbb{N} \to \{0,1\}\}$, equipped with the metric $d(f_1, f_2) = 2^{-n(f_1,f_2)}$, where $n(f_1, f_2)$ is the least n with $f_1(n) \neq f_2(n)$. Suppose, as in the examples constructed by Ted Slaman and John Steel [19], we have an action of $\mathbb{F}_2$ by isometries on this space relative to this metric: That is to say, for all $f_1, f_2 \in 2^{\mathbb{N}}$ and $\sigma \in \mathbb{F}_2$ we have $d(f_1, f_2) = d(\sigma \cdot f_1, \sigma \cdot f_2)$.

Then for each $s \in 2^{<\mathbb{N}}$ a finite binary sequence, we let

$$V_s = \{f \in 2^{\mathbb{N}} : f \supset s\}.$$

Taking $\mathcal{B}_n = \{V_s : \text{length}(s) \leq n\}$ we have witnesses to modularity.

We will in the future refer to this space with this action as X_{ss}.

Alexander Kechris has suggested a different way of thinking about these classes of examples. If we order the atoms of the various $\mathcal{B}_n$'s under inclusion, thereby obtaining a tree structure, then we can identify the space with a subset of the infinite branches, and view the group as acting by isomorphisms. In this way modular actions can be identified with actions by automorphisms of a rooted tree.

In the next theorem we equip $2^{\mathbb{F}_2} =_{\rm df} \{0,1\}^{\mathbb{F}_2}$ with the product measure, μ, and the shift action on the right: $(\tau \cdot f)(\sigma) = f(\sigma\tau)$. We let $E_{\mathbb{F}_2}$ denote this equivalence relation.

Theorem 3.6. *Let E be of modular type on $(X, \mathcal{B})$ and $M \subset 2^{\mathbb{F}_2}$ of full measure. Then there is no countable to one measurable*

$$\theta : M \to X$$

such that for all $f_1, f_2 \in 2^{\mathbb{F}_2}$,

$$f_1 E_{\mathbb{F}_2} f_2 \Rightarrow \theta(f_1) E \theta(f_2).$$

Proof. Note first of all that we may assume that on some positive measure set $M_0 \subset M$ we have that $\theta|_{M_0}$ is one-to-one. There are various ways to see this: One approach is to appeal to the uniformization theorem for subsets of the plane with countable sections, and argue that $\theta[M]$ is Borel and admits a Borel right inverse $\rho : \theta[M] \to M$; the image of ρ has conull saturation, and hence positive measure; since $\theta \circ \rho(y) = y$ all $y \in \text{Dom}(\rho)$, we can simply take the image of ρ as our positive set.

For the time being I wish to make a drastic simplifying assumption: θ is injective *everywhere*. This will ease some of the notation, and we can return to the further problems faced in the general case after having seen the main ideas. In actual fact it will turn out that the general argument is only slightly more involved and requires only minor regearing.

Let G be a countable group acting by Borel automorphisms with $E = E_G$, $\mathcal{B}_0 \subset \mathcal{B}_1 \subset ... \subset \mathcal{B}_n \subset \mathcal{B}_{n+1} \subset ...\mathcal{B}$ with $\bigcup \mathcal{B}_n$ generating $\mathcal{B}$ and each $\mathcal{B}_n$ a finite G-invariant algebra. We let $A(\mathcal{B}_n)$ denote the atoms of the algebra $\mathcal{B}_n$.

For $B \in \mathcal{B}$ we let

$$\hat{B} = \theta^{-1}[B].$$

We can then define $\hat{\mathcal{B}}$ to be $\{\hat{B} : B \in \mathcal{B}\}$, $\hat{\mathcal{B}_n} = \{\hat{B} : B \in \mathcal{B}_n\}$, and $A(\hat{\mathcal{B}_n})$ to be the atoms of $\hat{\mathcal{B}_n}$. Note that off of a measure zero set we still have $\hat{\mathcal{B}}$ generating the Borel algebra of $2^{\mathbb{F}_2}$. (See 15.2 of [11]. This is the place where we are using θ injective.)

For $x \in \{a, a^{-1}, b, b^{-1}\}$, $f \in 2^{\mathbb{F}_2}$, let $g_{f,x} \in G$ be such that

$$g_{f,x} \cdot \theta(f) = \theta(x \cdot f).$$

Note that this assignment $f \mapsto g_{f,x}$ can be chosen to be measurable, for any such x.

Claim (I): $\forall x \in \{a, a^{-1}, b, b^{-1}\} \forall \epsilon > 0 \exists M \subset 2^{\mathbb{F}_2} \exists n \in \mathbb{N}$ such that $\mu(M) > 1 - \epsilon$ and $\forall \hat{B} \in A(\hat{\mathcal{B}_n}) \forall f_1, f_2 \in \hat{B} \cap M$,

$$g_{f_1,x} = g_{f_2,x};$$

in other words, the function $f \mapsto g_{f,x}$ depends only on which $\hat{B} \in A(\hat{\mathcal{B}_n})$ the point f resides. □

Proof of Claim. One approach is to fix $F \subset G$ finite such that off of a set of measure less than $\epsilon/2$ we have each $g_{f,x} \in F$; we then go on and let $(M_g)_{g \in F}$ be such that for all $f \in M_g$,

$$g_{f,x} = g$$

and $\mu(\bigcup_{g \in F} M_g) > 1 - \epsilon/2$. Since $\bigcup \hat{\mathcal{B}}_n$ generates $\hat{\mathcal{B}}$ as a σ-algebra, we have that $\bigcup_{n \in \mathbb{N}} \hat{\mathcal{B}}_n$ is dense in the measure algebra. Thus for each $g \in \mathbb{F}$ we may choose $n_g \in \mathbb{N}$ and $\hat{B}_g \in \hat{\mathcal{B}}_{n_g}$ with

$$\mu(\hat{B}_g \Delta M_g) < \frac{\epsilon}{2|F|}.$$

Thus taking

$$n = \max\{n_g : g \in F\}$$

and

$$M = (\bigcup_{g \in F} M_g) \setminus (\bigcup_{g \in F} \hat{B}_g \Delta M_g),$$

we are done. □

Applying this last claim repeatedly we may produce sets $N_2 \subset N_1 \subset 2^{\mathbb{F}_2}$, $n \in \mathbb{N}$, and

$$g : A(\hat{\mathcal{B}_n}) \times \{a, a^{-1}, b, b^{-1}\} \to G,$$
$$(\hat{B}, x) \mapsto g_{\hat{B},x},$$

such that

(i) for all $\tau \in \mathbb{F}_2$ with[2] $d(\tau, e) \leq 3$ and all $f \in N_2$ we have $\tau \cdot f \in N_1$;
(ii) $\mu(N_2) > 1 - (10^{-9})$;
(iii) if $\hat{B} \in A(\hat{\mathcal{B}_n})$, $f \in \hat{B} \cap N_1$, $x \in \{a, a^{-1}, b, b^{-1}\}$, then

$$g_{\hat{B},x} = g_{f,x}.$$

We may further assume that $\mu(\hat{B}) < 10^{-8}$ for all $\hat{B} \in A(\hat{\mathcal{B}_n})$.

[2] We write $d(\tau_0, \tau_1)$ to indicate the distance of τ_0 from τ_1 in the Cayley graph; thus $d(\tau, e) \leq 3$ indicates that τ can be written as a word in $\{a, a^{-1}, b, b^{-1}, e\}$ of length at most 3.

Claim (II): If $x \in \{a, a^{-1}, b, b^{-1}\}$, $B_1, B_2 \in A(\mathcal{B}_n)$, and $x \cdot [\hat{B}_1 \cap N_1] \cap \hat{B}_2 \cap N_1 \neq 0$, then

(i) $x \cdot [\hat{B}_1 \cap N_1] \subset \hat{B}_2$;
(ii) $x^{-1} \cdot [\hat{B}_2 \cap N_1] \subset \hat{B}_1$;
(iii) $x \cdot [\hat{B}_1] \supset \hat{B}_2 \cap N_1$.

Proof of Claim. (i) Choose $f_0 \in \hat{B}_1 \cap N_1 \cap x^{-1} \cdot [\hat{B}_2]$. Then $g_{f_0,x} = g_{\hat{B}_1,x}$, and since

$$g_{\hat{B}_1,x} \cdot \theta(f_0) = \theta(x \cdot f_0) \in B_2$$

and G permutes $A(\mathcal{B}_n)$, we have $g_{\hat{B}_1,x} \cdot B_1 = B_2$. Thus for all $f \in \hat{B}_1 \cap N_1$ we have

$$\theta(x \cdot f) = g_{f,x} \cdot \theta(f) = g_{\hat{B}_1,x} \cdot \theta(f) \in B_2,$$

$$\therefore x \cdot f \in \hat{B}_2.$$

(ii) Choose $f_0 \in \hat{B}_1 \cap N_1 \cap x^{-1} \cdot [\hat{B}_2 \cap N_1]$. Letting $f_1 = x \cdot f_0$ we have

$$f_1 \in \hat{B}_2 \cap N_1,$$

$$x^{-1} \cdot f_1 \in \hat{B}_1,$$

and we finish by applying (i) but with x^{-1} taking the place of x and $\hat{B}_1$ and $\hat{B}_2$ exchanging roles we are done.

(iii) If $f_0 \in \hat{B}_2 \cap N_1$, then $x^{-1} \cdot f_0 \in \hat{B}_1$ by (ii). $\square$

Definition. Let us say that $\hat{B} \in A(\hat{\mathcal{B}}_n)$ is *good* if

$$\frac{\mu(\hat{B} \setminus N_2)}{\mu(\hat{B})} < 10^{-4};$$

if it is not good, then we say it is *bad*.

It follows from $\mu(N_2) > 1 - (10^{-9})$ that

$$\mu(\bigcup\{\hat{B} \in A(\hat{\mathcal{B}}_n) : \hat{B} \text{ is bad}\} \cup (2^{\mathbb{F}_2} \setminus N_2) < 10^{-4}.$$

The remainder of the proof has the following overarching form. The last claim more or less states that $\mathbb{F}_2$ comes close to permuting the good elements of $A(\hat{\mathcal{B}}_n)$. But then if we look at the centers associated to the characteristic functions of the good elements of $A(\hat{\mathcal{B}}_n)$, we obtain a set which is overly $\mathbb{F}_2$-invariant, and a contradiction ensues.

Let

$$N_3 = \{f \in 2^{\mathbb{F}_2} : \forall \tau \in \mathbb{F}_2 (d(\tau, e) \leq 3 \Rightarrow \tau \cdot f \in N_2 \setminus \bigcup\{\hat{B} \in A(\hat{\mathcal{B}}_n) : \hat{B} \text{ bad}\})\}.$$

Note that $\mu(N_3) > 1 - (4 \times 3 \times 3 \times 10^{-4}) > 1 - 10^{-2}$.

Claim (III): If $f_0 \in N_3 \cap \hat{B}_1, \tau \in \mathbb{F}_2, d(\tau, e) \leq 3, \tau \cdot f_0 \in \hat{B}_2$, then

$$\frac{\mu(\hat{B}_1)}{\mu(\hat{B}_2)} \in (\frac{10^4 - 1}{10^4}, \frac{10^4}{10^4 - 1})$$

and

$$\mu(\tau \cdot \hat{B}_1 \Delta \hat{B}_2) < 4 \cdot 10^{-4} \min\{\mu(\hat{B}_1), \mu(\hat{B}_2)\}.$$

Proof of Claim. Let us suppose $\tau = x_0x_1x_2$, each $x_i \in \{a, a^{-1}, b, b^{-1}\}$; let us fix $\hat{C}_1, \hat{C}_2 \in A(\hat{\mathcal{B}}_n)$ containing $x_2 \cdot f$ and $x_1x_2 \cdot f$. Then by (iii) of the last claim, $x_2 \cdot \hat{B}_1 \supset \hat{C}_1 \cap N_1$, $x_1 \cdot \hat{C}_1 \supset \hat{C}_2 \cap N_1$, and $x_0\hat{C}_2 \supset \hat{B}_2 \cap N_1$. But for any $f' \in \hat{B}_2 \cap N_2$ we have

$$\tau^{-1} \cdot f' = x_2^{-1}x_1^{-1}x_0^{-1} \cdot f', x_1^{-1}x_0^{-1} \cdot f', x_0^{-1} \cdot f' \in N_1;$$

tracking backwards we obtain $x_0^{-1} \cdot f' \in \hat{C}_2 \cap N_1$, $x_1^{-1}x_0^{-1} \cdot f' \in \hat{C}_1 \cap N_1$, and finally $\tau^{-1} \cdot f' \in \hat{B}_1$. Thus, bearing in mind that $\tau \cdot f \in \hat{B}_2$ implying $\hat{B}_2$ good, we in general obtain

$$\tau \cdot [\hat{B}_1] \supset \hat{B}_2 \cap N_2$$

$$\therefore (\mu(\hat{B}_2 \setminus (\tau \cdot \hat{B}_1)) < 10^{-4}\mu(\hat{B}_2)$$

$$\therefore \mu(\hat{B}_2) - \mu(\hat{B}_1) = \mu(\hat{B}_2) - \mu(\tau \cdot \hat{B}_1) < 10^{-4}\mu(\hat{B}_2).$$

Similar reasoning implies $\tau^{-1} \cdot [\hat{B}_2] \supset \hat{B}_1 \cap N_2$,

$$\therefore \hat{B}_2 \supset \tau \cdot [\hat{B}_1 \cap N_2]$$

$$\therefore \mu((\tau \cdot \hat{B}_1) \setminus \hat{B}_2) < 10^{-4}\mu(\hat{B}_1).$$

Then as in the previous paragraph we have

$$\therefore \mu(\hat{B}_1) - \mu(\hat{B}_2) < 10^{-4}\mu(\hat{B}_1).$$

□

Definition. For $\hat{B} \subset 2^{\mathbb{F}_2}$ with $\mu(\hat{B}) \in (0,1)$ we define $\gamma_{\hat{B}} \in L^2(2^{\mathbb{F}_2})$ by

$$\gamma_{\hat{B}}(f) = \frac{\sqrt{(1-\mu(\hat{B}))}}{\sqrt{(\mu(\hat{B}))}}$$

for $f \in \hat{B}$ and

$$\gamma_{\hat{B}}(f) = \frac{-\sqrt{(\mu(\hat{B}))}}{\sqrt{(1-\mu(\hat{B}))}}$$

for $f \notin \hat{B}$. Note then that

$$\|\gamma_{\hat{B}}\|^2_{L^2} = \int (\gamma_{\hat{B}})^2$$

$$= \int_{\hat{B}} \frac{1-\mu(\hat{B})}{\mu(\hat{B})} + \int_{2^{\mathbb{F}_2} \setminus \hat{B}} \frac{\mu(\hat{B})}{1-\mu(\hat{B})} = 1$$

and that

$$\langle \gamma_{\hat{B}}, 1 \rangle = \int_{\hat{B}} ((1-\mu(\hat{B}))/\mu(\hat{B}))^{\frac{1}{2}} - \int_{2^{\mathbb{F}_2} \setminus \hat{B}} (\mu(\hat{B})/(1-\mu(\hat{B})))^{\frac{1}{2}}$$

$$= \sqrt{(\mu(\hat{B}))}\sqrt{(1-\mu(\hat{B}))} - \sqrt{(1-\mu(\hat{B}))}\sqrt{(\mu(\hat{B}))} = 0.$$

We therefore have a unit vector which is orthogonal to the constant functions.

We say that σ_0 is a *center* (or *weak center*) for $f \in 2^{\mathbb{F}_2}$ if for $\hat{B} \in A(\hat{\mathcal{B}}_n)$ containing f we have that σ_0 is a center (respectively, weak center) for $\pi(\gamma_{\hat{B}})$, the weight associated to the element of $A(\hat{\mathcal{B}}_n)$ in which f falls.

Claim (IV): If $f \in N_3, \tau \in \mathbb{F}_2, d(\tau, e) \leq 3, f \in \hat{B}_1, \tau \cdot f \in \hat{B}_2, \hat{B}_1, \hat{B}_2 \in A(\hat{\mathcal{B}}_n)$, then

$$\|\tau \cdot \gamma_{\hat{B}_1} - \gamma_{\hat{B}_2}\|_{L^2} = \|\gamma_{\tau \cdot \hat{B}_1} - \gamma_{\hat{B}_2}\|_{L^2} < 3 \cdot 10^{-2}.$$

Proof of Claim. We may partition $2^{\mathbb{F}_2}$ into the sets $\tau \cdot \hat{B}_1 \Delta \hat{B}_2, \tau \cdot \hat{B}_1 \cap \hat{B}_2, 2^{\mathbb{F}_2} \setminus (\tau \cdot \hat{B}_1 \cup \hat{B}_2)$ and let ψ_1, ψ_2, ψ_3 be the functions with support equal to those sets respectively and such that

$$\gamma_{\tau \cdot \hat{B}_1} - \gamma_{\hat{B}_2} = \psi_1 + \psi_2 + \psi_3.$$

It suffices to find suitable bounds on $\|\psi_1\|_{L^2}, \|\psi_2\|_{L^2}, \|\psi_3\|_{L^3}$. For ease of expression we assume $\mu(\hat{B}_1) \leq \mu(\hat{B}_2)$; this is harmless; the argument under the reverse inequality is symmetrical. Recall that we have

$$\mu(\hat{B}_2) < \frac{10^4}{10^4 - 1}\mu(\hat{B}_1)$$

by the last claim.

We calculate upper bounds on $\|\psi_1\|_{L^2}, \|\psi_2\|_{L^2}, \|\psi_3\|_{L^2}$ in turn.

$$\begin{aligned}
\|\psi_1\|_{L^2} &= (\int_{\tau \cdot \hat{B}_1 \Delta \hat{B}_2} \psi_1^2)^{1/2} \leq [(4 \cdot 10^{-4}\mu(\hat{B}_1))(\frac{1}{\sqrt{(\mu(\hat{B}_1))}})^2)^{\frac{1}{2}} \\
&= \sqrt{(4 \cdot 10^{-4})} = 2 \cdot 10^{-2}. \\
\|\psi_2\|_{L^2} &= (\int_{\tau \cdot \hat{B}_1 \cap \hat{B}_2} \psi_2^2)^{\frac{1}{2}} \leq \sqrt{(\mu(\hat{B}_2))}[\frac{\sqrt{(1-\mu(\hat{B}_1))}}{\sqrt{(\mu(\hat{B}_1))}} - \frac{\sqrt{(1-\mu(\hat{B}_2))}}{\sqrt{(\mu(\hat{B}_2))}}] \\
&\leq \sqrt{(\mu(\hat{B}_2))}[\frac{\sqrt{(1-\mu(\hat{B}_1))}}{\sqrt{(\mu(\hat{B}_1))}} - \frac{\sqrt{(1-\mu(\hat{B}_2))}}{\sqrt{(\mu(\hat{B}_2))}}][\frac{\sqrt{(1-\mu(\hat{B}_1))}}{\sqrt{(\mu(\hat{B}_1))}} \\
&\qquad + \frac{\sqrt{(1-\mu(\hat{B}_2))}}{\sqrt{(\mu(\hat{B}_2))}}](\sqrt{(\hat{B}_2)}) \\
&= \mu(\hat{B}_2)[\frac{(1-\mu(\hat{B}_1))}{(\mu(\hat{B}_1))} - \frac{(1-\mu(\hat{B}_2))}{(\mu(\hat{B}_2))} \\
&= \mu(\hat{B}_2)(\frac{\mu(\hat{B}_2) - \mu(\hat{B}_1)\mu(\hat{B}_2) - \mu(\hat{B}_1) + \mu(\hat{B}_1)\mu(\hat{B}_2)}{\mu(\hat{B}_1)\mu(\hat{B}_2)}) \\
&= \frac{\mu(\hat{B}_2) - \mu(\hat{B}_1)}{\mu(\hat{B}_1)} < \frac{10^{-4}\mu(\hat{B}_1)}{\mu(\hat{B}_1)} = 10^{-4}. \\
\|\psi_3\|_{L^2} &= ((\int_{2^{\mathbb{F}_2} \setminus (\tau \cdot \hat{B}_1 \cup \hat{B}_2)} [\frac{\sqrt{(\mu(\hat{B}_1))}}{\sqrt{(1-\mu(\hat{B}_1))}} - \frac{\sqrt{(\mu(\hat{B}_2))}}{\sqrt{(1-\mu(\hat{B}_2))}}]^2)^{\frac{1}{2}} \\
&\leq [(\frac{\sqrt{(\mu(\hat{B}_1))}}{\sqrt{(1-\mu(\hat{B}_1))}})^2 + (\frac{\sqrt{(\mu(\hat{B}_2))}}{\sqrt{(1-\mu(\hat{B}_2))}})^2]^{\frac{1}{2}} \\
&< (2\mu(\hat{B}_1) + 2\mu(\hat{B}_2))^{\frac{1}{2}} < (2 \cdot 10^{-8} + 2 \cdot 10^{-8})^{\frac{1}{2}} = 2 \cdot 10^{-4}.
\end{aligned}$$

Claim (V): If $f \in N_3, \tau \in \mathbb{F}_2, d(\tau, e) \leq 3$, σ_0 a center for f, then $\sigma_0\tau^{-1}$ is a weak center for $\tau \cdot f$. □

Proof of Claim. We fix $\hat{B}_1, \hat{B}_2 \in A(\hat{\mathcal{B}_n})$ with $f \in \hat{B}_1$ and $\tau \cdot f \in \hat{B}_2$. By definition we then have σ_0 a center for $\pi(\gamma_{\hat{B}_1})$. By Lemma 3.5 (iv) we have that $\sigma_0 \cdot \tau^{-1}$ is a center for $\pi(\tau \cdot \gamma_{\hat{B}_1}) = \pi(\gamma_{\tau \cdot \hat{B}_1})$. By Claim (IV) we have $\|\gamma_{\tau \cdot \hat{B}_1} - \gamma_{\hat{B}_2}\|_{L^2} < \frac{1}{12}$, and hence by Lemma 3.5 (v) $\sigma_0\tau^{-1}$ is a weak center for $\pi(\gamma_{\hat{B}_2})$ and hence by definition a weak center for $\tau \cdot f$. □

Definition. For $x \in \{a, a^{-1}, b, b^{-1}\}$ we let A_x be the set of $f \in N_3$ such that either the identity e is a center of f or some reduced word of the form σx is a center of f. For τ a reduced word we let B_τ be the set of $f \in 2^{\mathbb{F}_2}$ such that some reduced $\sigma\tau$ is a weak center for f.

Each $f \in N_3$ is a member of at least one A_x by Lemma 3.1, and hence we may fix A_x with $\mu(A_x) > \frac{1}{5}$. From the last claim we have that if $f \in A_x \cap N_3$, and τ does *not* begin with x, and $d(\tau, e) \leq 3$, then

$$\tau \cdot f \in B_{x\tau^{-1}} \subset B_{\tau^{-1}}.$$

There are 27 possibilities for a reduced word τ of length 3 not starting with x, and for each such τ we have

$$\mu(B_{x\tau^{-1}}) \geq \mu(A_x) > \frac{1}{5}.$$

Since $\frac{27}{5} > 2$, there is some $f \in 2^{\mathbb{F}_2}$ in three different $B_{x\tau_1^{-1}}, B_{x\tau_2^{-1}}, B_{x\tau_3^{-1}}$, each $x\tau_i$ reduced, each $d(\tau_i, e) = 3$. This flatly contradicts Lemma 3.3.

So much for the proof of the theorem under the simplifying assumption that θ is one-to-one everywhere.

In the general case we must first pass to M_0 of positive measure on which θ is injective. Since the action of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$ is mixing we may find invertible maps

$$T, S : M_0 \to M_0$$

such that at each $x \in M_0$ there will be $n(x), m(x) > 0$ with

$$T(x) = a^{n(x)} \cdot x,$$
$$S(x) = b^{m(x)} \cdot x,$$

and thus we have $\langle T, S\rangle \cong \mathbb{F}_2$.

Now we trot through the argument above, but selectively replacing $\langle a, b\rangle$ with $\langle T, S\rangle$. We still calculate the centers with respect to the Cayley graph on $\mathbb{F}_2 = \langle a, b\rangle$, but we use only words from $\langle T, S\rangle$ to shift.

We amend the definitions from above by setting $\hat{B} = M_0 \cap \theta^{-1}[B]$ for $B \in \mathcal{B}$; as in Claim (I) we may find an n such that for all $\hat{B} \in A(\hat{\mathcal{B}}_n)$, all $x \in \{T, T^{-1}, S, S^{-1}\}$, and f_1, f_2 in a subset of M_0 with relatively large measure, $g_{f_1,x} = g_{f_2,x}$, where $f \mapsto g_{f,x}$ has been chosen measurably with the requirement that $\theta(x.f) = g_{f,x} \cdot \theta(x)$.

Our definitions of N_2, N_1 are parallel to the ones before, but now asking that $N_2 \subset N_1 \subset M_0$ and $\mu(N_2) > (1 - 10^{-8})\mu(M_0)$ and having x range over $\{T, T^{-1}, S, S^{-1}\}$. In Claim (II) we have x again range over $\{T, T^{-1}, S, S^{-1}\}$ and in Claim (III) we have τ range over words of length less than three built from $\{T, T^{-1}, S, S^{-1}\}$. The main idea here is that if τ is an irreducible word in $\{T, T^{-1}, S, S^{-1}\}$ and $f \in M_0$, then there will be a corresponding $\bar{\tau}_f$ built from $\{a, a^{-1}, b, b^{-1}\}$ with $\bar{\tau}_f \cdot f = \tau \cdot f$ and various other natural properties: $\sigma = \tau$ if and only if $\bar{\sigma}_f = \bar{\tau}_f$; σ and τ are incomparable (i.e. neither extends the other) if and only if $\bar{\sigma}_f$ is incomparable with $\bar{\tau}_f$.

With these and other natural adjustments the proof passes through as before.

Kechris pointed out a more elegant approach to the last step of the argument, which passes from one-to-one somewhere to the specific case of one-to-one everywhere. We begin by assuming we have a homomorphism[3] θ from $E_{\mathbb{F}_2}$ as in the

[3]A Borel function θ is said to be a *homomorphism* from an equivalence relation F to an equivalence relation F' if $x_1 F x_2$ always implies $\theta(x_1) F' \theta(x_2)$.

statement of the theorem; we are aiming for a contradiction. We then choose Borel $\rho : \theta[M] \times \mathbb{N} \twoheadrightarrow M$ such that $\{\rho(y, n) : n \in \mathbb{N}\}$ enumerates the preimage of any $y \in \theta[M]$. We let $G \times \mathbb{Z}$ act on $X \times 2^{\mathbb{N}}$ with the product of the original action of G on X and the odometer action of $\mathbb{Z}$ on $2^{\mathbb{N}}$ (the generator acts by adding one with carry, but in fact any modular action of $\mathbb{Z}$ or any other countably infinite group will do); this action will still be modular since the odometer action of $\mathbb{Z}$ is modular. We let $(z_n)_{n\in\mathbb{N}}$ enumerate some orbit in $2^{\mathbb{N}}$ under the action of $\mathbb{Z}$. We now define a new homomorphism

$$\hat{\theta} : M \to X \times 2^{\mathbb{N}}$$

given by

$$\hat{\theta}(x) = (\theta(x), z_n),$$

where n is least such that $\rho(\theta(x), n) = x$. This new homomorphism is one-to-one everywhere, and falls into the proof above.

There is one obvious generalization which the argument accommodates.

Definition. Let $(Z, \mathcal{C}, \nu)$ be a standard Borel probability space and let $Z^{\mathbb{F}_2}$ be the product space consisting of all

$$f : \mathbb{F}_2 \to Z$$

with the product measure and the action defined by

$$\sigma \cdot f(\tau) = f(\tau\sigma).$$

This space with this action and measure is said to be the *Bernoulli shift* of $\mathbb{F}_2$ on $Z^{\mathbb{F}_2}$.

Theorem 3.7. *Let E be of modular type on $(X, \mathcal{B})$, $(Z, \mathcal{C}, \nu)$ a standard Borel probability space, and $M \subset Z^{\mathbb{F}_2}$ of full measure. Then there is no countable to one measurable*

$$\theta : M \to X$$

such that $f_1 E_{\mathbb{F}_2} f_2 \Rightarrow \theta(f_1) E \theta(f_2)$.

Proof. We let $(C_\ell)_{\ell\in\mathbb{N}}$ be a generating algebra of ν-independent sets in Z, each having measure one-half. For $S \subset \mathbb{N} \times \mathbb{F}_2$ finite we define

$$\psi_S : Z^{\mathbb{F}_2} \to \{-1, 1\},$$
$$f \mapsto (-1)^{|\{(\ell,\sigma)\in S : f(\sigma)\in C_\ell\}|}.$$

With this and other painless modifications the proof goes through as before. □

4. Borel reducibility

Definition. For E and F equivalence relations on standard Borel $(X, \mathcal{B})$ and $(Y, \mathcal{C})$, we say that E *is Borel reducible to* F, written $E \leq_B F$, if there is a Borel function $\theta : X \to Y$ with

$$x_1 E x_2 \Leftrightarrow \theta(x_1) F \theta(x_2).$$

We write $E <_B F$ if $E \leq_B F$ holds, but $F \leq_B E$ fails.

We say that E is *treeable* if it is countable and there is an acyclic graph on X, which is Borel in the sense of having its collection of vertices Borel as a subset of $X \times X$ in the product Borel structure, whose connected components form the E equivalence classes. E is *hyperfinite* if it can be written as an increasing union of Borel equivalence relations which have all equivalence classes finite.

Any free Borel action of $\mathbb{F}_2$ is treeable; one simply copies the Cayley graph across the various components of $E_{\mathbb{F}_2}$. An equivalence relation with countable classes is hyperfinite if and only if it is Borel reducible to E_0. A group is amenable if and only if it has a free action by measure preserving transformations on a standard Borel probability space whose resulting orbit equivalence relation is hyperfinite.

For these and other related facts the reader should refer to [13].

Theorem 4.1 (Jackson, Kechris, Louveau; see [13]). *There is a universal treeable equivalence relation; that is to say, there is a treeable equivalence relation $E_{\mathcal{T}\infty}$ such that for all F with countable classes we have*

$$F \leq_B E_{\mathcal{T}\infty}$$

if and only if F is treeable.

Not all treeable equivalence relations are universal in their sense. E_0 is treeable, but not $\leq_B$-universal. But in answer to a question raised in [13]:

Theorem 4.2. *There is an equivalence relation E on standard Borel $(X, \mathcal{B})$ with*

$$E_0 <_B E <_B E_{\mathcal{T}\infty}.$$

Proof. Let E be a modular equivalence relation, arising from a free ergodic action of $\mathbb{F}_2$ as in [19], as discussed following the original definition of modular in the previous section. Since it arises from a free Borel action of $\mathbb{F}_2$, it is treeable; since there is an invariant measure and $\mathbb{F}_2$ is non-amenable, it is not Borel reducible to E_0. To see that it is not $\leq_B$-universal among treeable equivalence relations we apply Theorem 3.6. □

John Clemens pointed out a further application of Theorem 3.6. He begins by considering the equivalence relation $E_{\mathbb{F}_2}^{X_{ss}} \times E_0$, where, as before, $\mathbb{F}_2$ acts in a measure preserving fashion on X_{ss} with modular type. This resulting product equivalence relation is still modular, and hence in particular $E_{\mathcal{T}\infty}$ is not Borel reducible to $E_{\mathbb{F}_2}^{X_{ss}} \times E_0$; on the other hand by [1] the product equivalence relation is not treeable and thus we also have $E_{\mathbb{F}_2}^{X_{ss}} \times E_0$ not Borel reducible to $E_{\mathcal{T}\infty}$.

In this way we have an example of a countable Borel equivalence relation which is not $\leq_B$-comparable with $E_{\mathcal{T}\infty}$. As Clemens puts it, $E_{\mathcal{T}\infty}$ is not a *node* among the countable Borel equivalence relations.

There is one manner in which the definition of modular could be relaxed. Instead of asking that each $\mathcal{B}_n$ be finite we ask only that it be countable and generated as an algebra by its atoms.

Definition. An equivalence relation E on a standard Borel space $(X, \mathcal{B})$ is *loosely modular* if there is a countable group G acting by Borel automorphisms on X with $E = E_G$ and there is an increasing sequence of countable algebras

$$\mathcal{B}_0 \subset \mathcal{B}_1 \subset \ldots \subset \mathcal{B}_n \subset \ldots \subset \mathcal{B}$$

such that

(a) each $B \in \mathcal{B}_n$ is a finite Booelan combination of the atoms in $\mathcal{B}_n$;

(b) each $\mathcal{B}_n$ is permuted by the action of G.

However, in the context of equivalence relations and actions on finite measure spaces, this generalization, although painless, buys relatively little new. We leave the following proposition as an exercise for the reader.

Proposition 4.3. *Let E on $(X, \mathcal{B})$ be loosely modular, F an ergodic equivalence relation on a standard Borel probability space $(Y, \mathcal{C}, \mu)$, and*

$$\theta : Y \to X$$

a countable to one measurable map with

$$y_1 F y_2 \Rightarrow \theta(y_1) E \theta(y_2).$$

Then there is a modular $\hat{E}$ on $(\hat{X}, \hat{\mathcal{B}})$ and a countable to one measurable $\hat{\theta} : Y \to \hat{X}$ with $y_1 F y_2 \Rightarrow \hat{\theta}(x_1) \hat{E} \hat{\theta}(x_2)$.

Thus, if F refuses measurable reduction to a modular equivalence relation, then it likewise refuses measurable reduction to any loosely modular equivalence relation.

An example of a loosely modular equivalence relation is given by any countable group of permutations of $\mathbb{N}$ acting by right composition on the space of injections from $\mathbb{N}$ to some countable set. It follows then in particular that there are countable Borel equivalence relations which are not reducible to, say, the group of recursive permutations acting on the space of injections $\mathbb{N} \hookrightarrow \mathbb{N}$.

Since the first draft of this paper, Kechris in [12] wrote his own treatment of this material which draws out the representation theoretic connections and in which he formalizes the notion of *anti-modular*.

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