ASYMPTOTIC BEHAVIOUR OF ARITHMETICALLY COHEN-MACAULAY BLOW-UPS

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ABSTRACT. This paper addresses problems on arithmetic Macaulayfications of projective schemes. We give a surprising complete answer to a question posed by Cutkosky and Herzog. Let $Y$ be the blow-up of a projective scheme $X = \text{Proj} R$ along the ideal sheaf of $I \subset R$. It is known that there are embeddings $Y \cong \text{Proj} k[[I^e]]$ for $c \geq d(I) e + 1$, where $d(I)$ denotes the maximal generating degree of $I$, and that there exists a Cohen-Macaulay ring of the form $k[[I^e]]$ which gives an arithmetic Macaulayfication of $X$ if and only if $H^0(Y, \mathcal{O}_Y) = k$, $H^i(Y, \mathcal{O}_Y) = 0$ for $i = 1, \ldots, \dim Y - 1$, and $Y$ is equidimensional and Cohen-Macaulay. We show that under these conditions, there are well-determined invariants $\varepsilon$ and $e_0$ such that $k[[I^e]]$ is Cohen-Macaulay for all $c > d(I) e + \varepsilon$ and $e > e_0$, and that these bounds are the best possible. We also investigate the existence of a Cohen-Macaulay Rees algebra of the form $R[[I^e]]$. If $R$ has negative $a^*$-invariant, we prove that such a Cohen-Macaulay Rees algebra exists if and only if $\pi_* \mathcal{O}_Y = \mathcal{O}_X$, $R^i \pi_* \mathcal{O}_Y = 0$ for $i > 0$, and $Y$ is equidimensional and Cohen-Macaulay. Moreover, these conditions imply the Cohen-Macaulayness of $R[[I^e]]$ for all $c > d(I) e + \varepsilon$ and $e > e_0$.

INTRODUCTION

Let $X$ be a projective scheme over a field $k$. An arithmetic Macaulayfication of $X$ is a proper birational morphism $\pi : Y \to X$ such that $Y$ has an arithmetically Cohen-Macaulay embedding, i.e. there exists a Cohen-Macaulay standard graded $k$-algebra $A$ such that $Y \cong \text{Proj} A$. Inspired by the problem of resolution of singularities, it was asked when $X$ has an arithmetic Macaulayfication. The local version of this problem (arithmetic Macaulayfication of local rings) has been extensively studied in the literature and recently solved by Kawasaki [22]. An important aspect of the global problem is to determine, given a proper birational morphism $Y \to X$, if $Y$ has an arithmetically Cohen-Macaulay embedding, and if it does, which embeddings of $Y$ are arithmetically Cohen-Macaulay.

Let $R$ be a standard graded $k$-algebra and let $I \subset R$ be a homogeneous ideal such that $X = \text{Proj} R$ and $Y$ is the blow-up of $X$ along the ideal sheaf of $I$. It was observed by Cutkosky and Herzog [9] that $Y \cong \text{Proj} k[[I^e]]$ for $c \geq d(I) e + 1$, for...
where \((I^c)_c\) denotes the vector space of forms of degree \(c\) of the ideal power \(I^c\) and \(d(I)\) is the maximal degree of the elements of a homogeneous basis of \(I\). In other words, \(Y\) can be embedded into a projective space by the complete linear system \(|cE_0 - eE|\), where \(E\) denotes the exceptional divisor and \(E_0\) is the pull-back of a general hyperplane in \(X\). By [26] we know that there exists a Cohen-Macaulay ring \(k[(I^c)_c]\) for \(c \geq d(I)e + 1\) if and only if \(Y\) satisfies the following conditions:

- \(Y\) is equidimensional and Cohen-Macaulay,
- \(H^0(Y, O_Y) = k\) and \(H^i(Y, O_Y) = 0\) for \(i = 1, \ldots, \dim Y - 1\).

In the first part of this paper, we study the problem of which values of \(c\) and \(e\) is \(k[(I^c)_c]\) a Cohen-Macaulay ring. This problem originated from a beautiful result of Geramita, Gimigliano and Pitteloud [13] which shows that if \(I\) is the defining ideal of a set of fat points in a projective space over a field of characteristic zero, then \(k[I]\) is a Cohen-Macaulay ring for all \(c \geq \reg(I)\), where \(\reg(I)\) is the Castelnuovo-Mumford regularity of \(I\). This result initiated the study on the Cohen-Macaulayness of algebras of the form \(k[(I^c)_c]\) first in [21] and then in [9, 23, 20, 10]. In particular, Cutkosky and Herzog [9] showed that if \(I\) is a locally complete intersection ideal, then there exists a constant \(\delta\) such that \(k[(I^c)_c]\) is Cohen-Macaulay for \(c \geq \delta e\). They raised the question of when there is a linear bound on \(c\) and \(e\) ensuring that \(k[(I^c)_c]\) is a Cohen-Macaulay ring.

Our results will give a complete answer to this question. We show that if the above two conditions are satisfied, then there exist well-determined invariants \(\varepsilon\) and \(e_0\) such that \(k[(I^c)_c]\) is a Cohen-Macaulay ring for all \(c > d(I)e + \varepsilon\) and \(e > e_0\) (Theorem 2.2). The invariant \(e_0\) is a projective version of the \(a^*-\)invariant, which is the largest non-vanishing degree of the graded local cohomology modules [23, 32]. The invariant \(\varepsilon\) comes from the asymptotic linearity of the Castelnuovo-Mumford regularity of powers of ideals [31, 10, 23, 34]. We will see that the bounds \(c > d(I)e + \varepsilon\) and \(e > e_0\) are the best possible (Theorem 2.3 and Example 2.5). The existence of linear bounds on \(c\) and \(e\) is not hard to prove. The novelty we claim here is that an explicit description for the best possible bounds is obtained. Moreover, if the Rees algebra \(R[It]\) is locally Cohen-Macaulay on \(X\), then \(e_0 = 0\) and we can replace the second condition by the weaker condition that \(H^0(X, O_X) = k\) and \(H^i(X, O_X) = 0\) for \(i = 1, \ldots, \dim X - 1\) (Theorem 2.4). These results strengthen and unify all previously-known results on the Cohen-Macaulayness of \(k[(I^c)_c]\) which were obtained by different methods.

In the second part of this paper, we investigate the more difficult question of when \(Y\) is an arithmetically Cohen-Macaulay blow-up of \(X\); that is, when there exists a standard graded \(k\)-algebra \(R\) and an ideal \(J \subset R\), such that \(X = \Proj R\), \(Y\) is the blow-up of \(X\) along the ideal sheaf of \(J\), and \(R[It]\) is a Cohen-Macaulay ring. Given \(R\) and \(I\), we will concentrate on ideals \(J \subset I\) which are generated by the elements of \((I^c)_c\). It is obvious that \((I^c)\) and \(J\) define the same ideal sheaf for \(c \geq d(I)e\). Rees algebras of the form \(R[It]\) \((e = 1)\) have been studied first for the defining ideal of a set of points in [15] and then for locally complete intersection ideals in [8], where it was shown that there exists a constant \(\lambda\) such that \(R[It]\) is a Cohen-Macaulay ring for \(c \geq \lambda\). This leads to the problem of whether there is a constant \(\delta\) such that the Rees algebra \(R[(I^c)_c]\) is a Cohen-Macaulay ring for \(c \geq \delta e\).

If \(a^*(R) < 0\) (e.g. if \(R\) is a polynomial ring) we solve this problem by showing that there exists a Cohen-Macaulay ring \(R[[(I^c)_c]]\) with \(c \geq d(I)e\) if and only if the
following conditions are satisfied:

- $Y$ is equidimensional and Cohen-Macaulay,
- $\pi_*\mathcal{O}_Y = \mathcal{O}_X$, $R^i\pi_*\mathcal{O}_Y = 0$ for $i > 0$.

In particular, these conditions imply that $\mathcal{R}[I^e_e]$ is a Cohen-Macaulay ring for all $c > d(I)e + e$ and $e > e_0$ (Theorem 3.4). From this it follows that there exists a Cohen-Macaulay algebra of the form $\mathcal{R}[I, t]$ with $c \geq d(I)$ if and only if $\mathcal{R}[I]$ is locally Cohen-Macaulay on $X$ and that $e_0 = 0$ in this case (Corollary 3.8). We would like to point out that this phenomenon does not hold in general. In fact, there exist examples with $a^*(R) \geq 0$ such that $\mathcal{R}[I^e_e]$ is a Cohen-Macaulay ring, whereas $\mathcal{R}[I^e_e]$ is not a Cohen-Macaulay ring for any $c > d(I)e$ (Example 3.5). Using the above result we obtain several new classes of Cohen-Macaulay Rees algebras. Furthermore, we show that if $H^n(X, \mathcal{O}_X) = k$ and $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$, then $Y$ is an arithmetically Cohen-Macaulay blow-up of $X$ if and only if $Y$ is locally arithmetical Cohen-Macaulay on $X$ (Theorem 3.12).

Our approach is based on the facts that the Rees algebra $\mathcal{S} = \mathcal{R}[I]$ has a natural bi-gradation and that $k[(I^e)_c]$ can be viewed as a diagonal subalgebra of $S$. As a consequence, the Cohen-Macaulayness of $k[(I^e)_c]$ can be characterized by means of the sheaf cohomology $H(Y, \mathcal{O}_Y(m, n))$. Using Leray spectral sequence and Serre-Grothendieck correspondence, we may pass from this sheaf cohomology to the local cohomology of $I^n$ and of $\omega_n$, where $\omega_n = \bigoplus_{n \in \mathbb{Z}} \omega_n$ denotes the graded canonical module of $S$. It was shown recently that there are linear bounds for the vanishing of the local cohomology of $I^n$ and $\omega_n$ ([31][10][23][51]). It turns out that these linear bounds yield a linear bound on $c$ and $e$ such that $k[(I^e)_c]$ is a Cohen-Macaulay ring. The Cohen-Macaulayness of the Rees algebra $\mathcal{R}[I^e_e]$ can be studied similarly by using a recent result of Hyry [19] which characterizes the Cohen-Macaulayness of a standard bi-graded algebra by means of sheaf cohomology.

The paper is organized as follows. In Section 1, we introduce the notion of a projective $a^*$-invariant which governs how sheaf cohomology behaves through blowing up morphisms. The material in this section is interesting on its own right. In Section 2, we study the Cohen-Macaulayness of rings of the form $k[(I^e)_c]$ which correspond to projective embeddings of $Y$. The last section of the paper deals with the problem of when $Y$ is an arithmetically Cohen-Macaulay blow-up of $X$.

For unexplained notation and facts we refer the reader to the books [3][5][17].

1. $a^*$-invariants

Let $R$ be an arbitrary commutative noetherian ring. Let $S = \bigoplus_{n \geq 0} S_n$ be a finitely generated graded algebra over $R$. We shall always use $S_+ = \bigoplus_{n > 0} S_n$ to denote the ideal generated by the homogeneous elements of positive degrees of $S$. Given any finitely generated graded $S$-module $F$, the local cohomology module $H^i_{S_+}(F)$ is also a graded $S$-module. It is well known that $H^i_{S_+}(F)_n = 0$ for $n \gg 0$, $i \geq 0$. Put

$$a_i(F) = \begin{cases} -\infty & \text{if } H^i_{S_+}(F)_n = 0, \\ \max\{n | H^i_{S_+}(F)_n \neq 0\} & \text{if } H^i_{S_+}(F)_n \neq 0. \end{cases}$$

Note that $a(F) := a_{\dim F}(F)$ is called the $a$-invariant of $F$ if $S$ is a standard graded algebra over a field. The $a^*$-invariant of $F$ is defined to be

$$a^*(F) := \max\{a_i(F) | i \geq 0\}.$$
This invariant was introduced in \cite{32} and \cite{29} in order to control the vanishing of graded local cohomology modules with different supports. It is closely related to the Castelnuovo-Mumford regularity via the equality
\[ \text{reg}(F) = \max \{ a_i(F) + i \mid i \geq 0 \}. \]

Here we are interested in the case when \( R \) is a standard graded algebra over a field \( k \) and \( S = R[It] \) is the Rees algebra of a homogeneous ideal \( I \subset R \) with \( \text{ht} I \geq 1 \). This Rees algebra has a natural grading with \( S_n = I^n t^n \). Let \( \omega_S = \bigoplus_{n \in \mathbb{Z}} \omega_n \) denote the canonical graded module of \( S \).

**Lemma 1.1.** Let \( S = R[It] \) be as above. If \( S \) is a Cohen-Macaulay ring, then \( a^*(S) = -1 \) and \( a^*(\omega_S) = 0 \).

**Proof.** It is well known that \( \dim S = \dim R + 1 \). Since \( S/I_+ = R \), we have \( \text{ht} I_+ = \dim S - \dim R = 1 \). This implies \( \text{grade} S_+ = 1 \). Hence \( a^*(S) \geq -1 \) by \cite{32} Corollary 2.3]. On the other hand, the Cohen-Macaulayness of \( S \) implies \( H^i_M(S) = 0 \) for \( i < \dim S \), where \( M \) denotes the maximal graded ideal of \( S \). By \cite{33} Corollary 3.2 we always have \( H^i_{I_+}(S)_n = 0 \) for \( n \geq 0 \). Hence \( H^i_M(S)_n = 0 \) for all \( n \geq 0 \) and \( i \geq 0 \). By \cite{19} Lemma 2.3] (or \cite{32} Corollary 2.8]), this implies \( H^i_{\omega}(S)_n = 0 \) for all \( n \geq 0 \) and \( i \geq 0 \). Therefore, \( a^*(S) = -1 \).

Since \( \omega_S \) is a Cohen-Macaulay module with \( \text{Hom}_S(\omega_S, \omega_S) \cong S \) \cite{2} Proposition 2], we also have \( H^i_M(\omega_S) = 0 \) for \( i < \dim S \) and, by local duality,
\[
H^i_{\dim S}(\omega_S)_n \cong \text{Hom}_S(\omega_S, \omega_S)_{-n} \cong S_{-n}.
\]

Since \( S_0 = R \neq 0 \) and \( S_{-n} = 0 \) for \( n > 0 \), we can conclude that \( a^*_X(\omega_S) = 0 \). \( \square \)

Let \( X = \text{Proj} R \). For each \( p \in X \), the homogeneous localization \( F_p \) is a finitely generated graded module over \( S_p \). Hence, we can define the *projective a*-invariant
\[
a^*_X(F) := \max \{ a^*(F_p) \mid p \in X \}.
\]

Note that \( H^i_{S_p}(F_p) = H^i_{\omega(S)}(F_p) \) (cf. \cite{29} Remark 2.2]). Then we always have \( a^*_X(F) \leq a^*(F) \). Hence \( a^*_X(F) \) is a finite number. Since \( a^*_X(F) \) is determined by the local structure of \( F \) on \( X \), it can easily be estimated in certain situations. As a demonstration, we show how to estimate \( a^*_X(F) \) in the following case which will play an important role in our further investigation.

We say that \( S \) is *locally Cohen-Macaulay on \( X \) if \( S_{(p)} \) is a Cohen-Macaulay ring for every \( p \in X \). This condition holds if, for instance, \( X \) is locally Cohen-Macaulay and \( I \) is locally a complete intersection.

**Proposition 1.2.** Let \( X = \text{Proj} R \) and \( S = R[It] \) be as above. Then \( a^*_X(S) \geq -1 \) and \( a^*_X(\omega_S) \geq 0 \). Equalities hold if \( S \) is locally Cohen-Macaulay on \( X \).

**Proof.** Let \( p \) be a minimal prime ideal in \( X \). Then \( R_{(p)} \) is an artinian ring. Since \( \mathfrak{p} \not
subset I \), we have \( I_{(p)} = I_{(p)}[t] \). Hence \( S_{(p)} = R_{(p)}[t] \) is a Cohen-Macaulay ring. By Lemma 1.1, this implies \( a^*(S_{(p)}) = -1 \) and \( a^*(\omega_{S_{(p)}}) = 0 \). Hence \( a^*_X(S) \geq -1 \) and \( a^*_X(\omega_S) \geq 0 \). This proves the first statement. The second statement is an immediate consequence of Lemma 1.1. \( \square \)

Beside the natural \( \mathbb{N} \)-graded structure given by the degrees of \( t \), the Rees algebra \( S = R[It] \) also has a natural bi-gradation with
\[
S_{(m,n)} = (I^n)m_t^n
\]

for \((m, n) \in \mathbb{N}^2\). Let \(Y\) be the blow-up of \(X\) along the ideal sheaf of \(I\). Then \(Y = \text{Proj} \ S\) with respect to this bi-gradation. If \(F = \bigoplus_{(m,n) \in \mathbb{Z}^2} F_{(m,n)}\) is a finitely generated bi-graded \(S\)-module, then \(F\) is also an \(\mathbb{Z}\)-graded \(S\)-module with \(F_n = \bigoplus_{m \in \mathbb{Z}} F_{(m,n)}\). Let \(\widetilde{F}\) denote the sheaf associated to \(F\) on \(Y\). We write \(\widetilde{F}(n)\) and \(\widetilde{F}(m, n)\) to denote the twisted \(\mathcal{O}_Y\)-modules with respect to the \(\mathbb{N}\)-gradation and the \(\mathbb{N}^2\)-gradation of \(S\). Moreover, we denote by \(\widetilde{F}_n\) the sheafification of \(F_n\) on \(X\).

It turns out that \(a_X^*(F)\) is a measure for when we can pass from the sheaf cohomology of \(\widetilde{F}(m, n)\) on \(Y\) to that of \(\widetilde{F}_n(m)\) on \(X\).

**Proposition 1.3.** Let \(F\) be a finitely generated bi-graded \(S\)-module. For \(n > a_X^*(F)\) we have

\[
\begin{aligned}
(i) \ & \pi_*(\widetilde{F}(n)) = \widetilde{F}_n \quad \text{and} \quad R^i \pi_*(\widetilde{F}(n)) = 0 \quad \text{for} \quad i > 0, \\
(ii) \ & H^i(Y, \widetilde{F}(m, n)) \cong H^i(X, \widetilde{F}_n(m)) \quad \text{for all} \quad m \in \mathbb{Z} \quad \text{and} \quad i \geq 0.
\end{aligned}
\]

**Proof.** Since (i) is a local statement, we only need to show that it holds locally. Let \(p\) be a closed point of \(X\), and consider the restriction \(\pi_p\) of \(\pi\) over an affine open neighborhood \(\text{Spec} \mathcal{O}_{X,p}\) of \(p\)

\[
\pi_p : Y_p = Y \times_X \text{Spec} \mathcal{O}_{X,p} \to \text{Spec} \mathcal{O}_{X,p}.
\]

We have \(\widetilde{F}|_{Y_p} = \widetilde{F}(p)\), where \(\widetilde{F}(p)\) is the sheaf associated to \(F(p)\) on \(Y_p\). Thus,

\[
R^i \pi_*(\widetilde{F}(n))|_{\text{Spec} \mathcal{O}_{X,p}} = R^i \pi_p_*(\widetilde{F}(p)(n)) = H^i(Y_p, \widetilde{F}(p)(n))^-.
\]

On the other hand, we know by the Serre-Grothendieck correspondence that there are the exact sequence

\[
0 \to H^0_{S(p)^+}(F(p)_n) \to (F(p)_n) \to H^0(Y_p, \widetilde{F}(p)(n)) \to H^1_{S(p)^+}(F(p)_n) \to 0
\]

and the isomorphisms \(H^i(Y_p, \widetilde{F}(p)(n)) \cong H^{i+1}_{S(p)^+}(F(p)_n)\) for \(i > 0\). By the definition of \(a_X^*(F)\), we know that \(H^i_{S(p)^+}(F(p)_n)\) for \(n > a_X^*(F)\), \(i > 0\). Thus,

\[
R^i \pi_*(\widetilde{F}(n))|_{\text{Spec} \mathcal{O}_{X,p}} = H^i(Y_p, \widetilde{F}(p)(n))^- = \begin{cases} (\widetilde{F}(n)(p)) & \text{if} \quad i = 0, \\
0 & \text{if} \quad i > 0,
\end{cases}
\]

for \(n > a_X^*(F)\).

To show (ii) we first observe that \(\widetilde{F}(m, n) = \widetilde{F}(n) \otimes \pi^* \mathcal{O}_X(m)\). By the projection formula, we have

\[
R^i \pi_*(\widetilde{F}(m, n)) = R^i \pi_* (\widetilde{F}(n) \otimes \mathcal{O}_X(m)) = \begin{cases} \widetilde{F}_n(m) & \text{if} \quad i = 0, \\
0 & \text{if} \quad i > 0,
\end{cases}
\]

Hence the conclusion follows from the Leray spectral sequence

\[
H^i(X, R^j \pi_*(\widetilde{F}(m, n))) \Rightarrow H^{i+j}(Y, \widetilde{F}(m, n)).
\]

\[\square\]

Let \(Y\) be the blow-up of a projective scheme \(X\) along an ideal sheaf \(\mathcal{I}\). We say that \(Y\) is **locally arithmetic Cohen-Macaulay on \(X\)** if there exist \(R\) and \(I\) such that \(X = \text{Proj} \ R, \mathcal{I} = I\) and \(S = R[It]\) is locally Cohen-Macaulay on \(X\).

**Corollary 1.4.** Assume that \(Y\) is locally arithmetic Cohen-Macaulay on \(X\). Then

(i) \ \pi_* \mathcal{O}_Y = \mathcal{O}_X \quad \text{and} \quad R^i \pi_* \mathcal{O}_Y = 0 \quad \text{for} \quad i > 0,

(ii) \ \begin{align*}
H^i(Y, \mathcal{O}_Y(m, 0)) & \cong H^i(X, \mathcal{O}_X(m)) \quad \text{for all} \quad m \in \mathbb{Z}, \ i \geq 0.
\end{align*}
Proof. With the above notations we have \( a^*_\chi(S) = -1 \) by Proposition [1.2]. Hence the conclusion follows from Proposition [1.3] by taking \( F = S \) and \( n = 0 \). \( \square \)

For each \( n \), the graded \( R \)-module \( F_n \) has an \( a^* \)-invariant \( a^*(F_n) \), which controls the vanishing of \( H^i(X, \widetilde{F}_n(m)) \) by the Grothendieck-Serre correspondence. On the other hand, since \( F \) is a finitely generated graded module over \( S = R[I] \), there exists a number \( n_0 \) such that \( F_n = I^{n-n_0}F_{n_0} \) for \( n \geq n_0 \). It was recently discovered that for any finitely generated graded \( R \)-module \( E \), the Castelnuovo-Mumford regularity \( \text{reg}(I^nE) \) is bounded by a linear function on \( n \) with slope \( d(I) \) [34, Theorem 2.2] (see also [10, 23] for the case \( R \) is a polynomial ring). By definition, we always have

\[
a^*(I^nE) \leq \max\{a_i(I^nE) + i | i \geq 0\} = \text{reg}(I^nE).
\]

Therefore, \( a^*(F_n) \) is bounded above by a linear function of the form \( d(I)n + \varepsilon \) for \( n \geq 1 \).

We will denote by \( \varepsilon(I) \) the smallest non-negative number such that

\[
a^*(I^n) \leq d(I)n + \varepsilon(I)
\]

for all \( n \geq 1 \). Since \( \omega_S = \bigoplus_{n \in \mathbb{Z}} \omega_n \) is a finitely generated bi-graded \( S \)-module, there is a similar bound for \( a^*(\omega_n) \). Note that the \( R \)-graded module \( \omega_n \) is also called an adjoint-type module of \( I \) because of its relationship to the adjoint ideals [20]. We will denote by \( \varepsilon^*(I) \) the smallest non-negative number such that

\[
a_i(\omega_n) \leq d(I)n + \varepsilon^*(I)
\]

for \( i \geq 2 \) and \( n \geq 1 \).

The meaning of these invariants will become more apparent in the next sections. Here we content ourselves with the following observations.

Lemma 1.5. With the above notations we have

(i) \( H^0(X, \widetilde{S}_n(m)) = S_{(m,n)} \) and \( H^i(X, \widetilde{S}_n(m)) = 0 \) for \( i > 0 \) and \( m > d(I)n + \varepsilon(I) \),

(ii) \( H^i(X, \widetilde{\omega}_n(m)) = 0 \) for \( i > 0 \) and \( m > d(I)n + \varepsilon^*(I) \).

Proof. Since \( S_n \cong I^n \), we have \( H^i_{R_+}(S_n)_m = 0 \) for \( i \geq 0, m > d(I)n + \varepsilon(I) \) and \( n \geq 1 \). Hence the first statement follows from the Serre-Grothendieck correspondence, which gives the exact sequence

\[
0 \to H^0_{R_+}(S_n)_m \to S_{(m,n)} \to H^0(X, \widetilde{S}_n(m)) \to H^1_{R_+}(S_n)_m \to 0
\]

and the isomorphisms

\[
H^i(X, \widetilde{S}_n(m)) \cong H^{i+1}_{R_+}(S_n)_m
\]

for \( i > 0 \). The second statement can be similarly proved. \( \square \)

2. Arithmetically Cohen-Macaulay embeddings of blow-ups

Let \( X \) be a projective scheme over a field \( k \). Let \( Y \to X \) be the blowing up of \( X \) along an ideal sheaf \( \mathcal{I} \). We say that \( Y \) has an arithmetically Cohen-Macaulay embedding if there exists a Cohen-Macaulay standard graded \( k \)-algebra \( A \) such that \( Y \cong \text{Proj} \, A \).

Let \( R \) be a finitely generated standard graded \( k \)-algebra and let \( I \subset R \) be a homogeneous ideal such that \( X = \text{Proj} \, R \) and \( \mathcal{I} \) is the ideal sheaf associated to \( I \). Let \( S = R[I] \) be the Rees algebra of \( R \) with respect to \( I \). It is well known
that $Y \cong \text{Proj } k{(I^c)}_c$ for $c \geq d(I)e + 1$ and $e \geq 1$, where $k{(I^c)}_c$ is the algebra generated by all forms of degree $c$ of the ideal power $I^c$ and $d(I)$ denotes the largest degree of a minimal set of homogeneous generators of $I$ (cf. [9, Lemma 1.1]). There is the following simple criterion for the existence of a Cohen-Macaulay algebra $k{(I^c)}_c$ (which is at the same time a criterion for the existence of an arithmetically Cohen-Macaulay embedding).

**Lemma 2.1** ([26, Corollary 3.5]). There exists a Cohen-Macaulay ring $k{(I^c)}_c$ for $c \geq d(I)e + 1$ if and only if the following conditions are satisfied:

(i) $Y$ is equidimensional and Cohen-Macaulay,

(ii) $H^0(Y, \mathcal{O}_Y) = k$ and $H^i(Y, \mathcal{O}_Y) = 0$ for $i = 1, \ldots, \dim Y - 1$.

The proof of [26] used a deep result on the relationship between the local cohomology modules of a bi-graded algebra and its diagonal subalgebras [7]. However, the above lemma simply follows from the basic fact that (i) and (ii) are equivalent to the existence of an arithmetically Cohen-Macaulay Veronese embedding of $Y$ (cf. [8, Lemma 1.1]). In fact, the Veronese subalgebras of $k[L]$ are exactly the algebras of the form $k{(I^c)}_c$ for $c \geq d(I) + 1$, $e \geq 1$. We notice that the statements of [26 Corollary 3.5] and [8, Lemma 1.1] missed the equidimensional condition.

In this section we will determine for which values of $c$ and $e$ is $k{(I^c)}_c$ a Cohen-Macaulay ring. First, we show that there are well-determined invariants $\varepsilon$ and $e_0$ such that $k{(I^c)}_c$ is a Cohen-Macaulay ring for all $c > d(I)e + \varepsilon$ and $e > e_0$.

**Theorem 2.2.** Let $R$ be a standard graded $k$-algebra and let $I \subset R$ be a homogeneous ideal with $\text{ht } I \geq 1$. Let $Y$ be the blow-up of $X = \text{Proj } R$ along the ideal sheaf of $I$ and $S = R[It]$. Assume that

(i) $Y$ is equidimensional and Cohen-Macaulay,

(ii) $H^0(Y, \mathcal{O}_Y) = k$ and $H^i(Y, \mathcal{O}_Y) = 0$ for $i = 1, \ldots, \dim Y - 1$.

Then $k{(I^c)}_c$ is a Cohen-Macaulay ring for $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$ and $e > \max\{a_X^\varepsilon(S), a_X^{\varepsilon^*}(\omega_S)\}$.

Note first that we always have $\max\{a_X^\varepsilon(S), a_X^{\varepsilon^*}(\omega_S)\} \geq 0$ by Proposition [1,2] and $\max\{\varepsilon(I), \varepsilon^*(I)\} \geq 0$ by the definition of $\varepsilon(I)$ and $\varepsilon^*(I)$.

**Proof.** Let $A = k{(I^c)}_c$. Since $c \geq de + 1$, we have $Y \cong \text{Proj } A$ [9, Lemma 1.1]. On the other hand, the Rees algebra $S = R[It]$ has a natural bi-graduation with $S_{(m,n)} = (I^m)t^n$ and $Y = \text{Proj } S$. Moreover, we may view $A$ as a diagonal subalgebra of $S$; that is, $A = \bigoplus_{n \in \mathbb{N}} S_{(en, cn)}$ [7, Lemma 1.2]. From this it follows that $A(n) = O_Y(cn, en)$. Therefore, the Serre-Grothendieck correspondence yields the exact sequence

$$0 \longrightarrow H^0_{A^+}(A) \longrightarrow A \longrightarrow \bigoplus_{n \in \mathbb{Z}} H^0(Y, O_Y(cn, en)) \longrightarrow H^1_{A^+}(A) \longrightarrow 0$$

and the isomorphisms

$$\bigoplus_{n \in \mathbb{Z}} H^i(Y, O_Y(cn, en)) \cong H^{i+1}_{A^+}(A)$$
for $i \geq 1$. It is well known that $A$ is a Cohen-Macaulay ring if and only if $H^i_{A_+}(A) = 0$ for $i \neq \dim A$. Therefore, $A$ is a Cohen-Macaulay ring if we can show

$$H^0(\mathcal{O}_Y(cn, en)) = A_n = \begin{cases} 0 & \text{for } n < 0, \\ k & \text{for } n = 0, \\ (I^n)_c & \text{for } n > 0, \end{cases}$$

$$H^i(\mathcal{O}_Y(cn, en)) = 0 \ (i = 1, ..., \dim Y - 1).$$

For $n = 0$, this follows from the assumption $H^0(\mathcal{O}_Y) = k$ and $H^i(\mathcal{O}_Y) = 0$ for $i = 1, ..., \dim Y - 1$.

For $n > 0$ we have $cn > d(I)en + \varepsilon(I)n \geq d(I)en + \varepsilon(I)$ and $en > a_X^*(S)n \geq a_X^*(S)$. Hence, using Proposition 1.3 and Lemma 1.5 we get

$$H^0(\mathcal{O}_Y(cn, en)) = H^0(X, I^{en}(cn)) = (I^n)_c,$$

$$H^i(\mathcal{O}_Y(cn, en)) = H^i(X, I^{en}(cn)) = 0, \ i = 1, ..., \dim Y - 1.$$

For $n < 0$ we have

$$H^i(\mathcal{O}_Y(cn, en)) = H^{\dim Y-i}(\mathcal{O}_Y(-cn, -en))$$

for $i \geq 0$. Serre duality can be applied here because $Y$ is equidimensional and Cohen-Macaulay. Since $-cn > -d(I)en - \varepsilon(I)n \geq -d(I)en + \varepsilon(I)$ and $-en > -a_X^*(\omega_S)n \geq a_X^*(\omega_S)$, using Proposition 1.3 and Lemma 1.5 we get

$$H^{\dim Y-i}(\mathcal{O}_Y(-cn, -en)) = H^{\dim Y-i}(X, (\omega_S)_{-cn}(-cn)) = 0$$

for $i < \dim Y$. So we get $H^i(\mathcal{O}_Y(cn, en)) = 0$ for all $n < 0$ and $i = 0, ..., \dim Y - 1$. The proof of Theorem 2.2 is now complete. \qed

The following theorem shows that the bound $e > \max\{a_X^*(S), a_X^*(\omega_S)\}$ of Theorem 2.2 is the best possible.

**Theorem 2.3.** Let the notations and assumptions be as in Theorem 2.2. Let

$$e_0 = \max\{a_X^*(S), a_X^*(\omega_S)\}.$$  

Then $k[(I^n)_c]$ is not a Cohen-Macaulay ring for $c \gg 0$ if $e_0 \geq 1$.

**Proof.** Let $A = k[(I^n)_c]$ for $c \gg 0$. As we have seen in the proof of Theorem 2.2, $A$ is not Cohen-Macaulay if $H^0(\mathcal{O}_Y(c, e_0)) \neq (I^n)_c$ or $H^i(\mathcal{O}_Y(c, e_0)) \neq 0$ for some $i = 1, ..., \dim Y - 1$.

We shall first consider the case $e_0 = a_X^*(S)$. Let $q$ be the smallest integer such that $e_0 = \max\{a_q(\mathcal{O}_p)\mid p \in X\}$. Then

$$H^i_{S(\mathcal{O}_p)}(S(\mathcal{O}_p))_{e_0} = 0, \ i < q, \ \text{for all } p \in X,$$

$$H^q_{S(\mathcal{O}_p)}(S(\mathcal{O}_p))_{e_0} \neq 0 \ \text{for some } p \in X.$$

It is a classical result that there exists $\dim R_{(p)}$ elements in $I_{(p)}$ which generates an ideal with the same radical as $I_{(p)}$. The same also holds for the ideal $S_{(p)} = I_{(p)}t$. From this it follows that $H^i_{S(\mathcal{O}_p)}(E) = 0$ for any $R_{(p)}$-module $E$ (cf. [4, Corollary 3.3.3]). Hence

$$q \leq \max\{\dim R_{(p)}\mid p \in X\} = \dim Y.$$


Let $Y_p = Y \times_X \text{Spec} \mathcal{O}_{X,p}$. The Serre-Grothendieck correspondence yields the exact sequence

$$0 \to H^0_{S(p)}(S_p)_{c_0} \to (S_p)_{c_0} \to H^0(Y_p, \tilde{S}_p(e_0)) \to H^1_{S(p)}(S_p)_{c_0} \to 0,$$

and isomorphisms $H^i(Y_p, \tilde{S}_p(e_0)) \cong H^{i+1}_{S(p)}(S_p)_{c_0}$, $i \geq 1$.

If $q \leq 1$, then $H^0(Y_p, \tilde{S}_p(e_0)) \neq (S_p)_{c_0} = I^c_\mathfrak{p}$ for some $\mathfrak{p} \in X$. From this it follows, as in the proof of Proposition 1.3, that $\pi_*(\mathcal{O}_Y(e_0)) \neq \tilde{I}^{c_0}$. But $\pi_*(\mathcal{O}_Y(e_0))(c)$ and $\tilde{I}^{c_0}(c)$ are generated by global sections for $c \gg 0$. Therefore, by the projection formula we have

$$H^0(X, \pi_*(\mathcal{O}_Y(c, e_0))) = H^0(X, \pi_*(\mathcal{O}_Y(e_0))(c)) \neq H^0(X, \tilde{I}^{c_0}(c)) = (I^c_\mathfrak{e})_c$$

for $c \gg 0$. Moreover,

$$H^0(Y, \mathcal{O}_Y(c, e_0)) = H^0(X, \pi_*(\mathcal{O}_Y(c, e_0))).$$

Hence $H^0(Y, \mathcal{O}_Y(c, e_0)) \neq (I^c_\mathfrak{e})_c$.

If $q \geq 2$, then the Serre-Grothendieck sequence implies $H^1(Y_p, \tilde{S}_p(e_0)) = 0$ for all $\mathfrak{p} \in X$, $0 < i < q - 1$, and $H^{q-1}(Y_p, \tilde{S}_p(e_0)) \neq 0$ for some $\mathfrak{p} \in X$. From this it follows, as in the proof of Proposition 1.3, that

$$R^i \pi_*(\mathcal{O}_Y(e_0)) = 0 \text{ for } 0 < i < q - 1,$$

$$R^{q-1} \pi_*(\mathcal{O}_Y(e_0)) \neq 0.$$

By the projection formula, we have

$$R^i \pi_*(\mathcal{O}_Y(c, e_0)) = R^i \pi_*(\mathcal{O}_Y(e_0)) \otimes \mathcal{O}_X(c) = 0 \text{ for } 0 < i < q - 1,$$

$$R^{q-1} \pi_*(\mathcal{O}_Y(c, e_0)) = R^{q-1} \pi_*(\mathcal{O}_Y(e_0)) \otimes \mathcal{O}_X(c) \neq 0.$$

Since $\pi_*(\mathcal{O}_Y(c, e_0)) = \pi_*(\mathcal{O}_Y(e_0))(c)$, we also have $H^{q-1}(X, \pi_*(\mathcal{O}_Y(c, e_0))) = 0$ for $c \gg 0$. Therefore, using Leray spectral sequence

$$H^i(X, R^i \pi_*(\mathcal{O}_Y(m, e_0))) \Rightarrow H^{i+j}(Y, \mathcal{O}_Y(m, e_0))$$

we can deduce that

$$H^{q-1}(Y, \mathcal{O}_Y(c, e_0)) = H^0(X, R^{q-1} \pi_*(\mathcal{O}_Y(c, e_0)))$$

for $c \gg 0$. But $R^{q-1} \pi_*(\mathcal{O}_Y(c, e_0))$ is generated by global sections for $c \gg 0$. So we get $H^{q-1}(Y, \mathcal{O}_Y(c, e_0)) \neq 0$.

Let us now consider the case $e_0 = a_Y(\omega_S)$. Let $q$ be the smallest integer such that $e_0 = \max\{a_q(\omega_S)_p) \mid p \in X\}$. For $p \in X$ we have $(\omega_S)_p = \bigoplus_{n \geq 0} H^0(Y_p, \omega_{Y_p}(n))$ (see [20] 2.5.2(1) and 2.6.2). From this it follows that $[\mathcal{H}^q_{(\omega_S)_p}]_n = 0$ for $n > 0$, $i = 0, 1$. Since $e_0 > 0$, this implies $q > 1$. Similarly as in the first case, we can also show that $q \leq \dim Y$ and that $H^{q-1}(Y, \omega_Y(c, e_0)) \neq 0$ for $c \gg 0$. By Serre duality we get

$$H^{\dim Y-q+1}(Y, \mathcal{O}_Y(-c, e_0)) = H^{q-1}(Y, \omega_Y(c, e_0)) \neq 0$$

for $c \gg 0$. This completes the proof of Theorem 2.3 \hfill \Box

We shall see later in Example 2.20 that the bound $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$ of Theorem 2.22 is sharp.

Now we want to study the problem when there exists a Cohen-Macaulay ring of the form $k[(I^c_\mathfrak{e})_c]$ for $c \geq 1$. 

Theorem 2.4. Let $R$ be an equidimensional standard graded $k$-algebra and let $I$ be a homogeneous ideal of $R$ with $\text{ht} I \geq 1$. Let $X = \text{Proj} R$ and $S = R[It]$. Assume that $S$ is locally Cohen-Macaulay on $X$. Then, there exists a Cohen-Macaulay ring $k[(I^*)_i]$ with $c \geq d(I)e + 1$ if and only if $H^i(X, \mathcal{O}_X) = k$ and $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, \ldots, \dim X - 1$. In particular, this condition implies that $k[(I^*)_i]$ is a Cohen-Macaulay ring for $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$ and $e \geq 1$.

Proof. Let $Y$ be the blow-up of $X$ along the ideal sheaf of $I$. The assumption implies that $Y$ is equidimensional and Cohen-Macaulay. Since $S$ is locally Cohen-Macaulay over $X$, $Y$ is locally arithmetic Cohen-Macaulay over $X$. Applying Corollary 1.4, we have $H^0(Y, \mathcal{O}_Y) = H^0(X, \mathcal{O}_X)$ and $H^i(Y, \mathcal{O}_Y) = H^i(X, \mathcal{O}_X)$ for $i > 0$. Therefore, the first statement follows from Lemma 2.1. Moreover, we have $\max\{a_X(S), a_X^*(\omega_S)\} = 0$ by Proposition 1.2. Hence the second statement follows from Theorem 2.2. 

Note that the condition $H^0(X, \mathcal{O}_X) = k$ and $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, \ldots, \dim X - 1$ is satisfied if $R$ is a Cohen-Macaulay ring.

The following example shows that the bound $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$ is sharp.

Example 2.5. Let $R = k[x_0, x_1, x_2]$ and $I = (x_1^2, x_2^2, x_3, x_4)$. It is easy to see that $S = R[It]$ is locally Cohen-Macaulay on $X = \text{Proj} R$. We have $I^n = (x_1, x_2)^{4n}$ for all $n \geq 2$. We have

$$a^*(I^n) = \begin{cases} 4 & \text{if } n = 1, \\ 4n - 1 & \text{if } n \geq 2. \end{cases}$$

From this it follows that $\varepsilon(I) = 0$. To compute $\varepsilon^*(I)$ we approximate $I$ by the ideal $J = (x_1, x_2)^4$. Put $S^* = R[It]$. Then we have the exact sequence

$$0 \to R[It] \to R[It] \to k \to 0.$$

From this it follows that $\omega_S = \omega_{S^*}$. Note that $S^*$ is a Veronese subring of the ring $T = R[(x_1, x_2)t]$ and that $T$ is a Gorenstein ring with $\omega_T = T(-2)$. Then $\omega_{S^*} = \bigoplus_{n \geq 1} (x_1, x_2)^{4n-2}$. We have

$$a^*(\omega_n) = a^*((x_1, x_2)^{4n-2}) = 4n - 3$$

for $n \geq 1$. Hence $\varepsilon^*(I) = 0$. By Theorem 2.4 these facts imply that $k[(I^*)_i]$ is Cohen-Macaulay for $c > 4e$ and $e \geq 1$ (which can be also verified directly). On the other hand, for $c = 4$ and $e = 1$, the ring $k[I_4] = k[x_1^4, x_2^4, x_1x_2, x_1^2, x_2^2]$ is not Cohen-Macaulay.

There have been various criteria for the Cohen-Macaulayness of Rees algebras (cf. [33, 18, 27, 80, 1, 21, 28]), so that one can construct various classes of ideals $I$ for which $S$ is locally Cohen-Macaulay on $X$. We list here only the most interesting applications of Theorem 2.4.

Corollary 2.6. Let $R$ be a Cohen-Macaulay standard graded $k$-algebra. Let $I \subset R$ be a homogeneous ideal with $\text{ht} I \geq 1$ which is a locally complete intersection. Then $k[(I^*)_i]$ is a Cohen-Macaulay ring for all $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$ and $e \geq 1$.

Proof. Let $X = \text{Proj} R$. The assumption on $I$ means that $I_p$ is a complete intersection ideal in $R_p$ for $p \in X$. Therefore, $R_p[I_p]t$ is Cohen-Macaulay for all $p \in X$. Hence, $S = R[It]$ is locally Cohen-Macaulay on $X$. The result follows from Theorem 2.4. \qed
Corollary 2.7. Let \( R \) be a polynomial ring over a field \( k \) of characteristic zero and \( I \subset R \) a non-singular homogeneous ideal with \( \text{ht} \ I \geq 1 \). Then, \( k[[I^n]]_e \) is a Cohen-Macaulay ring for \( c > d(I)e + \varepsilon(I) \) and \( e \geq 1 \).

Proof. The assumption implies that \( I \) is locally a complete intersection. Hence \( S = R[I] \) is locally Cohen-Macaulay on \( X = \text{Proj} \ R \). Let \( Y = \text{Proj} \ S \). Then \( Y \) is a projective non-singular scheme. Let \( m, n \) be positive integers with \( m \geq d(I)n + 1 \). Then \( \mathcal{O}_Y(m, n) \) is a very ample invertible sheaf on \( Y \) because \( Y \cong \text{Proj} \ k[[I^n]]_m \) [9, Lemma 1.1]. Let \( \omega_Y = \widetilde{\omega}_S \). Then \( H^i(Y, \omega_Y(m, n)) = 0 \) for \( i \geq 1 \) by Kodaira’s vanishing theorem. On the other hand, we have

\[
H^i(Y, \omega_Y(m, n)) = H^i(X, (\omega_S)_n(m))
\]

by Proposition 1.3. Therefore, \( H^i(X, (\omega_S)_n(m)) = 0 \) for \( i \geq 1 \). Using the Serre-Grothendieck correspondence we can deduce that

\[
H^i_{R_e}((\omega_S)_n)_m = 0 \quad \text{for} \quad i \geq 2.
\]

Hence \( \varepsilon'(I) = 0 \). Now, the conclusion follows from Corollary 2.6. \( \square \)

Remark 2.8. A similar result to Theorem 2.4 was already given by Cutkosky and Herzog [9, Theorem 4.1] when \( R \) is Cohen-Macaulay. Their result shows the existence of a constant \( \delta \) such that \( k[[I^n]]_e \) is Cohen-Macaulay for \( c \geq \delta e, e > 0 \), under some assumptions on the associated graded ring \( \bigoplus_{n \geq 0} I^n/I^{n+1} \). It is not hard to see that these assumptions imply \( \max \{ a^1_X(S), a^1_Y(\omega_S) \} \leq 0 \) (see [9, Lemma 2.1 and Lemma 2.2]). Hence their result is also a consequence of Theorem 2.2.

Similar statements to the above two corollaries were also given in [9] but without any information on the slope \( \delta \).

It is not easy to compute \( \varepsilon(I) \) explicitly, even when \( I \) is a non-singular ideal in a polynomial ring. By a famous result of Bertram, Ein and Lazarsfeld [3] we know that if \( I \) is the ideal of a smooth complex variety cut out scheme-theoretically by hypersurfaces of degree \( d_1 \geq \cdots \geq d_m \), then

\[
a_i(I^n) \leq d_1n + d_2 + \cdots + d_m - \text{ht} \ I
\]

for \( i \geq 2 \) and \( n \geq 1 \). However, we do not know any bound for \( a_1(I^n) \) in terms of \( d_1, \ldots, d_m \). It would be of interest to find such a bound. In general, if we happen to know the minimal free resolution of \( S \) over a bi-graded polynomial ring, then we can estimate \( \varepsilon(I) \) in terms of the shifts of syzygy modules of the resolution [10].

In the case when \( I \) is the defining ideal of a scheme of fat points, we know an explicit bound for \( a^*(I^n) \), namely \( a^*(I^n) \leq \text{reg}(I)n \) for all \( n \geq 1 \) [6, 13]. As a consequence, we immediately obtain the following result of Geramita, Gimigliano and Pitteloud.

Corollary 2.9 ([13, Theorem 2.4]). Let \( R \) be a polynomial ring over a field \( k \) of characteristic zero, and let \( I \subset R \) be the defining ideal of a scheme of fat points in \( \text{Proj} \ R \). Then, \( k[[I^n]]_e \) is a Cohen-Macaulay ring for \( c > \text{reg}(I)e \) and \( e \geq 1 \).

Proof. By definition, the ideal \( I \) has the form \( I = \bigcap_{i=1}^s p_i^{m_i} \), where \( p_i \) is the defining prime ideal of a closed point in \( X = \text{Proj} \ R \) and \( m_i \in \mathbb{N} \). Then \( R_{(p)}[I_{(p)}] \) is Cohen-Macaulay for all \( p \in X \). In fact, we may assume that \( p = p_i \) for some \( i \). Then \( p \) is a complete intersection and \( R_{(p)}[I_{(p)}] = R_{(p)}[p_i^{m_i}; t] \) is a Veronese subalgebra of \( R_{(p)}[p_i(t)] \). Since \( R_{(p)}[p_i(t)] \) is a Cohen-Macaulay ring, so is \( R_{(p)}[I_{(p)}] \). Thus,
$S = R[It]$ is locally Cohen-Macaulay on $X$. This argument also shows that $Y = \text{Proj} S$ is smooth. Using the Kodaira vanishing theorem we can show, as in the proof of Corollary \[2.7\] that $\varepsilon^*(I) = 0$. The conclusion now follows from the proof of Theorem \[2.4\] when we replace the slope $d(I)$ by $\text{reg}(I) \geq d(I)$ and $\varepsilon(I)$ by $0$ because of the bound $a^*(I^n) \leq \text{reg}(I)n$. \hfill\(\Box\)

It was asked in [7] whether there exists a Cohen-Macaulay ring $k[(I_e)_c]$ for $c \gg 0$ if $R$ is a polynomial ring and $R[It]$ is Cohen-Macaulay. This question has been positively settled in [25, Theorem 4.5]. We can make this result more precise as follows.

**Corollary 2.10.** Let $R$ be a Cohen-Macaulay standard graded $k$-algebra. Let $I \subset R$ be a homogeneous ideal with $\text{ht} I \geq 1$ such that $R[It]$ is Cohen-Macaulay. Then $k[(I_e)_c]$ is a Cohen-Macaulay ring for all $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$ and $e \geq 1$.

### 3. Arithmetically Cohen-Macaulay blow-ups

Let $X$ be a projective scheme over a field $k$. Let $\pi : Y \to X$ be the blowing up of $X$ along an ideal sheaf $T$. We say that $Y$ is an *arithmetically Cohen-Macaulay blow-up* of $X$ if there is a standard graded $k$-algebra $R$ and a homogeneous ideal $J \subset R$ with $\text{ht} J \geq 1$ such that $X = \text{Proj} R$, $T = \tilde{J}$, and $R[It]$ is a Cohen-Macaulay ring. The aim of this section is to characterize arithmetically Cohen-Macaulay blow-ups.

Let $R$ be a finitely generated standard graded $k$-algebra, and let $I$ be a homogeneous ideal of $R$ with $\text{ht} I \geq 1$, such that $X = \text{Proj} R$ and $I = \tilde{I}$. Let $d(I)$ denote the maximal degree of the elements of a homogeneous basis of $I$. For any ideal $J$ generated by $(I_e)_c$ with $c \geq d(I)e$ we have $J_n = (I_e)_n$ for all $n \geq c$ so that $T^e = \tilde{J}$. Hence $Y = \text{Proj} R[It]$. The Rees algebra $R[(I^e)_c] = R[It]$ is called a *truncated Rees algebra of $I^e$* \[3, 5, 8\]. We may strengthen the problem on the characterization of arithmetically Cohen-Macaulay blow-ups by asking the question: When does there exist a Cohen-Macaulay truncated Rees algebra $R[(I^e)_c]$? To solve this problem we shall need the following result of Hyry.

Let $T$ be a standard bi-graded algebra over a field $k$, that is, $T$ is generated over $k$ by the elements of degree $(1,0)$ and $(0,1)$. Let $M$ denote the maximal graded ideal of $T$ and define

$$a^1(T) := \max\{m| \text{there is } n \text{ such that } H^\text{dim} M T_{(m,n)} \neq 0\},$$

$$a^2(T) := \max\{n| \text{there is } m \text{ such that } H^\text{dim} M T_{(m,n)} \neq 0\}.$$

**Theorem 3.1** \[19, Theorem 2.5\]. Let $T$ be a standard bi-graded $k$-algebra with $a^1(T), a^2(T) < 0$. Let $Y = \text{Proj} T$. Then $T$ is Cohen-Macaulay if and only if the following conditions are satisfied:

$$H^0(Y,T_{(m,n)}) \cong T_{(m,n)} \text{ for } m, n \geq 0,$$

$$H^i(Y,T_{(m,n)}) = 0 \text{ for } m, n \geq 0, \ i > 0,$$

$$H^i(Y,T_{(m,n)}) = 0 \text{ for } m, n < 0, \ i < \text{dim} T - 2.$$

Let $J \subset R$ be an arbitrary ideal generated by forms of degree $c$ and put $T = R[It]$. Then $T$ can be equipped with another bi-grading given by

$$T_{(m,n)} = \langle J^c \rangle_{m+cn} \langle n \rangle_n$$

for $(m,n) \in \mathbb{N}^2$. With this bi-grading, $T$ is a standard bi-graded $k$-algebra. Comparing with the natural bi-grading of $T$ considered in the preceding sections, we
Lemma 3.2. Let $T = R[\{J_t\}]$ be as above. Then

(i) $a^1(T) \leq \max\{a^*(J^n) - nc | n \geq 0\}$,

(ii) $a^2(T) < 0$.

Proof. To prove (i) we will show more, namely, that $H^i_M(T)_{(m,n)} = 0$ for $m > \max\{a^*(J^n) - nc | n \geq 0\}$ and $i \geq 0$. Let $T_1$ denote the ideal of $T$ generated by the homogeneous elements of degree $(1,0)$. Then, by [13, Lemma 2.3], we only need to show that $H^i_{T_1}(T)_{(m,n)} = 0$ for $m > \max\{a^*(J^n) - nc | n \geq 0\}$ and $i \geq 0$. Since $T_1$ is generated by $R_+$, we always have

$$H^i_{T_1}(T)_{(m,n)} = \begin{cases} 0 & \text{for } n < 0, \\ H^i_{R_+}(J^n)_{m+nc} & \text{for } n \geq 0. \end{cases}$$

But $H^i_{R_+}(J^n)_{m+nc} = 0$ for $m + nc > a^*(J^n)$, $n \geq 0$. Therefore, $H^i_{T_1}(T)_{(m,n)} = 0$ for $m > \max\{a^*(J^n) - nc | n \geq 0\}$, as required.

To prove (ii) we first observe that $a^2(T) = \max\{n | H^i_M(T)_n \neq 0\}$, where the $\mathbb{Z}$-gradation comes from the natural grading $T_n = J^n t^n$, $n \geq 0$. Therefore, the conclusion $a^2(T) < 0$ follows from [33, Corollary 3.2].

Corollary 3.3. Let $R$ be a standard graded $k$-algebra with $a^*(R) < 0$ and let $I \subset R$ be a homogeneous ideal with $ht I \geq 1$. Let $T = R[\{I^e\}]$ for some fixed integers $c > d(I)e + \varepsilon(I)$ and $e \geq 1$. Then $a^1(T) < 0$ and $a^2(T) < 0$.

Proof. Let $J$ be the ideal of $R$ generated by $\{I^e\}$. By Lemma 3.2 we only need to prove that $a^*(J^n) < nc$ for $n \geq 0$. For $n = 0$, this follows from the assumption $a^*(R) < 0$. For $n \geq 1$, we will approximate $a^*(J)$ by $a^*(I^{cn})$. Since $J^n$ is generated by elements of degree $cn$ and since $cn > d(I)en \geq d(I^{cn})$, we have $(I^{cn}/J^n)_m = 0$ for $m \geq cn$. From this it follows that $H^n(I^{cn}/J^n) = I^{cn}/J^n$ and $H^i(I^{cn}/J^n) = 0$ for $i > 0$. Therefore, from the exact sequence

$$0 \rightarrow J^n \rightarrow I^{cn} \rightarrow I^{cn}/J^n \rightarrow 0$$

we can deduce that $H^i(J^n)_m = H^i(I^{cn})_m$ for $m \geq cn$ and $i \geq 0$. This implies

$$a^*(J^n) \leq \max\{cn - 1, a^*(I^{cn})\}.$$

By the definition of $\varepsilon(I)$ we have $a^*(I^{cn}) \leq d(I)en + \varepsilon(I) \leq cn - 1$. Therefore, $a^*(J^n) \leq cn - 1$ for $n \geq 1$.

We are now ready to give a necessary and sufficient condition for the existence of a Cohen-Macaulay truncated Rees algebra.

Theorem 3.4. Let $R$ be a standard graded $k$-algebra with $a^*(R) < 0$ and let $I \subset R$ be a homogeneous ideal with $ht I \geq 1$. Let $X = \text{Proj} R$, $S = R[\{I\}]$ and $Y = \text{Proj} S$. Then there exists a Cohen-Macaulay ring $R[\{I^e\}]$ with $c \geq d(I)e$ if and only if the following conditions are satisfied:

(i) $Y$ is equidimensional and Cohen-Macaulay,

(ii) $\pi_+ \mathcal{O}_Y = \mathcal{O}_X$ and $R^i \pi_+ \mathcal{O}_Y = 0$ for $i > 0$.

Moreover, these conditions imply that $R[\{I^e\}]$ is a Cohen-Macaulay ring for $c > d(I)e + \max\{\varepsilon(I), e^*(I)\}$ and $e > \max\{a_X^*(S), a_X^*(\omega_S)\}$. 

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Proof. Let $J$ be the ideal of $R$ generated by $(I^c)_c$ and $T = R[J^t]$ for a fixed pair of positive integers $c, e$ with $c \geq d(I)e$. Then $Y \cong \Proj T$. If $T$ is a Cohen-Macaulay ring, then (i) is obviously satisfied and $Y$ is locally arithmetic Cohen-Macaulay over $X$. (ii) follows from Corollary 3.4.

To prove the converse we equip $T$ with the aforementioned bi-gradation. Set $e_0 = \max\{a_X^0(S), a_X(\omega_S)\}$. We will use Theorem 3.1 to prove that $T$ is Cohen-Macaulay for $c > d(I)e + \max\{e(I), e^*(I)\}$ and $e > e_0$. By Corollary 3.3 we have $a^1(T) < 0$ and $a^2(T) < 0$. From the bi-gradation of $T$ we see that

$$T(m, n)^{\sim} = \mathcal{O}_Y(m + cn, en),$$

where $\mathcal{O}_Y(m + cn, n)$ denotes the twisted $\mathcal{O}_Y$-module with respect to the natural bi-gradation of $S$. If $\pi_\ast \mathcal{O}_Y = \mathcal{O}_X$ and $R^i\pi_\ast \mathcal{O}_Y = 0$ for $i > 0$, then we can show as in the proof of Proposition 1.3 that

$$H^i(Y, \mathcal{O}_Y(m, 0)) = H^i(X, \mathcal{O}_X(m))$$

for $i \geq 0$. Since $a^\ast(R) < 0$, we have $H^i_R(R)_m = 0$ for all $m \geq 0$ and $i \geq 0$. Using the Serre-Grothendieck correspondence between sheaf cohomology of $X$ and local cohomology of $R$ we can deduce that $H^0(X, \mathcal{O}_X(m)) = R_m$ and $H^i(X, \mathcal{O}_X(m)) = 0$ for $i > 0$. Therefore,

$$H^0(Y, \mathcal{O}_Y(m, 0)) = R_m = T_{(m, 0)},$$

$$H^i(Y, \mathcal{O}_Y(m, 0)) = 0, \ i > 0.$$

For $m \geq 0$ and $n > 0$ we have $m + cn > d(I)e + \varepsilon(I)$. Therefore, using Proposition 1.3 and Lemma 1.5 we get

$$H^0(Y, \mathcal{O}_Y(m + cn, en)) = T_{(m, n)},$$

$$H^i(Y, \mathcal{O}_Y(m + cn, en)) = 0, \ i > 0,$$

for $e > e_0$. For $m, n < 0$ we can show, similarly as above, that

$$H^i(Y, \omega_Y(-m - cn, -cn)) = 0$$

for $i > 0$ and $e > e_0$. If $Y$ is equidimensional and Cohen-Macaulay, we can apply Serre duality and obtain

$$H^i(Y, \mathcal{O}_Y(m + cn, en)) = 0, \ i < \dim Y.$$

Passing from $\mathcal{O}_Y(m + cn, en)$ to $T(m, n)^{\sim}$ we get

$$H^0(Y, T_{(m, n)}) \cong T_{(m, n)}$$

for $m, n \geq 0$, $i > 0$,

$$H^i(Y, T_{(m, n)}) = 0$$

for $m, n < 0, \ i < \dim T - 2$.

By Theorem 3.1 these conditions imply that $T$ is a Cohen-Macaulay ring. The proof of Theorem 3.4 is now complete.

The following example shows that the condition $a^\ast(R) < 0$ is not necessary for the existence of a Cohen-Macaulay truncated Rees algebra. It also shows that in general, the existence of a Cohen-Macaulay truncated Rees algebra does not imply the existence of a linear bound on $c$ ensuring the Cohen-Macaulayness of $R[(I^c)_c]$. 

Example 3.5. Take $R = k[x, y, z]/(xy^2 - z^3)$, the coordinate ring of a cuspidal plane curve, and $I = (x) \subseteq R$, a homogeneous ideal with $\text{ht } I = 1$. Then $R$ is a two-dimensional Cohen-Macaulay ring with $a^*(R) = 0$. It is obvious that $R[(I^e)_e,t] = R[I]$ is a Cohen-Macaulay ring for $e \geq 1$. For $c > e$ we have $R[(I^e)_e,t] \cong R[(x, y, z)_{c-e}]$. It is easy to check that the reduction number of the ideal $(x, y, z)_{c-e}$ is greater than 1. By [14], this implies that $R[(x, y, z)_{c-e}]$ is not Cohen-Macaulay for any $c > e$.

Now we will show that the bound $e > e_0$ in Theorem 3.4 is once again best possible.

Theorem 3.6. Let the notations and assumptions be as in Theorem 3.4. Let
$$e_0 = \max\{a^*_X(S), a^*_X(\omega S)\}.$$ Then $R[(I^{e_0})_e,t]$ is not a Cohen-Macaulay ring for $c \geq d(I)e_0$ if $e_0 \geq 1$.

Proof. Let $T = R[(I^{e_0})_e,t]$ for some $c \geq d(I)e_0$, and suppose $e_0 \geq 1$. Note that $(I^{e_0})_e$ and $T^{e_0}$ defines the same ideal sheaf in $\mathcal{O}_X$. Consider the natural $\mathbb{N}$-grading of $T$ and $S$ given by the degree of $t$. For any $p \in X$, the ring $T(p)$ is isomorphic to the $e_0$-th Veronese subring of $S(p)$. Hence
$$H^1_{T(p)}(T(p))_1 = H^1_{S(p)}((S(p))_{e_0}),$$
$$H^1_{T(p)}((\omega T)(p))_1 = H^1_{S(p)}((\omega S)(p))_{e_0},$$
for $i \geq 0$. By the definition of $e_0$ there exists $p \in X$ and $i \geq 0$ such that either $H^1_{S(p)}((S(p))_{e_0} \neq 0$ or $H^1_{S(p)}((\omega S)(p))_{e_0} \neq 0$. Therefore, $\max\{a^*(T), a^*(\omega_T)\} \geq 1$. By Lemma 1.4 this implies that $T$ is not a Cohen-Macaulay ring. □

From Theorem 3.4 we can derive the following sufficient condition for the existence of a truncated Cohen-Macaulay Rees algebra.

Theorem 3.7. Let $R$ be an equidimensional standard graded $k$-algebra with $a^*(R) < 0$ and let $I \subseteq R$ be a homogeneous ideal with $\text{ht } I \geq 1$. Let $X = \text{Proj } R$ and $S = R[I]$. Assume that $S$ is locally Cohen-Macaulay on $X$. Then $R[(I^e)_e,t]$ is a Cohen-Macaulay ring for $c > d(I)e + \max\{c(I), e^*(I)\}$ and $e \geq 1$.

Proof. It is obvious that the assumptions imply that $Y$ is equidimensional and Cohen-Macaulay. The condition $\pi_*\mathcal{O}_Y = \mathcal{O}_X$ and $R^i\pi_*\mathcal{O}_Y = 0$ for $i > 0$ follows from Corollary 1.4. Hence the conclusion follows from Theorem 3.4. □

The above condition is also a necessary condition for the existence of a truncated Cohen-Macaulay Rees algebra of the form $R[I_e,t]$ ($e = 1$).

Corollary 3.8. Let $R$ be a standard graded $k$-algebra with $a^*(R) < 0$ and let $I \subseteq R$ be a homogeneous ideal with $\text{ht } I \geq 1$. Let $X = \text{Proj } R$ and $S = R[I]$ and $e > d(I)$ if and only if $S$ is locally Cohen-Macaulay on $X$.

Proof. By Theorem 3.7 we only need to show that if $R[I_e,t]$ is a Cohen-Macaulay ring for some $c \geq d(I)$, then $S$ is locally Cohen-Macaulay on $X$. But this is obvious because $(I_e)$ and $I$ define the same ideal sheaf and $R[I_e,t]$ is locally Cohen-Macaulay on $X$. □

Using Theorem 3.7 we obtain several classes of Cohen-Macaulay Rees algebras.
Corollary 3.9 (cf. [8] Corollary 2.2.1(2)) for the case $e = 1$). Let $R$ be a Cohen-Macaulay standard graded $k$-algebra with $a(R) < 0$. Let $I \subset R$ be a homogeneous ideal with $\text{ht} \, I \geq 1$ which is locally a complete intersection. Then $R[(I^e)_e t]$ is a Cohen-Macaulay ring for all $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$ and $e \geq 1$.

**Proof.** As in the proof of Corollary 2.9, $S = R[It]$ is locally Cohen-Macaulay over $X = \text{Proj} \, R$. Since the assumption on $R$ implies $a^*(R) < 0$, the conclusion follows from Theorem 3.7. \hfill \Box

Corollary 3.10. Let $R$ be a polynomial ring over a field $k$ of characteristic zero and let $I \subset R$ be a non-singular homogeneous ideal. Then $R[(I^e)_e t]$ is a Cohen-Macaulay ring for all $c > d(I)e + \varepsilon(I)$ and $e \geq 1$.

**Proof.** We have seen in the proof of Corollary 2.7 that $\varepsilon^*(I) = 0$. Hence the assertion follows from Corollary 3.9. \hfill \Box

Corollary 3.11 (cf. [15] Theorem 2.4 for the case $e = 1$). Let $R$ be a polynomial ring over a field $k$ of characteristic zero and let $I \subset R$ be the defining ideal of a scheme of fat points in $\text{Proj} \, R$. Then $R[(I^e)_e t]$ is a Cohen-Macaulay ring for $c > \text{reg}(I)e$.

**Proof.** The proof follows from Theorem 3.7 with the same lines of arguments as in the proof of Corollary 2.9. \hfill \Box

Now we will use Theorem 3.7 to find a criterion for arithmetically Cohen-Macaulay blow-ups. Recall that the blow-up $Y$ of a projective scheme $X$ along an ideal sheaf $I$ is said to be locally arithmetic Cohen-Macaulay on $X$ if there exist a standard graded algebra $R$ over a field and a homogeneous ideal $I \subset R$ such that $X = \text{Proj} \, R$, $I = I$ and $S = R[It]$ is locally Cohen-Macaulay on $X$.

**Theorem 3.12.** Let $X$ be a projective scheme over a field $k$ such that $H^0(X, \mathcal{O}_X) = k$ and $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$. Let $Y \longrightarrow X$ be a proper birational morphism. Then $Y$ is an arithmetically Cohen-Macaulay blow-up if and only if $Y$ is equidimensional and locally arithmetic Cohen-Macaulay on $X$.

**Proof.** Suppose $Y$ is an arithmetically Cohen-Macaulay blow-up of $X$. Let $R$ be a standard graded algebra over $k$, and let $I$ be a homogeneous ideal of $R$, such that $X = \text{Proj} \, R$, $Y$ is the blow-up of $X$ along the ideal sheaf $I$, and $S = R[It]$ is a Cohen-Macaulay ring. Then, $\mathcal{O}_{X,x}[It] = S(p)$ is obviously Cohen-Macaulay for all $p \in X$. Thus, $Y$ is locally arithmetic Cohen-Macaulay on $X$.

Conversely, suppose $Y$ is equidimensional and locally arithmetic Cohen-Macaulay on $Y$. Then there exist a standard graded $k$-algebra $R$ and a homogeneous ideal $I \subset R$ such that $X = \text{Proj} \, R$, $Y$ is the blow-up of $X$ along the ideal sheaf of $I$, and $R[It]$ is locally Cohen-Macaulay on $X$. The assumption on the sheaf cohomology of $X$ implies that $H^i_{R[x]}(R) = 0$ for $i \geq 0$. Without restriction we may replace $R$ by a suitable Veronese subalgebra and obtain $H^i_{R[x]}(R) = 0$ for all $n \geq 0$ or, equivalently, $a^*(R) < 0$. Now we may apply Theorem 3.7 to find a Cohen-Macaulay Rees algebra $R[It]$ with $c > 0$. Since the ideal $(I_c)$ defines the same ideal sheaf $I$, we can conclude that $Y$ is an arithmetically blow-up of $X$. \hfill \Box
Note added in proof

After the manuscript was sent for publication, we were informed that Olga Lavila-Vidal obtained similar results in her thesis (ArXiv:math.AC/0407041) as our Theorem 2.4 and Corollary 2.6. She did not have an explicit description for the constant term as we did though.

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