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THE COHOMOLOGY OF THE STEENROD ALGEBRA AND REPRESENTATIONS OF THE GENERAL LINEAR GROUPS

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Dedicated to Professor Nguyễn Hữu Anh on the occasion of his sixtieth birthday

Abstract. Let Tr_k be the algebraic transfer that maps from the coinvariants of certain GL_k-representations to the cohomology of the Steenrod algebra. This transfer was defined by W. Singer as an algebraic version of the geometrical transfer $tr_k : \pi_*^S((B\mathbb{V}_k)_+) \to \pi_*^S(S^0)$. It has been shown that the algebraic transfer is highly nontrivial, more precisely, that Tr_k is an isomorphism for $k = 1, 2, 3$ and that $Tr = \bigoplus_k Tr_k$ is a homomorphism of algebras.

In this paper, we first recognize the phenomenon that if we start from any degree d and apply Sq^0 repeatedly at most $(k-2)$ times, then we get into the region in which all the iterated squaring operations are isomorphisms on the coinvariants of the GL_k-representations. As a consequence, every finite Sq^0-family in the coinvariants has at most $(k-2)$ nonzero elements. Two applications are exploited.

The first main theorem is that Tr_k is not an isomorphism for $k \geq 5$. Furthermore, for every $k > 5$, there are infinitely many degrees in which Tr_k is not an isomorphism. We also show that if Tr_ℓ detects a nonzero element in certain degrees of $\mathrm{Ker}(Sq^0)$, then it is not a monomorphism and further, for each $k > \ell$, Tr_k is not a monomorphism in infinitely many degrees.

The second main theorem is that the elements of any Sq^0-family in the cohomology of the Steenrod algebra, except at most its first $(k-2)$ elements, are either all detected or all not detected by Tr_k, for every k. Applications of this study to the cases $k = 4$ and 5 show that Tr_4 does not detect the three families g, D_3 and p', and that Tr_5 does not detect the family $\{h_{n+1}g_n | \, n \geq 1\}$.

1. Introduction and statement of results

There have been several efforts, implicit or explicit, to analyze the Steenrod algebra by using modular representations of the general linear groups. (See Mùi [22, 23, 24], Madsen-Milgram [19], Adams-Gunawardena-Miller [3], Priddy-Wilkerson [27], Peterson [25], Wood [32], Singer [28], Priddy [26], Kuhn [15] and others.) In particular, one of the most direct attempts in studying the cohomology of the Steenrod algebra by means of modular representations of the general linear groups was the surprising work [28] by W. Singer, which introduced a homomorphism, the so-called

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algebraic transfer, mapping from the coinvariants of certain representations of the general linear group to the cohomology of the Steenrod algebra.

Let $\mathbb{V}_k$ denote a k-dimensional $\mathbb{F}_2$-vector space, and let $PH_*(B\mathbb{V}_k)$ denote the primitive subspace consisting of all elements in $H_*(B\mathbb{V}_k)$ that are annihilated by every positive-degree operation in the mod 2 Steenrod algebra, $\mathcal{A}$. Throughout the paper, the homology is taken with coefficients in $\mathbb{F}_2$. The general linear group $GL_k := GL(\mathbb{V}_k)$ acts regularly on $\mathbb{V}_k$ and therefore on the homology and cohomology of $B\mathbb{V}_k$. Since the two actions of $\mathcal{A}$ and GL_k upon $H^*(B\mathbb{V}_k)$ commute with each other, there are inherited actions of GL_k on $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} H^*(B\mathbb{V}_k)$ and $PH_*(B\mathbb{V}_k)$. In [28], W. Singer defined the algebraic transfer

$$Tr_k : \mathbb{F}_2 \underset{GL_k}{\otimes} PH_d(B\mathbb{V}_k) \to \mathrm{Ext}_{\mathcal{A}}^{k,k+d}(\mathbb{F}_2, \mathbb{F}_2)$$

as an algebraic version of the geometrical transfer $tr_k : \pi_*^S((B\mathbb{V}_k)_+) \to \pi_*^S(S^0)$ to the stable homotopy groups of spheres.

It has been proved that Tr_k is an isomorphism for $k = 1, 2$ by Singer [28] and for $k = 3$ by Boardman [4]. Among other things, these data together with the fact that $Tr = \bigoplus_k Tr_k$ is an algebra homomorphism (see [28]) show that Tr_k is highly nontrivial. Therefore, the algebraic transfer is expected to be a useful tool in the study of the mysterious cohomology of the Steenrod algebra, $\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$.

Directly calculating the value of Tr_k on any nonzero element is difficult (see [28], [4], [11]). In this paper, our main idea is to exploit the relationship between the algebraic transfer and the squaring operation Sq^0. It is well known (see [18]) that there are squaring operations Sq^i ($i \geq 0$) acting on the cohomology of the Steenrod algebra that share most of the properties with Sq^i on the cohomology of spaces. However, Sq^0 is not the identity. On the other hand, there is an analogous squaring operation Sq^0, the Kameko one, acting on the domain of the algebraic transfer and commuting with the classical Sq^0 on $\mathrm{Ext}_{\mathcal{A}}^k(\mathbb{F}_2, \mathbb{F}_2)$ through the algebraic transfer. We refer to Section 2 for its precise meaning.

The key point is that the behaviors of the two squaring operations do not agree in infinitely many certain degrees, called *k-spikes*. A k-spike degree is a number that can be written as $(2^{n_1} - 1) + \cdots + (2^{n_k} - 1)$, but cannot be written as a sum of less than k terms of the form $(2^n - 1)$. (See a discussion of this notion after Definition 3.1.) The following result is originally due to Kameko [13]: If m is a k-spike, then

$$\widetilde{Sq}^0 : PH_*(B\mathbb{V}_k)_{\frac{m-k}{2}} \to PH_*(B\mathbb{V}_k)_m$$

is an isomorphism of GL_k-modules, where $\widetilde{Sq}^0$ is certain GL_k-homomorphism such that $Sq^0 = 1 \underset{GL_k}{\otimes} \widetilde{Sq}^0$. (See Section 2 for an explanation of $\widetilde{Sq}^0$.)

We recognize two phenomena on the universality and the stability of k-spikes: First, if we start from any degree d that can be written as $(2^{n_1} - 1) + \cdots + (2^{n_k} - 1)$, and apply the function δ_k with $\delta_k(d) = 2d + k$ repeatedly at most $(k-1)$ times, then we get a k-spike; second, k-spikes are mapped by δ_k to k-spikes. Therefore, we have

Theorem 1.1. *Let d be an arbitrary nonnegative integer. Then*

$$(\widetilde{Sq}^0)^{i-k+2} : PH_*(B\mathbb{V}_k)_{2^{k-2}d+(2^{k-2}-1)k} \to PH_*(B\mathbb{V}_k)_{2^i d+(2^i-1)k}$$

is an isomorphism of GL_k-modules for every $i \geq k-2$.

From the result of Carlisle and Wood [8] on the boundedness conjecture, one can see that, for any degree d, there exists t such that

$$(\widetilde{Sq}^0)^{i-t} : PH_*(B\mathbb{V}_k)_{2^t d+(2^t-1)k} \to PH_*(B\mathbb{V}_k)_{2^i d+(2^i-1)k}$$

is an isomorphism of GL_k-modules for every $i \geq t$. However, this result does not confirm how large t should be. Theorem 1.1 shows that a rather small number $t = k-2$ commonly serves for every degree d. It will be pointed out in Remark 6.5 that $t = k-2$ is the minimum number for this purpose.

An inductive property of k-spikes, which will also play a key role in the paper, is that if m is a k-spike, then $(2^n - 1 + m)$ is a $(k+1)$-spike for n big enough.

Two applications of the study will be exploited in this paper. The first application is the following theorem, which is one of the paper's main results.

Theorem 1.2. *Tr_k is not an isomorphism for $k \geq 5$. Furthermore, for every $k > 5$, there are infinitely many degrees in which Tr_k is not an isomorphism.*

That Tr_5 is not an isomorphism in degree 9 is due to Singer [28].

In order to prove this theorem, using the notion of k-spike, we introduce the concept of a critical element in $\mathrm{Ext}_{\mathcal{A}}^k(\mathbb{F}_2, \mathbb{F}_2)$ in such a way that if d is the stem of a critical element, then Tr_k is not an isomorphism either in degree d or in degree $2d+k$. Further, we show that if x is critical, then so is $h_n x$ for n big enough. Our inductive procedure starts with the initial critical element Ph_2 for $k = 5$.

Combining Theorem 1.2 and the results by Singer [28], Boardman [4] and Bruner–Hà–Hưng [7], we get

Corollary 1.3. (i) *Tr_k is an isomorphism for $k = 1, 2$ and 3.*
(ii) *Tr_k is not an isomorphism for $k \geq 4$.*
(iii) *For $k = 4$ and for each $k > 5$, there are infinitely many degrees in which Tr_k is not an isomorphism.*

Remarkably, we do not know whether the algebraic transfer fails to be a monomorphism or fails to be an epimorphism for $k > 5$. Therefore, Singer's conjecture is still open.

Conjecture 1.4 ([28])**.** Tr_k is a monomorphism for every k.

The following theorem is related to this conjecture.

Theorem 1.5. *If Tr_ℓ detects a critical element, then it is not a monomorphism and further, for each $k > \ell$, there are infinitely many degrees in which Tr_k is not a monomorphism.*

A family $\{a_i | \ i \geq 0\}$ of elements in $\mathrm{Ext}_{\mathcal{A}}^k(\mathbb{F}_2, \mathbb{F}_2)$ (or in $\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$) is called a *$Sq^0$-family* if $a_i = (Sq^0)^i(a_0)$ for every $i \geq 0$. Recall that, if $a \in \mathrm{Ext}_{\mathcal{A}}^{k,t}(\mathbb{F}_2, \mathbb{F}_2)$, then $t-k$ is called the stem of a, and denoted by $\mathrm{Stem}(a)$. The *root degree* of a is the maximum nonnegative integer r such that $\mathrm{Stem}(a)$ can be written in the form $\mathrm{Stem}(a) = 2^r d + (2^r - 1)k$, for some nonnegative integer d.

The second application of our study is the following theorem, which is also one of the paper's main results.

Theorem 1.6. *Let $\{a_i | \ i \geq 0\}$ be an Sq^0-family in $Ext_{\mathcal{A}}^k(\mathbb{F}_2, \mathbb{F}_2)$ and let r be the root degree of a_0. If Tr_k detects a_n for some $n \geq \max\{k-r-2, 0\}$, then it detects a_i for every $i \geq n$ and detects a_j modulo $Ker(Sq^0)^{n-j}$ for $\max\{k-r-2, 0\} \leq j < n$.*

An Sq^0-family is called *finite* if it has only finitely many nonzero elements. The existence of finite Sq^0-families in $\mathrm{Ext}^*_{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2)$ is well known, and that of finite Sq^0-families in $\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$ will be shown in Section 9.

The following is a consequence of Theorem 1.1 and Theorem 1.6.

Corollary 1.7. (i) *Every finite Sq^0-family in $\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$ has at most $(k-2)$ nonzero elements.*

(ii) *If Tr_k is a monomorphism, then it does not detect any element of a finite Sq^0-family in $Ext^k_{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2)$ with at least $(k-1)$ nonzero elements.*

The following is an application of Theorem 1.6 into the investigation of Tr_4.

Proposition 1.8. *Let $\{b_i|\ i\geq 0\}$ and $\{\overline{b}_i|\ i\geq 0\}$ be the Sq^0-families in $Ext^4_{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2)$ with b_0 one of the usual five elements $d_0, e_0, p_0, D_3(0), p'_0$, and $\overline{b}_0$ one of the usual two elements f_0, g_1.*

(i) *If Tr_4 detects b_n for some $n\geq 1$, then it detects b_i for every $i\geq 1$.*
(ii) *If Tr_4 detects $\overline{b}_n$ for some $n\geq 0$, then it detects $\overline{b}_i$ for every $i\geq 0$.*

Based on this event, we prove the following theorem by showing that Tr_4 does not detect g_1, $D_3(1)$, p'_1.

Theorem 1.9. *Tr_4 does not detect any element in the three Sq^0-families $\{g_i|\ i\geq 1\}$, $\{D_3(i)|\ i\geq 0\}$ and $\{p'_i|\ i\geq 0\}$.*

This theorem gives further negative information on Minami's ([21]) conjecture that the localization of the algebraic transfer given by inverting Sq^0 is an isomorphism. The first negative answer to this conjecture was given in Bruner–Hà–Hưng [7] by showing that the element in $(Sq^0)^{-1}\mathrm{Ext}^4_{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2)$ represented by the family $\{g_i|\ i\geq 1\}$ is not detected by $(Sq^0)^{-1}Tr_4$. From Theorem 1.9, the two elements in $(Sq^0)^{-1}\mathrm{Ext}^4_{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2)$ represented respectively by the two families $\{D_3(i)|\ i\geq 0\}$ and $\{p'_i|\ i\geq 0\}$ are also not detected by $(Sq^0)^{-1}Tr_4$.

Recently, T. N. Nam informed the author about his claim that Tr_4 does not detect $D_3(0)$.

Conjecture 1.10. Tr_4 is a monomorphism that detects all elements in $\mathrm{Ext}^4_{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2)$ except the ones in the three Sq^0-families $\{g_i|\ i\geq 1\}$, $\{D_3(i)|\ i\geq 0\}$ and $\{p'_i|\ i\geq 0\}$.

The following theorem would complete our knowledge in Corollary 1.3 on whether Tr_5 is not an isomorphism in infinitely many degrees.

Theorem 1.11. *If $h_{n+1}g_n$ is nonzero, then it is not detected by Tr_5.*

It has been claimed by Lin [16] that $h_{n+1}g_n$ is nonzero for every $n\geq 1$.

The paper is divided into nine sections and organized as follows. Section 2 is a recollection of the Kameko squaring operation. In Section 3, we explain the notion of k-spike and then study the Kameko squaring and its iterated operations in k-spike degrees. Section 4 deals with an inductive way of producing k-spikes, which plays a key role in the proofs of Theorems 1.1, 1.2, 1.5 and 1.6. In Section 5, based on the concept of critical element, we prove Theorems 1.2 and 1.5. Section 6 is devoted to the proofs of Theorems 1.1 and 1.6. Sections 7 and 8 are applications to the study of the fourth and the fifth algebraic transfers. Final remarks and conjectures are given in Section 9.

2. Preliminaries on the squaring operation

To make the paper self-contained, this section is a recollection of the Kameko squaring operation Sq^0 on $\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$. The most important property of the Kameko Sq^0 is that it commutes with the classical Sq^0 on $\mathrm{Ext}^*_{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2)$ (defined in [18]) through the algebraic transfer (see [4], [21]).

This squaring operation is constructed as follows.

As is well known, $H^*(B\mathbb{V}_k)$ is the polynomial algebra, $P_k := \mathbb{F}_2[x_1, ..., x_k]$, on k generators $x_1, ..., x_k$, each of degree 1. By dualizing,

$$H_*(B\mathbb{V}_k) = \Gamma(a_1, \dots, a_k)$$

is the divided power algebra generated by $a_1, \dots, a_k$, each of degree 1, where a_i is dual to $x_i \in H^1(B\mathbb{V}_k)$. Here the duality is taken with respect to the basis of $H^*(B\mathbb{V}_k)$ consisting of all monomials in $x_1, \dots, x_k$.

In [13] Kameko defined a homomorphism

$$\begin{array}{rccc} \widetilde{Sq}^0: & H_*(B\mathbb{V}_k) & \to & H_*(B\mathbb{V}_k), \\ & a_1^{(i_1)} \cdots a_k^{(i_k)} & \mapsto & a_1^{(2i_1+1)} \cdots a_k^{(2i_k+1)}, \end{array}$$

where $a_1^{(i_1)} \cdots a_k^{(i_k)}$ is dual to $x_1^{i_1} \cdots x_k^{i_k}$. The following lemma is well known.

Lemma 2.1. *$\widetilde{Sq}^0$ is a homomorphism of GL_k-modules.*

See e.g. [7] for a proof. Further, there are two well-known relations,

$$Sq_*^{2t+1}\widetilde{Sq}^0 = 0, \ \ Sq_*^{2t}\widetilde{Sq}^0 = \widetilde{Sq}^0 Sq_*^t.$$

See [10] for an explicit proof. Therefore, $\widetilde{Sq}^0$ maps $PH_*(B\mathbb{V}_k)$ to itself.

The Kameko Sq^0 is defined by

$$Sq^0 = 1 \underset{GL_k}{\otimes} \widetilde{Sq}^0 : \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k) \to \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k).$$

The dual homomorphism $\widetilde{Sq}^0_* : P_k \to P_k$ of $\widetilde{Sq}^0$ is obviously given by

$$\widetilde{Sq}^0_*(x_1^{j_1} \cdots x_k^{j_k}) = \begin{cases} x_1^{\frac{j_1-1}{2}} \cdots x_k^{\frac{j_k-1}{2}}, & j_1, ..., j_k \text{ odd}, \\ 0, & \text{otherwise}. \end{cases}$$

Hence

$$\mathrm{Ker}(\widetilde{Sq}^0_*) = \overline{Even},$$

where $\overline{Even}$ denotes the vector subspace of P_k spanned by all monomials $x_1^{i_1} \cdots x_k^{i_k}$ with at least one exponent i_t even.

The following lemma is more or less obvious.

Lemma 2.2 ([7])**.** *Let k and d be positive integers. Suppose that each monomial $x_1^{i_1} \cdots x_k^{i_k}$ of P_k in degree $2d + k$ with at least one exponent i_t even is hit. Then*

$$\widetilde{Sq}^0_* : (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)_{2d+k} \to (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)_d$$

is an isomorphism of GL_k-modules.

Here, as usual, a polynomial is called *hit* if it is $\mathcal{A}$-decomposable in P_k.

A proof of this lemma is sketched as follows.

Let $s : P_k \to P_k$ be a right inverse of $\widetilde{Sq}^0_*$ defined by

$$s(x_1^{i_1} \cdots x_k^{i_k}) = x_1^{2i_1+1} \cdots x_k^{2i_k+1}.$$

It should be noted that s does not commute with the doubling map on $\mathcal{A}$, that is, in general,

$$Sq^{2t}s \neq sSq^t.$$

However, $\mathrm{Im}(Sq^{2t}s - sSq^t) \subset \overline{Even}$.

Let $\mathcal{A}^+$ denote the ideal of $\mathcal{A}$ consisting of all positive degree operations. Under the hypothesis of the lemma, we have

$$(\mathcal{A}^+P_k + \overline{Even})_{2d+k} \subset (\mathcal{A}^+P_k)_{2d+k}.$$

Therefore, the map

$$\begin{array}{rcl} \overline{s} : (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)_d & \to & (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)_{2d+k} \\ \overline{s}[X] & = & [sX] \end{array}$$

is a well-defined linear map. Further, it is the inverse of

$$\widetilde{Sq}^0_* : (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)_{2d+k} \to (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)_d.$$

Therefore, $\widetilde{Sq}^0_*$ is an isomorphism in degree $2d+k$.

3. The iterated squaring operations in k-spike degrees

The following notion, which is originally due to Kraines [14], formulates some special degrees that we will mainly be interested in.

Definition 3.1. A natural number m is called a *k-spike* if

(a) $m = (2^{n_1} - 1) + \cdots + (2^{n_k} - 1)$ with $n_1, ..., n_k > 0$, and

(b) m cannot be written as a sum of less than k terms of the form $(2^n - 1)$.

Note that *k-spike* is our terminology. Other authors write $\mu(m) = k$ to say m is a k-spike. (See e.g. Wood [33, Definition 4.4].)

One easily checks e.g. that 20 is a 4-spike, 27 is a 5-spike and 58 is a 6-spike.

Let $\alpha(m)$ denote the number of ones in the dyadic expansion of m. The following two lemmas are more or less obvious, but useful later.

Lemma 3.2. *Condition* (a) *in Definition* 3.1 *is equivalent to*

$$\alpha(m+k) \le k \le m, \;\; m \equiv k \ (\mathrm{mod}\ 2).$$

Proof. Suppose $m = (2^{n_1} - 1) + \cdots + (2^{n_k} - 1)$ with $n_1, ..., n_k > 0$. Then

$$m \ge k = (2^1 - 1) + \cdots + (2^1 - 1) \ \ (k \text{ terms}).$$

In addition, from $m + k = 2^{n_1} + \cdots + 2^{n_k}$ with $n_1, ..., n_k > 0$, it implies

$$\alpha(m+k) \le k \ \text{ and } \ m \equiv k \ (\mathrm{mod}\ 2).$$

The equality $\alpha(m+k) = k$ occurs if and only if $n_1, ..., n_k$ are different from each other.

Conversely, suppose that $\alpha(m+k) \le k \le m$ and $m \equiv k$ (mod 2). Let $i = \alpha(m+k)$. Then we have

$$m + k = 2^{m_1} + \cdots + 2^{m_i},$$

where $m_1, ..., m_i > 0$, as $m + k$ is even.

If at least one exponent $m_j > 1$, then we write $(m+k)$ as a sum of $(i+1)$ terms of 2-powers as follows:

$$m + k = 2^{m_1} + \cdots + 2^{m_j - 1} + 2^{m_j - 1} + \cdots + 2^{m_i}.$$

This procedure can be continued if at least one of the exponents $m_1, ..., m_j - 1, m_j - 1, ..., m_i$ is bigger than 1. After each step, the number of terms in the sum increases by 1. The procedure stops only in the case when the sum becomes $m+k = 2+\cdots+2$ with the number of terms $(m+k)/2 \geq 2k/2 = k$. In particular, we reached at some step a sum of exactly k terms

$$m + k = 2^{n_1} + \cdots + 2^{n_k}$$

with $n_1, ..., n_k > 0$, or equivalently

$$m = (2^{n_1} - 1) + \cdots + (2^{n_k} - 1).$$

The lemma is proved. $\square$

The following lemma helps to recognize k-spikes.

Lemma 3.3. *A natural number m is a k-spike if and only if*

(i) $\alpha(m + k) \leq k \leq m, \quad m \equiv k \,(\mathrm{mod}\ 2)$, *and*
(ii) $\alpha(m + i) > i$ *for* $1 \leq i < k$.

Proof. From Lemma 3.2, if m satisfies (i), then $m = (2^{n_1} - 1) + \cdots + (2^{n_k} - 1)$ with $n_1, ..., n_k > 0$. Also by Lemma 3.2, if m satisfies (ii), then it cannot be written as a sum of less than k terms of the form $(2^n - 1)$.

So, if m satisfies (i) and (ii), then it is a k-spike.

Conversely, suppose m is a k-spike. Then (i) holds by Lemma 3.2.

It suffices to show (ii). Suppose to the contrary that $\alpha(m + i) \leq i$ for some i with $1 \leq i < k$. We then have $\alpha(m + i) \leq i < k \leq m$. Let us consider the two cases.

Case 1: $m \equiv i \,(\mathrm{mod}\ 2)$. Then, by Lemma 3.2, we get $m = (2^{n_1} - 1) + \cdots + (2^{n_i} - 1)$ with $n_1, ..., n_i > 0$. This contradicts to the definition of a k-spike.

Case 2: $m \equiv i - 1 \,(\mathrm{mod}\ 2)$. It implies $i > 1$. Indeed, if $i = 1$, combining the hypothesis $\alpha(m + 1) \leq 1$ with the fact $m + 1$ is odd, we get $m + 1 = 1$. This contradicts the hypothesis that m is a natural number.

By Lemma 4.3 below, we have

$$\alpha(m + (i - 1)) = \alpha(m + i) - 1 \leq i - 1.$$

As $m \equiv i - 1 \,(\mathrm{mod}\ 2)$, we apply Lemma 3.2 again to see that m can be written as a sum of $(i - 1)$ terms of the form $(2^n - 1)$. This is also a contradiction.

Combining the two cases, we see that if m is a k-spike, then (i) and (ii) hold. The lemma follows. $\square$

The following proposition is originally due to Kameko [13]. We give a proof of it to make the paper self-contained.

Proposition 3.4. *If m is a k-spike, then*

$$\widetilde{Sq}^0_* : (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)_m \to (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)_{\frac{m-k}{2}}$$

is an isomorphism of GL_k-modules.

Proof. By using Lemma 2.2, it suffices to show that any monomial R of P_k in degree m with at least one even exponent is hit. Such a monomial R can be written, up to a permutation of variables, in the form

$$R = x_1 \cdots x_i Q^2,$$

with $0 \leq i < k$, where Q is a monomial in degree $(m-i)/2$.

If $i = 0$, then $R = Q^2$ is simply in the image of Sq^1. (It is also in the image of $Sq^{\frac{m}{2}}$, as $R = Q^2 = Sq^{\frac{m}{2}}Q$.) So, it suffices to consider the case $0 < i < k$.

Let χ be the anti-homomorphism in the Steenrod algebra. The so-called χ-trick, which was known to Brown and Peterson in the mid-sixties, states that

$$uSq^n(v) \equiv \chi(Sq^n)(u)v \mod \mathcal{A}^+M,$$

for u, v in any $\mathcal{A}$-algebra M. (See also Wood [32].) In our case, it claims that

$$R = x_1 \cdots x_i Q^2 = x_1 \cdots x_i Sq^{\frac{m-i}{2}} Q$$

is hit if and only if $\chi(Sq^{\frac{m-i}{2}})(x_1 \cdots x_i)Q$ is. We will show $\chi(Sq^{\frac{m-i}{2}})(x_1 \cdots x_i) = 0$.

As $\mathcal{A}$ is a commutative coalgebra, χ is a homomorphism of coalgebras (see [20, Proposition 8.6]). Then we have the Cartan formula

$$\chi(Sq^n)(uv) = \sum_{i+j=n} \chi(Sq^i)(u)\chi(Sq^j)(v).$$

Furthermore, it is shown by Brown and Peterson in [5] that

$$\chi(Sq^n)(x_j) = \begin{cases} x_j^{2^q}, & \text{if } n = 2^q - 1 \text{ for some } q, \\ 0, & \text{otherwise,} \end{cases}$$

for x_j in degree 1.

So, in order to prove $\chi(Sq^{\frac{m-i}{2}})(x_1 \cdots x_i) = 0$ we need only to show that $\frac{m-i}{2}$ cannot be written in the form

$$\frac{m-i}{2} = (2^{\ell_1} - 1) + \cdots + (2^{\ell_i} - 1)$$

with $\ell_1, ..., \ell_i \geq 0$. This equation is equivalent to

$$m = (2^{\ell_1+1} - 1) + \cdots + (2^{\ell_i+1} - 1).$$

Since $0 < i < k$, this equality contradicts the hypothesis that m is a k-spike. The proposition is completely proved. $\square$

The following lemma is the base for an iterated application of Proposition 3.4.

Lemma 3.5. *If m is a k-spike, then so is $(2m + k)$.*

Proof. (a) From the definition of k-spike,

$$m = (2^{n_1} - 1) + \cdots + (2^{n_k} - 1),$$

for $n_1, ..., n_k > 0$. It implies that

$$2m + k = (2^{n_1+1} - 1) + \cdots + (2^{n_k+1} - 1).$$

So, $2m + k$ satisfies the first condition in the definition of k-spike.

(b) Also by this definition, we have

$$\alpha(m + k - j) > k - j,$$

for $1 \leq j < k$. Hence

$$\begin{aligned} \alpha(2m+k+(k-2j)) &= \alpha(2(m+k-j)) \\ &= \alpha(m+k-j) > k-j > k-2j, \\ \alpha(2m+k+(k-2j+1)) &= \alpha(2(m+k-j)+1) \\ &= \alpha(2(m+k-j))+1 \quad \text{(by Lemma 4.3)} \\ &= \alpha(m+k-j)+1 \\ &> (k-j)+1 > k-2j+1. \end{aligned}$$

Note that each i satisfying $1 \leq i < k$ can be written either in the form $i = k-2j$ (for $1 \leq j \leq \frac{k-1}{2}$) or in the form $i = k-2j+1$ (for $1 \leq j \leq \frac{k}{2}$). So, the above two inequalities show that

$$\alpha(2m+k+i) > i,$$

for $1 \leq i < k$. Thus, $2m+k$ satisfies the second condition in Definition 3.1.

Combining parts (a) and (b), we see that $2m+k$ is a k-spike. □

Remark 3.6. The converse of Lemma 3.5 is false. For instance, 27 is a 5-spike, whereas $11 = (27-5)/2$ is not.

Proposition 3.7. *If m is a k-spike, then*

$$(\widetilde{Sq}^0)^{i+1} : PH_*(B\mathbb{V}_k)_{\frac{m-k}{2}} \to PH_*(B\mathbb{V}_k)_{2^i m+(2^i-1)k}$$

is an isomorphism of GL_k-modules for every $i \geq 0$.

Proof. If m is a k-spike, then by the dual of Proposition 3.4, we have an isomorphism of GL_k-modules

$$\widetilde{Sq}^0 : PH_*(B\mathbb{V}_k)_{\frac{m-k}{2}} \to PH_*(B\mathbb{V}_k)_m.$$

On the other hand, from Lemma 3.5, if m is a k-spike, then so is $2^i m + (2^i-1)k$ for every $i \geq 0$. Hence, repeatedly applying the dual of Proposition 3.4, we get an isomorphism of GL_k-modules

$$(\widetilde{Sq}^0)^{i+1} : PH_*(B\mathbb{V}_k)_{\frac{m-k}{2}} \to PH_*(B\mathbb{V}_k)_{2^i m+(2^i-1)k}.$$

The proposition is proved. □

Corollary 3.8. *If m is a k-spike, then*

$$(Sq^0)^{i+1} : (\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k))_{\frac{m-k}{2}} \to (\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k))_{2^i m+(2^i-1)k}$$

is an isomorphism for every $i \geq 0$.

4. Recognition of k-spikes

In this section, we introduce an inductive way of producing k-spikes, which will play a key role in the proofs of Theorems 1.1, 1.2, 1.5 and 1.6 in the next two sections.

Lemma 4.1. *If m is a k-spike, then $(2^n - 1 + m)$ is a $(k+1)$-spike for every n with $2^n \geq m+k-1$.*

To prove this lemma, we need the following two lemmas.

Lemma 4.2. *If* $2^n \geq a$, *then*

$$\alpha(2^n - 1 + a) \geq \alpha(a).$$

Proof. The proof proceeds by induction on $\alpha(a)$. If $\alpha(a) = 1$, then a is a power of 2, say $a = 2^p \leq 2^n$. We have

$$2^n - 1 + 2^p = 2^n + (2^p - 1) = 2^n + (2^{p-1} + \cdots + 2^0).$$

Thus $\alpha(2^n - 1 + 2^p) = 1 + p \geq 1 = \alpha(a)$.

Suppose inductively that the lemma is valid for $\alpha(a) = t$. We now consider the case $\alpha(a) = t + 1 > 1$. That is,

$$a = 2^{n_{t+1}} + 2^{n_t} + \cdots + 2^{n_1} \quad \text{with} \quad n_{t+1} > n_t > \cdots > n_1.$$

Set $b = 2^{n_t} + \cdots + 2^{n_1} < 2^{n_{t+1}}$; then $a = 2^{n_{t+1}} + b$, and $\alpha(b) = t$. From $2^n \geq a$, it implies $2^n > 2^{n_{t+1}}$. Therefore, we obtain

$$\begin{aligned} \alpha(2^n - 1 + a) &= \alpha(2^n + 2^{n_{t+1}} - 1 + b) \\ &= 1 + \alpha(2^{n_{t+1}} - 1 + b) \\ &\geq 1 + \alpha(b) \quad \text{(by the inductive hypothesis)} \\ &= 1 + t = \alpha(a). \end{aligned}$$

The lemma is proved. □

The following lemma is an obvious observation.

Lemma 4.3. *If* e *is an even number, then*

$$\alpha(e + 1) = \alpha(e) + 1.$$

Proof of Lemma 4.1. (a) Since $m = (2^{n_1} - 1) + \cdots + (2^{n_k} - 1)$, we get

$$(2^n - 1) + m = (2^n - 1) + (2^{n_1} - 1) + \cdots + (2^{n_k} - 1).$$

So the first condition in Definition 3.1 holds for $(2^n - 1 + m)$.

(b) If $1 \leq i < k$, then $2^n \geq m + k - 1 \geq m + i$. By Lemma 4.2, we have

$$\alpha(2^n - 1 + m + i) \geq \alpha(m + i) > i.$$

The last inequality comes from the hypothesis that m is a k-spike.

Finally, we need to show $\alpha(2^n - 1 + m + k) > k$. Recall that, as m is a k-spike, then $m \equiv k \pmod 2$. Hence, $e = (2^n - 1) + m + (k - 1)$ is even. By Lemma 4.3, we have

$$\begin{aligned} \alpha(2^n - 1 + m + k) &= \alpha(2^n - 1 + m + (k - 1) + 1) \\ &= \alpha(2^n - 1 + m + (k - 1)) + 1. \end{aligned}$$

Now, applying Lemma 4.2 to the case $2^n \geq m + k - 1$, we get

$$\begin{aligned} \alpha(2^n - 1 + m + (k - 1)) + 1 &\geq \alpha(m + (k - 1)) + 1 \\ &> (k - 1) + 1 = k. \end{aligned}$$

The last inequality comes from the fact that m is a k-spike.

In summary, the second condition in Definition 3.1 holds for $(2^n - 1 + m)$.

Combining parts (a) and (b), we see that $(2^n - 1 + m)$ is a $(k + 1)$-spike. The lemma is proved. □

Remark 4.4. Lemma 4.1 cannot be improved in the meaning that the hypothesis $2^{n+1} \geq m+k-1$ does not imply $(2^n - 1 + m)$ to be a $(k+1)$-spike. This is the case of $k = 5$, $m = 27$ and $2^n = 16$, because $15 + 27 = 42$ is not a 6-spike.

The following corollary is a key point in the proof of Lemma 6.3 and therefore in the proofs of Theorems 1.1 and 1.6.

Corollary 4.5. *$2^k - k$ is a k-spike for every $k > 0$.*

Proof. We prove this by induction on k. The corollary holds trivially for $k = 1$.

Suppose inductively that $2^k - k$ is a k-spike. Then, as $2^k > (2^k - k) + k - 1$, applying Lemma 4.2 to the case $n = k$ and $m = 2^k - k$, we have

$$2^{k+1} - (k+1) = (2^k - 1) + (2^k - k)$$

to be a $(k+1)$-spike. The corollary follows. $\square$

5. The algebraic transfer is not an isomorphism for $k \geq 4$

We first briefly recall the definition of the algebraic transfer. Let $\widehat{P}_1$ be the submodule of $\mathbb{F}_2[x_1, x_1^{-1}]$ spanned by all powers x_1^i with $i \geq -1$. The usual $\mathcal{A}$-action on $P_1 = \mathbb{F}_2[x_1]$ is canonically extended to an $\mathcal{A}$-action on $\mathbb{F}_2[x_1, x_1^{-1}]$ (see Adams [2], Wilkerson [31]). $\widehat{P}_1$ is an $\mathcal{A}$-submodule of $\mathbb{F}_2[x_1, x_1^{-1}]$. The inclusion $P_1 \subset \widehat{P}_1$ gives rise to a short exact sequence of $\mathcal{A}$-modules:

$$0 \to P_1 \to \widehat{P}_1 \to \Sigma^{-1}\mathbb{F}_2 \to 0\,.$$

Let e_1 be the corresponding element in $\mathrm{Ext}^1_{\mathcal{A}}(\Sigma^{-1}\mathbb{F}_2, P_1)$. Singer set $e_k = e_1 \otimes \cdots \otimes e_1 \in \mathrm{Ext}^k_{\mathcal{A}}(\Sigma^{-k}\mathbb{F}_2, P_k)$ (k times). Then, he defined $Tr_k^* : \mathrm{Tor}^{\mathcal{A}}_k(\mathbb{F}_2, \Sigma^{-k}\mathbb{F}_2) \to \mathrm{Tor}^{\mathcal{A}}_0(\mathbb{F}_2, P_k) = \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k$ by $Tr_k^*(z) = e_k \cap z$. Its image is a submodule of $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)^{GL_k}$. The k-th algebraic transfer is defined to be the dual of Tr_k^*.

We will need to apply the following theorem by D. Davis [9].

Let h_n be the nonzero element in $\mathrm{Ext}^{1,2^n}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$.

Theorem 5.1 ([9]). *If x is a nonzero element in $Ext^{k,k+d}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ with $4 \leq d \leq 2^j$, then $h_n x \neq 0$ for every $n \geq 2j + 1$.*

The following concept plays a key role in this section.

Definition 5.2. A nonzero element $x \in \mathrm{Ext}^k_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ is called *critical* if

(a) $Sq^0(x) = 0$, and
(b) $2\mathrm{Stem}(x) + k$ is a k-spike.

Note that, by Lemma 3.5, if $\mathrm{Stem}(x)$ is a k-spike, then so is $2\mathrm{Stem}(x) + k$.

Lemma 5.3. *If $x \in Ext^k_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ is critical, then so is $h_n x$ for every n with $2^n \geq \max\{4d^2, d+k\}$, where $d = Stem(x)$.*

Proof. First, we show that if x is critical, then $\mathrm{Stem}(x) > 0$. Indeed, suppose to the contrary that $\mathrm{Stem}(x) = 0$; then $x = h_0^k$. As x is critical, $Sq^0(x) = Sq^0(h_0^k) = h_1^k = 0$. This implies that $k \geq 4$, as h_1, h_1^2, h_1^3 all are nonzero, whereas $h_1^4 = 0$. However, $2\mathrm{Stem}(x) + k = k$ is not a k-spike for $k \geq 4$, because it can be written as a sum $k = 3 + 1 + \cdots + 1$ of $(k-2)$ terms of the form $(2^n - 1)$. This contradicts the definition of a critical element.

Now we have $\mathrm{Stem}(x) > 0$. Combining the fact that Sq^0 is a monomorphism in positive stems of $\mathrm{Ext}_{\mathcal{A}}^{k}(\mathbb{F}_2, \mathbb{F}_2)$ for $k \leq 4$, and that x is critical, we get $k > 4$. As x is a nonzero element of positive stem in $\mathrm{Ext}_{\mathcal{A}}^{k}(\mathbb{F}_2, \mathbb{F}_2)$ with $k > 4$, by the vanishing line theorem (see [1]) we have $\mathrm{Stem}(x) > 7$. Therefore, x satisfies the hypothesis of Theorem 5.1 that $d = \mathrm{Stem}(x) \geq 4$.

Let j be the smallest positive integer such that $2^j \geq d$. Then, the smallest positive integer i with $2^i \geq d^2$ should be either $2j$ or $2j-1$. From the hypothesis $2^n \geq 4d^2$, it implies that $2^{n-2} \geq d^2$. Hence, we get $n - 2 \geq i \geq 2j - 1$, or equivalently, $n \geq 2j + 1$.

Therefore, by Theorem 5.1, $h_n x \neq 0$ if $2^n \geq 4d^2$.

As Sq^0 is a homomorphism of algebras, we have

$$Sq^0(h_n x) = Sq^0(h_n) Sq^0(x) = Sq^0(h_n) \cdot 0 = 0.$$

Since x is critical, $m := 2d + k$ is a k-spike. We need to show that $2\mathrm{Stem}(h_n x) + (k+1)$ is a $(k+1)$-spike. We have

$$\mathrm{Stem}(h_n x) = 2^n - 1 + \mathrm{Stem}(x) = 2^n - 1 + d.$$

A routine calculation shows

$$\begin{aligned} 2\mathrm{Stem}(h_n x) + (k+1) &= 2(2^n - 1 + d) + (k+1) \\ &= 2^{n+1} - 2 + (2d + k) + 1 = 2^{n+1} - 1 + m. \end{aligned}$$

By Lemma 4.1, this number is a $(k+1)$-spike for every n with $2^{n+1} \geq m + k - 1 = 2(d+k) - 1$, or equivalently $2^n \geq d + k$.

In summary, $h_n x$ is critical for every n with

$$2^n \geq \max\{4d^2, d + k\}.$$

The lemma is proved. □

Remark 5.4. (a) Suppose $h_n x \neq 0$ although $2^n < 4(\mathrm{Stem}(x))^2$. If x is critical and $2^n \geq \mathrm{Stem}(x) + k$, then $h_n x$ is also critical.

(b) There is no critical element for $k \leq 4$, as Sq^0 is a monomorphism in positive stems of $\mathrm{Ext}_{\mathcal{A}}^{k}(\mathbb{F}_2, \mathbb{F}_2)$ for $k \leq 4$.

Proposition 5.5. (i) *For $k = 5$, there is at least one number, which is the stem of a critical element.*

(ii) *For each $k > 5$, there are infinitely many numbers, which are stems of critical elements.*

Proof. For $k = 5$, $Ph_2 \in \mathrm{Ext}_{\mathcal{A}}^{5,16}(\mathbb{F}_2, \mathbb{F}_2)$ is critical. Indeed, it is well known (see e.g. Tangora [30]) that $\mathrm{Ext}_{\mathcal{A}}^{5,32}(\mathbb{F}_2, \mathbb{F}_2) = 0$, so we get

$$Sq^0(Ph_2) = 0.$$

Further, by Lemma 3.3, $2\mathrm{Stem}(Ph_2) + 5 = 27$ is a 5-spike.

We can start the inductive argument of Lemma 5.3 with the initial critical element Ph_2. The proposition follows. □

The following theorem is also numbered as Theorem 1.2 in the Introduction.

Theorem 5.6. *Tr_k is not an isomorphism for $k \geq 5$. Furthermore, for every $k > 5$, there are infinitely many degrees in which Tr_k is not an isomorphism.*

Proof. In order to prove the theorem, by means of Proposition 5.5 it suffices to show that Tr_k is not an isomorphism either in degree d or in degree $2d+k$, where d denotes the stem of a critical element $x \in \mathrm{Ext}_{\mathcal{A}}^k(\mathbb{F}_2, \mathbb{F}_2)$.

We consider the following two cases:

Case 1: x is not in the image of Tr_k.

Then, Tr_k is not an epimorphism in degree d.

Case 2: $x = Tr_k(y)$ for some $y \in \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$.

From $x \neq 0$, it implies that $y \neq 0$. According to Boardman [4, Thm 6.9 and Cor 6.12] and Minami [21, Thm 4.4], we have a commutative diagram

$$\begin{tikzcd}
(\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k))_d \arrow[r, "Tr_k"] \arrow[d, "Sq^0"] & \mathrm{Ext}_{\mathcal{A}}^{k,k+d}(\mathbb{F}_2, \mathbb{F}_2) \arrow[d, "Sq^0"] \\
(\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k))_{2d+k} \arrow[r, "Tr_k"] & \mathrm{Ext}_{\mathcal{A}}^{k,2(k+d)}(\mathbb{F}_2, \mathbb{F}_2)
\end{tikzcd}$$

where the left vertical arrow is the Kameko Sq^0 and the right vertical one is the classical squaring operation.

As $m = 2d+k$ is a k-spike, by Corollary 3.8 the Kameko Sq^0 is an isomorphism. So, from $y \neq 0$, we have

$$z = Sq^0(y) \neq 0.$$

Now, by the commutativity of the diagram, we get

$$Tr_k(z) = Tr_k(Sq^0(y)) = Sq^0(Tr_k(y)) = Sq^0(x) = 0.$$

This means that Tr_k is not a monomorphism in degree $2d+k$. The theorem is completely proved. $\square$

Remark 5.7. (a) We can show that $\mathbb{F}_2 \underset{GL_5}{\otimes} PH_*(B\mathbb{V}_5)_{11} = 0$. It implies that Ph_2 is not detected by Tr_5.

(b) By Lemma 5.3, $h_n Ph_2$ is critical for every $n \geq 9$, as $\mathrm{Stem}(Ph_2) + 5 < 4(\mathrm{Stem}(Ph_2))^2 = 4 \cdot 11^2 = 484 < 2^9 = 512$. Also, by Remark 5.4, $h_n Ph_2$ is critical for $n = 4, 5, 6$, as it is nonzero (see [6]) and $2^4 \geq \mathrm{Stem}(Ph_2) + 5 = 16$. R. Bruner privately claimed $h_7 Ph_2 \neq 0$. It seems likely that $h_8 Ph_2 \neq 0$. If so, by the same argument, these two elements are also critical.

The following corollary is also numbered as Corollary 1.3 in the Introduction.

Corollary 5.8. (i) *Tr_k is an isomorphism for $k = 1, 2$ and 3.*

(ii) *Tr_k is not an isomorphism for $k \geq 4$.*

(iii) *For $k = 4$ and for each $k > 5$, there are infinitely many degrees in which Tr_k is not an isomorphism.*

This result is due to Singer [28] for $k = 1, 2$, to Boardman [4] for $k = 3$, and to Bruner–Hà–Hưng [7] for $k = 4$. That Tr_5 is not an isomorphism in degree 9 is also due to Singer [28]. The remaining part is shown by Theorem 5.6.

Our knowledge's gap on whether Tr_5 is not an isomorphism in infinitely many degrees will be studied in Section 8.

The following theorem is also numbered as Theorem 1.5 in the Introduction.

Theorem 5.9. *If Tr_ℓ detects a critical element, then it is not a monomorphism, and further, for each $k > \ell$, there are infinitely many degrees in which Tr_k is not a monomorphism.*

Proof. The proof proceeds by induction on $k \geq \ell$.

For $k = \ell$, suppose Tr_ℓ detects a critical element $x_\ell \in \mathrm{Ext}_{\mathcal{A}}^{\ell}(\mathbb{F}_2, \mathbb{F}_2)$. Then, by Case 2 in the proof of Theorem 5.6, Tr_ℓ is not a monomorphism in degree $2\mathrm{Stem}(x_\ell) + \ell$.

By means of this argument, it suffices to show that if Tr_k detects a critical element x_k, then Tr_{k+1} detects infinitely many critical elements, whose stems are different from each other.

From the hypothesis, $x_k = Tr_k(y_k)$ for some $y_k \in \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$. With ambiguity of notation, let h_n also denote the element in $\mathbb{F}_2 \underset{GL_1}{\otimes} PH_*(B\mathbb{V}_1)$, whose image under Tr_1 is the usual $h_n \in \mathrm{Ext}_{\mathcal{A}}^{1}(\mathbb{F}_2, \mathbb{F}_2)$. As $Tr = \bigoplus_k Tr_k$ is a homomorphism of algebras (see [28]), we have

$$Tr_{k+1}(h_n y_k) = Tr_1(h_n) Tr_k(y_k) = h_n x_k.$$

By Lemma 5.3, the element $h_n x_k$ is critical for every n with $2^n \geq \max\{4d^2, d+k\}$.

By the first part of the theorem, since Tr_{k+1} detects the critical element $h_n x_k$, it is not a monomorphism in degree $2\mathrm{Stem}(h_n x_k) + (k+1)$ for every n with $2^n \geq \max\{4d^2, d+k\}$. Thus, Tr_{k+1} is not a monomorphism in infinitely many degrees. The theorem follows. $\square$

6. The stability of the iterated squaring operations

The following theorem, which is also numbered as Theorem 1.1 in the Introduction, shows that Sq^0 is eventually isomorphic on $\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$. More precisely, it claims that if we start from any degree d of this module, and apply Sq^0 repeatedly at most $(k-2)$ times, then we get into the region, in which all the iterated squaring operations are isomorphisms.

Theorem 6.1. *Let d be an arbitrary nonnegative integer. Then*

$$(\widetilde{Sq}^0)^{i-k+2} : PH_*(B\mathbb{V}_k)_{2^{k-2}d+(2^{k-2}-1)k} \to PH_*(B\mathbb{V}_k)_{2^i d+(2^i-1)k}$$

is an isomorphism of GL_k-modules for every $i \geq k-2$.

In the theorem, for $k = 1$ we take the convention that $2^{1-2}d + (2^{1-2}-1)k = d$. Let us denote

$$(Sq^0)^{-1}(\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k))_d = \varinjlim_i \{\cdots \xrightarrow{Sq^0} (\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k))_{2^i d+(2^i-1)k} \xrightarrow{Sq^0} \cdots\}.$$

The following corollary is an immediate consequence of Theorem 6.1.

Corollary 6.2. *Let d be an arbitrary nonnegative integer. Then,*

(i) *The following iterated operation is an isomorphism for every $i \geq k-2$:*

$$(Sq^0)^{i-k+2} : \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)_{2^{k-2}d+(2^{k-2}-1)k} \to \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)_{2^i d+(2^i-1)k}.$$

(ii)

$$(Sq^0)^{-1}(\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k))_d \cong (\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k))_{2^{k-2}d+(2^{k-2}-1)k}.$$

(iii) *If $d = 2^{k-2}d' + (2^{k-2}-1)k$ for some nonnegative integer d', then*

$$(Sq^0)^{-1}(\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k))_d \cong (\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k))_d.$$

In order to prove Theorem 6.1, we need the following lemma. Let δ_k denote the function given by $\delta_k(d) = 2d + k$.

Lemma 6.3. *If d is a nonnegative integer with $\alpha(d+k) \leq k$, then $\delta_k^{k-1}(d) = 2^{k-1}d + (2^{k-1}-1)k$ is a k-spike.*

Proof. The lemma holds trivially for $k = 1$. Indeed, from the hypothesis $\alpha(d+1) \leq 1$ it implies that $d = 2^n - 1$ for some n. Then $\delta_1^0(d) = d = 2^n - 1$ is an 1-spike.

We now consider the case of $k \geq 2$. First, we observe that $k \leq 2^{k-1}d + (2^{k-1}-1)k \equiv k \pmod 2$ and

$$\alpha(2^{k-1}d + (2^{k-1}-1)k + k) = \alpha(2^{k-1}(d+k)) = \alpha(d+k) \leq k.$$

By Lemma 3.2, $\delta_k^{k-1}(d) = 2^{k-1}d+(2^{k-1}-1)k$ satisfies condition (a) of Definition 3.1. So, in order to prove the lemma, it suffices to show that

$$\alpha(2^{k-1}d + (2^{k-1}-1)k + i) > i \quad \text{for } 1 \leq i < k.$$

We now work modulo 2^{k-1}. First, we have

$$2^{k-1}d + (2^{k-1}-1)k \equiv (2^{k-1}-1)k \pmod{2^{k-1}}.$$

Let $k = 2^{n_t} + \cdots + 2^{n_1}$ be the dyadic expansion of k with $n_t > \cdots > n_1$. We get

$$(2^{k-1}-1)k = 2^{k-1}(2^{n_t} + \cdots + 2^{n_2}) + (2^{k-1+n_1} - (2^{n_t} + \cdots + 2^{n_1})).$$

Thus

$$\begin{aligned}
(2^{k-1}-1)k &\equiv 2^{k-1+n_1} - (2^{n_t} + \cdots + 2^{n_1}) \pmod{2^{k-1}} \\
&\equiv 2^{k-1} - (2^{n_t} + \cdots + 2^{n_1}) \pmod{2^{k-1}} \\
&\equiv 2^{k-1} - k \pmod{2^{k-1}},
\end{aligned}$$

where $2^{k-1} - k \geq 0$ because $k \geq 2$.

As a consequence, we get

$$2^{k-1}d + (2^{k-1}-1)k + i \equiv 2^{k-1} - k + i \pmod{2^{k-1}}$$

for $1 \leq i < k$. Since $k \geq 2$ and $d \geq 0$ we have

$$2^{k-1}d + (2^{k-1}-1)k + i \geq (2^{k-1}-1)2 + 1 > 2^{k-1}.$$

From this inequality it implies that, in the dyadic expansion of $2^{k-1}d + (2^{k-1}-1)k + i$, there is at least one nonzero term 2^n with $n \geq k-1$. On the other hand, as $2^{k-1} - k + i < 2^{k-1}$ for $1 \leq i < k$, the dyadic expansion of $2^{k-1} - k + i$ is just a combination of the 2-powers $2^0, 2^1, ..., 2^{k-2}$. Therefore, in order to prove

$$\alpha(2^{k-1}d + (2^{k-1}-1)k + i) > i$$

for $1 \leq i < k$, we need only to show that

$$\alpha(2^{k-1} - k + i) \geq i.$$

From Corollary 4.5, $2^{k-1}-(k-1)$ is a $(k-1)$-spike. Then we have

$$\alpha(2^{k-1}-(k-1)+j)>j$$

for $1\le j<k-1$. Set $i=j+1$; we get

$$\alpha(2^{k-1}-k+i)\ge i$$

for $2\le i<k$. In addition, it is obvious that

$$\alpha(2^{k-1}-k+1)\ge 1.$$

In summary, we have shown that

$$\alpha(2^{k-1}-k+i)\ge i$$

for $1\le i<k$. The lemma is proved. □

Remark 6.4. (a) Lemma 6.3 cannot be improved in the meaning that the number $\delta_k^{k-2}(d)=2^{k-2}d+(2^{k-2}-1)k$ is not a k-spike in general.

Indeed, taking $d=2^t+1-k$ with t big enough so that $d\ge 0$, we have

$$\alpha(2^{k-2}d+(2^{k-2}-1)k+(k-1))=\alpha(2^{t+k-2}+(2^{k-2}-1))=k-1.$$

By Lemma 3.3, $2^{k-2}d+(2^{k-2}-1)k$ is not a k-spike.

(b) However, a number could be a k-spike although it is not of the form $\delta_k^{k-1}(d)$ for any nonnegative integer d. For instance, this is the case of the following numbers with $k=4$:

$$\begin{array}{lll}\mathrm{Stem}(e_2)=80, & \mathrm{Stem}(f_1)=40, & \mathrm{Stem}(p_2)=144,\\ \mathrm{Stem}(D_3(2))=256, & \mathrm{Stem}(p_2')=288, & \end{array}$$

where $e_2, f_1, p_2, D_3(2), p_2'$ are the usual elements in $\mathrm{Ext}^4_{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2)$. This observation will be helpful in the proof of Proposition 7.2 below.

Proof of Theorem 6.1. According to Wood's theorem [32] (it was originally Peterson's conjecture), the primitive part $PH_*(B\mathbb{V}_k)$ is concentrated in the degrees d with $\alpha(d+k)\le k$. This fact together with the equality

$$\alpha(\delta_k^i(d)+k)=\alpha(2^i(d+k))=\alpha(d+k)$$

show that, if $\alpha(d+k)>k$, then the domain and the target of the homomorphism in the theorem are both zero.

If $\alpha(d+k)\le k$, then the theorem is an immediate consequence of Lemma 6.3 and Proposition 3.7. The theorem is proved. □

Remark 6.5. Let $k=5$ and $d=0$. As $\delta_5^{5-2}(0)=35$, Theorem 6.1 claims that

$$(\widetilde{Sq}^0)^{i-3}: PH_*(B\mathbb{V}_5)_{35}\to PH_*(B\mathbb{V}_5)_{5(2^i-1)}$$

is an isomorphism of GL_5-modules for $i\ge 3$. In the final section we will see that

$$Sq^0:\mathbb{F}_2\underset{GL_5}{\otimes}PH_*(B\mathbb{V}_5)_{15}\to\mathbb{F}_2\underset{GL_5}{\otimes}PH_*(B\mathbb{V}_5)_{35}$$

is not a monomorphism. This shows that Theorem 6.1 cannot be improved in the meaning that $(k-2)$ is, in general, the minimum times that we must repeatedly apply Sq^0 to get into "the isomorphism region" of the iterated squaring operations.

A family $\{a_i|\ i\ge 0\}$ of elements in $\mathrm{Ext}^k_{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2)$ is called an Sq^0-*family* if $a_i=(Sq^0)^i(a_0)$ for every $i\ge 0$. An Sq^0-family in $\mathbb{F}_2\underset{GL_k}{\otimes}PH_*(B\mathbb{V}_k)$ is similarly defined.

Definition 6.6. Let $a_0 \in \mathrm{Ext}_{\mathcal{A}}^k(\mathbb{F}_2, \mathbb{F}_2)$. The *root degree* of a_0 is the maximum nonnegative integer r such that $\mathrm{Stem}(a_0)$ can be written in the form

$$\mathrm{Stem}(a_0) = \delta_k^r(d) = 2^r d + (2^r - 1)k,$$

for some nonnegative integer d.

The following theorem is also numbered as Theorem 1.6 in the Introduction.

Theorem 6.7. *Let $\{a_i | \ i \geq 0\}$ be an Sq^0-family in $Ext_{\mathcal{A}}^k(\mathbb{F}_2, \mathbb{F}_2)$ and let r be the root degree of a_0. If Tr_k detects a_n for some $n \geq \max\{k-r-2, 0\}$, then it detects a_i for every $i \geq n$ and detects a_j modulo $Ker(Sq^0)^{n-j}$ for $\max\{k-r-2, 0\} \leq j < n$.*

Proof. It is easy to see that

$$\alpha(\mathrm{Stem}(a_i) + k) = \alpha(2^i(\mathrm{Stem}(a_0) + k)) = \alpha(\mathrm{Stem}(a_0) + k).$$

Suppose $\alpha(\mathrm{Stem}(a_0) + k) > k$; then we have $\alpha(\mathrm{Stem}(a_i) + k) > k$ for every $i \geq 0$. By Wood's theorem [32] (it was originally Peterson's conjecture), $PH_*(B\mathbb{V}_k)_t = 0$ in any degree t with $\alpha(t + k) > k$. So, all elements of the family $\{a_i | \ i \geq 0\}$ are not detected by Tr_k.

Now we consider the case where $\alpha(\mathrm{Stem}(a_0) + k) \leq k$. We observe that

$$\alpha(\mathrm{Stem}(a_0) + k) = \alpha(2^r(d + k)) = \alpha(d + k) \leq k.$$

Set $q = \max\{k - r - 2, 0\}$, and we have

$$\mathrm{Stem}(a_{q+1}) = \delta_k^{q+1}(\mathrm{Stem}(a_0)) = \delta_k^{q+r+1}(d).$$

Note that

$$q + r + 1 = \max\{k - r - 2, 0\} + r + 1 \geq (k - r - 2) + r + 1 = k - 1.$$

So, by Lemmas 6.3 and 3.5, $\mathrm{Stem}(a_{q+1})$ is a k-spike.

According to Theorem 6.1, if $c = \mathrm{Stem}(a_q)$, then

$$(\widetilde{Sq}^0)^{i-q} : PH_*(B\mathbb{V}_k)_c \to PH_*(B\mathbb{V}_k)_{2^{i-q}c+(2^{i-q}-1)k}$$

is an isomorphism of GL_k-modules for every $i \geq q$.

Suppose Tr_k detects a_n with $n \geq q$, that is, $a_n = Tr_k(\widetilde{a}_n)$ for some $\widetilde{a}_n$ in $\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$. If $i \geq n$, then we set $\widetilde{a}_i = (Sq^0)^{i-n}(\widetilde{a}_n)$. As the squaring operations commute with each other through the algebraic transfer, we have

$$\begin{aligned} a_i &= (Sq^0)^{i-n}(a_n) = (Sq^0)^{i-n} Tr_k(\widetilde{a}_n) \\ &= Tr_k(Sq^0)^{i-n}(\widetilde{a}_n) = Tr_k(\widetilde{a}_i). \end{aligned}$$

Thus, a_i is detected by Tr_k for every $i \geq n$.

Next we consider j with $\max\{k - r - 2, 0\} \leq j < n$. Then we set

$$\widetilde{a}_j = [(Sq^0)^{n-j}]^{-1}(\widetilde{a}_n).$$

This makes sense, as it is shown above that $(Sq^0)^{n-j}$ is isomorphic in the degree of $\widetilde{a}_j$. Again, as the squaring operations commute with each other through the algebraic transfer, we have

$$\begin{aligned} (Sq^0)^{n-j} Tr_k(\widetilde{a}_j) &= Tr_k(Sq^0)^{n-j}(\widetilde{a}_j) = Tr_k(\widetilde{a}_n) \\ &= a_n = (Sq^0)^{n-j}(a_j). \end{aligned}$$

As a consequence, we get

$$Tr_k(\widetilde{a}_j) = a_j \pmod{\mathrm{Ker}(Sq^0)^{n-j}}.$$

This means that Tr_k detects a_j modulo $\mathrm{Ker}(Sq^0)^{n-j}$. The theorem is proved. $\square$

Remark 6.8. (a) Under the hypothesis of Theorem 6.7, let

$$a'_i = Tr_k(Sq^0)^{i-n}(\widetilde{a}_n)$$

for every $i \geq \max\{k-r-2, 0\}$ no matter whether $i \geq n$ or $i < n$. Then we get a new Sq^0-family $\{a'_i|\ i \geq \max\{k-r-2,0\}\}$, whose every element is detected by Tr_k and

$$a'_i = \begin{cases} a_i, & \text{if } i \geq n, \\ a_i \pmod{\mathrm{Ker}(Sq^0)^{n-i}}, & \text{if } i < n. \end{cases}$$

The new Sq^0-family is called the *adjustment* of the original one.

(b) Theorem 6.7 is still valid and can be shown by the same proof if we replace $\max\{k-r-2,0\}$ by any number q such that $\mathrm{Stem}(a_{q+1})$ is a k-spike. This remark will be useful in the proof of Proposition 7.2 for the case $k = 4$.

Corollary 6.9. *Let $\{a_i|\ i \geq 0\}$ be an Sq^0-family in $Ext^k_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ and let r be the root degree of a_0. Suppose the classical Sq^0 is a monomorphism in the stems of the elements $\{a_i|\ i \geq \max\{k-r-2, 0\}\}$. If Tr_k detects a_n for some $n \geq \max\{k-r-2,0\}$, then it detects a_i for every $i \geq \max\{k-r-2,0\}$.*

An Sq^0-family is called *finite* if it has only finitely many nonzero elements, *infinite* if all of its elements are nonzero. The following is also numbered as Corollary 1.7 in the Introduction.

Corollary 6.10. (i) *Every finite Sq^0-family in $\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$ has at most $(k-2)$ nonzero elements.*

(ii) *If Tr_k is a monomorphism, then it does not detect any element of a finite Sq^0-family in $Ext^k_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ with at least $(k-1)$ nonzero elements.*

Proof. (i) Suppose that $\{\widetilde{a}_i|\ i \geq 0\}$ is an Sq^0-family in $\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$ with at least $(k-1)$ nonzero elements. Then $\widetilde{a}_0, \widetilde{a}_1, ..., \widetilde{a}_{k-2}$ are its first $(k-1)$ nonzero elements. Set $d = \deg(\widetilde{a}_0)$; then $\deg(\widetilde{a}_{k-2}) = 2^{k-2}d + (2^{k-2}-1)k$. Therefore, by Corollary 6.2,

$$(Sq^0)^{i-k+2} : \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)_{2^{k-2}d+(2^{k-2}-1)k} \to \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)_{2^i d+(2^i-1)k}$$

is an isomorphism for every $i \geq k-2$. Therefore, from $\widetilde{a}_{k-2} \neq 0$ it implies that $\widetilde{a}_i = (Sq^0)^{i-k+2}(\widetilde{a}_{k-2})$ is nonzero for every $i \geq k-2$. Thus, the Sq^0-family is infinite.

(ii) Let $a_0, a_1, ..., a_{k-2}$ be the last $(k-1)$ nonzero elements of the given finite Sq^0-family in $\mathrm{Ext}^k_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$. As a_{k-2} is the last nonzero element in the Sq^0-family, we have $Sq^0(a_{k-2}) = 0$. Set $d = \mathrm{Stem}(a_0)$; then by Lemma 6.3, $2\mathrm{Stem}(a_{k-2}) + k = 2^{k-1}d + (2^{k-1}-1)k$ is a k-spike. So, a_{k-2} is critical.

Suppose to the contrary that Tr_k detects some (nonzero) element in the Sq^0-family. Then, as the squaring operations commute with each other through the algebraic transfer, Tr_k also detects the critical element a_{k-2}. According to Theorem 5.9, this contradicts the hypothesis that Tr_k is a monomorphism. The corollary is proved. $\square$

7. On behavior of the fourth algebraic transfer

This section is an application of the previous section into the study of Tr_4. We refer to [30], [6], [17] for an explanation of the generators of $\operatorname{Ext}_{\mathcal{A}}^4(\mathbb{F}_2, \mathbb{F}_2)$.

It has been known (see [17]) that the graded module $\operatorname{Ext}_{\mathcal{A}}^4(\mathbb{F}_2, \mathbb{F}_2)$ is generated by $h_i h_j h_\ell h_m$, $h_i c_j$, $d_i, e_i, f_i, g_{i+1}, p_i, D_3(i), p'_i$ and subject to the relations

$$\begin{aligned} &h_i h_{i+1} = 0, \quad h_i h_{i+2}^2 = 0, \quad h_i^3 = h_{i-1}^2 h_{i+1}, \\ &h_i^2 h_{i+3}^2 = 0, \quad h_i c_j = 0 \qquad \text{for } i = j-1, j, j+2, j+3. \end{aligned}$$

The following is also numbered as Conjecture 1.10 in the Introduction.

Conjecture 7.1. *Tr_4 is a monomorphism that detects all elements in $\operatorname{Ext}_{\mathcal{A}}^4(\mathbb{F}_2, \mathbb{F}_2)$ except the ones in the three Sq^0-families $\{g_i|\ i \geq 1\}$, $\{D_3(i)|\ i \geq 0\}$ and $\{p'_i|\ i \geq 0\}$.*

That Tr_4 does not detect the family $\{g_i|\ i \geq 1\}$ is due to Bruner–Hà–Hưng [7]. Recently, T. N. Nam informed the author about his claim that Tr_4 does not detect the element $D_3(0)$.

The following proposition, which is also numbered as Proposition 1.8 in the Introduction, is an attempt to prepare for a proof of Conjecture 7.1.

Proposition 7.2. *Let $\{b_i|\ i \geq 0\}$ and $\{\overline{b}_i|\ i \geq 0\}$ be the Sq^0-families in $Ext_{\mathcal{A}}^4(\mathbb{F}_2, \mathbb{F}_2)$ with b_0 one of the usual five elements $d_0, e_0, p_0, D_3(0), p'_0$, and $\overline{b}_0$ one of the usual two elements f_0, g_1.*

(i) *If Tr_4 detects b_n for some $n \geq 1$, then it detects b_i for every $i \geq 1$.*
(ii) *If Tr_4 detects $\overline{b}_n$ for some $n \geq 0$, then it detects $\overline{b}_i$ for every $i \geq 0$.*

Proof. Although the stems of b_2 and $\overline{b}_1$ cannot be written as $\delta_4^3(d)$ for some non-negative integer d (except for $b_2 = d_2$ and $\overline{b}_1 = g_2$), it is easy to check by using Lemma 3.3 that they all are 4-spikes.

Following part (b) of Remark 6.8, we can show this proposition by the same argument as given in the proof of Theorem 6.7. Furthermore, as Sq^0 is a monomorphism in positive stems of $\operatorname{Ext}_{\mathcal{A}}^4(\mathbb{F}_2, \mathbb{F}_2)$ (see e.g. [17]), the proposition has the strong formulation as in Corollary 6.9.

The proposition is proved. □

By means of Proposition 7.2, to prove Conjecture 7.1 it suffices to show that:

(1) Tr_4 detects d_0, e_0, f_0, p_0;
(2) Tr_4 does not detect $g_1, D_3(1), p'_1$; and
(3) Tr_4 is a monomorphism.

The following theorem is also numbered as Theorem 1.9 in the Introduction.

Theorem 7.3. *Tr_4 does not detect any element in the three Sq^0-families $\{g_i|\ i \geq 1\}$, $\{D_3(i)|\ i \geq 0\}$ and $\{p'_i|\ i \geq 0\}$.*

Outline of the proof. First, we show that $\mathbb{F}_2 \underset{GL_4}{\otimes} PH_*(B\mathbb{V}_4)$ is zero in degree 20. Therefore, Tr_4 does not detect g_1 of stem 20 and, by Proposition 7.2, does not detect any element in the Sq^0-family $\{g_i|\ i \geq 1\}$. (Note again that this part of the theorem is due to Bruner–Hà–Hưng [7].)

Second, as the stems of $D_3(1)$ and p'_1 are respectively 126 and 142, we focus to the GL_4-module $PH_*(B\mathbb{V}_4)$ in degrees 126 and 142. By routine computations, we show that $PH_*(B\mathbb{V}_4)$ has dimension 80 and 285 in degrees 126 and 142, respectively, and further that $\mathbb{F}_2 \underset{GL_4}{\otimes} PH_*(B\mathbb{V}_4)$ is of dimension 1 in these two degrees.

Note that, as Tr_1 detects the family $\{h_n |\ n \geq 0\}$ (see [28]), the homomorphism of algebras $Tr = \bigoplus_k Tr_k$ detects the subalgebra generated by the family $\{h_n |\ n \geq 0\}$. So, Tr_4 definitely sends the two generators of its domain in degrees 126 and 142 to the nonzero elements $h_0^2 h_6^2$ and $h_0^2 h_4 h_7$, respectively. Therefore, the two elements $D_3(1)$ and p_1' of, respectively, stems $126, 142$ are not detected by Tr_4.

The theorem is proved by combining this fact and Proposition 7.2. □

8. An observation on the fifth algebraic transfer

From Corollary 5.8, the following conjecture naturally comes up.

Conjecture 8.1. There are infinitely many degrees in which Tr_5 is not an isomorphism.

The fact that g_n is not detected by Tr_4 and that $Tr = \bigoplus_k Tr_k$ is a homomorphism of algebras do not imply that $h_i g_n$ is not detected by Tr_5. For instance, $h_0 g_1 = h_2 e_0$ and $h_1 g_1 = h_2 f_0$ are presumably detected by Tr_5, as e_0 and f_0 are expectedly detected by Tr_4.

The purpose of this section is to prove the following, which is also numbered as Theorem 1.11 in the Introduction.

Theorem 8.2. *If $h_{n+1} g_n$ is nonzero, then it is not detected by Tr_5.*

Outline of the proof. We first observe that, as Sq^0 is a homomorphism of algebras, $\{h_{n+1} g_n |\ n \geq 1\}$ is an Sq^0-family, that is,

$$(Sq^0)^{n-1}(h_2 g_1) = h_{n+1} g_n,$$

for every $n \geq 1$.

Next, using Lemma 3.3 we easily show that $\mathrm{Stem}(h_2 g_1) = 23$ is not a 5-spike, but $\delta_5(23) = 2 \cdot 23 + 5 = 51$ is. So, by Proposition 3.7,

$$(\widetilde{Sq}^0)^i : PH_*(B\mathbb{V}_5)_{23} \to PH_*(B\mathbb{V}_k)_{2^i \cdot 23 + (2^i - 1)5}$$

is an isomorphism of GL_5-modules for every $i \geq 0$.

In addition, a routine computation shows that $PH_*(B\mathbb{V}_5)$ is of dimension 1245 in degree 23, and further that

$$\mathbb{F}_2 \underset{GL_5}{\otimes} PH_*(B\mathbb{V}_5)_{23} = 0.$$

As a consequence, we get

$$\mathbb{F}_2 \underset{GL_5}{\otimes} PH_*(B\mathbb{V}_5)_{2^i \cdot 23 + (2^i - 1)5} = 0,$$

for every $i \geq 0$. So, the domain of Tr_5 is zero in the degree that equals

$$\mathrm{Stem}(h_{n+1} g_n) = 2^{n-1} \cdot 23 + (2^{n-1} - 1)5$$

for every $n \geq 1$.

Therefore, if $h_{n+1} g_n$ is nonzero, then it is not detected by Tr_5. The theorem is proved. □

Corollary 8.3. *If $h_{n+1} g_n$ is nonzero for every $n \geq 1$, then there are infinitely many degrees, namely the degrees of $h_{n+1} g_n$ for $n \geq 1$, in which Tr_5 is not an epimorphism.*

The corollary's hypothesis is claimed to be true by Lin [16]. So, Conjecture 8.1 is established.

Remark 8.4. As $h_3g_2 = h_5g_1$ (see [30]) and Sq^0 is a homomorphism of algebras, Theorem 8.2 also shows that if $h_{n+4}g_n$ is nonzero, then it is not detected by Tr_5.

Which elements in $\mathrm{Ext}^5_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ are detected by Tr_5?

This question can be partially answered by using the fact that $Tr = \bigoplus_k Tr_k$ is an algebra homomorphism and the information on elements detected by Tr_k for $k \leq 4$. For instance, $h_3D_3(0) = h_0d_2$ (see [6]) is presumably detected by Tr_5, as h_0 is detected by Tr_1 and d_2 is expectedly detected by Tr_4 (see Conjecture 7.1).

Based on Theorem 6.7 and concrete calculations, the following conjecture presents some "new" families, which are expectedly detected by Tr_5.

Conjecture 8.5. Tr_5 detects every element in the Sq^0-families initiated by the classes $n, x, h_0g_2, D_1, H_1, h_1D_3(0), h_2D_3(0), Q_3, h_4D_3(0), h_6g_1, h_0g_3$ of stems 31, 37, 44, 52, 62, 62, 64, 67, 76, 83, 92, respectively.

Conjectures 8.5 and 7.1, together with the fact that $Tr = \bigoplus_k Tr_k$ is an algebra homomorphism, predict that Tr_5 detects all Sq^0-families initiated by the classes of stems < 125, except possibly the three families, which are respectively initiated by Ph_1, Ph_2 and h_0p'. Since $Sq^0(Ph_1) = h_2g_1$, every element of the Sq^0-family initiated by Ph_1 is not detected by Tr_5 (see [28] for Ph_1 and Theorem 8.2 for $h_{n+1}g_n$). It has been known that Tr_5 does not detect the Sq^0-family of exactly one nonzero element $\{Ph_2\}$ (see Remark 5.7). We have no prediction on whether the Sq^0-family initiated by h_0p' of stem 69 is detected or not.

9. Final remarks

Remark 9.1. We still do not know whether Tr_k fails to be a monomorphism or fails to be an epimorphism for $k > 5$. If Singer's Conjecture 1.4 that Tr_k is a monomorphism for every k is true, then by Theorem 1.5 the algebraic transfer does not detect the kernel of Sq^0 in k-spike degrees.

This leads us to the study of the kernel of Sq^0 in $\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$. The map

$$\widetilde{Sq}^0 : PH_*(B\mathbb{V}_k) \to PH_*(B\mathbb{V}_k)$$

is obviously injective. Taking this observation together with Corollary 3.8 into account, one would expect that the Kameko map

$$Sq^0 = 1 \underset{GL_k}{\otimes} \widetilde{Sq}^0 : \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k) \to \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$$

is also a monomorphism. However, this is false. Indeed, $PH_*(B\mathbb{V}_5)$ has dimension 432 and 1117 in degrees 15 and 35, respectively, while $\mathbb{F}_2 \underset{GL_5}{\otimes} PH_*(B\mathbb{V}_5)$ has dimension 2 and 1 in degrees 15 and 35, respectively. We obtained this claim by using a computer program of S. Shpectorov written in GAP.

As in the proof of Theorem 5.9, let h_n also denote the element in $\mathbb{F}_2 \underset{GL_1}{\otimes} PH_*(B\mathbb{V}_1)$, whose image under the isomorphism Tr_1 is the usual $h_n \in \mathrm{Ext}^1_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$. In the following remark, we will use the product of $\bigoplus_k(\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k))$ defined by Singer in [28] and his result that $Tr = \bigoplus_k Tr_k : \bigoplus_k(\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)) \to \mathrm{Ext}^*_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ is a homomorphism of algebras.

Remark 9.2. (a) Let $t_5 = h_0^4 h_4 \in (\mathbb{F}_2 \underset{GL_5}{\otimes} PH_*(B\mathbb{V}_5))_{15}$. Then $Sq^0(t_5) = 0$ and $Tr_5(t_5) = h_0^4 h_4 \neq 0$.

(b) If $t_k \in \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$ is a positive degree element with $Sq^0(t_k) = 0$ and $Tr_k(t_k) \neq 0$, then $Sq^0(h_n t_k) = 0$ and $Tr_{k+1}(h_n t_k) \neq 0$ for every n with $2^n \geq 4(\deg(t_k))^2$.

"Proof of Remark 9.2*".* Part (a) of this proof proceeds under two hypotheses:

(i) $\mathbb{F}_2 \underset{GL_5}{\otimes} PH_*(B\mathbb{V}_5)$ has respectively dimension 2 and 1 in degrees 15 and 35. (This is known by a computer calculation as written above.)

(ii) $d_0 \in \mathrm{Im}(Tr_4)$. (This is a part of Conjecture 1.10. When this paper was being revised, Lê M. Hà privately informed the author that he proved this claim.)

It would be better to write a direct proof in the framework of invariant theory for the fact $Sq^0(h_0^4 h_4) = 0$ in $\mathbb{F}_2 \underset{GL_5}{\otimes} PH_*(B\mathbb{V}_5)$.

(a) As is well known, $\mathrm{Ext}_{\mathcal{A}}^{5,5+15}(\mathbb{F}_2, \mathbb{F}_2) = \mathrm{Span}\{h_0^4 h_4, h_1 d_0\}$. Combining the fact that $Tr = \bigoplus_k Tr_k$ is an algebra homomorphism with the one that h_n is in the image of Tr_1 for every n, and d_0 is in the image of Tr_4, we conclude that $h_0^4 h_4$ and $h_1 d_0$ are both in the image of Tr_5. On the other hand, the domain of Tr_5 has dimension 2 in degree 15. So, Tr_5 is an isomorphism in degree 15. As $\mathbb{F}_2 \underset{GL_5}{\otimes} PH_*(B\mathbb{V}_5)$ has respectively dimension 2 and 1 in degrees 15 and 35, there exists a nonzero element $t_5 \in \mathbb{F}_2 \underset{GL_5}{\otimes} PH_*(B\mathbb{V}_5)$ in degree 15 such that $Sq^0(t_5) = 0$. Since Tr_5 is an isomorphism in degree 15, $Tr_5(t_5) \neq 0$.

Next, we show that $Tr_5(t_5) = h_0^4 h_4$. Indeed, we suppose to the contrary that $Tr_5(t_5) = \lambda h_0^4 h_4 + h_1 d_0$, for some $\lambda \in \mathbb{F}_2$. Then, as $Sq^0(h_0^4) = h_1^4 = 0$ in $\mathrm{Ext}_{\mathcal{A}}^*(\mathbb{F}_2, \mathbb{F}_2)$ and Sq^0 is an algebra homomorphism, we have

$$Sq^0 Tr_5(t_5) = \lambda Sq^0(h_0^4 h_4) + Sq^0(h_1 d_0) = Sq^0(h_1 d_0) = h_2 d_1.$$

Since Tr_5 commutes with the squaring operations, we get

$$Tr_5 Sq^0(t_5) = Sq^0 Tr_5(t_5) = h_2 d_1 \neq 0.$$

This contradicts the above conclusion that $Sq^0(t_5) = 0$. Therefore,

$$Tr_5(t_5) \neq \lambda h_0^4 h_4 + h_1 d_0$$

for any $\lambda \in \mathbb{F}_2$. Combining this with the fact that $Tr_5(t_5) \neq 0$ in $\mathrm{Span}\{h_0^4 h_4, h_1 d_0\}$, we get $Tr_5(t_5) = h_0^4 h_4$.

With ambiguity of notation, we also have $Tr_5(h_0^4 h_4) = h_0^4 h_4 = Tr_5(t_5)$. As Tr_5 is an isomorphism in degree 15, we obtain $t_5 = h_0^4 h_4$.

(b) As Sq^0 is an algebra homomorphism, we have

$$Sq^0(h_n t_k) = Sq^0(h_n) Sq^0(t_k) = 0.$$

On the other hand, as $Tr = \bigoplus_k Tr_k$ is also an algebra homomorphism, we get

$$Tr_{k+1}(h_n t_k) = Tr_1(h_n) Tr_k(t_k) = h_n Tr_k(t_k).$$

As shown in the proof of Lemma 5.3, a consequence of Davis' Theorem 5.1 claims that, if $Tr_k(t_k) \neq 0$, then $h_n Tr_k(t_k) \neq 0$ for every n with

$$2^n \geq 4(\mathrm{Stem}(Tr_k(t_k)))^2 = 4(\deg(t_k))^2.$$

The remark is proved. □

As an immediate consequence, we have

Corollary 9.3. (i) *$Ker(Sq^0) \cap (\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k))$ is nonzero for $k = 5$ and has an infinite dimension for $k > 5$.*
(ii) *For $k = 5$, Tr_k detects a nonzero element in the kernel of Sq^0, and for each $k > 5$, it detects infinitely many nonzero elements in this kernel.*

It has been known (see [28], [4]) that Sq^0 is injective on $\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$ for $k \leq 3$.

Conjecture 9.4. Sq^0 is a monomorphism in positive degrees of $\mathbb{F}_2 \underset{GL_4}{\otimes} PH_*(B\mathbb{V}_4)$. In other words, Sq^0 is a monomorphism in positive degrees of $\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$ if and only if $k \leq 4$.

The following is an analogue of Corollary 6.2 and is related to Corollary 6.10.

Conjecture 9.5 (Sq^0 is eventually isomorphic on the Ext groups)**.** Let $\mathrm{Im}(Sq^0)^i$ denote the image of $(Sq^0)^i$ on $\mathrm{Ext}^k_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$. There is a number t depending on k such that

$$(Sq^0)^{i-t} : \mathrm{Im}(Sq^0)^t \to \mathrm{Im}(Sq^0)^i$$

is an isomorphism for every $i > t$.

In other words, $\mathrm{Ker}(Sq^0)^i = \mathrm{Ker}(Sq^0)^t$ on $\mathrm{Ext}^k_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ for every $i > t$. As a consequence, any finite Sq^0-family in $\mathrm{Ext}^k_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ has at most t nonzero elements.

Is the conjecture true for $t = k - 2$?

An observation on the known generators of the Ext groups supports the above conjecture with t much smaller than $k - 2$.

It also leads us to the question on whether Sq^0 is an isomorphism on

$$\mathrm{Im}(Sq^0)^t \subset \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$$

for some $t < k - 2$. (This question has an affirmative answer given by Corollary 6.2 for $t = k - 2$.)

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