

SAARI'S CONJECTURE FOR THE COLLINEAR n -BODY PROBLEM

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ABSTRACT. In 1970 Don Saari conjectured that the only solutions of the Newtonian n -body problem that have constant moment of inertia are the relative equilibria. We prove this conjecture in the collinear case for any potential that involves only the mutual distances. Furthermore, in the case of homogeneous potentials, we show that the only collinear and non-zero angular momentum solutions are homographic motions with central configurations.

1. INTRODUCTION

In a series of papers published in the 1970s, Don Saari investigated the boundedness of solutions in the Newtonian n -body problem. In one of the earliest, [11], he proved that if a solution has constant potential for any finite time interval, then it has both constant potential and constant moment of inertia at all times. In the same paper he stated what is now known as Saari's conjecture: *Every solution of the Newtonian n -body problem that has constant moment of inertia is a relative equilibrium.*

During a recent visit at the University of Victoria, Don told us that in the 1970s he had found a simple proof of his conjecture in the 3-body case. Unfortunately he never published it, lost his notes, and could not remember the details. Although many people tried to solve his conjecture, nobody has yet succeeded. In the late 1970s and early 1980s, Julian Palmore published two papers, [8, 9], in which he claimed to have a proof, but his arguments do not survive a careful scrutiny. In October 2002, Chris McCord gave a talk at the Midwest Dynamical Systems Conference held at the University of Cincinnati, where he presented a proof in the case of three bodies of equal masses, [6]. Also, Rick Moeckel has announced a computer-assisted proof for the Newtonian 3-body case. At the time this article was written, their papers had not yet been published.

Recently the interest in Saari's conjecture has grown even more due to the variational proof given by Alain Chenciner and Richard Montgomery for the existence of the figure eight solution, numerically discovered by Christopher Moore, [2], in 1993. Carles Simó studied this orbit numerically, [12], and found out that its moment of inertia is very close to being a constant all along the motion. This fact prompted Chenciner to ask for analytical arguments that the figure eight solution has no constant moment of inertia.

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In this paper we make some modest steps towards understanding Saari's conjecture. In particular we prove that the conjecture is true in the collinear case for any number of bodies and for any potential that involves only the mutual distances. This generalizes a result obtained by Pizzetti in 1904, [10, 14], for the Newtonian 3-body problem. Our result is not only more general but the proof we provide is much simpler than the one of the Italian mathematician. Our presentation is organized as follows.

In Section 2 we introduce the equations of motion and several definitions. In Section 3 we solve the collinear case, i.e., the one in which the bodies are on a line that rotates in the plane of motion. We first prove that every non-zero angular momentum and collinear solution of the n -body problem given by any potential that depends only on the mutual distances alone, is homographic. This implies that if the moment of inertia is constant, the only solutions of this type are the relative equilibria. Moreover, we show that the only non-zero angular momentum and collinear solutions of the n -body problem given by a homogeneous potential of degree $\alpha \neq -2$ are the homographic motions with central configurations. We end our paper with Section 4, in which we give a geometrical interpretation of Saari's Conjecture in the Newtonian case.

2. EQUATIONS OF MOTION

Consider the planar motion of n interacting bodies P_1, \dots, P_n . Let the mass and position (with respect to the center of mass) of the the body P_i be given by m_i and $\mathbf{r}_i = (x_i, y_i)$. Let $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ be the distance between the i th and the j th bodies and denote $x = (\mathbf{r}_1, \dots, \mathbf{r}_n) \in \mathbb{R}^{2n}$ and $\dot{x} = (\dot{\mathbf{r}}_1, \dots, \dot{\mathbf{r}}_n) \in \mathbb{R}^{2n}$. Take $L = T - U$ as the Lagrangian function of the system under discussion, where U and T are the potential and the kinetic energy. Then the Euler-Lagrange equations for the body P_i are

$$(1) \quad m_i \ddot{\mathbf{r}}_i = -\frac{\partial U}{\partial \mathbf{r}_i}.$$

With the scalar product

$$(2) \quad \langle x, \dot{x} \rangle = \sum_{i=1}^n m_i \mathbf{r}_i \cdot \dot{\mathbf{r}}_i$$

and the notation

$$(3) \quad \nabla = (m_1^{-1} \nabla_1, \dots, m_n^{-1} \nabla_n),$$

the equations of motion take the simpler form

$$(4) \quad \dot{x} = y, \quad \dot{y} = -\nabla U,$$

where ∇_i is the i th gradient component.

The moment of inertia and the kinetic energy can be written as

$$(5) \quad I = \langle x, x \rangle, \quad T = \frac{1}{2} \langle y, y \rangle.$$

A *central configuration* of the n -body problem is a configuration $x \in \mathbb{R}^{2n}$ that satisfies the algebraic equations

$$(6) \quad \nabla U(x) = \omega^2 \nabla I$$

for some constant ω (see [7, 13, 14] for more details).

A given solution $x = x(t)$ of the n -body problem is called *homographic* if, in the barycentric coordinate system, the configuration of the bodies is similar to itself when t varies. By this we mean that there exist a scalar $\nu = \nu(t) > 0$, an orthogonal 2-matrix $\Omega = \Omega(t)$ such that for every i and t one has

$$(7) \quad \mathbf{r}_i = \nu(t)\Omega(t)\mathbf{r}_i^0,$$

where $\mathbf{r}_i, \nu, \Omega$ correspond to an arbitrary t and \mathbf{r}_i^0 denotes \mathbf{r}_i at some initial instant $t = t_0$.

There are two limiting types of homographic solutions. One appears when the configuration is dilating without rotation, i.e., $\Omega(t)$ is the identity matrix. Such solutions are called *homothetic*. The other shows up when the configuration is rotating without dilatation, i.e., $\nu(t) = 1$. These particular homographic solutions are characterized by

$$(8) \quad \mathbf{r}_i = \Omega(t)\mathbf{r}_i^0$$

and are called *relative equilibria*. In such cases the system rotates about the center of mass as a rigid body, the angular velocity is constant and the mutual distances do not changes when t varies. The relative equilibria are the solutions we are interested in, and they make the object of Saari's Conjecture.

In the following sections we will deal with two types of potentials that are more general than the Newtonian one: first, when the potential energy is a function of the mutual distances only and, second, when the potential is a homogeneous function of degree α . In the latter case the Lagrangian function is of the form

$$(9) \quad L = T - U = \frac{1}{2} \sum_{i=1}^n m_i \dot{\mathbf{r}}_i^2 + \beta \sum_{i>j} m_i m_j |\mathbf{r}_i - \mathbf{r}_j|^\alpha,$$

where $\beta = 1$ if $\alpha < 0$, whereas $\beta = -1$ if $\alpha > 0$. The corresponding Hamiltonian function is

$$(10) \quad H = T + U = \sum_{i=1}^n \frac{\mathbf{p}_i^2}{2m_i} - \beta \sum_{i>j} m_i m_j |\mathbf{r}_i - \mathbf{r}_j|^\alpha.$$

The Lagrangian and Hamiltonian above describe the Newtonian n -body problem when $\alpha = -1$ and $\beta = G$, where G is the gravitational constant.

3. THE COLLINEAR N-BODY PROBLEM

In this section we will prove Saari's Conjecture in the collinear case for any potential that depends on the mutual distances alone. By the collinear case we mean the one in which the bodies are on a line that rotates in the plane around the center of mass of the bodies.

Let \mathbf{K} be the total angular momentum and let \mathbf{K}_i be the angular momentum of the body P_i about the center of mass. Denote by \mathbf{M}_i the moment of the forces for the body P_i . \mathbf{F}_{ij} is the force acting on the body P_i while interacting with P_j . It is well known (see [14]), and easy to check, that every collinear solution is planar. Assume that the potential U is function of the mutual distances alone. Note that $\mathbf{F}_{ij} = \nabla_{\mathbf{r}_{ij}} U$. In this setting, we can prove the following result.

Theorem 1. *Every collinear and non-zero angular momentum solution of the n -body problem given by a potential that depends only on the mutual distances is homographic.*

Since \mathbf{r}_i and \mathbf{F}_{ij} are collinear,

$$(11) \quad \dot{\mathbf{K}}_i = \mathbf{M}_i = \sum_{\substack{j=0 \\ j \neq i}}^n m_j \mathbf{r}_i \times \mathbf{F}_{ij} = 0.$$

Consequently \mathbf{K}_i is a constant vector. If K_i is the component of \mathbf{K}_i orthogonal to the plane of motion, then $K_i = c_i$, where c_i is a constant. We can also write the component of the angular momentum orthogonal to the plane of motion as

$$(12) \quad K_i = c_i = m_i r_i^2(t) \omega(t),$$

where the angular velocity ω is the same for all the bodies, since they belong to a line. Let us note that the vectors r_i and the angular velocity ω may depend on time.

Now consider the ratio between the components of the angular momentum of any two bodies P_i and P_j . Using (12), we get

$$(13) \quad \frac{c_i}{c_j} = \frac{m_i r_i^2}{m_j r_j^2},$$

consequently the ratio between distances is

$$(14) \quad \frac{r_i(t)}{r_j(t)} = \sqrt{\frac{m_j c_i}{m_i c_j}}.$$

Consequently the geometrical configuration of the n bodies remains similar to itself as t varies. This concludes the proof.

The particular Newtonian case of Theorem 1 was proved by Pizzetti in 1904 for the 3-body case, [10, 14]. Unaware of his result, we found the above shorter, simpler and more general proof. Using Theorem 1, we can now confirm Saari's conjecture in the collinear case for any number n of bodies.

Corollary 1. *Consider the n -body problem given by a potential that depends on the mutual distances alone. Then the only solutions that are collinear, have non-zero angular momentum, and constant moment of inertia are the relative equilibria.*

Proof. The component of the total angular momentum orthogonal to the plane of motion can be written as

$$(15) \quad K = \sum_{i=0}^N K_i = \omega \sum_{i=0}^N m_i r_i^2 = \omega I = C,$$

where I is the moment of inertia. Since I and K are constant, ω is constant too. The fact that mutual distances are also constant follows immediately from (12). Hence we showed that if the moment of inertia is constant, the corresponding solutions are relative equilibria. This completes the proof.

Corollary 1 completely characterizes the collinear solutions with constant moment of inertia, but Theorem 1 is not the strongest result that can be proved. Indeed, in 1767 Leonhard Euler showed that if in the Newtonian case three bodies of arbitrary masses are arranged initially on a line, as in Figure 1, such that the ratio AB/BC of their mutual distances has a certain value given by a formula depending on the masses, and if suitable initial velocities are assigned to the particles, then they will move periodically on ellipses remaining on a line at all times. Such motions are called homographic motions with central configurations.

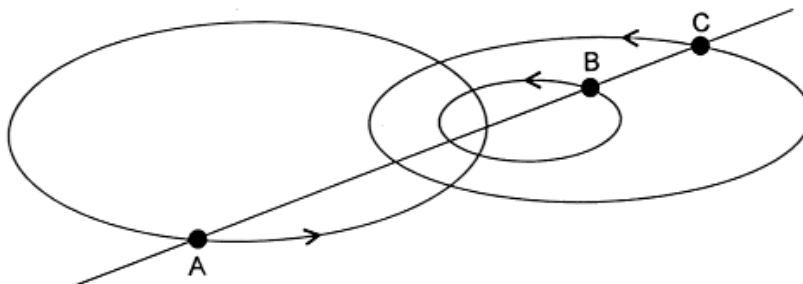


FIGURE 1. The Eulerian solution of the three-body problem.

This leads to a natural question: “Are there any collinear orbits given by homographic solutions with central configurations?” It turns out that the answer is positive if the potential is homogeneous of degree $\alpha \neq -2$. More precisely, we have the following result.

Theorem 2. *The only collinear and non-zero angular momentum solutions of the n -body problem given by a homogeneous potential of degree $\alpha \neq -2$, are the homographic motions with central configurations. In particular if the moment of inertia is constant, the solutions are relative equilibria.*

Proof. The first part of the proof follows immediately from Theorem 1 and Proposition 2.14 in [1], which shows that if the potential is homogeneous of degree $\alpha \neq -2$, then the homographic solutions are homographic with central configurations. The second part follows from Corollary 1. This concludes the proof.

We further present an alternative proof of Theorem 2, which uses Sundman’s inequality. Differentiating the expression of the angular momentum of the body P_i (see (12)) with respect to the time variable and using (11), we find that

$$(16) \quad \dot{K}_i = 2m_i r_i \dot{r}_i \omega(t) + m_i r_i^2 \dot{\omega}(t) = 0.$$

This implies that

$$(17) \quad \dot{r}_i = -\frac{r_i}{2} \frac{\dot{\omega}}{\omega}.$$

With these preparations, we can rewrite the kinetic energy,

$$(18) \quad T = \frac{1}{2} \sum_{i=1}^N (m_i \dot{r}_i^2 + m_i r_i^2 \omega^2),$$

in the form

$$(19) \quad T = \frac{C}{2} \left(\frac{\dot{\omega}^2}{4\omega^3} + \omega \right),$$

where we replaced \dot{r}_i with the expression in (17) and I with C/ω .

Now consider Sundman’s inequality

$$(20) \quad 2TI - J^2 \geq |C|^2,$$

where

$$(21) \quad J = \sum_{i=1}^N m_i \mathbf{r}_i \cdot \dot{\mathbf{r}}_i = \sum_{i=1}^N m_i r_i \dot{r}_i$$

(see [1, 14] for a derivation). Using (17), we can write

$$(22) \quad J = - \sum_{i=1}^N m_i r_i^2 \frac{\dot{\omega}}{2\omega} = -I \frac{\dot{\omega}}{2\omega}$$

and since $C = I\omega$, we get

$$(23) \quad J = -\frac{\dot{\omega}}{2\omega^2} C.$$

With the help of equations (19) and (23), we can write the left-hand side of Sundman's inequality as

$$(24) \quad 2TI - J^2 = \frac{C^2}{\omega} \left(\frac{\dot{\omega}^2}{4\omega^3} + \omega \right) - \left(-\frac{\dot{\omega}^2 C}{2\omega^2} \right)^2 = C^2,$$

which becomes an equality in this case. Therefore the solutions are homographic (as proved in [1]) and since U is homogeneous of degree $\alpha \neq -2$ ([1], Proposition 2.14), they are either solutions with central configurations or rigid motions. In the latter case the moment of inertia I is constant and we can show as in Corollary 1 that the solutions are relative equilibria. This concludes the alternative proof.

Let us remark that the homogeneity of the potential is an essential hypothesis of the theorem since there are quasi-homogeneous potentials (see [4]), in particular the Lennard-Jones potential, which have central configurations that depend on the mutual distances and hence, in those cases, different and more complex types of motions might occur.

4. A GEOMETRICAL INTERPRETATION

In this section we will give a geometrical interpretation of the above results in the Newtonian case. Our discussions with Don Saari revealed that he had developed a similar geometric understanding of the problem long before we found ours.

The moment of inertia can be written in terms of the relative distances as

$$(25) \quad I = \frac{1}{2M} \sum_{i=1}^n \sum_{j=1}^n m_i m_j r_{ij}^2.$$

For three collinear bodies, the potential energy and the moment of inertia contain only three terms, so it is natural to try to represent the manifolds $I = \text{constant}$ and $U = \text{constant}$ in the space of relative distances.

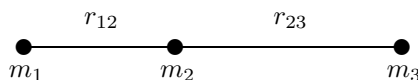


FIGURE 2. Three bodies on a line for a fixed ordering of the masses.

Moreover since the bodies are on a line, they are subject to additional constraints. For instance if one fixes the ordering of the masses as in Figure 2, the condition is

that $r_{13} = r_{12} + r_{23}$, which means that the bodies are confined to a plane in the space of relative distances. Since three bodies can be ordered in three ways, the corresponding conditions define three planes in the space of mutual distances, as we illustrate in Figure 3, which depicts the intersection of the manifolds $U = \text{constant}$, $I = \text{constant}$, and the three planes that give the collinear configurations. It is now easy to see that, once ordering the bodies, the collinear relative equilibria correspond to the case in which the intersection of $U = \text{constant}$ and $I = \text{constant}$ restricted to the plane is just one point.

From the purely geometric point of view there are three possible cases for the outcome of the above intersection: empty, one point, or two points. But by Moulton's theorem we know that for each ordering there is only one relative equilibrium and thus the relative equilibrium must correspond to the case when the intersection of $I = \text{constant}$ and $U = \text{constant}$ is tangent to the plane (see Figure 3). This is an alternative proof of Saari's Conjecture in the collinear case of three bodies.

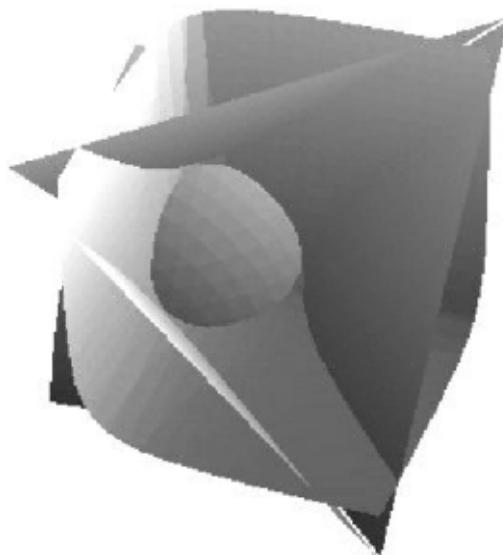


FIGURE 3. Intersection of the manifolds $U = \text{constant}$, $I = \text{constant}$ with the three planes in the case of equal masses.

However, this proof cannot be generalized to $n > 3$. Indeed, the geometry becomes more complicated as the number of bodies increases. It is first easy to find the following recurrence relation for the number $M(n)$ of the mutual distances of n bodies:

$$(26) \quad M(n+1) = M(n) + n.$$

Since $M(3) = 3$, this recurrence relation can be solved; it leads to the formula

$$(27) \quad M(n) = \frac{n(n-1)}{2}.$$

The number of linear relations that must be verified so that the n bodies are on a line can also be found. The recurrence relation that gives the number $R(n)$ of

linear relations is

$$(28) \quad R(n+1) = 2R(n) - R(n-1) + 1.$$

The initial conditions $R(2) = 0$ and $R(3) = 1$ lead to the formula

$$(29) \quad R(n) = \frac{n(n-1)}{2} - (n-1).$$

This means that we have $M(n)$ unknowns, $R(n)$ linear relations, and two more relations given by $U = \text{constant}$ and $I = \text{constant}$. Consequently if $n > 3$ the number of mutual distances is larger than the number of relations. In general this implies that the problem could have infinitely many solutions, unless the intersection is degenerate, in which case the intersection reduces to one point. As we have shown in the previous section, Saari's Conjecture is true in the general case, so the intersection is indeed one point. For $n > 3$, however, it is impossible to draw this conclusion only from geometrical considerations. But it is interesting to remark that the dynamics of the problem leads to this unlikely geometric configuration.

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NOTE ADDED IN PROOF

At a conference held at the beginning of June 2004 in Cala Gonone, Sardinia, the first author gave a talk on Saari's Conjecture. At the end of the talk, Rick Moeckel asked what happens in the rectilinear case (when all the bodies move on a fixed line). The author answered that collisions always occur, so the potential becomes infinite and the moment of inertia cannot be constant, therefore the conjecture is false. But he could not remember the collision proof on the spot. Rick Moeckel found an alternative argument the next day, using McGehee coordinates. The proof the first author thought about can be found on page 45 of

F. Diacu, Singularities of the N -body Problem, in *Classical and Celestial Mechanics—The Recife Lectures*, pp. 35–62, H. Cabral and F. Diacu, editors, Princeton Univ. Press, Princeton, NJ, 2002.

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