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POINCARÉ-HOPF INEQUALITIES

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Abstract. In this article the main theorem establishes the necessity and sufficiency of the Poincaré-Hopf inequalities in order for the Morse inequalities to hold. The convex hull of the collection of all Betti number vectors which satisfy the Morse inequalities for a pre-assigned index data determines a Morse polytope defined on the nonnegative orthant. Using results from network flow theory, a scheme is provided for constructing all possible Betti number vectors which satisfy the Morse inequalities for a pre-assigned index data. Geometrical properties of this polytope are described.

1. Introduction

The interplay between topology and dynamics dates back to Poincaré. In the early 20's, [7], Morse related the topology of the closed manifold M of dimension n to its dynamical data by a collection of inequalities. These Morse inequalities constitute a classical result which establishes relations between the number of non-degenerate critical points c_i of Morse index i of a smooth real valued function $f : M \to \mathbb{R}$ and the Betti numbers of M, $\gamma_i(M)$. This collection of inequalities are presented in (1.1) where one should read $h_i = c_i$. The function $f : M \to \mathbb{R}$ is called a Morse function and its gradient determines a smooth flow which we refer to as a Morse flow. In this sense, the inequalities can be viewed as relations between the number c_i of singularities of index i of the Morse flow and the Betti numbers of the phase space M.

In the early 70's, [3], Conley generalized these results to a theory with a more topological flavor and independent of the differentiable nature of the flow. In [3] the existence of a Lyapunov function associated to a flow on a manifold is proved. With respect to this function the flow maintains an underlying gradient-like behavior. However, in this setting, the dynamics is much richer and singularities can be exchanged for richer invariant sets (isolated invariant sets) for which the index introduced by Conley can be computed.

In this article, we assume that the flow on a closed connected orientable manifold M has a finite component chain recurrent set $\mathcal{R}$, where each component R_k is an

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isolated invariant set. We will work with the $\mathbb{Z}_2$ homology Conley index of R_k and denote its dimensions by $(h_0, \dots, h_n)_k$. In [3] it is proven that, given a continuous flow ϕ_t on M with $\mathcal{R} = \bigcup R_k$ and $h_j = \sum_k (h_j)_k$, the following generalized Morse inequalities hold:

(1.1)

$$\begin{array}{rll}
\gamma_n - \gamma_{n-1} + - \dots \pm \gamma_2 \pm \gamma_1 \pm \gamma_0 &= h_n - h_{n-1} + - \dots \pm h_2 \pm h_1 \pm h_0 & \text{(n)} \\
\gamma_{n-1} - \gamma_{n-2} + - \dots \pm \gamma_2 \pm \gamma_1 \pm \gamma_0 &\leq h_{n-1} - h_{n-2} + - \dots \pm h_2 \pm h_1 \pm h_0 & \text{(n} - 1) \\
\vdots & \vdots & \\
\gamma_j - \gamma_{j-1} + - \dots \pm \gamma_2 \pm \gamma_1 \pm \gamma_0 &\leq h_j - h_{j-1} + - \dots \pm h_2 \pm h_1 \pm h_0 & \text{(j)} \\
\gamma_{j-1} - \gamma_{j-2} + - \dots \pm \gamma_2 \pm \gamma_1 \pm \gamma_0 &\leq h_{j-1} - h_{j-2} + - \dots \pm h_2 \pm h_1 \pm h_0 & \text{(j} - 1) \\
\vdots & \vdots & \\
\gamma_2 - \gamma_1 + \gamma_0 &\leq h_2 - h_1 + h_0 & (2) \\
\gamma_1 - \gamma_0 &\leq h_1 - h_0 & (1) \\
\gamma_0 &\leq h_0 & (0)
\end{array}$$

We note that in this generalization the left-hand side continues to represent the topology of the phase space and the right-hand side the dynamics of the flow.

The novelty in this paper is a collection of inequalities which provides constraints on the dynamics without involving the topology of the manifold M, in other words without reference to the Betti numbers of M. We refer to these inequalities as the Poincaré-Hopf inequalities for closed manifolds, (1.2), (1.3) and (1.4), where h_j is as defined in the previous paragraph.

$$(1.2) \quad \left\{ \begin{array}{ll}
\left\{ \begin{array}{l}
n = 2i+1 \left\{ \begin{array}{l} -h_i \leq (h_{i+2} - h_{i-1}) - (h_{i+3} - h_{i-2}) + - \dots \\ \qquad \pm (h_{2i} - h_1) \pm (h_{2i+1} - h_0) \ \leq \ h_{i+1} \end{array} \right. \\[2ex]
n = 2i \left\{ \begin{array}{l} -h_i \leq (h_{i+1} - h_{i-1}) - (h_{i+2} - h_{i-2}) + - \dots \\ \qquad \pm (h_{2i-2} - h_2) \pm (h_{2i} - h_0) \ \leq \ h_i \end{array} \right.
\end{array} \right. & \text{(i)} \\[4ex]
\left\{ \begin{array}{l}
-h_j \leq (h_{n-(j-1)} - h_{j-1}) - (h_{n-(j-2)} - h_{j-2}) + - \dots \\ \qquad \pm (h_{n-1} - h_1) \pm (h_n - h_0) \ \leq \ h_{n-j} \\
\quad \vdots \\
-h_2 \leq (h_{n-1} - h_1) - (h_n - h_0) \ \leq \ h_{n-2} \\[2ex]
\left\{ \begin{array}{l} h_1 \geq h_0 - 1 \\ h_{n-1} \geq h_n - 1 \end{array} \right.
\end{array} \right. & \begin{array}{l} \text{(j)} \\ \\ \\ (2) \\ \\ (1) \end{array}
\end{array} \right.$$

In the case $n = 2i + 1$ we have

$$(1.3) \qquad \sum_{j=0}^{2i+1} (-1)^j h_j = 0$$

and in the case $n = 2i \equiv 2 \mod 4$ we have

$$(1.4) \qquad h_i - \sum_{j=0}^{i-1} (-1)^j (h_{2i-j} - h_j) \ \text{ be even.}$$

Surprisingly, it turns out that, whenever these inequalities are satisfied for a pre-assigned dynamical data $(h_0, \dots, h_n)$, it can be shown that there exists a collection of Betti numbers that satisfy the Morse inequalities with this same data.

Conversely, if the Morse inequalities are satisfied for $(h_0, \ldots, h_n)$ and $(\gamma_0, \ldots, \gamma_n)$, then $(h_0, \ldots, h_n)$ satisfies the Poincaré-Hopf inequalities. This is stated in Theorem 1.1 and is proved in Section 3.

A nonnegative integral vector $(\gamma_0, \gamma_1, \ldots, \gamma_{n-1}, \gamma_n)$ satisfying $\gamma_{n-k} = \gamma_k$, for $k = 0, \ldots, n$, $\gamma_0 = \gamma_n = 1$ (and in some cases it will be required that $\gamma_{n/2}$ be even if n is even), is called a Betti number vector.

Theorem 1.1. *A set of nonnegative numbers* $(h_0, h_1, \ldots, h_n)$ *satisfies the Poincaré-Hopf inequalities in* (1.2) *if and only if it satisfies the Morse inequalities* (1.1) *for some Betti number vector* $(\gamma_0, \gamma_1, \ldots, \gamma_{n-1}, \gamma_n)$.

This result is not merely a change of inequalities. One should note that the Morse inequalities involve $(h_0, \ldots, h_n)$ and $(\gamma_0, \ldots, \gamma_n)$, whereas the Poincaré-Hopf inequalities only involve $(h_0, \ldots, h_n)$.

This theorem has many applications. In particular, it can be used to obtain partial answers to the question of realizability of abstract Lyapunov graphs $L(h_0, \ldots, h_n)$ as flows on manifolds. Note that abstract Lyapunov graphs carry dynamical data and local topological invariants of level sets but no global topological information of the manifold on which it can be realized. Hence, one cannot verify the Morse inequalities for abstract Lyapunov graphs, however, we can verify the Poincaré-Hopf inequalities. For more details see [1] and [2].

In some sense the Poincaré-Hopf inequalities pre-process admissible data, that is, if $(h_0, \ldots, h_n)$ does not satisfy the Poincaré-Hopf inequalities, there is no closed n-manifold which admits $(h_0, \ldots, h_n)$ as its dynamical data. This follows from Theorem 1.1 and from the classical results of Morse [7].

The Poincaré-Hopf inequalities can also be used to prove the existence of critical points of index k from a priori knowledge of the existence of critical points of lower index and their duals. That is, these inequalities can also be used to give bounds on the numbers h_j with respect to alternating sums of h_s with $s < j$ and their duals h_{n-s}. In the case of Morse flows these inequalities provide bounds on the number of singularities c_j of Morse index j with respect to alternating sums of c_s with $s < j$ and their duals c_{n-s}.

In [1] the Poincaré-Hopf inequalities in all generality are introduced for flows on isolating blocks N and their Lyapunov graph L_N in order to ensure the continuation of L_N to a Morse type Lyapunov graph. These inequalities involve the Betti numbers of the exiting and entering boundaries of N. We refer to these inequalities as the Poincaré-Hopf inequalities for isolating blocks which will be presented in Section 2. We show in [2] that a particular case of the Poincaré-Hopf inequalities for isolating blocks (2.3), (2.4) and (2.5) are the Poincaré-Hopf inequalities for closed manifolds (1.2), (1.3) and (1.4).

An important role in the proofs of the results presented in this paper is played by a more elaborate classification of singularities that provides not only the information on its index, but also on its connectivity type. Given a nondegenerate singularity, a classical approach is to associate its Morse index j to it. More generally, one can associate to it the dimensions of the Conley homology indices, $h_j = 1$ and $h_k = 0$ for all $k \neq j$. In [4] these singularities are classified not only by their index, but also by the effect caused on the Betti numbers of the entering and exiting boundaries, N^+ and N^-, of an isolating block N containing the singularity. In other words, a singularity of Morse index j can increase (resp., decrease) the j-th (resp., $(j-1)$-th) Betti number of N^+ with respect to the j-th (resp., $(j-1)$-th) Betti number of

N^-. We refer to this singularity as j-disconnecting, or for short j-d (resp., $(j-1)$-connecting, or for short $(j-1)$-c). In the case $n = 2i$, a singularity of index i is β-i, if all Betti numbers are kept constant. If N is orientable it is shown in [4] that β-i occurs only in the case $n = 2i \equiv 0 \mod 4$.

The crucial step in the proof of Theorem 1.1 is to introduce a linear system, henceforth called an h^{cd}-system, given by (1.5) and (1.6). This system can be characterized by the dimensions of the Conley homology indices $(h_0, \ldots, h_n)$ and its unknowns are precisely $(h_1^c, h_1^d, \ldots, h_{n-1}^c, h_{n-1}^d)$, where h_j^d (resp., h_j^c) are the number of singularities $h_j = 1$ of type j-disconnecting (resp., of type $(j-1)$-connecting). Nonnegative integer solutions to this system correspond to different ways one can choose h_j^c and h_j^d for a pre-assigned index data $(h_0, \ldots, h_n)$.

This system was defined in a more general setting by a continuation algorithm in [1] and appears in Section 2 as (2.6) and (2.7). In [1] it was shown that (2.6) and (2.7) have a nonnegative solution if and only if the Poincaré-Hopf inequalities (2.3), (2.4) and (2.5) are satisfied. This is the subject of Proposition 2.1 stated in Section 2. The following corollary is proven to be a direct consequence of Proposition 2.1 in [2].

Corollary 1.2. *The systems* (1.5) *and* (1.6) *have nonnegative integral solutions* $(h_1^c, h_1^d, \ldots, h_{n-1}^c, h_{n-1}^d)$ *if and only if the Poincaré-Hopf inequalities* (1.2), (1.3), (1.4)*, for isolating blocks are satisfied.*

$$n = 2i+1 \begin{cases} h_0 - 1 - h_1^c = 0 \\ \{h_j = h_j^c + h_j^d, \quad j = 1, \ldots, 2i \\ h_{2i+1} - 1 - h_{2i}^d = 0 \\ \begin{cases} h_1^d - h_2^c - h_{2i}^c + h_{2i-1}^d = 0 \\ h_2^d - h_3^c - h_{2i-1}^c + h_{2i-2}^d = 0 \\ \vdots \\ h_i^d - h_{i+1}^c = 0 \end{cases} \end{cases} \tag{1.5}$$

$$n = 2i \begin{cases} h_0 - 1 - h_1^c = 0 \\ \{h_j = h_j^c + h_j^d + \beta^i, \quad j = 1, \ldots, 2i-1, \beta^i = 0 \textit{ if } j \neq i \textit{ and } 2i \not\equiv 0 \mod 4 \\ h_{2i} - 1 - h_{2i-1}^d = 0 \\ \begin{cases} h_1^d - h_2^c - h_{2i-1}^c + h_{2i-2}^d = 0 \\ h_2^d - h_3^c - h_{2i-2}^c + h_{2i-3}^d = 0 \\ \vdots \\ h_{i-1}^d - h_i^c - h_{i+1}^c + h_i^d = 0 \end{cases} \end{cases} \tag{1.6}$$

We also show in [1] that (2.6) and (2.7) and consequently the h^{cd}-system above, constitute a network-flow problem with possible additional constraints. The nature of the network involved allows a complete characterization of all possible solutions of the h^{cd}-system by means of one particular (complementary) solution of the system $(h_1^c, h_1^d, \ldots, h_{n-1}^c, h_{n-1}^d)$ and the simple circulations of the network.

A natural problem to consider is the calculation of all possible Betti number vectors that satisfy the Morse inequalities for a given pre-assigned index data $(h_0, \ldots, h_n)$. We furnish a method that makes this possible by establishing a relationship between the h^{cd} solutions of system (1.5), or (1.6), and the desired Betti number vectors. Thus the complementary solution of the h^{cd}-system and the simple circulations are enough to obtain all Betti number vectors that satisfy the Morse inequalities.

Since the nonnegative solutions of the Morse inequalities and the boundary and Poincaré duality conditions (3.6) and (3.7) determine a polyhedron, it is also natural to investigate how the integral vectors satisfying all these constraints (the Betti number vectors) relate to the whole polyhedron. First of all it is shown that this polyhedron is in fact limited, and henceforth called the Morse polytope. It is known that a polytope is the convex hull of its extreme points, or vertices, and hence completely characterized by its vertices. Frequently however, the convex hull of the integer valued vectors in the polytope is strictly (properly) contained in the polytope. It is surprising to verify that this is not the case here, since all vertices of the polytope considered are integral. In other words, the Morse polytope coincides with the convex hull of the Betti number vectors that satisfy the Morse inequalities.

Another surprising fact is the close relationship between the two sets of vectors: the nonnegative h^{cd}'s that solve the h^{cd}-system (2.6) or (2.7) and the Betti number vectors that solve the Morse inequalities, that, together with the characterization obtained for the Morse polytope, permits us to obtain all the Betti number vectors from the h^{cd} vectors and vice-versa.

This article is divided in the following sections. Section 2 will summarize the Poincaré-Hopf inequalities for isolating blocks obtained in [1]. In Section 3 we prove Theorem 1.1 in several stages: Subsection 3.1 in the $n = 2i + 1$ dimensional case; Subsection 3.2 in the $n = 2i \equiv 0 \mod 4$ dimensional case and in Subsection 3.3 the $n = 2i \equiv 2 \mod 4$ dimensional case. Lastly, Section 4 describes the Morse polytope $\mathcal{P}$ and presents additional geometric properties thereof. Sections 3 and 4 rely on results from integer programming theory and network flow theory.

2. Poincaré-Hopf inequalities

In this section we discuss the Poincaré-Hopf inequalities for isolating blocks. For more details see [1].

A set $S \subset M$ is invariant if $\phi_t(S) = S$ for all $t \in \mathbb{R}$. A compact set $N \subset M$ is an *isolating neighborhood* if $\text{inv}(N, \phi) = \{x \in N : \phi_t(x) \subset N,\ \forall\ t \in \mathbb{R}\} \subset \text{int}N$. A compact set N is an *isolating block* if $N^- = \{x \in N : \phi_{[0,t)}(x) \not\subset N, \forall t > 0\}$ is closed and $\text{inv}(N, \phi) \subset \text{int } N$. An invariant set S is called an *isolated invariant set* if it is a maximal invariant set in some isolating neighborhood N, that is, $S = \text{inv}(N, \phi)$.

A component R of $\mathcal{R}$ of the flow ϕ_t is an example of an invariant set. We will work under the hypothesis that $\mathcal{R}$ is the finite union of isolated invariant sets R_i. If f is a Lyapunov function associated to a flow and $c = f(R)$, then for $\varepsilon > 0$, the component of $f^{-1}[c - \varepsilon, c + \varepsilon]$ that contains R is an isolating neighborhood for R. Take

$$(N, N^-) = (f^{-1}[c - \varepsilon, c + \varepsilon], f^{-1}(c - \varepsilon))$$

as an index pair for R. The Conley index is defined as the homotopy type of N/N^-. Its homology is denoted by $CH_*(S)$ and its rank denoted by $h_* = \text{rank } CH_*(S)$. For more details see [3].

The Poincaré-Hopf inequalities for an isolated invariant set Λ in an isolating block N with entering set for the flow N^+ and exiting set for the flow N^- are obtained by analysis of long exact sequences of (N, N^+) and (N, N^-). This analysis can be found in [1].

Note that (N, N^-) is an index pair for Λ and (N, N^+) is an index pair for the isolated invariant set of the reverse flow, Λ'.

Consider the long exact sequences for the pairs (N, N^-) and (N, N^+):

$$0 \to H_n(N^-) \xrightarrow{i_n} H_n(N) \xrightarrow{p_n} H_n(N,N^-) \xrightarrow{\partial_n} H_{n-1}(N^-) \xrightarrow{i_{n-1}} \tag{2.1}$$

$$\to H_{n-1}(N) \xrightarrow{p_{n-1}} H_{n-1}(N,N^-) \xrightarrow{\partial_{n-1}} H_{n-2}(N^-) \xrightarrow{i_{n-2}}$$

$$\to H_{n-2}(N) \xrightarrow{p_{n-2}} H_{n-2}(N,N^-) \xrightarrow{\partial_{n-2}} \dots \xrightarrow{\partial_4} H_3(N^-) \xrightarrow{i_3}$$

$$\to H_3(N) \xrightarrow{p_3} H_3(N,N^-) \xrightarrow{\partial_3} H_2(N^-) \xrightarrow{i_2}$$

$$\to H_2(N) \xrightarrow{p_2} H_2(N,N^-) \xrightarrow{\partial_2} H_1(N^-) \xrightarrow{i_1}$$

$$\to H_1(N) \xrightarrow{p_1} H_1(N,N^-) \xrightarrow{\partial_1} H_0(N^-) \xrightarrow{i_0} H_0(N) \xrightarrow{p_0} H_0(N,N^-) \to 0$$

$$0 \to H_n(N^+) \xrightarrow{i'_n} H_n(N) \xrightarrow{p'_n} H_n(N,N^+) \xrightarrow{\partial'_n} H_{n-1}(N^+) \xrightarrow{i'_{n-1}} \tag{2.2}$$

$$\to H_{n-1}(N) \xrightarrow{p'_{n-1}} H_{n-1}(N,N^+) \xrightarrow{\partial'_{n-1}} H_{n-2}(N^+) \xrightarrow{i'_{n-2}}$$

$$\to H_{n-2}(N) \xrightarrow{p'_{n-2}} H_{n-2}(N,N^+) \xrightarrow{\partial'_{n-2}} \dots \xrightarrow{\partial'_4} H_3(N^+) \xrightarrow{i'_3}$$

$$\to H_3(N) \xrightarrow{p'_3} H_3(N,N^+) \xrightarrow{\partial'_3} H_2(N^+) \xrightarrow{i'_2}$$

$$\to H_2(N) \xrightarrow{p'_2} H_2(N,N^+) \xrightarrow{\partial'_2} H_1(N^+) \xrightarrow{i'_1}$$

$$\to H_1(N) \xrightarrow{p'_1} H_1(N,N^+) \xrightarrow{\partial'_1} H_0(N^+) \xrightarrow{i'_0} H_0(N) \xrightarrow{p'_0} H_0(N,N^+) \to 0$$

We will assume throughout our analysis that the Conley duality condition on the indices holds. That is, the isolated invariant sets Λ and Λ' have the property that

$$\operatorname{rank} H_j(N,N^-) = h_j \quad \text{and} \quad \operatorname{rank} H_j(N,N^+) = \overline{h}_j = h_{n-j}.$$

Denote $\operatorname{rank} H_0(N^-) = e^-$, $\operatorname{rank} H_0(N^+) = e^+$ and $\operatorname{rank}(H_j(N^{\pm})) = B_j^{\pm}$.

By simultaneously analyzing the following pairs of maps

$$\{[(p_i,\partial'_i),(p'_i,\partial_i)],\dots,[(p_2,\partial'_2),(p'_2,\partial_2)]\},$$

and analyzing p_1 and p'_1 we obtain the Poincaré-Hopf inequalities in all its generality, where h_j is the dimension of the homology Conley index and $B_j^- = \sum_{i=1}^{e^-}(\beta_j^-)_i$,

$B_j^+ = \sum_{i=1}^{e^+} (\beta_j^+)_i$, where $j \in \{1, \dots, n-2\}$.

(2.3)

$$\begin{cases} n = 2i+1 \begin{cases} h_i \geq -(B_{i-1}^+ - B_{i-1}^-) + (B_{i-2}^+ - B_{i-2}^-) + - \dots \pm (B_2^+ - B_2^-) \\ \qquad \pm (B_1^+ - B_1^-) - (h_{i+2} - h_{i-1}) + (h_{i+3} - h_{i-2}) + - \dots \\ \qquad \pm (h_{2i-1} - h_2) \pm (h_{2i} - h_1) \pm [(h_{2i+1} - h_0) + (e^+ - e^-)] \\ \\ h_{i+1} \geq -\left[-(B_{i-1}^+ - B_{i-1}^-) + (B_{i-2}^+ - B_{i-2}^-) + - \dots \pm (B_2^+ - B_2^-) \right. \\ \qquad \pm (B_1^+ - B_1^-) - (h_{i+2} - h_{i-1}) + (h_{i+3} - h_{i-2}) + - \dots \\ \qquad \left. \pm (h_{2i-1} - h_2) \pm (h_{2i} - h_1) \pm [(h_{2i+1} - h_0) + (e^+ - e^-)] \right] \end{cases} \\ \\ n = 2i \begin{cases} h_i \geq -(B_{i-1}^+ - B_{i-1}^-) + (B_{i-2}^+ - B_{i-2}^-) + - \dots \pm (B_2^+ - B_2^-) \\ \qquad \pm (B_1^+ - B_1^-) - (h_{i+1} - h_{i-1}) + (h_{i+2} - h_{i-2}) + - \dots \\ \qquad \pm (h_{2i-2} - h_2) \pm (h_{2i-1} - h_1) \pm [(h_{2i} - h_0) + (e^+ - e^-)] \\ \\ h_i \geq -\left[-(B_{i-1}^+ - B_{i-1}^-) + (B_{i-2}^+ - B_{i-2}^-) + - \dots \pm (B_2^+ - B_2^-) \right. \\ \qquad \pm (B_1^+ - B_1^-) - (h_{i+1} - h_{i-1}) + (h_{i+2} - h_{i-2}) + - \dots \\ \qquad \left. \pm (h_{2i-2} - h_2) \pm (h_{2i-1} - h_1) \pm [(h_{2i} - h_0) + (e^+ - e^-)] \right] \end{cases} \\ \qquad \vdots \\ \begin{cases} h_j \geq -(B_{j-1}^+ - B_{j-1}^-) + (B_{j-2}^+ - B_{j-2}^-) + - \dots \pm (B_2^+ - B_2^-) \pm (B_1^+ - B_1^-) \\ \qquad -(h_{n-(j-1)} - h_{j-1}) + (h_{n-(j-2)} - h_{j-2}) + - \dots \\ \qquad \pm (h_{n-1} - h_1) \pm [(h_n - h_0) + (e^+ - e^-)] \\ \\ h_{n-j} \geq -\left[-(B_{j-1}^+ - B_{j-1}^-) + (B_{j-2}^+ - B_{j-2}^-) + - \dots \pm (B_2^+ - B_2^-) \right. \\ \qquad \pm (B_1^+ - B_1^-) - (h_{n-(j-1)} - h_{j-1}) + (h_{n-(j-2)} - h_{j-2}) + - \dots \\ \qquad \left. \pm (h_{n-1} - h_1) \pm [(h_n - h_0) + (e^+ - e^-)] \right] \end{cases} \\ \qquad \vdots \\ \begin{cases} h_2 \geq -(B_1^+ - B_1^-) - (h_{n-1} - h_1) + (h_n - h_0) + (e^+ - e^-) \\ h_{n-2} \geq -\left[-(B_1^+ - B_1^-) - (h_{n-1} - h_1) + (h_n - h_0) + (e^+ - e^-) \right] \end{cases} \\ \\ \begin{cases} h_1 \geq h_0 - 1 + e^- \\ h_{n-1} \geq h_n - 1 + e^+ \end{cases} \end{cases}$$

Furthermore, the Poincaré-Hopf equality must be considered in the odd-dimensional case $n = 2i + 1$:

$$\mathcal{B}^+ - \mathcal{B}^- = e^- - e^+ + \sum_{j=0}^{2i+1} (-1)^j h_j, \tag{2.4}$$

where

$$\mathcal{B}^+ = \frac{(-1)^i}{2} B_i^+ \pm B_{i-1}^+ \pm \dots - B_1^+,$$
$$\mathcal{B}^- = \frac{(-1)^i}{2} B_i^- \pm B_{i-1}^- \pm \dots - B_1^-.$$

Moreover, in the even-dimensional case $n \equiv 2 \mod 4$, the condition that

$$h_i - \sum_{j=1}^{i-1}(-1)^{j+1}(B_j^+ - B_j^-) - \sum_{j=0}^{i-1}(-1)^j(h_{2i-j} - h_j) + (e^- - e^+) \text{ be even} \tag{2.5}$$

must be imposed.

The Poincaré-Hopf inequalities for isolating blocks will be the collection of constraints (2.3)–(2.5).

In [1] it is shown that

Proposition 2.1. *The systems* (2.6) *and* (2.7) *have nonnegative integral solutions* $(h_1^c, h_1^d, \ldots, h_{n-1}^c, h_{n-1}^d)$ *if and only if the Poincaré-Hopf inequalities* (2.3)*,* (2.4) *and* (2.5) *are satisfied.*

$$n = 2i+1 \left\{ \begin{array}{l} e^- - 1 - h_1^c = 0 \\ \{h_j = h_j^c + h_j^d, \ \ j = 1, \ldots, 2i \\ e^+ - 1 - h_{2i}^d = 0 \\ \left\{ \begin{array}{l} -(B_1^+ - B_1^-) + h_1^d - h_2^c - h_{2i}^c + h_{2i-1}^d = 0 \\ -(B_2^+ - B_2^-) + h_2^d - h_3^c - h_{2i-1}^c + h_{2i-2}^d = 0 \\ \vdots \\ \frac{-(B_i^+ - B_i^-)}{2} + h_i^d - h_{i+1}^c = 0 \end{array} \right. \end{array} \right. \tag{2.6}$$

(2.7)

$$n = 2i \left\{ \begin{array}{l} e^- - 1 - h_1^c = 0 \\ \{h_j = h_j^c + h_j^d + \beta^i, \ \ j = 1, \ldots, 2i-1, \beta^i = 0 \textit{ if } j \neq i \textit{ and } 2i \not\equiv 0 \mod 4 \\ e^+ - 1 - h_{2i-1}^d = 0 \\ \left\{ \begin{array}{l} -(B_1^+ - B_1^-) + h_1^d - h_2^c - h_{2i-1}^c + h_{2i-2}^d = 0 \\ -(B_2^+ - B_2^-) + h_2^d - h_3^c - h_{2i-2}^c + h_{2i-3}^d = 0 \\ \vdots \\ -(B_{i-1}^+ - B_{i-1}^-) + h_{i-1}^d - h_i^c - h_{i+1}^c + h_i^d = 0 \end{array} \right. \end{array} \right.$$

By taking $B_j^+ = B_j^- = 0$, for all $j \neq 0, n-1$, and $B_j^+ = B_j^- = 1$, for $j = 0, n-1$, $e^+ = h_n$, $e^- = h_0$, we have the Poincaré-Hopf inequalities for closed manifolds (1.2), (1.3) and (1.4), and Corollary 1.2. For details see [2].

3. Equivalence results

In this section we prove Theorem 1.1 in several stages: Subsection 3.1 in the $n = 2i+1$ dimensional case; Subsection 3.2 in the $n = 2i \equiv 0 \mod 4$ dimensional case and in Subsection 3.3 the $n = 2i \equiv 2 \mod 4$ dimensional case. Throughout the exposition we assume the result given by Corollary 1.2.

In the following we show that the h^{cd}-system has a nonnegative integral solution if and only if there exist nonnegative Betti number vectors that satisfy the Morse inequalities (1.1). In the case $n \equiv 0 \mod 4$ one additional hypothesis, namely that the pre-assigned index data $(h_0, \ldots, h_n)$ satisfy the condition that $\sum_{j=0}^n (-1)^{j+1} h_j$ be even, is needed. Furthermore, the results are shown in a constructive fashion, that is, a recipe is given for constructing a nonnegative Betti number vector satisfying (1.1) from a nonnegative integral solution to the appropriate h^{cd}-system and vice versa.

3.1. **Case n odd.** Suppose there are nonnegative integers $(h_0, h_1, \ldots, h_{2i+1}, h_1^c, h_1^d, \ldots, h_{2i}^c, h_{2i}^d)$, where $i \geq 1$, that satisfy the linear system

$$\text{(3.1)} \quad \begin{cases} h_1^c = h_0 - 1, & \\ h_j^c + h_j^d = h_j, & \text{for } j = 1, \ldots, 2i, \\ h_{2i}^d = h_{2i+1} - 1, & \\ h_j^d - h_{j+1}^c + h_{2i-j}^d - h_{2i-j+1}^c = 0, & \text{for } j = 1, \ldots, i-1, \\ h_i^d - h_{i+1}^c = 0. & \end{cases}$$

Fix the nonnegative integers $(h_0, h_1, \ldots, h_{2i+1})$ that form part of a solution of (3.1). Then, for this fixed $(h_0, h_1, \ldots, h_{2i+1})$, there exists a nonnegative integral $(h_1^c, h_1^d, h_2^c, h_2^d, \ldots, h_{2i}^c, h_{2i}^d)$ that solves (3.1) and thus the equivalent system below, obtained by multiplying the odd equations in (3.1) by -1:

$$\text{(3.2)} \quad \begin{cases} -h_1^c = -(h_0 - 1), & \\ (-1)^{j+1}(h_j^c + h_j^d) = (-1)^{j+1} h_j, & \text{for } j = 1, \ldots, 2i, \\ h_{2i}^d = h_{2i+1} - 1, & \\ (-1)^j (h_j^d - h_{j+1}^c + h_{2i-j}^d - h_{2i-j+1}^c) = 0, & \text{for } j = 1, \ldots, i-1, \\ (-1)^i (h_i^d - h_{i+1}^c) = 0. & \end{cases}$$

It is shown in [1] that the h^{cd}-system (3.2) constitutes a (feasibility) network-flow problem: to find a nonnegative flow (h^{cd} vector) satisfying flow balance contraints (equations of the h^{cd}-system).[1] The coefficient matrix of (3.2) is the node-arc incidence matrix of a digraph. The rows are associated with the nodes of a digraph while the columns are associated with the arcs. If the j-th arc (column) leaves node k and enters node ℓ, then the j-th column has a -1 in row k, a 1 in row ℓ and zeros elsewhere. The j-th element of the right-hand-side vector of system (3.2) is the *node constant* associated with node j. Variables represent flows along arcs. The j-th equation states a flow balance condition: flow into node j - flow out of node j = node constant. The network contains a chain of $i-1$ cycles of length four. The arcs in the j-th cycle are associated with variables h_{j+1}^d, h_{2i-j}^c, h_{2i-j}^d and h_{j+1}^c, and the orientation of the first two arcs is opposite to the orientation of the last two, with respect to an arbitrary orientation of the cycle. The nodes in the j-th cycle are associated with equations $j+2$, $2i+1-j$, $2i+2+j$ and $2i+3+j$ of (3.2). Thus the node associated with the $(2i+2+j)$-th equation of (3.2) constitutes the intersection of cycles $j-1$ and j. The arc sequence associated with $(h_1^c, h_1^d, h_{2i}^c, h_{2i}^d)$, the variables still unaccounted for, form a nonoriented path that is adjacent to the first cycle. The intersection of this path and the first cycle is the node associated with equation $2i+3$ of (3.2). Arcs corresponding to flow variables $(h_1^c, h_1^d, h_2^c, h_2^d, \ldots, h_{2i}^c, h_{2i}^d)$, in this order, form an Eulerian nonoriented path covering the whole digraph. This path has a zig-zag shape in the planar embedding of the digraph exemplified in Figure 1 for the case $i = 3$. The node constants are written inside the respective nodes.

The network-flow problem (3.2) may be decomposed into i independent network-flow problems. Figure 2 depicts the networks arising from the decomposition of the generic $i = 3$ instance shown in Figure 1. Such a decomposition gives rise to one path-network and $i-1$ cycle-networks. The former has a unique solution or none at all, while the latter may admit infinite solutions. Notice that, if $(\bar{h}_2^c, \bar{h}_2^d, \bar{h}_5^c, \bar{h}_5^d)$

[1]The usual network-flow problem is an optimization, so one seeks the feasible flow that optimizes some linear function of the flow.

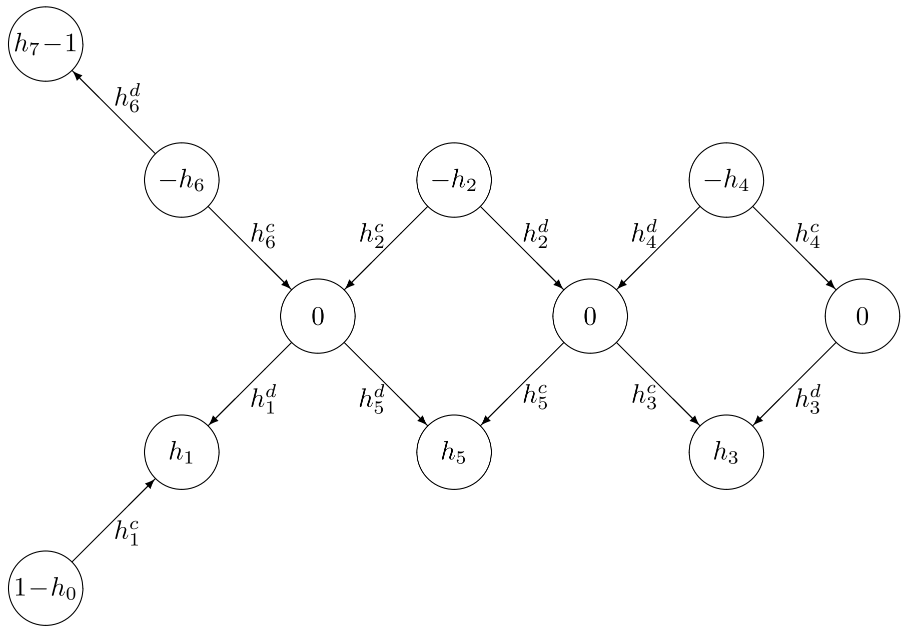


FIGURE 1. Network instance, for $i = 3$.

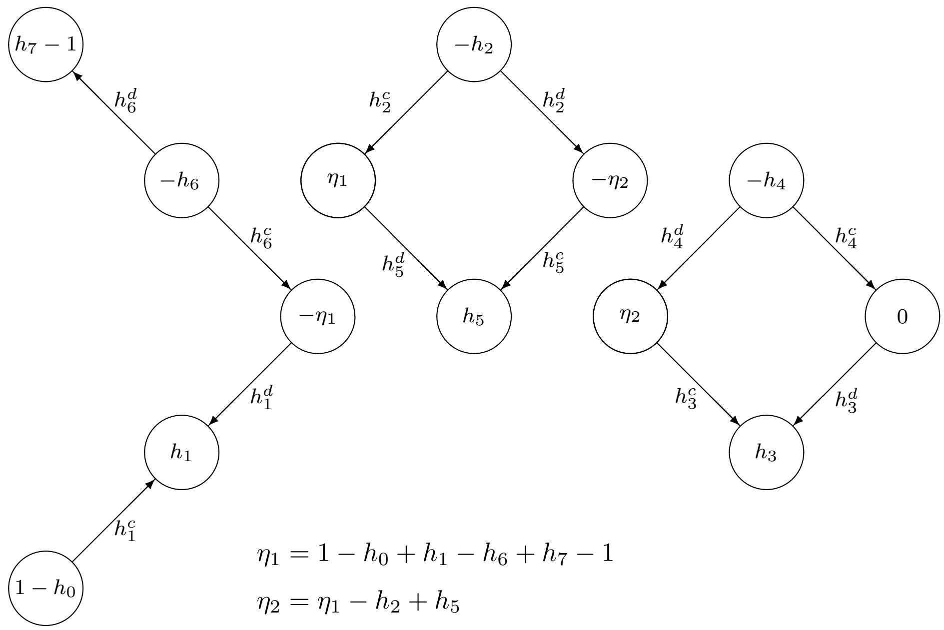
is a solution to the network-flow problem associated with the network in the middle of Figure 2, then $(\bar{h}_2^c, \bar{h}_2^d, \bar{h}_5^c, \bar{h}_5^d) + m(-1, 1, 1, -1)$ is yet another solution for $-\min\{\bar{h}_5^c, \bar{h}_2^d\} \leq m \leq \min\{\bar{h}_2^c, \bar{h}_5^d\}$. The flow $(h_2^c, h_2^d, h_5^c, h_5^d) = (-1, 1, 1, -1)$ is called a *circulation*, since it is a solution of the homogeneous version of the equation set of the network-flow problem, obtained by setting all node constants to zero. Thus a circulation satisfies the condition: flow into node j = flow out of node j, for every node of the network.

Lemma 3.1. *If the h^{cd}-system* (3.2) *has a nonnegative solution, then there exists a unique nonnegative integral solution that satisfies the complementarity condition*

$$h_j^c h_{n-j}^d = 0, \quad \textit{for } j = 2, \ldots, i. \tag{3.3}$$

Proof. It was shown in [1] that the existence of a nonnegative solution of (3.2) implies the existence of a nonnegative integral solution, say $\bar{h}^{cd}$, and that this solution may be partitioned into i subvectors, solutions of independent network-flow problems. If $(\bar{h}_j^c, \bar{h}_j^d, \bar{h}_{2i+1-j}^c, \bar{h}_{2i+1-j}^d)$ is a nonnegative integral solution of the j-th such subproblem (associated with the $(j-1)$-th cycle of the original network), for $2 \leq j \leq i$, then $(\bar{h}_j^c, \bar{h}_j^d, \bar{h}_{2i+1-j}^c, \bar{h}_{2i+1-j}^d) + \min\{\bar{h}_j^c, \bar{h}_{2i+1-j}^d\}(-1, 1, 1, -1)$ is yet another nonnegative integral solution of the subproblem which satisfies the complementarity condition (3.3). These updated subvectors make up the desired solution of the original network-flow problem, which is clearly unique. □

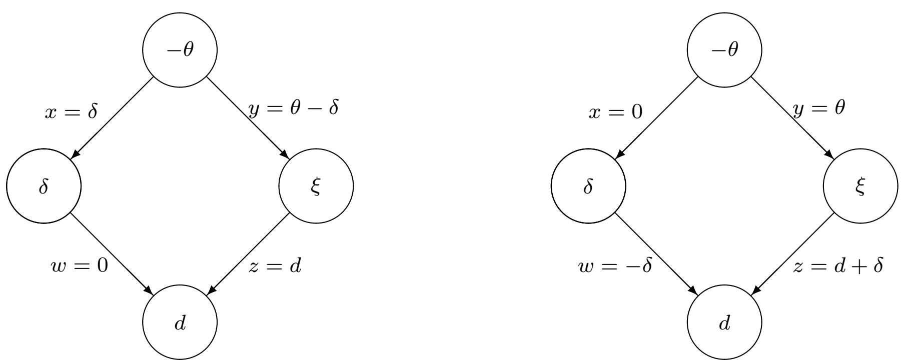
Lemma 3.1 implies that, if the generic cycle-network-flow problem of Figure 3 has a nonnegative solution, then either both solutions shown are equal and nonnegative (if $\delta = 0$) or precisely one of them is nonnegative and satisfies the complementarity condition (3.3), so the construction of such a solution is a straightforward task.



$$\eta_1 = 1 - h_0 + h_1 - h_6 + h_7 - 1$$

$$\eta_2 = \eta_1 - h_2 + h_5$$

FIGURE 2. Decomposition of network in Figure 1.



Hypothesis: $-\theta + \delta + d + \xi = 0$

FIGURE 3. Complementary flows of a generic cycle-network.

Furthermore, if the node constants are integral, the complementary solution also is. Such a solution is of special interest in the present work, in connection with the definition of the function $\Gamma : \mathbb{R}^{4i} \to \mathbb{R}^{2i+2}$. The vector $\Gamma(h^{cd})$ is defined as follows:

$$\Gamma_0(h^{cd}) = \Gamma_{2i+1}(h^{cd}) = 1,$$
$$\Gamma_j(h^{cd}) = \begin{cases} h^d_j - h^c_{j+1}, & \text{if } 1 \le j < i, \\ h^d_i, & \text{if } j = i, \\ h^c_{i+1}, & \text{if } j = i+1, \\ -h^d_{j-1} + h^c_j, & \text{if } i+2 \le j \le 2i. \end{cases} \tag{3.4}$$

Proposition 3.2. *Suppose that, for a given set of nonnegative integers* $(h_0, h_1, \dots, h_{2i}, h_{2i+1})$*, system* (3.2) *has a nonnegative solution. Let* h^{*cd} *be the nonnegative integral solution that satisfies the complementarity condition* (3.3)*. Then* $\Gamma(h^{*cd})$ *is a Betti number vector that satisfies the Morse inequalities* (1.1)*.*

Proof. It is convenient to rewrite the Morse inequalities (1.1) in the following more compact form:

$$\sum_{j=0}^{2i+1-k} (-1)^{j+1}\gamma_j \begin{cases} = \displaystyle\sum_{j=0}^{2i+1} (-1)^{j+1} h_j, & \text{if } k = 0, \\ \ge \displaystyle\sum_{j=0}^{2i+1-k} (-1)^{j+1} h_j, & \text{if } 1 \le k \le 2i+1,\ k \text{ odd}, \\ \le \displaystyle\sum_{j=0}^{2i+1-k} (-1)^{j+1} h_j, & \text{if } 1 \le k \le 2i+1,\ k \text{ even}. \end{cases} \tag{3.5}$$

We also recall the conditions a nonnegative integral vector must satisfy in order to be a Betti number vector:

$$\gamma_0 = \gamma_n = 1, \tag{3.6}$$
$$\gamma_j = \gamma_{n-j}, \quad \text{for } 1 \le j \le i. \tag{3.7}$$

Let $\gamma = \Gamma(h^{*cd})$. Integrality of γ follows easily from the integrality of h^{*cd}. Furthermore, equation $2i + 2 + j$ of (3.2) implies

a) If $1 \le j < i$

$$(-1)^j({h^*}^d_j - {h^*}^c_{j+1} + {h^*}^d_{2i-j} - {h^*}^c_{2i-j+1}) = (-1)^j(\gamma_j - \gamma_{2i+1-j}) = 0. \tag{3.8}$$

b) If $j = i$

$$(-1)^i({h^*}^d_i - {h^*}^c_{i+1}) = (-1)^i(\gamma_i - \gamma_{i+1}) = 0. \tag{3.9}$$

Equations (3.8), (3.9) and the definition of γ_0 and γ_{2i+1} given in (3.4) imply that the vector γ satisfies the boundary and duality conditions (3.6)–3.7).

Given that (3.7) is already established, the nonnegativity of γ is obtained if we show that either γ_j or γ_{2i+1-j}, for $0 \le j \le i$, is nonnegative. This is trivially true for $j = 0$ and i. Consider $1 \le j \le i - 1$. By definition and (3.7),

$$\gamma_j = {h^*}^d_j - {h^*}^c_{j+1} = -{h^*}^d_{2i-j} + {h^*}^c_{2i+1-j} = \gamma_{2i+1-j}.$$

From (3.3) we have that ${h^*}^c_{j+1}{h^*}^d_{2i-j} = 0$. If ${h^*}^c_{j+1} = 0$, then $\gamma_{2i+1-j} = \gamma_j = {h^*}^d_j - {h^*}^c_{j+1} = {h^*}^d_j \geq 0$. If ${h^*}^d_{2i-j} = 0$, then $\gamma_j = \gamma_{2i+1-j} = -{h^*}^d_{2i-j} + {h^*}^c_{2i+1-j} = {h^*}^c_{2i+1-j} \geq 0$. Therefore γ is nonnegative. In the following it will be shown that it also satisfies (3.5).

If the network-flow problem (3.2) has a solution, then the sum of the node constants must be zero, since the sum of the flow balance equations produces a zero term on the left-hand side (all columns of the coefficient matrix have precisely two nonzero elements, 1 and −1). Thus $0 = (1-h_0)+\sum_{j=1}^{2i}(-1)^{j+1}h_j+h_{2i+1}-1 = \sum_{j=0}^{2i+1}(-1)^{j+1}h_j$. Now, if γ satisfies (3.6) and (3.7), then

$$\begin{aligned}\sum_{j=0}^{2i+1}(-1)^{j+1}\gamma_j &= \sum_{j=0}^{i}(-1)^{j+1}\gamma_j + \sum_{j=i+1}^{2i+1}(-1)^{j+1}\gamma_j \\ &= \sum_{j=0}^{i}(-1)^{j+1}\gamma_j + \sum_{j=i+1}^{2i+1}(-1)^{j+1}\gamma_j \\ &= \sum_{j=0}^{i}(-1)^{j+1}\gamma_j + \sum_{k=0}^{i}(-1)^{k}\gamma_{2i+1-k} \\ &= \sum_{j=0}^{i}(-1)^{j+1}(\gamma_j - \gamma_{2i+1-j}) \\ &= 0.\end{aligned}$$

Therefore γ satisfies the first equation in (3.5).

Now consider the sum of equations 1 through $2i+2-\ell$ of (3.2), where $1 \leq \ell \leq 2i$:

$$\begin{aligned} &-{h^*}^c_1 + \sum_{j=1}^{2i+1-\ell}(-1)^{j+1}({h^*}^c_j + {h^*}^d_j) = \\ &{h^*}^d_1 + \sum_{j=2}^{2i+1-\ell}(-1)^{j+1}({h^*}^c_j + {h^*}^d_j) = \\ &\sum_{j=1}^{2i-\ell}(-1)^{j+1}{h^*}^d_j + \sum_{j=2}^{2i+1-\ell}(-1)^{j+1}{h^*}^c_j + (-1)^{2i+2-\ell}{h^*}^d_{2i+1-\ell} = \\ &\sum_{j=1}^{2i-\ell}(-1)^{j+1}({h^*}^d_j - {h^*}^c_{j+1}) + (-1)^{2i+2-\ell}{h^*}^d_{2i+1-\ell} = \\ &\qquad -(h_0-1) + \sum_{j=1}^{2i+1-\ell}(-1)^{j+1}h_j.\end{aligned}$$

The last equality implies

$$-1 + \sum_{j=1}^{2i-\ell}(-1)^{j+1}({h^*}^d_j - {h^*}^c_{j+1}) + (-1)^{2i+2-\ell}{h^*}^d_{2i+1-\ell} = \sum_{j=0}^{2i+1-\ell}(-1)^{j+1}h_j. \tag{3.10}$$

We consider the following three cases when calculating the partial alternate sum of the first $2i+2-\ell$ components of γ.

a) $1 \leq \ell \leq i$

$$\begin{aligned}\sum_{j=0}^{2i+1-\ell}(-1)^{j+1}\gamma_j &= -1+\sum_{j=1}^{i-1}(-1)^{j+1}({h^*}_j^d - {h^*}_{j+1}^c) + (-1)^{i+1}{h^*}_i^d + (-1)^{i+2}{h^*}_{i+1}^c \\ &\quad + \sum_{j=i+2}^{2i+1-\ell}(-1)^{j+1}(-{h^*}_{j-1}^d + {h^*}_j^c) \\ &= -1+\sum_{j=1}^{i}(-1)^{j+1}({h^*}_j^d - {h^*}_{j+1}^c) + \sum_{j=i+1}^{2i-\ell}(-1)^{j}(-{h^*}_j^d + {h^*}_{j+1}^c) \\ &= -1+\sum_{j=1}^{2i-\ell}(-1)^{j+1}({h^*}_j^d - {h^*}_{j+1}^c). \qquad (3.11)\end{aligned}$$

Substituting (3.11) in (3.10) we obtain

$$\sum_{j=0}^{2i+1-\ell}(-1)^{j+1}\gamma_j + (-1)^{2i+2-\ell}{h^*}_{2i+1-\ell}^d = \sum_{j=0}^{2i+1-\ell}(-1)^{j+1}h_j. \qquad (3.12)$$

Taking into account the nonnegativity of h^{*cd}, equation (3.12) implies

$$\sum_{j=0}^{2i+1-\ell}(-1)^{j+1}\gamma_j \begin{cases} \geq \sum_{j=0}^{2i+1-\ell}(-1)^{j+1}h_j, & \text{if } 1 \leq \ell \leq i,\ \ell \text{ odd},\\ \leq \sum_{j=0}^{2i+1-\ell}(-1)^{j+1}h_j, & \text{if } 1 \leq \ell \leq i,\ \ell \text{ even},\end{cases} \qquad (3.13)$$

which means that γ defined in (3.4) satisfies the inequalities in (3.5), for $1 \leq \ell \leq i$.

b) $\ell = i+1$

$$\begin{aligned}\sum_{j=0}^{2i+1-\ell}(-1)^{j+1}\gamma_j &= -1+\sum_{j=1}^{i-1}(-1)^{j+1}({h^*}_j^d - {h^*}_{j+1}^c) + (-1)^{i+1}{h^*}_i^d \\ &= -1+\sum_{j=1}^{2i-\ell}(-1)^{j+1}({h^*}_j^d - {h^*}_{j+1}^c) + (-1)^{2i+2-\ell}{h^*}_{2i+1-\ell}^d. \qquad (3.14)\end{aligned}$$

From (3.14) and (3.10) we conclude that, for $\ell = i+1$,

$$\sum_{j=0}^{i}(-1)^{j+1}\gamma_j = \sum_{j=0}^{i}(-1)^{j+1}h_j, \qquad (3.15)$$

and thus γ also satisfies (3.5) for $\ell = i+1$.

c) $i+2 \leq \ell \leq 2i$

$$\begin{aligned}\sum_{j=0}^{2i+1-\ell}(-1)^{j+1}\gamma_j &= -1+\sum_{j=1}^{2i+1-\ell}(-1)^{j+1}({h^*}_j^d - {h^*}_{j+1}^c) \\ &= -1+\sum_{j=1}^{2i-\ell}(-1)^{j+1}({h^*}_j^d - {h^*}_{j+1}^c) \qquad (3.16)\\ &\quad + (-1)^{2i+2-\ell}({h^*}_{2i+1-\ell}^d - {h^*}_{2i+2-\ell}^c).\end{aligned}$$

Using (3.16) and (3.10) we obtain

$$\sum_{j=0}^{2i+1-\ell}(-1)^{j+1}\gamma_j+(-1)^{2i+2-\ell}{h^*}^{c}_{2i+2-\ell}=\sum_{j=0}^{2i+1-\ell}(-1)^{j+1}h_j, \tag{3.17}$$

which implies (3.13), given the nonnegativity of ${h^*}^{cd}$.

The first equation in (3.2), $-{h^*}^{c}_{1}=-(h_0-1)$ and the fact that ${h^*}^{c}_{1}\geq 0$ imply $h_0\geq 1=\gamma_0$, so the last inequality in (3.5) is also true.

Thus we have established that γ defined in (3.4) satisfies constraints in (3.5). □

The next proposition establishes the converse of Proposition 3.2, i.e., if there is a Betti number vector that satisfies the Morse inequalities, then the h^{cd}-system has a nonnegative integral solution. The proof is also constructive, and, as before, it is based on the existence of particular solutions, namely Betti number vectors that satisfy (1.1) and saturate the middle inequality therein.

Lemma 3.3. *If γ is Betti number vector which satisfies the Morse inequalities* (1.1), *then γ^* where $\gamma^*_j=\gamma_j$ for $0\leq j\leq 2i+1$, $i\neq j\neq i+1$, and γ^*_i, γ^*_{i+1} satisfy*

$$\mathbb{Z}\ni\gamma^*_i=\gamma^*_{i+1}\in[0,U], \tag{3.18}$$

where

$$\begin{aligned}U&=(-1)^{i+1}\left(\sum_{j=0}^{i}(-1)^{j+1}h_j-\sum_{j=0}^{i-1}(-1)^{j+1}\gamma^*_j\right)\\&=h_i+(-1)^{i+1}\sum_{j=0}^{i-1}(-1)^{j+1}(h_j-\gamma^*_j)\end{aligned}$$

is a Betti number vector that solves (1.1).

In particular, if $\gamma^*_i=\gamma^*_{i+1}=U$, then γ^* saturates the middle inequality in (1.1), *i.e.,*

$$\sum_{j=0}^{i}(-1)^{j+1}\gamma^*_j=\sum_{j=0}^{i}(-1)^{j+1}h_j. \tag{3.19}$$

Proof. Taking into account the fact that all linear inequalities in (3.5) except the middle one contain the difference $\pm(\gamma_i-\gamma_{i+1})$ or contain neither γ_i nor γ_{i+1}, all inequalities in (3.5)–(3.7) are satisfied by γ^* as long as the following conditions hold:

$$\gamma^*_i=\gamma^*_{i+1}, \tag{3.20}$$

$$(-1)^{i+1}\sum_{j=0}^{i}(-1)^{j+1}\gamma^*_j\leq(-1)^{i+1}\sum_{j=0}^{i}(-1)^{j+1}h_j, \tag{3.21}$$

$$\gamma^*_i,\gamma^*_{i+1}\geq 0. \tag{3.22}$$

It follows that γ^* will be a Betti number vector that solves the Morse inequalities (1.1) as long as γ_i^* and γ_{i+1}^* are integers satisfying

$$\begin{aligned} 0 \le \gamma_i^* = \gamma_{i+1}^* &\le (-1)^{i+1}\left(\sum_{j=0}^{i}(-1)^{j+1}h_j - \sum_{j=0}^{i-1}(-1)^{j+1}\gamma_j^*\right) \\ &= h_i + (-1)^{i+1}\sum_{j=0}^{i-1}(-1)^{j+1}(h_j - \gamma_j^*) = U, \end{aligned}$$

which is the content of (3.18). The interval $[0, U]$ is nonempty (it contains the current value of γ_i) and has integral-valued endpoints, since γ is, by assumption, integral. Setting γ_i^* and γ_{i+1}^* equal to U we assure that γ^* is a Betti number vector that satisfies (1.1) and (3.19). □

The function $H^{cd} : \mathbb{R}^{2i+2} \to \mathbb{R}^{4i}$ will provide a construction rule for a solution of the h^{cd}-system. The vector $H^{cd}(\gamma)$ is defined as follows:

$$H^d_{2i+1-\ell}(\gamma) = (-1)^\ell \sum_{j=0}^{2i+1-\ell}(-1)^{j+1}(h_j - \gamma_j), \quad \text{for } 1 \le \ell \le i, \tag{3.23}$$

$$H^c_{2i+2-\ell}(\gamma) = (-1)^\ell \sum_{j=0}^{2i+1-\ell}(-1)^{j+1}(h_j - \gamma_j), \quad \text{for } i+2 \le \ell \le 2i+1, \tag{3.24}$$

$$H^d_\ell(\gamma) = \gamma_\ell + H^c_{\ell+1}(\gamma), \quad \text{for } 1 \le \ell \le i-1, \tag{3.25}$$

$$H^c_\ell(\gamma) = \gamma_\ell + H^d_{\ell-1}(\gamma), \quad \text{for } i+2 \le \ell \le 2i, \tag{3.26}$$

$$H^d_i(\gamma) = \gamma_i, \tag{3.27}$$

$$H^c_{i+1}(\gamma) = \gamma_{i+1}. \tag{3.28}$$

Proposition 3.4. *Suppose that, for a given set of nonnegative integers* $(h_0, h_1, \ldots, h_{2i}, h_{2i+1})$*, there are Betti number vectors that satisfy the Morse inequalities* (1.1)*. Let* γ^* *be a Betti number vector which satisfies* (1.1) *and* (3.19)*. Then* $H^{cd}(\gamma^*)$ *is a nonnegative integral solution of the* h^{cd}*-system* (3.2)*.*

Proof. Let $h^{cd} = H^{cd}(\gamma^*)$. Integrality of h^{cd} follows easily from the integrality of γ^* and h. Rewriting (3.5) as

$$(-1)^k \sum_{j=0}^{2i+1-k}(-1)^{j+1}\gamma_j \begin{cases} = \displaystyle\sum_{j=0}^{2i+1}(-1)^{j+1}h_j, & \text{if } k = 0, \\ \le (-1)^k \displaystyle\sum_{j=0}^{2i+1-k}(-1)^{j+1}h_j, & \text{if } 1 \le k \le 2i+1, \end{cases} \tag{3.29}$$

we conclude that the components of h^{cd} defined in (3.23) and (3.24) are nonnegative. These facts, on the other hand, together with the hypothesis $\gamma^* \ge 0$, imply that the components defined in (3.25)–(3.28) are also nonnegative.

We must now verify that h^{cd} satisfies the constraints in (3.1). Equation (3.24) for $\ell = 2i+1$ and (3.6) imply

$$h^c_1 = (-1)^{2i+1}(-1)^1(h_0 - \gamma_0^*) = h_0 - 1,$$

thus h^{cd} satisfies the first equation in (3.1). Equation (3.23) for $\ell = 1$, (3.6) and the first equation of (3.5) imply

$$\begin{aligned} h^d_{2i} &= (-1)^1 \sum_{j=0}^{2i} (-1)^{j+1}(h_j - \gamma^*_j) \\ &= -\left(\sum_{j=0}^{2i+1} (-1)^{j+1}(h_j - \gamma^*_j) - (h_{2i+1} - \gamma^*_{2i+1}) \right) \\ &= h_{2i+1} - \gamma^*_{2i+1} \\ &= h_{2i+1} - 1, \end{aligned}$$

which implies the $(2i+2)$-th equation of (3.1).

The last i equations of (3.1),

$$\begin{aligned} h^d_j - h^c_{j+1} + h^d_{2i-j} - h^c_{2i+1-j} &= (\gamma^*_j + h^c_{j+1}) - h^c_{j+1} + h^d_{2i-j} - (\gamma^*_{2i+1-j} + h^d_{2i-j}) \\ &= \gamma^*_j - \gamma^*_{2i+1-j} = 0, \qquad \text{for } 1 \le j \le i-1 \end{aligned}$$

$$h^d_i - h^c_{i+1} = \gamma^*_i - \gamma^*_{i+1} = 0,$$

are validated using (3.25), (3.26) and (3.7).

Let $i + 2 \le \ell \le 2i$. Summing the appropriate equations in (3.24) we obtain

$$\begin{aligned} h^c_{2i+2-\ell} + h^c_{2i+1-\ell} &= (-1)^\ell \left(\sum_{j=0}^{2i+1-\ell} (-1)^{j+1}(h_j - \gamma^*_j) - \sum_{j=0}^{2i-\ell} (-1)^{j+1}(h_j - \gamma^*_j) \right) \\ &= (-1)^\ell (-1)^{2i+2-\ell}(h_{2i+1-\ell} - \gamma^*_{2i+1-\ell}) \\ &= h_{2i+1-\ell} - \gamma^*_{2i+1-\ell}, \end{aligned}$$

which implies, using (3.25), the $(2i+2-\ell)$-th equation of (3.1):

$$\gamma^*_{2i+1-\ell} + h^c_{2i+2-\ell} + h^c_{2i+1-\ell} = h^d_{2i+1-\ell} + h^c_{2i+1-\ell} = h_{2i+1-\ell}.$$

Thus equations $(1+j)$ of (3.1), for $1 \le j \le i-1$, are satisfied by h^{cd}.

The $(1+i)$-th equation of (3.1) follows from (3.24), (3.27) and (3.19):

$$\begin{aligned} h^c_i + h^d_i &= (-1)^{i+2} \sum_{j=0}^{i-1} (-1)^{j+1}(h_j - \gamma^*_j) \; + \; \gamma^*_i \\ &= (-1)^{i+2} \sum_{j=0}^{i-1} (-1)^{j+1}(h_j - \gamma^*_j) + h_i + (-1)^{i+1} \sum_{j=0}^{i-1} (-1)^{j+1}(h_j - \gamma^*_j) \\ &= h_i, \end{aligned}$$

and the $(1+(i+1))$-th equation of (3.1) follows from (3.23), (3.28) and (3.19):

$$\begin{aligned} h^c_{i+1} + h^d_{i+1} &= \gamma_{i+1} + (-1)^i \sum_{j=0}^{i+1} (-1)^{j+1}(h_j - \gamma^*_j) \\ &= h_i + (-1)^{i+1} \sum_{j=0}^{i-1} (-1)^{j+1}(h_j - \gamma^*_j) + (-1)^i \sum_{j=0}^{i+1} (-1)^{j+1}(h_j - \gamma^*_j) \\ &= h_i + (-1)^i \left((-1)^{i+1}(h_i - \gamma^*_i) + (-1)^{i+2}(h_{i+1} - \gamma^*_{i+1})\right) \\ &= h_i - (h_i - \gamma^*_i) + (h_{i+1} - \gamma^*_{i+1}) \\ &= h_{i+1}. \end{aligned}$$

Finally, we show that h^{cd} satisfies the remaining equations of (3.1): equations $(i+2+j)$, for $1 \leq j \leq i-1$. Let $1 \leq \ell \leq i-1$. Using (3.23) we have

$$\begin{aligned} h^d_{2i+1-\ell} + h^d_{2i-\ell} &= (-1)^\ell \sum_{j=0}^{2i+1-\ell} (-1)^{j+1}(h_j - \gamma^*_j) + (-1)^{\ell+1} \sum_{j=0}^{2i-\ell} (_1)^{j+1}(h_j - \gamma^*_j) \\ &= (-1)^\ell (-1)^{2i+2-\ell}(h_{2i+1-\ell} - \gamma^*_{2i+1-\ell}). \end{aligned}$$

Therefore, taking into account definition (3.26),

$$h^d_{2i+1-\ell} + \gamma^*_{2i+1-\ell} + h^d_{2i-\ell} = h^d_{2i+1-\ell} + h^c_{2i+1-\ell} = h_{2i+1-\ell}.$$

□

Propositions 3.2 and 3.4 imply Theorem 1.1 for the case n odd.

3.2. **Case $n \equiv 0 \mod 4$.** Assume $n = 2i$, where $i \geq 2$ is even. Suppose there are nonnegative integers $(h_0, h_1, \ldots, h_{2i+1}, h^c_1,\ h^d_1, \ldots, h^c_{2i}, h^d_{2i})$ that satisfy the linear h^{cd}-system

$$(3.30)\qquad \begin{cases} h^c_1 = h_0 - 1, & \\ h^c_j + h^d_j = h_j, & \text{for } j = 1, \ldots, i-1, \\ h^c_i + \beta + h^d_i = h_i, & \\ h^c_j + h^d_j = h_j, & \text{for } j = i+1, \ldots, 2i-1, \\ h^d_{2i-1} = h_{2i} - 1, & \\ h^d_j - h^c_{j+1} + h^d_{2i-1-j} - h^c_{2i-j} = 0, & \text{for } j = 1, \ldots, i-1. \end{cases}$$

Fix $(h_0, h_1, \ldots, h_{2i})$ at nonnegative integer values such that (3.30) has a nonnegative integral solution $(h^c_1, h^d_1, h^c_2, h^d_2, \ldots, h^c_{2i-1}, h^d_{2i-1})$. Then this latter vector satisfies (3.31) below, equivalent to (3.30), obtained by multiplying by -1 the odd equations up to $i+1$ and the even equations thereafter:

$$(3.31)\qquad \begin{cases} -h^c_1 = -(h_0 - 1), & \\ (-1)^{j+1}(h^c_j + h^d_j) = (-1)^{j+1}h_j, & \text{for } j = 1, \ldots, i-1, \\ -h^c_i - \beta - h^d_i = -h_i, & \\ (-1)^j(h^c_j + h^d_j) = (-1)^j h_j, & \text{for } j = i+1, \ldots, 2i-1, \\ h^d_{2i-1} = h_{2i} - 1, & \\ (-1)^j(h^d_j - h^c_{j+1} + h^d_{2i-1-j} - h^c_{2i-j}) = 0, & \text{for } j = 1, \ldots, i-1. \end{cases}$$

System (3.31) may be decomposed (see [1]) into two independent systems: (3.32) and (3.33). System (3.32) is a network-flow problem defined on a digraph whose incidence matrix is the coefficient matrix of (3.2) for $n = 2i-1$. Thus for the case $i = 4$ the digraph corresponding to system (3.32) has precisely the structure depicted in Figure 1, but of course with different labels on arcs and nodes. In the general case, the digraph with incidence matrix given by the coefficient matrix of (3.32) has $i-2$ cycles of length four plus a nonoriented path also of length four. The j-th cycle contains arcs associated with variables h^d_{j+1}, h^c_{2i-1-j}, h^d_{2i-1-j} and h^c_{j+1}, and the orientation of the first two arcs is opposite to the orientation of the last two, with respect to an arbitrary orientation of the cycle. Nodes of the j-th cycle are associated with equations $j+2$, $2i+2+j$, $2i-j$ and $2i+1+j$. Figure 4 depicts the decomposition of the problem for the case $i = 4$. Notice that the picture on the right of the figure, representing equations (3.33), departs from the conventions of the classical network-flow problem, because the flows associated with the downward pointing arcs are associated with the same variable. Nevertheless we may easily adapt the proof of Lemma 3.1 to this case.

(3.32)
$$\begin{cases} -h^c_1 = -(h_0 - 1), & \\ (-1)^{j+1}(h^c_j + h^d_j) = (-1)^{j+1} h_j, & \text{for } j = 1, \dots, i-1, \\ (-1)^j (h^c_j + h^d_j) = (-1)^j h_j, & \text{for } j = i+1, \dots, 2i-1, \\ h^d_{2i-1} = h_{2i} - 1, & \\ (-1)^j (h^d_j - h^c_{j+1} + h^d_{2i-1-j} - h^c_{2i-j}) = 0, & \text{for } j = 1, \dots, i-2, \\ -h^d_{i-1} + h^c_{i+1} = -\sum_{j=0}^{i-1} (-1)^{j+1} h_j - \sum_{j=i+1}^{2i} (-1)^j h_j. & \end{cases}$$

(3.33)
$$\begin{cases} -h^c_i - \beta - h^d_i = -h_i, \\ h^c_i - h^d_i = \sum_{j=0}^{i-1} (-1)^{j+1} h_j + \sum_{j=i+1}^{2i} (-1)^j h_j. \end{cases}$$

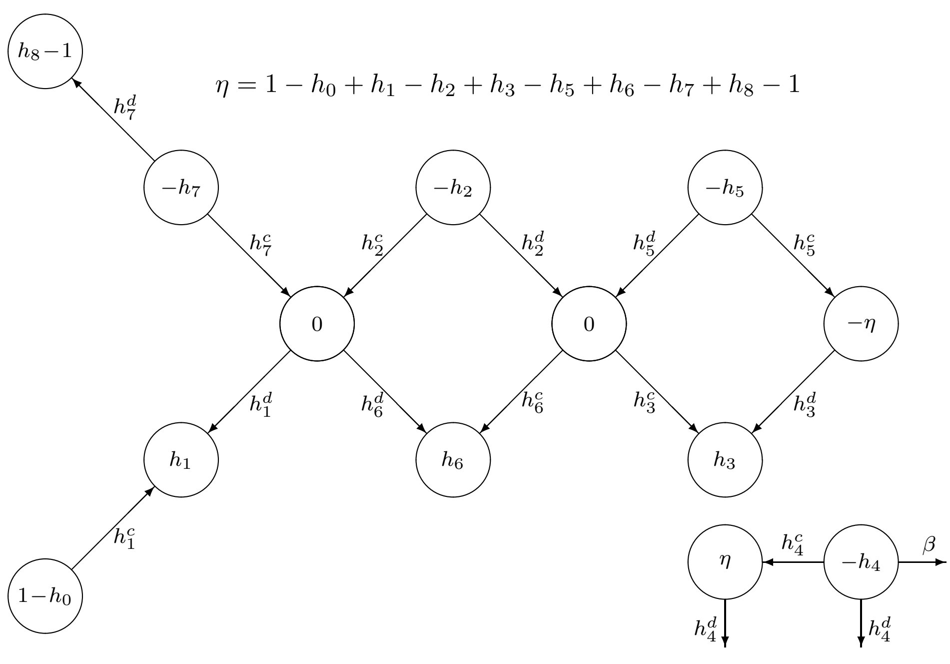


FIGURE 4. Decomposition of problem instance for $n = 2i = 8$.

Lemma 3.5. *If the h^{cd}-system* (3.31) *has a nonnegative solution, then there exists a unique nonnegative integral solution that satisfies the complementarity condition* (3.3). *Furthermore, if $\sum_{j=0}^{2i}(-1)^{j+1}h_j$ is even, then the β component of the complementarity solution is even.*

Proof. If the h^{cd}-system has a nonnegative solution, then it has a nonnegative integral solution, say $\bar{h}^{cd}$. Furthermore, this solution may be partitioned into i subvectors, solutions of independent systems. Subvector $(\bar{h}_j^c, \bar{h}_j^d, \bar{h}_{2i+1-j}^c, \bar{h}_{2i+1-j}^d)$, for $2 \le j \le i-1$, is the solution of the j-th independent system, a cycle network-flow problem of the type depicted in Figure 3. Subvector $(\bar{h}_i^c, \bar{\beta}, \bar{h}_i^d)$ solves system (3.33). These facts were established in [1]. We have already seen in the proof of Lemma 3.1 that $(\bar{h}_j^c, \bar{h}_j^d, \bar{h}_{2i+1-j}^c, \bar{h}_{2i+1-j}^d)$ may be altered to produce another nonnegative integral subvector that satisfies (3.3). It is also easy to verify that $(\hat{h}_i^c, \hat{\beta}, \hat{h}_i^d) = (\bar{h}_i^c, \bar{\beta}, \bar{h}_i^d) + \min\{\bar{h}_i^c, \bar{h}_i^d\}(-1, 2, -1)$ is a nonnegative integral solution of system (3.33) that satisfies (3.3). Figure 5 shows the two solutions of (3.33) where (3.3) holds. They coincide, in case $\eta = 0$, or precisely one of them is nonnegative, otherwise.

$h_i^d = 0,\ h_i^c = \eta$

$$\beta = h_i - \eta = \sum_{j=0}^{2i}(-1)^j h_j - 2\sum_{j=i+1}^{2i}(-1)^j h_j$$

$h_i^d = -\eta,\ h_i^c = 0$

$$\beta = h_i + \eta = \sum_{j=0}^{2i}(-1)^j h_j - 2\sum_{j=0}^{i-1}(-1)^j h_j$$

$$\eta = \sum_{j=0}^{i-1}(-1)^{j+1} h_j + \sum_{j=i+1}^{2i}(-1)^j h_j$$

FIGURE 5. Possible complementary solutions of (3.33).

Finally, from the formulas for β shown in Figure 5, it follows that this component of the nonnegative complementary solution of (3.33) is even if and only if the alternate sum of h_j's, $\sum_{j=0}^{2i}(-1)^{j+1}h_j$, is even. Thus there is a unique solution of the h^{cd}-system (3.31) that satisfies (3.3), formed by the updated subvectors. □

The function Γ needs to be redefined for the $n \equiv 0 \mod 4$ case:

$$\Gamma_0(h^{cd}) = \Gamma_{2i}(h^{cd}) = 1,$$

(3.34)
$$\Gamma_j(h^{cd}) = \begin{cases} h_j^d - h_{j+1}^c, & \text{if } 1 \le j \le i-1, \\ \beta, & \text{if } j = i, \\ -h_{j-1}^d + h_j^c, & \text{if } i+1 \le j \le 2i-1. \end{cases}$$

Proposition 3.6. *Let h^{*cd} be the complementary solution of* (3.31), *for a given set of nonnegative integers $(h_0, h_1, \dots, h_{2i})$. Then $\Gamma(h^{*cd})$ is a Betti number vector that satisfies the Morse inequalities* (1.1). *Furthermore, the i-th component of any Betti number vector that solves* (1.1) *is even if and only if $\sum_{j=0}^{2i}(-1)^j h_j$ is even.*

Proof. The Morse inequalities are rewritten below for the case $n \equiv 0 \mod 4$.

$$\sum_{j=0}^{2i-k}(-1)^{j+1}\gamma_j \begin{cases} = \sum_{j=0}^{2i}(-1)^{j+1}h_j, & \text{if } k=0, \\ \le \sum_{j=0}^{2i-k}(-1)^{j+1}h_j, & \text{if } 1 \le k \le 2i,\ k \text{ odd}, \\ \ge \sum_{j=0}^{2i-k}(-1)^{j+1}h_j, & \text{if } 1 \le k \le 2i,\ k \text{ even}. \end{cases} \tag{3.35}$$

If γ is a Betti number vector satisfying (3.35), then

$$\sum_{j=0}^{2i}(-1)^{j+1}\gamma_j = 2\sum_{j=0}^{i-1}(-1)^{j+1}\gamma_j + (-1)^{i+1}\gamma_i = \sum_{j=0}^{2i}(-1)^{j+1}h_j.$$

Therefore $\gamma_i = \sum_{j=0}^{2i}(-1)^j h_j - 2\sum_{j=0}^{i-1}(-1)^{j+1}\gamma_j$ is even if and only if $\sum_{j=0}^{2i}(-1)^j h_j$ is also even.

Let $\gamma = \Gamma(h^{*cd})$. Notice that, from (3.34) and Lemma 3.5, γ_i is even if and only if $\sum_{j=0}^{2i}(-1)^{j+1}h_j$ is even. We show in the sequence that γ is a Betti number vector satisfying (3.35).

The integrality of h^{*cd} implies the integrality of γ. Equation $2i+1+j$, for $1 \le j \le i-1$, of (3.31) implies

$$\begin{aligned} (-1)^j({h^*}^d_j - {h^*}^c_{j+1} + {h^*}^d_{2i-1-j} - {h^*}^c_{2i-j}) = \\ (-1)^j(\gamma_j - \gamma_{2i-j}) = 0, \end{aligned} \tag{3.36}$$

which means that γ satisfies (3.7). Condition (3.6) is true by definition.

Nonnegativity of γ_j is trivial for $j=0$, i and $2i$, and, for $1 \le j \le 2i-1$, follows from (3.7) and (3.3), as in the proof of Proposition 3.2. It remains to be shown that it also satisfies (3.35). The proof is analogous to that of Proposition 3.2, thus it is convenient to define the system (3.37), equivalent to (3.30), obtained by multiplying odd equations by -1:

(3.37)

$$\begin{cases} -h^c_1 = -(h_0 - 1), & \\ (-1)^{j+1}(h^c_j + h^d_j) = (-1)^{j+1}h_j, & \text{for } j = 1,\dots,i-1, \\ -h^c_i - \beta - h^d_i = -h_i, & \\ (-1)^{j+1}(h^c_j + h^d_j) = (-1)^{j+1}h_j, & \text{for } j = i+1,\dots,2i-1, \\ -h^d_{2i-1} = -(h_{2i} - 1), & \\ (-1)^{j+1}(h^d_j - h^c_{j+1} + h^d_{2i-1-j} - h^c_{2i-j}) = 0, & \text{for } j = 1,\dots,i-1. \end{cases}$$

Since the systems (3.30) and (3.37) are equivalent, h^{*cd} also solves (3.37). Adding equations 1 through $2i+1$ of (3.37) we obtain

$$\begin{aligned}
-h^{*c}_{\ 1} + \sum_{j=1}^{2i-1}(-1)^{j+1}(h^{*c}_{\ j} + h^{d}_{j}) - h^{*d}_{\ 2i-1} - \beta^* &= \\
h^{*d}_{\ 1} + \sum_{j=2}^{2i-2}(-1)^{j+1}(h^{*c}_{\ j} + h^{*d}_{\ j}) + h^{*c}_{\ 2i-1} - \beta^* &= \\
\sum_{j=1}^{2i-2}(-1)^{j+1}h^{*d}_{\ j} + \sum_{j=2}^{2i-1}(-1)^{j+1}h^{*c}_{\ j} - \beta^* &= \\
\sum_{j=1}^{2i-2}(-1)^{j+1}h^{*d}_{\ j} + \sum_{j=1}^{2i-2}(-1)^{j}h^{*c}_{\ j+1} - \beta^* &= \\
\sum_{j=1}^{2i-2}(-1)^{j+1}(h^{*d}_{\ j} - h^{*c}_{\ j+1}) - \beta^* &= -(h_0 - 1) \\
&\quad + \sum_{j=1}^{2i-1}(-1)^{j+1}h_j - (h_{2i} - 1) \\
&= 2 + \sum_{j=0}^{2i}(-1)^{j+1}h_j. \qquad (3.38)
\end{aligned}$$

The alternate sum of γ's, according to (3.34), gives

$$\begin{aligned}
\sum_{j=0}^{2i}(-1)^{j+1}\gamma_j &= -1 + \sum_{j=1}^{i-1}(-1)^{j+1}(h^{*d}_{\ j} - h^{*c}_{\ j+1}) - \beta^* \\
&\quad + \sum_{j=i+1}^{2i-1}(-1)^{j+1}(-h^{*d}_{\ j-1} + h^{*c}_{\ j}) - 1 \\
&= -2 + \sum_{j=1}^{i-1}(-1)^{j+1}(h^{*d}_{\ j} - h^{*c}_{\ j+1}) + \sum_{j=i}^{2i-2}(-1)^{j+1}(h^{*d}_{\ j} - h^{*c}_{\ j+1}) - \beta^* \\
&= -2 + \sum_{j=1}^{2i-2}(-1)^{j+1}(h^{*d}_{\ j} - h^{*c}_{\ j+1}) - \beta^*. \qquad (3.39)
\end{aligned}$$

Comparing (3.38) and (3.39) we conclude that

$$\sum_{j=0}^{2i}(-1)^{j+1}\gamma_j = \sum_{j=0}^{2i}(-1)^{j+1}h_j, \qquad (3.40)$$

that is, γ satisfies the first equation in (3.35).

When calculating the partial sum of equations 1 through $2i+1-\ell$ of (3.37), there are the following two cases to consider.

a) $1 \leq \ell \leq i$

$$-{h^*}^c_1 + \sum_{j=1}^{2i-\ell}(-1)^{j+1}({h^*}^c_j + {h^*}^d_j) - \beta^* =$$

$${h^*}^d_1 + \sum_{j=2}^{2i-\ell}(-1)^{j+1}({h^*}^c_j + {h^*}^d_j) - \beta^* =$$

$$\sum_{j=1}^{2i-\ell}(-1)^{j+1}{h^*}^d_j - \sum_{j=1}^{2i-1-\ell}(-1)^{j+1}{h^*}^c_{j+1} - \beta^* =$$

$$\sum_{j=1}^{2i-1-\ell}(-1)^{j+1}({h^*}^d_j - {h^*}^c_{j+1}) + (-1)^{2i+1-\ell}{h^*}^d_{2i-\ell} - \beta^* = -(h_0 - 1) + \sum_{j=1}^{2i-\ell}(-1)^{j+1}h_j$$

$$= 1 + \sum_{j=0}^{2i-\ell}(-1)^{j+1}h_j. \tag{3.41}$$

b) $i + 1 \leq \ell \leq 2i$

$$-{h^*}^c_1 + \sum_{j=1}^{2i-\ell}(-1)^{j+1}({h^*}^c_j + {h^*}^d_j) =$$

$${h^*}^d_1 + \sum_{j=2}^{2i-\ell}(-1)^{j+1}({h^*}^c_j + {h^*}^d_j) =$$

$$\sum_{j=1}^{2i-\ell}(-1)^{j+1}{h^*}^d_j - \sum_{j=1}^{2i-1-\ell}(-1)^{j+1}{h^*}^c_{j+1} =$$

$$\sum_{j=1}^{2i-\ell}(-1)^{j+1}({h^*}^d_j - {h^*}^c_{j+1}) + (-1)^{2i+1-\ell}{h^*}^c_{2i+1-\ell} = -(h_0 - 1) + \sum_{j=1}^{2i-\ell}(-1)^{j+1}h_j$$

$$= 1 + \sum_{j=0}^{2i-\ell}(-1)^{j+1}h_j. \tag{3.42}$$

Likewise, we consider two possibilities for the partial sum $\sum_{j=0}^{2i-\ell}(-1)^{j+1}\gamma_j$:
a) $1 \leq \ell \leq i$

$$\begin{aligned}\sum_{j=0}^{2i-\ell}(-1)^{j+1}\gamma_j &= -1 + \sum_{j=1}^{i-1}(-1)^{j+1}({h^*}^d_j - {h^*}^c_{j+1}) - \beta^* \\ &\quad + \sum_{j=i+1}^{2i-\ell}(-1)^{j+1}(-{h^*}^d_{j-1} + {h^*}^c_j) \\ &= -1 + \sum_{j=1}^{i-1}(-1)^{j+1}({h^*}^d_j - {h^*}^c_{j+1}) + \sum_{j=i}^{2i-1-\ell}(-1)^{j+1}({h^*}^d_j - {h^*}^c_{j+1}) - \beta^* \\ &= -1 + \sum_{j=1}^{2i-1-\ell}(-1)^{j+1}({h^*}^d_j - {h^*}^c_{j+1}) - \beta^*. \end{aligned} \tag{3.43}$$

Comparing (3.43) and (3.41) we conclude that

$$\sum_{j=0}^{2i-\ell}(-1)^{j+1}\gamma_j + (-1)^{2i+1-\ell}{h^*}^d_{2i-\ell} = \sum_{j=0}^{2i-\ell}(-1)^{j+1}h_j. \tag{3.44}$$

Using the fact that $h^{*cd} \geq 0$, (3.44) implies

$$\sum_{j=0}^{2i-\ell}(-1)^{j+1}\gamma_j \begin{cases} \leq \sum_{j=0}^{2i-\ell}(-1)^{j+1}h_j, & \text{if } 1 \leq \ell \leq i,\ \ell \text{ odd}, \\ \geq \sum_{j=0}^{2i-\ell}(-1)^{j+1}h_j, & \text{if } 1 \leq \ell \leq i,\ \ell \text{ even}, \end{cases} \tag{3.45}$$

which means γ satisfies the inequalities in (3.35), for $1 \leq \ell \leq i$.

b) $i+1 \leq \ell \leq 2i-1$

$$\sum_{j=0}^{2i-\ell}(-1)^{j+1}\gamma_j = -1 + \sum_{j=1}^{2i-\ell}(-1)^{j+1}({h^*}^d_j - {h^*}^c_{j+1}). \tag{3.46}$$

Comparing (3.46) and (3.42) we conclude that

$$\sum_{j=0}^{2i-\ell}(-1)^{j+1}\gamma_j + (-1)^{2i+1-\ell}{h^*}^c_{2i+1-\ell} = \sum_{j=0}^{2i-\ell}(-1)^{j+1}h_j. \tag{3.47}$$

Using the nonnegativity of h^{*cd}, (3.47) implies that γ satisfies (3.45) for $i+1 \leq \ell \leq 2i-1$.

Finally, from (3.42) with $\ell = 2i$, we have $-{h^*}^c_1 = 1 - h_0$ and since, by hypothesis, ${h^*}^c_1 \geq 0$, then $-h_0 \leq -1 = -\gamma_0$ holds, the last inequality in (3.35).

We have thus established that γ satisfies constraints (3.35). □

The new function $H^{cd}(\gamma) = (H^c_1(\gamma),\ H^d_1(\gamma),\ \ldots,\ H^c_i(\gamma),\ B(\gamma),\ H^d_i(\gamma),\ \ldots,\ H^c_{2i-1}(\gamma),\ H^d_{2i-1}(\gamma))$, given by

$$H^d_{2i-\ell}(\gamma) = (-1)^{\ell+1}\sum_{j=0}^{2i-\ell}(-1)^{j+1}(h_j - \gamma_j), \quad \text{for } 1 \leq \ell \leq i, \tag{3.48}$$

$$H^c_{2i+1-\ell}(\gamma) = (-1)^{\ell+1}\sum_{j=0}^{2i-\ell}(-1)^{j+1}(h_j - \gamma_j), \quad \text{for } i+1 \leq \ell \leq 2i, \tag{3.49}$$

$$H^d_\ell(\gamma) = \gamma_\ell + h^c_{\ell+1}, \qquad \text{for } 1 \leq \ell \leq i-1, \tag{3.50}$$

$$H^c_\ell(\gamma) = \gamma_\ell + h^d_{\ell-1}, \qquad \text{for } i+1 \leq j \leq 2i-1, \tag{3.51}$$

$$B(\gamma) = \gamma_i \tag{3.52}$$

will provide a construction rule for solutions of the h^{cd}-system (3.30).

Proposition 3.7. *If, for a given set of nonnegative integers* $(h_0, h_1, \ldots, h_{2i}, h_{2i})$*, there is Betti number vector* γ *that satisfies the Morse inequalities* (1.1)*, then* $H^{cd}(\gamma)$ *is a nonnegative integral solution of the* h^{cd}*-system* (3.30)*.*

Proof. Let $h^{cd} = H^{cd}(\gamma)$. Clearly, h^{cd} given by (3.48)–(3.52) is integral, since so are γ and h. Rewriting (3.35) as

$$(-1)^{k+1}\sum_{j=0}^{2i-k}(-1)^{j+1}\gamma_j \begin{cases} = \displaystyle\sum_{j=0}^{2i}(-1)^{j+1}h_j, & \text{if } k=0, \\ \le (-1)^{k+1}\displaystyle\sum_{j=0}^{2i-k}(-1)^{j+1}h_j, & \text{if } 1\le k\le 2i, \end{cases} \tag{3.53}$$

we conclude that the components of h^{cd} defined in (3.48) and (3.49) are nonnegative. This fact, on the other hand, together with the hypothesis $\gamma \ge 0$, imply that the components defined in (3.50)–(3.52) are also nonnegative.

From (3.49) with $\ell = 2i$ we obtain

$$h_1^c = (-1)^{2i+1}(-1)^{0+1}(h_0 - \gamma_0) = h_0 - 1,$$

that is, the first equation in (3.30). On the other hand, (3.48) with $\ell = 1$ gives

$$\begin{aligned} h_{2i-1}^d &= (-1)^{1+1}\sum_{j=0}^{2i-1}(-1)^{j+1}(h_j-\gamma_j) \\ &= \sum_{j=0}^{2i}(-1)^{j+1}(h_j-\gamma_j) - (-1)^{2i+1}(h_{2i}-\gamma_{2i}) \\ &= h_{2i}-1, \end{aligned}$$

which is the $(2i+1)$-th equation of (3.30).

The last $i-1$ equations of (3.30) are easily verified:

$$\begin{aligned} h_j^d - h_{j+1}^c + h_{2i-1-j}^d - h_{2i-j}^c &= (\gamma_j + h_{j+1}^c) - h_{j+1}^c + h_{2i-1-j}^d - (\gamma_{2i-j} + h_{2i-1-j}^d) \\ &= \gamma_j - \gamma_{2i-j} = 0, \qquad \text{for } 1 \le j \le i-1. \end{aligned}$$

Let $i+1 \le \ell \le 2i-1$. Summing the appropriate equations in (3.49) we obtain

$$\begin{aligned} h_{2i+1-\ell}^c + h_{2i-\ell}^c &= (-1)^{\ell+1}\left(\sum_{j=0}^{2i-\ell}(-1)^{j+1}(h_j-\gamma_j) - \sum_{j=0}^{2i-\ell-1}(-1)^{j+1}(h_j-\gamma_j)\right) \\ &= (-1)^{\ell+1}(-1)^{2i+1-\ell}(h_{2i-\ell}-\gamma_{2i-\ell}) \\ &= h_{2i-\ell}-\gamma_{2i-\ell}, \end{aligned}$$

which implies, using (3.50), the $(2i+1-\ell)$-th equation of (3.30):

$$\gamma_{2i-\ell} + h_{2i+1-\ell}^c + h_{2i-\ell}^c = h_{2i-\ell}^d + h_{2i-\ell}^c = h_{2i-\ell}.$$

Thus equations $(1+j)$ of (3.30), for $1 \le j \le i-1$, are satisfied by h^{cd}.

Verifying the $(i+1)$-th equation of (3.30):

$$\begin{aligned} h_i^c + \beta + h_i^d &= (-1)^{i+2}\sum_{j=0}^{2i-(i+1)}(-1)^{j+1}(h_j-\gamma_j) + \gamma_i + (-1)^{i+1}\sum_{j=0}^{i}(-1)^{j+1}(h_j-\gamma_j) \\ &= (-1)^{i+1}(-1)^{i+1}(h_i-\gamma_i)+\gamma_i \\ &= h_i. \end{aligned}$$

Finally, letting $1 \leq \ell \leq i-1$, we have

$$\begin{aligned} h^d_{2i-\ell} + h^d_{2i-1-\ell} &= (-1)^{\ell+1}\left(\sum_{j=0}^{2i-\ell}(-1)^{j+1}(h_j-\gamma_j) - \sum_{j=0}^{2i-1-\ell}(-1)^{j+1}(h_j-\gamma_j)\right) \\ &= (-1)^{\ell+1}(-1)^{2i+1-\ell}(h_{2i-\ell}-\gamma_{2i-\ell}) \\ &= h_{2i-\ell}-\gamma_{2i-\ell}, \end{aligned}$$

which implies, using (3.50),

$$h^d_{2i-\ell} + h^d_{2i-1-\ell} + \gamma_{2i-\ell} = h^d_{2i-\ell} + h^c_{2i-\ell} = h_{2i-\ell},$$

which completes the verification that h^{cd} satisfies (3.30). □

Propositions 3.6 and 3.7 imply Theorem 1.1 for the case $n \equiv 0 \mod 4$.

3.3. **Case $n \equiv 2 \mod 4$.** In this section we assume $n = 2i$, where $i \geq 3$ is odd. Suppose there are nonnegative integers $\{h_0, h_1, \dots, h_{2i+1}, h^c_1, h^d_1, \dots, h^c_{2i}, h^d_{2i}\}$ that satisfy the linear system

$$\text{(3.54)} \qquad \begin{cases} h^c_1 = h_0 - 1, & \\ h^c_j + h^d_j = h_j, & \text{for } j = 1,\dots,2i-1, \\ h^d_{2i-1} = h_{2i} - 1, & \\ h^d_j - h^c_{j+1} + h^d_{2i-1-j} - h^c_{2i-j} = 0, & \text{for } j = 1,\dots,i-1. \end{cases}$$

If system (3.54) has a solution, then so does system (3.55), obtained from (3.54) by adding variable β in the $i+1$-th equation, since any solution of (3.54) may be transformed into a solution of (3.55) by setting β to zero. The systems are clearly not equivalent, since (3.55) may admit solutions when (3.54) has none.

$$\text{(3.55)} \qquad \begin{cases} h^c_1 = h_0 - 1, & \\ h^c_j + h^d_j = h_j, & \text{for } j = 1,\dots,i-1, \\ h^c_i + \beta + h^d_i = h_i, & \\ h^c_j + h^d_j = h_j, & \text{for } j = i+1,\dots,2i-1, \\ h^d_{2i-1} = h_{2i} - 1, & \\ h^d_j - h^c_{j+1} + h^d_{2i-1-j} - h^c_{2i-j} = 0, & \text{for } j = 1,\dots,i-1. \end{cases}$$

Notice that system (3.55) has the same structure as system (3.30). But, in order for (3.54) to have a solution, the sum $\sum_{j=0}^{i-1}(-1)^{j+1}h_j + \sum_{j=i}^{2i}(-1)^j h_j$ must be even; see [1]. Thus we may extend almost exactly the results stated for the case $n \equiv 0 \mod 4$. Thus, in particular, (3.55) admits a nonnegative integral solution that satisfies (3.3) if (3.54) has a nonnegative solution, and the function $\Gamma(h^{cd})$ defined in (3.34) gives the recipe for construction of a Betti number vector that solves (1.1) given a complementary solution of (3.55).

Proposition 3.8. *Suppose, for a given set of nonnegative integers* $(h_0, h_1, \dots, h_{2i})$, *that the system* (3.54) *has a nonnegative integral solution. Let* h^{*cd} *be a complementary solution of* (3.55)*. Then* $\Gamma(h^{*cd})$ *is a Betti number vector that solves* (1.1), *and such that* $\Gamma_i(h^{*cd})$ *is even.*

Proposition 3.9. *If, for a given set of nonnegative integers* $(h_0, h_1, \dots, h_{2i}, h_{2i})$, *there is a Betti number vector* γ, *with* γ_i *even, that satisfies* (1.1), *then the* h^{cd}-*system* (3.54) *has a nonnegative integral solution.*

Proof. By Proposition 3.7, $h^{cd} = H^{cd}(\gamma)$ given by (3.48)–(3.52) is a nonnegative integral solution of (3.55). By construction, $\beta = 2k$, for a nonnegative integer k. We claim $h^{cd} = (h_1^c, h_1^d, \ldots, h_i^c + k,\ h_i^d + k, \ldots, h_{2i-1}^c, h_{2i-1}^d)$ is a nonnegative integral solution of (3.54). In order to see that we only need to check the equations containing h_i^c and h_i^d:

$$\begin{aligned} h_i &= h_i^c + \beta + h_i^d \\ &= h_i^c + 2k + h_i^d \\ &= (h_i^c + k) + (h_i^d + k), \\ 0 &= h_{i-1}^d - h_i^c + h_i^d - h_{i+1}^c \\ &= h_{i-1}^d - h_i^c - k + k + h_i^d - h_{i+1}^c \\ &= h_{i-1}^d - (h_i^c + k) + (h_i^d + k) - h_{i+1}^c. \end{aligned}$$

□

Propositions 3.8 and 3.9 establish the proof of Theorem 1.1 for the case $n \equiv 2 \mod 4$.

4. Polytopes

In this section we study the Morse polyhedron $\mathcal{P}$ restricted to the nonnegative orthant, i.e., the set of nonnegative $\gamma = (\gamma_0, \ldots, \gamma_n)$ satisfying the Morse inequalities (1.1) for a pre-assigned index data $(h_0, \ldots, h_n)$, and the duality conditions $\gamma_k = \gamma_{n-k}$, for $k = 0, \ldots, n$. We will see that this polyhedron bears a close relationship with the polyhedron $\mathcal{H}^{cd}$ constituted by the nonnegative solutions of the h^{cd}-system.

First of all we point out that $\mathcal{P}$ is a polytope, i.e., a bounded polyhedron. This follows from the restriction of nonnegativity on γ and from the fact that taking all possible pairs of consecutive inequalities in (1.1), we obtain $\gamma_j \leq h_j$, for $j = 1, \ldots, n$. Therefore $0 \leq \gamma_j \leq h_j$, for $0 \leq j \leq n$, for every $\gamma \in \mathcal{P}$. It is also easy to see that $\mathcal{H}^{cd}$ also defines a polytope, since the associated problem may be interpreted as network-flow problem (with added constraints in the case n even) with no directed cycles.

It was shown in [1] that all vertices of $\mathcal{H}^{cd}$ are integral, a straightforward result considering the identification of the h^{cd}-system as a network-flow problem. We will show that this is also true for $\mathcal{P}$, a somewhat surprising result. This implies that $\mathcal{P}$ is the convex hull of the integral vectors in $\mathcal{P}$, or, equivalently, that $\mathcal{P}$ is the convex hull of the collection of Betti number vectors which satisfy the Morse inequalities. Finally we will see that there is not, necessarily, a 1-1 relationship between integral vectors in $\mathcal{H}^{cd}$ and $\mathcal{P}$. While every integral γ in $\mathcal{P}$ may be obtained from an integral h^{cd} in $\mathcal{H}^{cd}$, the converse is not necessarily true. This result is useful since we may easily construct the solutions of the h^{cd}-system, from the complementary solution and the circulations.

4.1. Case n odd.

In this section we explore the geometry of $\mathcal{P}$ for the case n odd. In particular we show that all its vertices are integral and describe the two faces that contain all vertices. Given that the second face is the projection of the first one onto a hyperplane, knowledge of the first face gives all information about the polytope, since it is the convex hull of this face and its projection. The results are illustrated with an example.

In order to simplify the exposition, boundary and duality conditions (3.6)–(3.7) will be used to eliminate more than half the variables, namely γ_0, γ_{i+1}, ..., γ_{2i+1}. Using equations (3.6)–(3.7), we have that

$$\sum_{j=0}^{2i+1-k}(-1)^{j+1}\gamma_j = -1 + \sum_{j=1}^{\min\{k-1,2i+1-k\}}(-1)^{j+1}\gamma_j.$$

Also note that these same conditions imply that the constraints corresponding to $k = 0$, 1 and $2i+1$ in (3.5) represent, in fact, constraints on the pre-assigned index data $h = (h_0, \ldots, h_n)$, grouped on the first row of (4.1). There is a 1-to-1 correspondence between the nonnegative $\gamma = (\gamma_0, \ldots, \gamma_{2i+1})$ satisfying (3.5)–(3.7) and the nonnegative $\gamma^r = (\gamma_1, \ldots, \gamma_i)$ satisfying (4.1) below. Furthermore, $\gamma^r \in \mathcal{P}^r$ is a vertex[2] of $\mathcal{P}^r$ if and only if the corresponding γ is a vertex of $\mathcal{P}$. Thus, instead of $\mathcal{P}$, we may consider the polytope $\mathcal{P}^r = \{\gamma^r \in \mathbb{R}^i \mid \text{constraints in (4.1)}\}$.

(4.1)

$$\begin{aligned}
0 &= \sum_{j=0}^{2i+1}(-1)^{j+1}h_j,\\
0 &\le h_0 - 1,\\
0 &\le h_{2i+1} - 1,\\
\sum_{j=1}^{\min\{k-1,2i+1-k\}}(-1)^{j+1}\gamma_j &\le 1 + \sum_{j=0}^{2i+1-k}(-1)^{j+1}h_j, \text{ for } 2 \le k \le 2i,\ k \text{ even},\\
\sum_{j=1}^{\min\{k-1,2i+1-k\}}(-1)^{j+1}\gamma_j &\ge 1 + \sum_{j=0}^{2i+1-k}(-1)^{j+1}h_j, \text{ for } 2 \le k \le 2i,\ k \text{ odd},\\
\gamma^r &\ge 0.
\end{aligned}$$

Proposition 4.1. *The polytope $\mathcal{P}^r$ given by* (4.1) *satisfies the following properties:*

(1) *The vertices of $\mathcal{P}^r$ are integral.*
(2) *Each vertex of $\mathcal{P}^r$ belongs to one of the faces*

$$\mathcal{F}_t = \{\gamma^r \in \mathcal{P}^r \mid \sum_{j=1}^{i}(-1)^{j+1}\gamma_j = 1 + \sum_{j=1}^{i}(-1)^{j+1}h_j\}$$

or

$$\mathcal{F}_0 = \{\gamma^r \in \mathcal{P}^r \mid \gamma_i = 0\}.$$

(3) *The face $\mathcal{F}_0$ is the projection of $\mathcal{F}_t$ onto the hyperplane $\{\gamma^r \in \mathbb{R}^i \mid \gamma_i = 0\}$.*
(4) *Each (integral) nonnegative γ^r in $\mathcal{F}_t$ corresponds to an (integral) nonnegative h^{cd} satisfying* (3.1).

Proof. (1) Let A be the coefficient matrix of the system of inequalities in $\mathcal{P}^r$, excepting the nonnegativity inequalities. Variable γ_j shows up only in constraints $3+j$ through $3+2i-j$, with coefficient $(-1)^{j+1}$. Thus the $0,1$ matrix $\tilde{A} = (\tilde{a}_{ij}) = (|a_{ij}|)$ has the consecutive ones property, which implies that it is totally unimodular; see [5, 6]. Since A is obtained from $\tilde{A}$ by multiplying even columns by -1, A is also totally unimodular. Finally note that the right-hand-side elements in

[2] A vertex of a convex set is a vector in the set that may not be expressed as a nontrivial convex combination of two distinct vectors also belonging to the convex set.

the inequalities that define polytope $\mathcal{P}^r$ are clearly integral. Thus the polytope $\mathcal{P}^r$ defined by (4.1) has integral vertices, or equivalently, it is the convex hull of the integral vectors satisfying the inequalities in (4.1).

(2) Let $\bar{\gamma}^r \in \mathcal{P}^r$ and let $M\bar{\gamma}^r = q$ be the constraints that are tight at $\bar{\gamma}^r$. Then $\bar{\gamma}^r$ is vertex of $\mathcal{P}$ if and only if the rank of M is i. Now M contains precisely i columns, which implies that it cannot contain a column of zeros. Since there are only two inequalities containing γ_i, one of these must be tight at $\bar{\gamma}$, otherwise the i-th column of M will be the zero vector. Therefore if $\bar{\gamma}$ is a vertex it must belong to one of the faces $\mathcal{F}_t = \{\gamma^r \in \mathcal{P}^r \mid (-1)^{i+1} \sum_{j=1}^{i} (-1)^{j+1}\gamma_j = (-1)^{i+1}(1 + \sum_{j=1}^{i}(-1)^{j+1}h_j)\}$ or $\mathcal{F}_0 = \{\gamma^r \in \mathcal{P}^r \mid \gamma_i = 0\}$.

(3) If $\gamma^r \in \mathcal{F}_0$, then, from Lemma 3.3, γ^{*r} given by $\gamma^*_j = \gamma_j$, for $1 \le j \le i-1$, and $\gamma^*_i = 0$ also belongs to $\mathcal{P}^r$, and is clearly in $\mathcal{F}_0$.

(4) If (integral) $\gamma^r \in \mathcal{P}^r$ belongs to $\mathcal{F}_t$, then the nonnegative γ given by $\gamma_0 = \gamma_{2i+1} = 1$ and $\gamma_{2i+1-j} = \gamma_j$, for $j = 1, \dots, i$, satisfies (3.5)–(3.7) and saturates the inequality corresponding to $k = i+1$ in (3.5). But then h^{cd} given by equations (3.23)–(3.28) is an (integral) nonnegative vector satisfying (3.1), as shown in the proof of Proposition 3.4. □

Suppose the h^{cd}-system (3.2) admits nonnegative solutions. Let h^{*cd} be the nonnegative integral solution that also satisfies the complementarity conditions (3.3). Let $\gamma^* = \Gamma(h^{*cd})$ be the corresponding integral nonnegative solution of (3.5)–(3.7) given by (3.4). Its restriction γ^{*r} plays a special role in $\mathcal{P}^r$, as shown in the next proposition.

Proposition 4.2. *Suppose the system of equations* (3.2) *admits nonnegative solutions. Then the polytope $\mathcal{P}^r$ may be rewritten as*

$$\text{(4.2)} \quad \mathcal{P}^r = \{0 \le \gamma^r \in \mathbb{R}^i \mid (-1)^{k+1} \sum_{j=0}^{k} (-1)^{j+1}\gamma_j \le (-1)^{k+1} \sum_{j=0}^{k} (-1)^{j+1}\gamma^*_j, \ \textit{for } 1 \le k \le i\}.$$

Furthermore, γ^{*r} is a vertex of $\mathcal{P}^r$ and also its maximum vector, componentwise.

Proof. If (3.2) admits nonnegative solutions, the constraints $0 = \sum_{j=0}^{2i+1}(-1)^{j+1}h_j$, $0 \le h_0 - 1$ and $0 \le h_{2i+1} - 1$ are redundant and may be dropped. Now notice that

$$\min\{k-1, 2i+1-k\} = \begin{cases} k-1, & \text{if } 2 \le k \le i, \\ i = k-1 = 2i+1-k, & \text{if } k = i+1, \\ 2i+1-k, & \text{if } i+2 \le k \le 2i. \end{cases}$$

Therefore the partial sum $\sum_{j=1}^{\ell}(-1)^{j+1}\gamma_j$, for $1 \le \ell \le i-1$, appears twice in (4.1): when $k = \ell+1$ and when $k = 2i+1-\ell$. Since $\ell+1$ and $2i+1-\ell$ are either both odd or both even, we may collect these two inequalities as follows:

$$\sum_{j=1}^{\ell}(-1)^{j+1}\gamma_j \begin{cases} \ge 1 + \max\left\{\sum_{j=0}^{2i-\ell}(-1)^{j+1}h_j, \sum_{j=0}^{\ell}(-1)^{j+1}h_j\right\}, & \text{if } 1 \le \ell \le i-1, \ \ell \text{ even}, \\ \le 1 + \min\left\{\sum_{j=0}^{2i-\ell}(-1)^{j+1}h_j, \sum_{j=0}^{\ell}(-1)^{j+1}h_j\right\}, & \text{if } 1 \le \ell \le i-1, \ \ell \text{ odd}. \end{cases}$$

The fact that γ^* is a nonnegative integral solution of (3.5)–(3.7) implies that ${\gamma^*}^r \in \mathcal{P}^r$. Furthermore, using (3.7), equations (3.12) and (3.17), for $2 \le \ell \le i$ and $k = 2i + 2 - \ell$ (thus $i + 2 \le k \le 2i$), we obtain

$$\sum_{j=0}^{2i+1-\ell} (-1)^{j+1}\gamma_j^* + (-1)^{2i+2-\ell} {h^*}^d_{2i+1-\ell} = \sum_{j=0}^{\ell-1}(-1)^{j+1}\gamma_j^* + (-1)^{2i+2-\ell} {h^*}^d_{2i+1-\ell}$$

$$(4.3) \qquad = \sum_{j=0}^{2i+1-\ell} (-1)^{j+1} h_j,$$

$$\sum_{j=0}^{2i+1-k} (-1)^{j+1}\gamma_j^* + (-1)^{2i+2-k} {h^*}^c_{2i+2-k} = \sum_{j=0}^{\ell-1}(-1)^{j+1}\gamma_j^* + (-1)^{\ell} {h^*}^c_{\ell}$$

$$(4.4) \qquad = \sum_{j=0}^{\ell-1}(-1)^{j+1} h_j.$$

Since, by (3.3), ${h^*}^c_\ell {h^*}^d_{2i+1-\ell} = 0$, at least one of the two inequalities above involving the partial sum $\sum_{j=0}^{\ell-1}(-1)^{j+1}\gamma_j^*$, must be satisfied as equality. Thus ${\gamma^*}^r$ satisfies:

$$\sum_{j=0}^{\ell-1}(-1)^{j+1}\gamma_j^* = \begin{cases} \max\left\{ \sum_{j=0}^{2i+1-\ell} (-1)^{j+1} h_j, \sum_{j=0}^{\ell-1}(-1)^{j+1} h_j \right\}, & \text{if } 2 \le \ell \le i,\ \ell \text{ odd}, \\ \min\left\{ \sum_{j=0}^{2i+1-\ell} (-1)^{j+1} h_j, \sum_{j=0}^{\ell-1}(-1)^{j+1} h_j \right\}, & \text{if } 2 \le \ell \le i,\ \ell \text{ even}, \end{cases}$$

which imply (4.2) for $k \ne i$, taking into account that $\gamma_0^* = 1$. Finally, the inequality in (4.1) corresponding to $k = i$ is satisfied as equality by γ^*, as seen in Lemma 3.3, and thus

$$\sum_{j=0}^{i}(-1)^{j+1}\gamma_j^* = \sum_{j=0}^{i}(-1)^{j+1} h_j.$$

Odd and even cases may be combined with the appropriate multiplication, producing the desired inequalities, that, together with the nonnegative constraints, provide the alternative definition of $\mathcal{P}^r$ given in (4.2).

Finally, the inequality in (4.2) for $k = 1$ implies $\gamma_1 \le \gamma_1^*$, and, if we add inequalities $j - 1$ and j of (4.2) we obtain $\gamma_j \le \gamma_j^*$, for $1 \le j \le i$. Thus ${\gamma^*}^r$ is the maximum vector, componentwise, of $\mathcal{P}$. Then ${\gamma^*}^r$ must be a vertex of $\mathcal{P}^r$. □

Proposition 4.1 implies that each (integral) vector in $\mathcal{F}_t$ may be obtained from a corresponding h^{cd} satisfying (3.1) (or, equivalently, (3.2)), although this is not, in general, a 1-to-1 correspondence. Given the complementary solution ${h^*}^{cd}$ we may construct all integral vectors in $\mathcal{F}_t$ by successive additions of circulations thereto and computation of the corresponding γ^r using the function $\Gamma(h^{cd})$ defined in (3.4). The following example illustrates these facts in a concrete setting.

4.1.1. *Example.* Let $n = 2i + 1 = 7$, $(h_0, \ldots, h_7) = (2, 5, 11, 10, 5, 3, 3, 3)$. Using decomposition and the formula for complementary solutions summarized in Figure 3 we may obtain the complementary solution ${h^*}^{cd} = (1, 4, 3, 8, 5, 5, 5, 0, 3, 0, 1, 2)$, shown in Figure 6. All other nonnegative integral solutions of (3.2) may be obtained by adding integer multiples of the unit circulation along cycle 1, $\text{circ}^1 =$

$(0,0,1,-1,0,0,0,0,-1,1,0,0)$, and/or the unit circulation along cycle 2, $\mathrm{circ}^2 = (0,0,0,0,1,-1,-1,1,0,0,0,0)$. From h^{*cd} we obtain vector $\gamma^* = (1,1,3,5,5,3,1,1)$, and, using Proposition 4.2, the polytope $\mathcal{P}^r$ is given by the inequalities

$$\begin{array}{ccccccc} \gamma_1 & & & & & \leq & 1, \\ \gamma_1 & - & \gamma_2 & & & \geq & -2, \\ \gamma_1 & - & \gamma_2 & + & \gamma_3 & \leq & 3, \\ & & \gamma_1, & \gamma_2, & \gamma_3 & \geq & 0. \end{array}$$

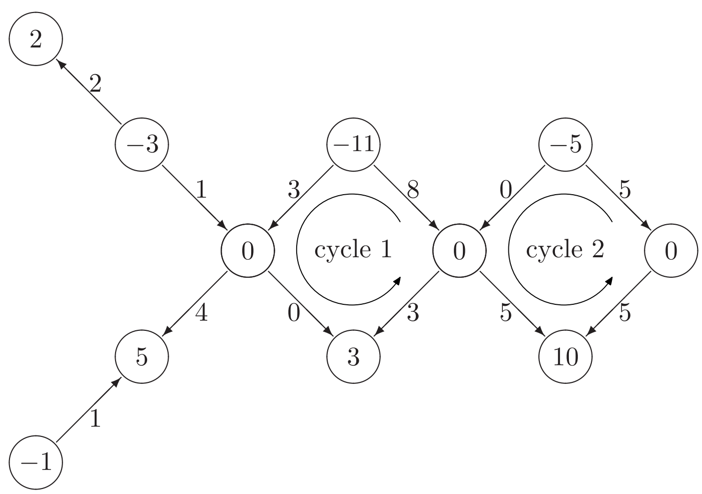


FIGURE 6. Solution h^{*cd} of example.

Figure 7 depicts four views of the polytope $\mathcal{P}^r$. In this example no inequality is redundant, and the polytope has six facets. Facet $\mathcal{F}_0$ is always hidden, but one can realize, perhaps with the aid of Figure 8, that it is indeed the projection of the top facet $\mathcal{F}_t$ on the $\gamma_3 = 0$ plane. The maximum vector γ^{*r} is labeled on the first view of $\mathcal{P}^r$.

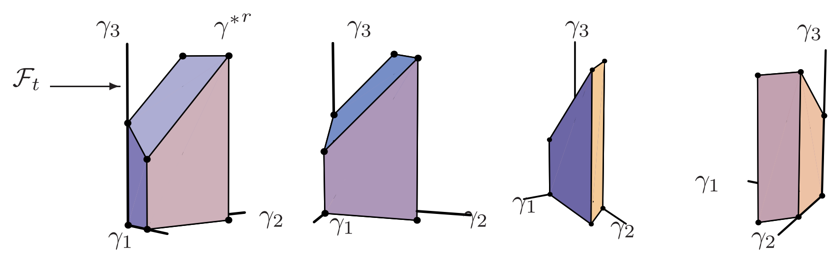


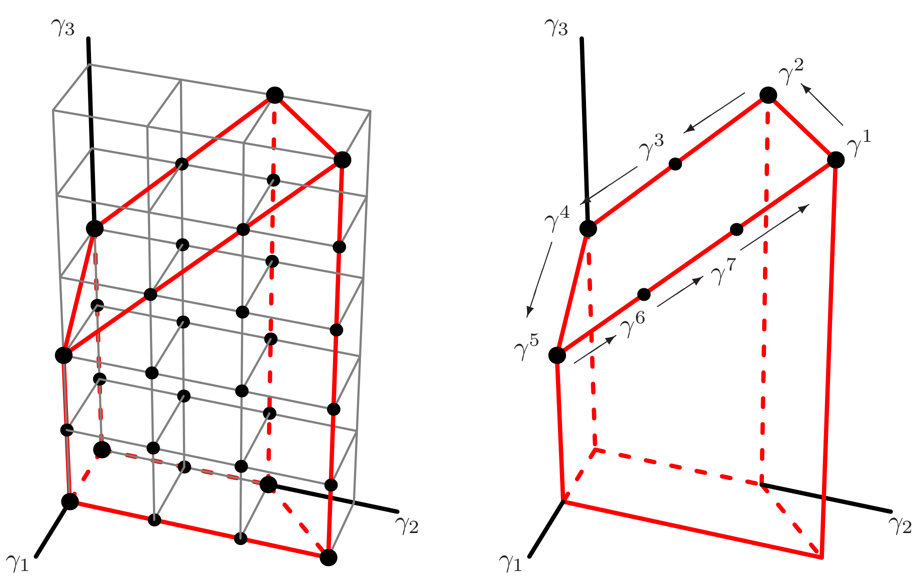
FIGURE 7. Four views of the polytope $\mathcal{P}^r$.

In order to see the relationship between the solutions h^{cd} obtained as we add circulations and the corresponding γ^r, it is more convenient to look at the frame of the polytope $\mathcal{P}^r$, depicted in Figure 8(a), together with the lattice determined by the integral vectors in the nonnegative orthant. The nonnegative integral points belonging to $\mathcal{P}^r$ are emphasized. Denote by $\gamma^1 = \gamma^{*r}$, γ^2, …, γ^7 the integral vectors belonging to $\mathcal{F}_t$, in counterclockwise order. Notice that adding circ^1 to h^{cd} has the effect of subtracting 1 from the components γ_1 and γ_2 of γ. Similarly, adding circ^2 to h^{cd} has the effect of subtracting 1 from γ_2 and γ_3. Figure 8(b) focuses on $\mathcal{F}_t$ and the integral points therein. The following diagram summarizes the operations with circulations on h^{cd} needed to generate all points in $\mathcal{F}_t$. Notice there is not a 1-to-1 correspondence between nonnegative integral h^{cd} satisfying (3.2) and the nonnegative integral γ^r in $\mathcal{F}_t$. For instance, the vector $h^{cd(4)} + \text{circ}^2 = (1, 4, 4, 7, 8, 2, 2, 3, 2, 1, 1, 2)$ is a valid nonnegative integral solution of (3.2), but the corresponding γ^r violates the nonnegative constraint $\gamma_2 \geq 0$.

$$
\begin{array}{ccccccccc}
\gamma^{*r} = \gamma^1 & \longrightarrow & \gamma^2 & \longrightarrow & \gamma^3 & \longrightarrow & \gamma^4 & \longrightarrow & \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \\
h^{*cd} = h^{cd(1)} & \xrightarrow{+\text{circ}^1} & h^{cd(2)} & \xrightarrow{+\text{circ}^2} & h^{cd(3)} & \xrightarrow{+\text{circ}^2} & h^{cd(4)} & \xrightarrow[-\text{circ}^1]{+\text{circ}^2} &
\end{array}
$$

$$
\begin{array}{ccccccccc}
\longrightarrow & \gamma^5 & \longrightarrow & \gamma^6 & \longrightarrow & \gamma^7 & \longrightarrow & \gamma^1 \\
 & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
\longrightarrow & h^{cd(5)} & \xrightarrow{-\text{circ}^2} & h^{cd(6)} & \xrightarrow{-\text{circ}^2} & h^{cd(7)} & \xrightarrow{-\text{circ}^2} & h^{cd(1)}
\end{array}
$$

The knowledge of γ^1, …, γ^7 in fact makes it possible to compute all the integral points in $\mathcal{P}^r$, since they lie in one of the segments $[\gamma^i, \hat{\gamma}_i]$, for $i = 1, \ldots, 7$, where $\hat{\gamma}^i_j = \gamma^i_j$, for $j \neq 3$, and $\hat{\gamma}_3 = 0$.



(a) Lattice of integral points in $\mathcal{P}^r$. (b) Integral points in facet $\mathcal{F}_t$ of $\mathcal{P}^r$.

FIGURE 8. Frame of polytope $\mathcal{P}^r$.

4.2. **Case $n \equiv 0 \mod 4$.** Suppose $n = 2i$, where $i \geq 2$ is even. Consider the linear system in $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{2i}) \geq 0$ constituted by the Morse inequalities (1.1), or, equivalently, (3.35), and the boundary and duality constraints (3.6) and (3.7), where we assume the pre-assigned index data $(h_0, \ldots, h_{2i})$ is such that $\sum_{j=0}^{2i}(-1)^j h_j$ is even.

As in the odd case, we will use some of the constraints defining $\mathcal{P}$ to eliminate variables, but the derivation is more involved in this case. This will enable us to work with an equivalent polytope $\mathcal{P}^r$, embedded in a lower-dimensional space.

The first equality in (3.35) may be used to eliminate γ_i from the system. This is accomplished by substituting

$$\gamma_i = (-1)^{i+1}\left(\sum_{j=0}^{2i}(-1)^{j+1}h_j - \sum_{j=0}^{i-1}(-1)^{j+1}\gamma_j - \sum_{j=i+1}^{2i}(-1)^{j+1}\gamma_j\right) \tag{4.5}$$

in the equations containing γ_i in (3.35), i.e., those corresponding to $k = 1, \ldots, i$. The constraint corresponding to a generic $k \in \{1, \ldots, i\}$ is thus transformed:

$$\begin{aligned}
&(-1)^{k+1}\left(\sum_{\substack{j=0\\ j\neq i}}^{2i-k}(-1)^{j+1}\gamma_j + (-1)^{2(i+1)}\left(-\sum_{j=0}^{i-1}(-1)^{j+1}\gamma_j - \sum_{j=i+1}^{2i}(-1)^{j+1}\gamma_j)\right)\right)\\
&= -(-1)^{k+1}\sum_{j=2i+1-k}^{2i}(-1)^{j+1}\gamma_j\\
&\leq (-1)^{k+1}\left(\sum_{j=0}^{2i-k}(-1)^{j+1}h_j - (-1)^{2(i+1)}\sum_{j=0}^{2i}(-1)^{j+1}h_j\right)\\
&= -(-1)^{k+1}\sum_{j=2i+1-k}^{2i}(-1)^{j+1}h_j.
\end{aligned}$$

Using the duality conditions (3.7) the above constraint becomes

$$\begin{aligned}
-(-1)^{k+1}\sum_{j=2i+1-k}^{2i}(-1)^{j+1}\gamma_j &= -(-1)^{k+1}\sum_{j=2i+1-k}^{2i}(-1)^{j+1}\gamma_{2i-j}\\
&= -(-1)^{k+1}\sum_{j=0}^{k-1}(-1)^{j+1}\gamma_j\\
&\leq -(-1)^{k+1}\sum_{j=2i+1-k}^{2i}(-1)^{j+1}h_j,
\end{aligned}$$

and $\gamma_i \geq 0$ implies

$$(-1)^{i+1}\left(\sum_{j=0}^{2i}(-1)^{j+1}h_j - \sum_{j=0}^{i-1}(-1)^{j+1}\gamma_j - \sum_{j=i+1}^{2i}(-1)^{j+1}\gamma_j\right) =$$

$$(-1)^{i+1}\left(\sum_{j=0}^{2i}(-1)^{j+1}h_j - \sum_{j=0}^{i-1}(-1)^{j+1}\gamma_j - \sum_{j=i+1}^{2i}(-1)^{j+1}\gamma_{2i-j}\right) =$$

$$(-1)^{i+1}\left(\sum_{j=0}^{2i}(-1)^{j+1}h_j - \sum_{j=0}^{i-1}(-1)^{j+1}\gamma_j - \sum_{\ell=0}^{i-1}(-1)^{j+1}\gamma_\ell\right) =$$

$$(-1)^{i+1}\left(\sum_{j=0}^{2i}(-1)^{j+1}h_j - 2\sum_{j=0}^{i-1}(-1)^{j+1}\gamma_j\right) \geq 0.$$

Therefore there is a 1-to-1 relationship between nonnegative solutions $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{2i})$ of (3.35), (3.6) and (3.7) and the solutions $\gamma^r = (\gamma_1, \ldots, \gamma_{i-1})$ of (4.6). Thus we may simplify our study of $\mathcal{P}$ by considering $\mathcal{P}^r = \{\gamma^r \in \mathbb{R}^{i-1} \mid$ constraints in (4.6)$\}$ instead.

(4.6)

$$\begin{aligned}
0 &\geq 1 - h_{2i},\\
\sum_{j=1}^{k}(-1)^{j+1}\gamma_j &\begin{cases} \leq 1 + \displaystyle\sum_{j=2i-k}^{2i}(-1)^{j+1}h_j, & \text{if } 1 \leq k \leq i-1,\ k \text{ odd},\\ \geq 1 + \displaystyle\sum_{j=2i-k}^{2i}(-1)^{j+1}h_j, & \text{if } 1 \leq k \leq i-1,\ k \text{ even},\end{cases}\\
(-1)^i\sum_{j=1}^{i-1}(-1)^{j+1}\gamma_j &\geq (-1)^i\left(1 + \frac{1}{2}\sum_{j=0}^{2i}(-1)^{j+1}h_j\right),\\
\sum_{j=1}^{2i-k}(-1)^{j+1}\gamma_j &\begin{cases} \leq 1 + \displaystyle\sum_{j=0}^{2i-k}(-1)^{j+1}h_j, & \text{if } i+1 \leq k \leq 2i-1,\ k \text{ odd},\\ \geq 1 + \displaystyle\sum_{j=0}^{2i-k}(-1)^{j+1}h_j, & \text{if } i+1 \leq k \leq 2i-1,\ k \text{ even},\end{cases}\\
0 &\geq 1 - h_0,\\
\gamma_j &\geq 0, \qquad\qquad \text{for } 1 \leq j \leq i-1.
\end{aligned}$$

Proposition 4.3. *The polytope $\mathcal{P}^r$ defined by* (4.6) *has integral vertices and each (integral) γ^r in the polytope corresponds to an (integral) nonnegative h^{cd} satisfying* (3.30). *Each vertex of $\mathcal{P}^r$ belongs to one of three faces:*

$$\mathcal{F}_t = \left\{\gamma \in \mathcal{P}^r \mid \sum_{j=1}^{i-1}(-1)^{j+1}\gamma_j = 1 + \min\left\{\sum_{j=0}^{i-1}(-1)^{j+1}h_j, \sum_{j=i+1}^{2i}(-1)^{j+1}h_j\right\}\right\},$$

$$\mathcal{F}_b = \left\{\gamma \in \mathcal{P}^r \mid \sum_{j=1}^{i-1}(-1)^{j+1}\gamma_j = 1 + \frac{1}{2}\sum_{j=0}^{2i}(-1)^{j+1}h_j\right\},$$

$$\mathcal{F}_0 = \{\gamma^r \in \mathcal{P}^r \mid \gamma_{i-1} = 0\}.$$

Proof. The matrix of coefficients A corresponding to the inequalities in (4.6), excepting the nonnegativity ones, is a $0, \pm 1$ matrix. It is is easy to see that the $0, 1$ matrix $\tilde{A} = (|a_{ij}|)$ has the consecutive ones property and thus is totally unimodular. Since A is obtained from $\tilde{A}$ by multiplying the even columns by -1, it is also totally unimodular. Taking into account the fact that the right-hand-side of (4.6) is integral, we conclude that all vertices in $\mathcal{P}^r$ are integral.

The correspondence between integral points of $\mathcal{P}^r$ and h^{cd} satisfying (3.30) mimics the proof of item (4) of Proposition 4.1.

The argument for the last assertion is similar to the one given in the proof of item (2) of Proposition 4.1. In order for a $\bar{\gamma}^r$ in $\mathcal{P}^r$ to be a vertex, the set of saturated constraints at $\bar{\gamma}^r$ must include one of the four inequalities involving γ_{i-1} in (4.6), repeated below for convenience. Notice we use the fact that i is even.

$$\sum_{j=1}^{i-1}(-1)^{j+1}\gamma_j \leq 1 + \sum_{j=i+1}^{2i}(-1)^{j+1}h_j, \tag{4.7}$$

$$\sum_{j=1}^{i-1}(-1)^{j+1}\gamma_j \leq 1 + \sum_{j=0}^{i-1}(-1)^{j+1}h_j, \tag{4.8}$$

$$\sum_{j=1}^{i-1}(-1)^{j+1}\gamma_j \geq 1 + \frac{1}{2}\sum_{j=0}^{2i}(-1)^{j+1}h_j, \tag{4.9}$$

$$\gamma_{i-1} \geq 0. \tag{4.10}$$

Inequalities (4.7)–(4.8) are equivalent to the inequality below.

$$\sum_{j=1}^{i-1}(-1)^{j+1}\gamma_j \leq 1 + \min\left\{\sum_{j=i+1}^{2i}(-1)^{j+1}h_j, \sum_{j=0}^{i-1}(-1)^{j+1}h_j\right\}. \tag{4.11}$$

Since one of the three inequalities (4.11), (4.9) or (4.10), must be tight at a vertex, the vertex must belong to $\mathcal{F}_t$, $\mathcal{F}_b$ or $\mathcal{F}_0$, respectively. □

The next proposition is the analogue of Proposition 4.2 for the $n \equiv 0 \mod 4$ case. As before we single out the vector γ^* corresponding to the complementary solution h^{*cd} of (3.31).

Proposition 4.4. *Assume* (3.31) *has a nonnegative solution. Let h^{*cd} be its complementary solution. Let $\gamma^* = \Gamma(h^{*cd})$, given by* (3.34)*, be the Betti number vector that solves the Morse inequalities* (1.1)*. The polytope $\mathcal{P}^r$ may be recast as*

$$\mathcal{P}^r = \{0 \leq \gamma^r \mid \tag{4.12}$$

$$\left.\begin{array}{l} (-1)^{k+1}\displaystyle\sum_{j=0}^{k}(-1)^{j+1}\gamma_j \leq (-1)^{k+1}\sum_{j=0}^{k}(-1)^{j+1}\gamma_j^*, \quad \textit{for } 1 \leq k \leq i-1 \\ \displaystyle\sum_{j=0}^{i-1}(-1)^{j+1}\gamma_j \geq \frac{1}{2}\sum_{j=0}^{2i}(-1)^{j+1}h_j \end{array}\right\}.$$

Furthermore, γ^* (resp., γ^{*r}) is a vertex and the maximum vector in $\mathcal{P}$ (resp. $\mathcal{P}^r$), componentwise.

Proof. If we assume (3.31) has a solution, the inequalities $0 \geq 1-h_{2i}$ and $0 \geq 1-h_0$ are redundant and may be eliminated. Grouping together the remaining inequalities in (3.35) we have

$$\sum_{j=1}^{k}(-1)^{j+1}\gamma_j \begin{cases} \leq \; 1+\min\left\{\displaystyle\sum_{j=2i-k}^{2i}(-1)^{j+1}h_j, \sum_{j=0}^{k}(-1)^{j+1}h_j\right\} & \text{if } 1\leq k\leq i-1, \; k \text{ odd},\\ \geq \; 1+\max\left\{\displaystyle\sum_{j=2i-k}^{2i}(-1)^{j+1}h_j, \sum_{j=0}^{k}(-1)^{j+1}h_j\right\} & \text{if } 1\leq k\leq i-1, \; k \text{ even},\end{cases}$$
$$\sum_{j=1}^{i-1}(-1)^{j+1}\gamma_j \;\geq\; 1+\frac{1}{2}\sum_{j=0}^{2i}(-1)^{j+1}h_j$$
$$\gamma_j \;\geq\; 0, \qquad \text{for } 1\leq j\leq i-1. \tag{4.13}$$

Now γ^* satisfies the first equation in (3.35) and (3.44). Using the first equation in (3.35) to eliminate γ_i^* in (3.44), and then using the duality conditions (3.7), we have

$$-\sum_{j=2i+1-k}^{2i}(-1)^{j+1}\gamma_j^* + (-1)^{2i+1-k}h_{2i-k}^d = -\sum_{j=0}^{k-1}(-1)^{j+1}\gamma_j^* + (-1)^{2i+1-k}h_{2i-k}^d$$
$$= -\sum_{j=2i+1-k}^{2i}(-1)^{j+1}h_j, \quad \text{for } 1\leq k\leq i. \tag{4.14}$$

Equation (3.47) may be rewritten as

$$\sum_{j=0}^{k-1}(-1)^{j+1}\gamma_j^* + (-1)^k h_k^c = \sum_{j=0}^{k-1}(-1)^{j+1}h_j, \quad \text{for } 2\leq k\leq i. \tag{4.15}$$

Equations (4.14)–(4.15), the facts that h^{*cd} satisfies (3.3), $\gamma_0^* = 1$ and that γ^{*r} satisfies (4.6) imply

$$\min\left\{\sum_{j=2i-k}^{2i}(-1)^{j+1}h_j, \sum_{j=0}^{k}(-1)^{j+1}h_j\right\} = \sum_{j=0}^{k}(-1)^{j+1}\gamma_j^*, \quad \text{for } 1\leq k\leq i-1, \; k \text{ odd},$$

and

$$\max\left\{\sum_{j=2i-k}^{2i}(-1)^{j+1}h_j, \sum_{j=0}^{k}(-1)^{j+1}h_j\right\} = \sum_{j=0}^{k}(-1)^{j+1}\gamma_j^*, \; \text{for } 1\leq k\leq i-1, \; k \text{ even}.$$

Substituting the above expressions in (4.13) we obtain (4.12).

Concerning the last assertion of the proposition, notice that the first inequality in (4.13) reads $\gamma_1 \leq \gamma_1^*$ and inequalities corresponding to $k=\ell-1$ and $k=\ell$ imply $\gamma_\ell \leq \gamma_\ell^*$, for $2\leq \ell\leq i-1$. Therefore γ^{*r} is the maximum vector, componentwise, of $\mathcal{P}^r$. This implies it is a vertex of $\mathcal{P}^r$. Finally, (4.5) and (4.12) imply, using the fact that i is even,

$$\gamma_i = 2\left(\sum_{j=0}^{i-1}(-1)^{j+1}\gamma_j - \sum_{j=0}^{2i}(-1)^{j+1}h_j\right) \leq 2\left(\sum_{j=0}^{i-1}(-1)^{j+1}\gamma_j^* - \sum_{j=0}^{2i}(-1)^{j+1}h_j\right),$$

which gives an upper bound for γ_i, achieved by γ_i^*. Thus γ^* is the maximum vector in $\mathcal{P}$, componentwise, and therefore a vertex thereof. □

The next example illustrates the relationship between the nonnegative integral solutions h^{cd} of (3.31) and the integral vectors γ^r of $\mathcal{P}^r$.

4.2.1. *Example.* Let $n = 2i = 8$ and $(h_0, \ldots, h_8) = (3, 5, 7, 8, 5, 2, 2, 2, 2)$. Figure 9 gives an unorthodox representation for the constraints (3.31). Notice that the two arcs dangling down from the two rightmost nodes represent the same variable, namely h_4^d. This deviates from the network-flow problem framework. Thus, while circ^1 and circ^2 are associated with cycles 1 and 2 as in the last example, there is a third "circulation", circ^3, whose support is given by $(h_4^c, \beta, h_4^d) = (1, -2, 1)$. Nevertheless, if we keep this anomaly in mind, we can still use the picture to quickly compute all the different solutions, which may be obtained by adding integer multiples of circ^i, for $1 \leq i \leq 3$, to the complementary solution h^{*cd} depicted in Figure 9.

$$
\begin{array}{rcrcrcl}
\gamma_1 & & & & & \leq & 1, \\
\gamma_1 & - & \gamma_2 & & & \geq & -1, \\
\gamma_1 & - & \gamma_2 & + & \gamma_3 & \leq & 1, \\
\gamma_1 & - & \gamma_2 & + & \gamma_3 & \geq & 0, \\
 & & \gamma_1, & \gamma_2, & \gamma_3 & \geq & 0.
\end{array}
$$

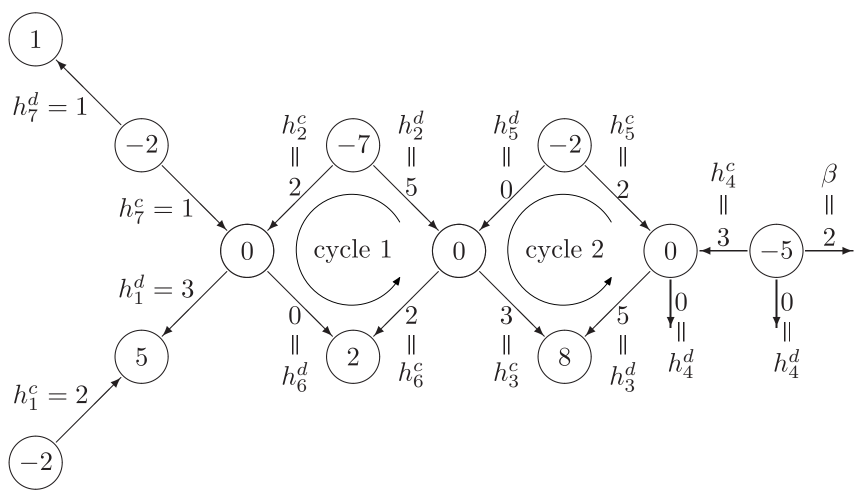
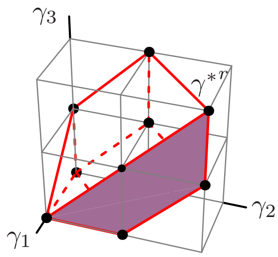
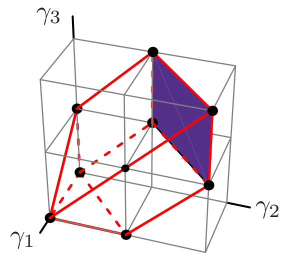


FIGURE 9. Solution h^{*cd} of example.

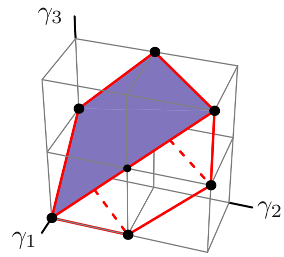
Polytope $\mathcal{P}^r$ corresponding to the given data is shown in Figure 10. All integral γ^r may be obtained starting at the maximum vector γ^{*r} and adding appropriate integer multiples of the vectors $(-1, -1, 0)$, $(0, -1, -1)$, $(0, 0, -1)$, $(1, 0, -1)$ and



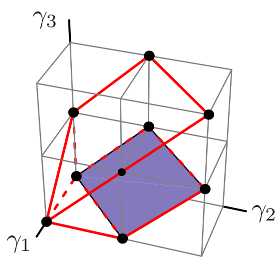
(a) $\mathcal{F}^1 = \{\gamma^r \in \mathcal{P}^r \mid \gamma_1 = 1\}$.



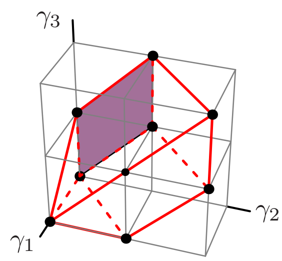
(b) $\mathcal{F}^2 = \{\gamma^r \in \mathcal{P}^r \mid \gamma_1 - \gamma_2 = -1\}$.



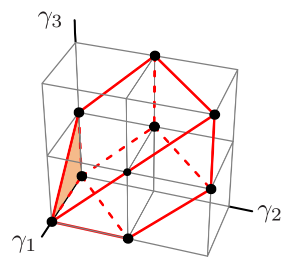
(c) $\mathcal{F}^3 = \mathcal{F}_t = \{\gamma^r \in \mathcal{P}^r \mid \gamma_1 - \gamma_2 + \gamma_3 = 1\}$.



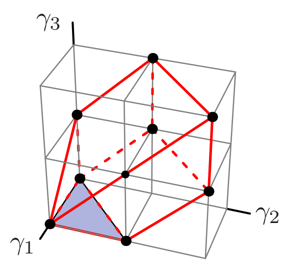
(d) $\mathcal{F}^4 = \mathcal{F}_b = \{\gamma^r \in \mathcal{P}^r \mid \gamma_1 - \gamma_2 + \gamma_3 = 0\}$.



(e) $\mathcal{F}^5 = \{\gamma^r \in \mathcal{P}^r \mid \gamma_1 = 0\}$.



(f) $\mathcal{F}^6 = \{\gamma^r \in \mathcal{P}^r \mid \gamma_2 = 0\}$.



(g) $\mathcal{F}^7 = \mathcal{F}_0 = \{\gamma^r \in \mathcal{P}^r \mid \gamma_3 = 0\}$.

FIGURE 10. Facets of $\mathcal{P}^r$ and integer grid.

$(-1, 0, 0)$. The corresponding operation on h^{cd} is shown below:

$$\begin{array}{ccc} \overline{\gamma} & \xrightarrow{(-1,-1,0)} & \hat{\gamma} \\ \updownarrow & & \updownarrow \\ \overline{h}^{cd} & \xrightarrow{+\text{circ}^1} & \hat{h}^{cd} \end{array} \qquad \begin{array}{ccc} \overline{\gamma} & \xrightarrow{(0,-1,-1)} & \hat{\gamma} \\ \updownarrow & & \updownarrow \\ \overline{h}^{cd} & \xrightarrow{+\text{circ}^2} & \hat{h}^{cd} \end{array} \qquad \begin{array}{ccc} \overline{\gamma} & \xrightarrow{(0,0,-1)} & \hat{\gamma} \\ \updownarrow & & \updownarrow \\ \overline{h}^{cd} & \xrightarrow{+\text{circ}^3} & \hat{h}^{cd} \end{array}$$

$$\begin{array}{ccc} \overline{\gamma} & \xrightarrow{\qquad (1,0,-1) \qquad} & \hat{\gamma} \\ \updownarrow & & \updownarrow \\ \overline{h}^{cd} & \xrightarrow{+\text{circ}^2-\text{circ}^1} & \hat{h}^{cd} \end{array} \qquad \begin{array}{ccc} \overline{\gamma} & \xrightarrow{\qquad (1,0,0) \qquad} & \hat{\gamma} \\ \updownarrow & & \updownarrow \\ \overline{h}^{cd} & \xrightarrow{-\text{circ}^1+\text{circ}^2-\text{circ}^2} & \hat{h}^{cd} \end{array}$$

4.3. **Case $n \equiv 2 \mod 4$.** Suppose $n = 2i$, where $i \geq 3$ is odd, and $\sum_{j=0}^{2i}(-1)^{j+1}h_j$ is even. The version (3.35) of the Morse inequalities (1.1) are valid for this case and may be manipulated as in section 4.2 in order to obtain the smaller equivalent system (4.6) in $\gamma^r = (\gamma_1, \ldots, \gamma_{i-1})$. Propositions 4.3 and 4.4 thus also hold for the case $n \equiv 2 \mod 4$, with the slightly different construction rule for h^{cd} corresponding to a given γ satisfying (3.35) given at the end of the proof of Proposition 3.9.

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