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CANONICAL VARIETIES WITH NO CANONICAL AXIOMATISATION

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ABSTRACT. We give a simple example of a variety **V** of modal algebras that is canonical but cannot be axiomatised by canonical equations or first-order sentences. We then show that the variety **RRA** of representable relation algebras, although canonical, has no canonical axiomatisation. Indeed, we show that every axiomatisation of these varieties involves infinitely many non-canonical sentences.

Using probabilistic methods of Erdős, we construct an infinite sequence $G_0, G_1, \ldots$ of finite graphs with arbitrarily large chromatic number, such that each G_n is a bounded morphic image of G_{n+1} and has no odd cycles of length at most n. The inverse limit of the sequence is a graph with no odd cycles, and hence is 2-colourable. It follows that a modal algebra (respectively, a relation algebra) obtained from the G_n satisfies arbitrarily many axioms from a certain axiomatisation of **V** (**RRA**), while its canonical extension satisfies only a bounded number of them. First-order compactness will now establish that **V** (**RRA**) has no canonical axiomatisation. A variant of this argument shows that all axiomatisations of these classes have infinitely many non-canonical sentences.

1. INTRODUCTION

A *relation algebra* is an algebra of the form $\mathcal{A} = (A, +, -, 0, 1, 1', \breve{\ }, ;)$, satisfying certain equations laid down by Tarski in [34]. Relation algebras were devised as abstract approximations to the true algebras of binary relations, rather as boolean algebras are abstract analogues of algebras of unary relations. Formally, a relation algebra is said to be *representable* if it embeds into an algebra of binary relations of the form

$$\mathcal{B} = (\wp(E),\ \cup,\ E \setminus -,\ \emptyset,\ E,\ Id_U,\ -^{-1},\ |\ ),$$

where U is a set, $E \subseteq U \times U$ is an equivalence relation on U, $\wp(E)$ is the power set of E, $\cup$ is union of relations, $R \mapsto E \setminus R$ is complementation relative to E, $\emptyset$ and E are constants denoting the smallest and largest relations in $\mathcal{B}$, Id_U is the identity relation $\{(x, x) : x \in U\}$ on U, R^{-1} is the converse $\{(x, y) : (y, x) \in R\}$ of R, and

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$R \mid S$ is the composition or relative product $\{(x, y) : \exists z((x, z) \in R \text{ and } (z, y) \in S)\}$ of R and S. (If $U = \emptyset$, $\mathcal{B}$ is degenerate, with just one element.)

RRA denotes the class of representable relation algebras. This class has shown itself over the years to be rather wild. In [25], Lyndon showed that not every relation algebra is representable. In [26], he gave an explicit (infinite) axiomatisation of **RRA**, and similar axioms obtained by games will be used below. But in [31], building on work of Jónsson and Lyndon [20, 27], Monk showed that **RRA** is not finitely axiomatisable in first-order logic. Since then, many 'negative' results about axiomatisations of **RRA** have been established. For example, it has no axiomatisation using equations with finitely many variables [21], and no axiomatisation by Sahlqvist equations (proved in [37] using results in [17], and implied by the results of the current paper). The problem of whether a finite relation algebra is representable is undecidable [15], so **RRA** is not finitely axiomatisable in second-order or higher-order logic. Other 'negative' results about **RRA** are known (see, e.g., [15]).

While questions concerning **RRA** tend to have negative answers, the class does have a few more positive aspects. It is a variety [35], and indeed a conjugated discriminator variety. Also it is *canonical.* This means that if $\mathcal{A}$ is a representable relation algebra, then so is its canonical extension $\mathcal{A}^\sigma$ — the algebra consisting of all sets of ultrafilters of $\mathcal{A}$, with operations induced from those of $\mathcal{A}$. The notion of canonical extension has been important since Stone's representation theorem for boolean algebras [33], which showed that any boolean algebra can be represented as an algebra of unary relations by embedding it into its canonical extension. Canonical extensions are central in the theory of boolean algebras with operators (BAOs) [22]; the analogous notion of canonical frame is fundamental in modal logic [3]. A variety of BAOs is said to be canonical if it is closed under taking canonical extensions. The analogous notion in modal logic is that of a canonical logic. Here, we take this to mean that the logic in question is valid in all its canonical frames: that is, those for languages of arbitrary cardinality. Canonicity is important in proving completeness (see [3]).

Canonicity of **RRA** was first shown by Monk. His proof used relation algebra reducts of cylindric algebras and is unpublished, but was reported briefly in [30, theorem 2.12]. The first full published proof is in [28], using 'n-dimensional relation algebras', and a proof using saturation was given in [14] and [15, theorem 3.36]. It can also be proved using a general phenomenon in the duality theory of BAOs, namely, that a variety of BAOs is canonical if it is generated by a class of relational structures that is closed under ultraproducts [12].

An equation ε (or, more generally, a first-order sentence) such that for any algebra $\mathcal{A}$ of its signature, $\mathcal{A} \models \varepsilon$ implies $\mathcal{A}^\sigma \models \varepsilon$, is said to be *canonical.* Any set of equations defines a variety, and conversely, any variety is definable by equations [2]. Clearly, any set of canonical equations defines a canonical variety; but the converse of this is not immediate. Let $\mathbf{V}$ be a canonical variety of BAOs, and let Σ be a set of equations that axiomatises $\mathbf{V}$. Then $\mathcal{A} \models \Sigma$ implies $\mathcal{A}^\sigma \models \Sigma$. But it is not apparent that $\mathcal{A} \models \varepsilon$ implies $\mathcal{A}^\sigma \models \varepsilon$ for each individual equation $\varepsilon \in \Sigma$. Of course it may be possible to replace Σ by an equivalent set of canonical equations. This is always possible if $\mathbf{V}$ is finitely axiomatisable, since in BAOs, any finite set of equations is equivalent to a single equation. But **RRA** is not finitely axiomatisable, and in principle it could be that, although canonical, it has *no* axiomatisation by canonical equations. Note that the question of whether it does or not, which is listed as problem 2 in [15, chapter 21], is of a different nature than ones concerning syntactic aspects of possible axiomatisations of **RRA**, such as being Sahlqvist or using a given number of variables. Canonicity of an equation is a 'semantic' property, preserved under logical equivalence. (There is now a sizeable body of work showing undecidability of semantic properties: see, e.g., [4], [5, Chapter 17] and the references therein, and [6, 23, 24, 36]. For undecidability of canonicity see, e.g., [23, theorem 9.6.1].)

In this paper we confirm (in theorem 6.8) an earlier conjecture of the second author that **RRA** has no canonical axiomatisation. We view this result as both an addition to our knowledge of relation algebras and as a contribution to the theory of canonicity. For readers with an interest in the latter rather than in the former, we also supply (in theorem 4.5) an explicit, simple example of a canonical variety **V** without canonical axiomatisation in a similarity type with only two diamonds (i.e., unary operators). Note that we could have applied results on simulations [24, 10] to prove the mere existence of canonical varieties without canonical axiomatisations in simpler similarity types. **V** is related to modal logics studied by Hughes in [18]. Viewed modally, it yields a canonical modal logic that cannot be axiomatised by formulas that are individually canonical. Apart from these outcomes of our investigations, we believe that our proof method, which was suggested by earlier work in atomless BAOs [39] and graphs [16], bears some interest of its own.

Our starting point is a simple observation: if a variety does have a canonical axiomatisation, then given *any* axiomatisation Σ of it, for each $\varphi \in \Sigma$ there is a finite subset $X_\varphi \subseteq \Sigma$ such that whenever $\mathcal{A} \models X_\varphi$ then $\mathcal{A}^\sigma \models \varphi$. (This is because by first-order compactness, φ will be logically implied by a finite subset C of the set of canonical axioms, and C will in turn be implied by some finite $X_\varphi \subseteq \Sigma$. So $\mathcal{A} \models X_\varphi \Rightarrow \mathcal{A} \models C \Rightarrow \mathcal{A}^\sigma \models C \Rightarrow \mathcal{A}^\sigma \models \varphi$.) So, crudely, there cannot be an arbitrarily large 'gap' between the axioms satisfied by $\mathcal{A}$ and by $\mathcal{A}^\sigma$. We show that **V** (above) and **RRA** have no canonical axiomatisation by showing that there is, after all, such a gap.

Using probabilistic methods of Erdős (see, e.g., [8, 7]), we construct an infinite sequence $\mathcal{G} = (G_0, G_1, \ldots)$ of finite graphs with arbitrarily large chromatic number, such that each G_n is a bounded morphic image of G_{n+1} and has no odd cycles of length at most n. $\mathcal{G}$ can then be used to construct a relation algebra $\mathcal{A}$. The game-theoretic axioms for **RRA** mentioned above are useful now. Rather as in [16], it can be shown by games that the number of axioms that $\mathcal{A}$ satisfies increases with the chromatic number of the graphs. Because the G_n have arbitrarily high chromatic number, $\mathcal{A}$ can be made to satisfy arbitrarily many of the axioms. But its canonical extension $\mathcal{A}^\sigma$ is constructible in a similar way from the inverse limit of $\mathcal{G}$. This is a graph with no odd cycles, and hence is 2-colourable. So $\mathcal{A}^\sigma$ satisfies only a bounded number of the axioms. It now follows that **RRA** has no canonical axiomatisation.

The remaining possibility that **RRA** can be axiomatised using canonical axioms plus a finite number of non-canonical ones can be eliminated by a similar argument. Since the non-canonical axioms would be implied by a finite number — say, k — game axioms, it suffices to find (for arbitrarily large finite k) a relation algebra $\mathcal{A}$ satisfying arbitrarily many of the axioms but with $\mathcal{A}^\sigma$ satisfying only the first k of them. This can easily be done by adding a finite complete graph to the G_n, to increase the chromatic number of the inverse limit of $\mathcal{G}$ sufficiently to ensure that $\mathcal{A}^\sigma$ does satisfy the first k axioms.

The proofs for $\mathbf{V}$ are similar but simpler, since it can be axiomatised by equations expressing directly that its algebras have infinite chromatic number.

It would be interesting to extend these results (and those of [16]) to the varieties $\mathbf{RA}_n$ and $\mathbf{S}\mathfrak{Ra}\mathbf{CA}_n$ of relation algebras ($n \geq 5$) and the varieties $\mathbf{RCA}_n$ and $\mathbf{RDf}_n$ of representable n-dimensional cylindric and diagonal-free algebras, respectively, and to many-dimensional modal logics such as $S5^n$ and K^n, where $n \geq 3$. It is hoped to address this in a later paper.

Our contribution to the study of canonicity can be phrased as follows. One may define a notion of chromatic number for a modal algebra, and essentially what we have done is construct algebras with arbitrarily large finite chromatic numbers but whose canonical extensions have bounded chromatic number. Our use of graphs is unusual in this field, and we hope that links between modal and algebraic logic and graph theory will deepen in the future.

Outline of paper. In section 2 we construct the required graphs. In section 3 we recall and introduce some technical material on inverse systems. With this background we can provide, in section 4, a simple example of a canonical variety with no axiomatisation using only finitely many non-canonical sentences. In section 5 we recall some results on relation algebras and games, and use compactness to provide a sufficient criterion for **RRA** to have no axiomatisation using finitely many non-canonical sentences. Finally, in section 6 we construct the relation algebras from the graphs, and show that they have the properties required by the criterion.

Notation. We identify a natural number n with $\{0, 1, \ldots, n-1\}$. We also identify (notationally) a structure or algebra with its domain. For a map $f : X \to Y$, $y \in Y$, and $Y' \subseteq Y$, we write $f^{-1}(y)$ for $\{x \in X : f(x) = y\}$, and $f^{-1}[Y']$ for $\{x \in X : f(x) \in Y'\}$.

Homomorphisms. The notion of homomorphism arises in this paper in three contexts: graphs, algebras, and relational structures. When these are construed in the natural way as first-order structures, our use of 'homomorphism' always boils down to the usual model-theoretic notion of a map between similar structures that preserves atomic formulas forwards.

2. Random graphs

A *graph* is a pair (G, E), where G is a non-empty set of 'nodes', and $E \subseteq G \times G$ is an irreflexive symmetric binary relation on G, called the *edge relation.* For background material on graphs and the probabilistic method used below, see, e.g., Diestel's book [7]. We identify (notationally) a graph with its set of nodes. So if G is a graph, $|G|$ denotes the cardinality of the set of nodes of G. An *edge* of G is a set $\{x, y\}$ such that $(x, y) \in E$. We write this edge as simply xy; note that xy is the *same* edge as yx. A graph G is *complete* if xy is an edge of G for all distinct nodes $x, y \in G$. The *disjoint union* of graphs (G_0, E_0) and (G_1, E_1) is the graph $((G_0 \times \{0\}) \cup (G_1 \times \{1\}), \{((x, i), (y, i)) : i < 2, (x, y) \in E_i\})$. In practice, we will identify (x, i) with x, and so regard two graphs as subgraphs of their disjoint union. For graphs G, H, a map $f : G \to H$ is a *homomorphism* if whenever xy is an edge of G, then $f(x)f(y)$ is an edge of H.

For $c \geq 3$, a *c-cycle* in a graph G is formally a (not necessarily induced) subgraph of G that is isomorphic to a graph with c nodes $n_1, \ldots, n_c$, say, and with edges precisely $n_i n_{i+1}$ (for $1 \leq i < c$) and $n_c n_1$. However, to simplify notation, we regard a c-cycle in G as a sequence $x_1, \ldots, x_c$ of distinct nodes of G such that $x_1x_2, \ldots, x_{c-1}x_c, x_cx_1$ are edges of G, on the understanding that $x_1, \ldots, x_c$ and $x_2, \ldots, x_c, x_1$ and $x_c, x_{c-1}, \ldots, x_1$ all denote the *same* cycle. (It would be simpler for our purposes if they did not, but we wish to conform to standard terminology.) A *cycle* is a c-cycle for some $c \geq 3$; its *length* is c. An *odd* (*even*) cycle is one of odd (even) length.

An *independent set* in a graph G is a set X of nodes of G such that for no $x, y \in X$ is xy an edge of G. For an integer k, a *k-colouring* of G is a partition of the nodes of G into $\leq k$ independent sets. (Letting each set correspond to a different colour, this means that we can colour the nodes of G using at most k colours so that the ends of each edge have different colours.) G is *k-colourable* if it has a k-colouring. Its *chromatic number,* $\chi(G)$, is the smallest k (if any) for which it has a k-colouring, and ∞ if there is no such k. We will need the following well-known facts.

Fact 2.1.

(1) *The chromatic number of a complete graph G is the number of nodes of G, if G is finite, and ∞, otherwise.*
(2) *Let G, H be graphs.*
  (a) *The chromatic number of the disjoint union of G and H is equal to* $\max(\chi(G), \chi(H))$.
  (b) *If there is a homomorphism $f : G \to H$, then $\chi(G) \leq \chi(H)$.*
(3) *A graph is* 2*-colourable iff it has no odd cycles. (For a proof see, e.g.,* [7, *proposition* 1.6.1]*.)*

However, chromatic number is not 'locally determined' in graphs: for any finite k, there is a finite graph G that is not k-colourable but has no cycles of length $\leq k$. (So all subgraphs of G of size $\leq k$ are 2-colourable.) This is a celebrated result of Erdős [8]; it was one of the first probabilistic constructions of graphs. Theorem 2.3 below is a modification of it. Our presentation relies heavily on that of Erdős' theorem given by Diestel in [7].

Theorem 2.3 will follow readily from the following proposition.

Proposition 2.2. *Let $c \geq 3$ and $k \geq 1$ be integers, and let $0 < \beta < \beta^+ < 1$. Let G be a finite graph with n nodes, say, and suppose that G has no independent set of size $> n\beta/k$ and no odd cycles of length $< c$. Then there is a finite graph G^+ with n^+ nodes (say), such that:*

(1) *there is a surjective graph homomorphism $\rho : G^+ \to G$,*
(2) *for each edge xy of G and $x' \in \rho^{-1}(x)$, there is $y' \in \rho^{-1}(y)$ such that $x'y'$ is an edge of G^+,*
(3) *G^+ has no independent set of size $> n^+\beta^+/k$,*
(4) *G^+ has no odd cycles of length $\leq c$.*

In modal terms, conditions (1) and (2) say roughly that ρ is a surjective *bounded morphism* (or surjective p-morphism). We were not able to show that G^+ has no independent set of size $> n^+\beta/k$, but only the (arbitrarily slightly) weaker condition (3). Of course we could assume that c is odd, but we will not need to do so until lemma 2.10.

The proposition will be proved below. We use it to deduce:

Theorem 2.3. *Let $k \geq 2$. There are finite graphs $G_0, G_1, \ldots$, and surjective homomorphisms $\rho_i : G_{i+1} \to G_i$ for $i < \omega$, such that for each i,*

(1) *for each edge xy of G_i and each $x' \in \rho_i^{-1}(x)$, there is $y' \in \rho_i^{-1}(y)$ such that $x'y'$ is an edge of G_{i+1},*
(2) *G_i has no odd cycles of length $\leq i$,*
(3) *$\chi(G_i) = k$.*

Proof. Choose real numbers $\beta_0, \beta_1, \ldots$ with $(k-1)/k \leq \beta_0 < \beta_1 < \cdots < 1$. We define finite graphs G_i and surjective homomorphisms $\rho_i : G_{i+1} \to G_i$ $(i < \omega)$, satisfying conditions (1) and (2) of the theorem and such that each G_i has no independent set of size $> |G_i|\beta_i/(k-1)$. Let K_k be a complete graph with k nodes. For $i = 0, 1, 2$, we let $G_i = K_k$ and let ρ_0 and ρ_1 be the identity map. Note that $|G_i|\beta_i/(k-1) \geq 1$, so G_i has no independent set of size $> |G_i|\beta_i/(k-1)$. Certainly it has no odd cycles of length $\leq i$. The remaining G_i (for $i = 3, 4, \ldots$) can be built by induction using proposition 2.2, taking 'k' to be $k-1$, 'c' to be i, 'β' to be β_{i-1}, 'β^+' to be β_i, 'G' to be G_{i-1}, noting that the hypotheses of the proposition are valid, and setting $G_i = G^+$.

Since each G_i has K_k as a homomorphic image, by fact 2.1(1, 2) its chromatic number is at most k. If some G_i had chromatic number $\leq k-1$, it would have an independent set of size $\geq |G_i|/(k-1)$. But we chose G_i with no independent set of size $> |G_i|\beta_i/(k-1)$, and $\beta_i < 1$. This is a contradiction, so each G_i has chromatic number k. □

Remark 2.4. The theorem shows that for any $k \geq 2$ and $c < \omega$, there is a finite graph with chromatic number exactly k, all of whose subgraphs of size c are 2-colourable.

Proof of proposition 2.2. The rest of this section is devoted to the proof of proposition 2.2. We adopt its hypotheses. Assume that the set of nodes of G is $n = \{0, 1, \ldots, n-1\}$. We first build a finite graph G' with nodes $n \times n'$ for some large $n' = \{0, \ldots, n'-1\}$ to be determined, with $\pi : G' \to G$ given by $\pi(i, j) = i$. Let $0 < \varepsilon < 1/c$. We build G' probabilistically, putting an edge between $x, y \in G'$ with probability p if $\pi(x)\pi(y)$ is an edge of G, and 0 otherwise, where

$$p = (n')^{\varepsilon - 1}. \tag{1}$$

The choices for each $\{x, y\}$ are made independently of one another. Our use of probability follows [7] closely, and readers may wish to refer to this for background information. Write $q = 1 - p$, and

$$\beta' = \frac{\beta + \beta^+}{2} > \beta. \tag{2}$$

Lemma 2.5. *The probability that G' has an independent set of size $> nn'\beta'/k$ tends to* 0 *as $n' \to \infty$.*

Proof. This is the most complicated lemma. We begin with a claim.

Claim. Let $X \subseteq G'$ with $|X| > nn'\beta'/k$. The probability that X is an independent set is at most $q^{\gamma n'^2}$, where

$$\gamma = \left(\frac{\beta' - \beta}{k - \beta}\right)^2 > 0. \tag{3}$$

Proof of the claim. Enumerate the nodes of G as $\{x_0, x_1, \ldots, x_{n-1}\}$ so that, letting $s_i = |X \cap \pi^{-1}(x_i)|$ for $i < n$, we have $s_0 \geq s_1 \geq \cdots \geq s_{n-1}$. Let

$$l = \lfloor n\beta/k \rfloor. \tag{4}$$

Then

$$\begin{array}{rcll} \sum_{i<l} s_i & \leq & ln' & \text{since } s_i \leq n' \text{ for all } i, \\ \sum_{l \leq i<n} s_i & \leq & (n-l)s_l & \text{since } s_i \leq s_l \text{ for all } i \text{ with } l \leq i < n. \end{array} \tag{5}$$

Now we obtain

$$\begin{array}{rcll} nn'\beta'/k & < & |X| & \\ & = & \sum_{i<n} s_i & \\ & = & \sum_{i<l} s_i + \sum_{l \leq i<n} s_i & \\ & \leq & ln' + (n-l)s_l & \text{(by (5))} \\ & = & ns_l + l(n' - s_l) & \\ & \leq & ns_l + (n\beta/k)(n' - s_l) & \text{(by (4), since } n' - s_l \geq 0) \\ & = & n(s_l(k-\beta) + n'\beta)/k. & \end{array}$$

So $n'\beta' < s_l(k-\beta) + n'\beta$. Since $k - \beta > 0$, we conclude that

$$s_l > n'(\beta' - \beta)/(k - \beta). \tag{6}$$

Now G has no independent set of size $> l$. So $\{x_0, x_1, \ldots, x_l\}$ is not independent; say $x_i x_j$ is an edge in it. Hence, for each $y \in X \cap \pi^{-1}(x_i)$ and $y' \in X \cap \pi^{-1}(x_j)$, the probability that yy' is not an edge of G' is q. There are s_i such y and s_j such y'. As each edge of G' is chosen independently of the rest, the probability that there is no edge in X from $\pi^{-1}(x_i)$ to $\pi^{-1}(x_j)$ is $q^{s_i s_j}$. By (6) and (3), $s_i s_j \geq s_l^2 > \gamma n'^2$. So the probability that X is an independent set is at most $q^{\gamma n'^2}$, as claimed.

Now, the probability P (say) that there is some independent set in G' of size $> nn'\beta'/k$ is at most the sum of the individual probabilities that X is independent, the sum being taken over all $X \subseteq G'$ of size $\lfloor nn'\beta'/k \rfloor + 1$. By the claim,

$$P \leq \binom{nn'}{\lfloor nn'\beta'/k \rfloor + 1} q^{\gamma n'^2} \leq (nn')^{\lfloor nn'\beta'/k \rfloor + 1} \cdot q^{\gamma n'^2}. \tag{7}$$

Observe that $e^x \geq 1 + x$ for all $x \in \mathbb{R}$, so

$$q = 1 - p \leq e^{-p}. \tag{8}$$

Hence, $P \leq (nn')^{\lfloor nn'\beta'/k \rfloor + 1} \cdot e^{-p\gamma n'^2} \leq (nn')^{2nn'\beta'/k} \cdot e^{-p\gamma n'^2}$ for large enough n'. Taking logs and using (1), we get

$$\begin{array}{rcl} \ln(P) & \leq & (2nn'\beta'/k)\ln(nn') - p\gamma n'^2 \\ & = & (2nn'\beta'/k)(\ln(n) + \ln(n')) - \gamma n'^{\varepsilon-1} n'^2 \\ & = & n'\big(2n\beta' \ln(n)/k + 2n\beta' \ln(n')/k - \gamma n'^{\varepsilon}\big). \end{array} \tag{9}$$

Since $\varepsilon, \gamma > 0$, the right-hand term $\gamma n'^{\varepsilon}$ dominates, and we get $\ln(P) \to -\infty$ and so $P \to 0$ as $n' \to \infty$, as required. $\square$

Let

$$\delta = n(\beta^+ - \beta)/2kc > 0. \tag{10}$$

This is independent of n'.

Lemma 2.6. *The probability that G' has $\geq n'\delta$ cycles of length c tends to 0 as $n' \to \infty$.*

Proof. Let E be the expected (or mean) number of cycles of length c in G'. This is by definition the sum over all G' of the probability of G', times the number of cycles of length c in G'. More formally, let $\mathcal{S}$ be the set of all sequences $(x_1, \ldots, x_c)$ of distinct nodes of G'. Let $\sim$ be the equivalence relation on $\mathcal{S}$ generated by the union of the relations $\{((x_1, \ldots, x_c), (x_2, \ldots, x_c, x_1)) : (x_1, \ldots, x_c) \in \mathcal{S}\}$ and $\{((x_1, \ldots, x_c), (x_c, x_{c-1}, \ldots, x_1)) : (x_1, \ldots, x_c) \in \mathcal{S}\}$. Two sequences are $\sim$-equivalent iff one arises from the other by a cyclic permutation and perhaps reversing the ordering. Write $\mathcal{C}$ for the set of $\sim$-equivalence classes. We think of the elements of $\mathcal{C}$ as *potential (c-)cycles:* the number of c-cycles in G' is the number of $\sim$-classes $(x_1, \ldots, x_c)/{\sim} \in \mathcal{C}$ such that $x_1, \ldots, x_c$ is a cycle in G'. Given G' and $C = (x_1, \ldots, x_c)/{\sim} \in \mathcal{C}$, define

$$\xi_{G'}(C) = \begin{cases} 1, & \text{if } x_1, \ldots, x_c \text{ is a cycle in } G', \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

This is well defined. The number of c-cycles in G' is $\sum_{C \in \mathcal{C}} \xi_{G'}(C)$. Then, writing $\mathbb{P}(G')$ for the probability of G', the definition of E becomes

$$E = \sum_{G'} \Big(\mathbb{P}(G') \cdot \sum_{C \in \mathcal{C}} \xi_{G'}(C) \Big). \tag{12}$$

Rearranging the sums in (12) shows that E is at most the number of ways of choosing $C = (x_1, \ldots, x_c)/{\sim} \in \mathcal{C}$, times the bound p^c on the probability $\mathbb{P}(C)$ that $x_1, \ldots, x_c$ is a cycle in G'. Indeed, by (12) and (1), we have

$$\begin{aligned} E &= \sum_{C \in \mathcal{C}} \Big(\sum_{G'} \mathbb{P}(G') \xi_{G'}(C) \Big) \\ &= \sum_{C \in \mathcal{C}} \mathbb{P}(C) \\ &\leq |\mathcal{C}| \cdot p^c \\ &\leq ((nn')^c / 2c) \cdot n'^{(\varepsilon - 1)c}, \\ &= n^c n'^{c\varepsilon} / 2c. \end{aligned} \tag{13}$$

Now the probability Q (say) that G' has $\geq n'\delta$ cycles of length c is at most $E/n'\delta$. (This is Markov's inequality; cf. [7, lemma 11.1.4].) For, if $Q > E/n'\delta$ and $\mathcal{G}$ is the set of G' with $\geq n'\delta$ cycles of length c, (12) yields

$$E \geq \sum_{G' \in \mathcal{G}} \Big(\mathbb{P}(G') \cdot \sum_{C \in \mathcal{C}} \xi_{G'}(C) \Big) \geq \sum_{G' \in \mathcal{G}} \mathbb{P}(G') \cdot n'\delta = Q \cdot n'\delta > E, \tag{14}$$

which is impossible. So by (13),

$$Q \leq E/n'\delta \leq \frac{n^c}{2c\delta} (n')^{c\varepsilon - 1}. \tag{15}$$

But $\varepsilon < 1/c$ so $c\varepsilon - 1 < 0$. This ensures that the right-hand side of (15) tends to 0 as $n' \to \infty$. □

Lemma 2.7. *Let R be the probability that there is an edge xy of G and a node $x' \in \pi^{-1}(x)$ with no edges in G' from x' into $\pi^{-1}(y)$. Then $R \to 0$ as $n' \to \infty$.*

Proof. R is bounded by the number of pairs (x, y) such that xy is an edge of G, times the number of nodes $x' \in \pi^{-1}(x)$, times the probability $q^{|\pi^{-1}(y)|}$ of there being no edge from x' to any node in $\pi^{-1}(y)$. So, using (8) again,

$$R \le n^2 \cdot n' \cdot q^{n'} \le n^2 n' e^{-pn'}. \tag{16}$$

Taking logs and using (1) gives

$$\begin{array}{rcl} \ln R & \le & 2\ln n + \ln n' - pn' \\ & = & 2\ln n + \ln n' - n'^{\varepsilon} \\ & & \to -\infty \text{ as } n' \to \infty. \end{array} \tag{17}$$

So $R \to 0$ as $n' \to \infty$, as required. □

So for large n', the probability of G' not having the properties of any of the three lemmas is non-zero. We may therefore take G' of finite size nn' for large n', such that:

(1) G' has no independent set of size $> nn'\beta'/k$, where β' is as defined in (2),
(2) G' has $< n'\delta$ cycles of length c, where δ is as defined in (10),
(3) $\pi : G' \to G$ is a surjective homomorphism, and for each edge xy of G, every node of $\pi^{-1}(x)$ is connected by a edge to some node in $\pi^{-1}(y)$.

It remains to remove any odd cycles of length $\le c$. The classical Erdős construction does this by deleting nodes from the cycles. We proceed differently since we wish to preserve condition (3) just cited. Let $\mathcal{E}$ denote the set of all edges of G' that lie in a cycle of G' of length c. Then

$$|\mathcal{E}| < cn'\delta. \tag{18}$$

Let $\mathbb{Z}_2$ denote the additive group $(\{0,1\},+)$ of order 2, with identity 0 and with $1+1=0$. For each edge e of G' define $\hat{e} \in \mathbb{Z}_2$ by

$$\hat{e} = \begin{cases} 1, & \text{if } e \in \mathcal{E}, \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

Let G^+ be the graph (based on $G' \times \mathbb{Z}_2$) whose nodes are all pairs (x',g) for $x' \in G'$, $g \in \mathbb{Z}_2$, and such that

$$(x',g)(y',h) \text{ is an edge of } G^+ \iff \begin{array}{c} x'y' \text{ is an edge of } G' \\ \text{and } g+h = \widehat{x'y'}. \end{array} \tag{20}$$

Note that this defines an edge relation (irreflexive and symmetric) on G^+. Essentially G^+ is a disjoint union of two copies of G', except that edges in c-cycles are 'twisted' to run between the copies, thereby 'unwinding' the cycles. Let $n^+ = |G^+| = 2nn'$. Let $\rho : G^+ \to G$ be defined by $\rho(x',g) = \pi(x')$.

We check that G^+, ρ have the properties required for proposition 2.2. Clearly, ρ is a surjective homomorphism : $G^+ \to G$.

Lemma 2.8. *For every edge xy of G, every node of $\rho^{-1}(x)$ is connected by an edge of G^+ to some node of $\rho^{-1}(y)$.*

Proof. Let $(x',g) \in G^+$ with $\rho(x',g) = x$. We know there is $y' \in G'$ with $\pi(y') = y$ and such that $x'y'$ is an edge of G'. Then $(y', g + \widehat{x'y'}) \in \rho^{-1}(y)$, and by (20), $(x',g)(y', g+\widehat{x'y'})$ is an edge of G^+. □

A homomorphic image of a graph with even cycles may be acyclic. For example, a 4-cycle will map homomorphically onto a single edge. So it is worth proving that this cannot happen with odd cycles.

Lemma 2.9. *Let H, H' be any graphs and $f : H \to H'$ a homomorphism. Let $l \geq 3$. If H has an odd cycle of length $\leq l$, then so does H'.*

Proof. By restricting f to such a cycle, we may suppose that $|H| \leq l$. By discarding nodes of H' not in $rng(f)$, we may also suppose that $|H'| \leq l$. If H' has no odd cycles of length $\leq l$, then it has no odd cycles at all, so by fact 2.1(3) it is 2-colourable. Then since f is a homomorphism, by fact 2.1(2) H is 2-colourable, which by fact 2.1(3) is impossible as H has an odd cycle. □

Lemma 2.10. *G^+ has no odd cycles of length $\leq c$.*

Proof. Since G has no odd cycles of length $< c$ and $\rho : G^+ \to G$ is a homomorphism, lemma 2.9 shows that G^+ has no odd cycles of length $< c$. Assume for contradiction that c is odd and $(x_1, g_1), \ldots, (x_c, g_c)$ is a cycle in G^+. Then by lemma 2.9 again, the induced subgraph of G with nodes $\rho(x_1, g_1), \ldots, \rho(x_c, g_c)$ has a cycle of some odd length $l \leq c$. Since G has no odd cycles of length $< c$, it follows that $l = c$, so $\rho(x_1, g_1), \ldots, \rho(x_c, g_c)$ are distinct. Therefore $x_1, \ldots, x_c$ are distinct nodes of G'. Because $(x_i, g_i) \mapsto x_i$ is a homomorphism, $x_1, \ldots, x_c$ is a cycle in G', and so the edges $x_1x_2, \ldots, x_{c-1}x_c, x_cx_1$ are in $\mathcal{E}$. By (20), we must have $g_1 + g_2 = \widehat{x_1x_2} = 1$, $\ldots$, $g_{c-1} + g_c = \widehat{x_{c-1}x_c} = 1$, and $g_c + g_1 = \widehat{x_cx_1} = 1$. So in $\mathbb{Z}_2$ we have

$$(g_1 + g_2) + \cdots + (g_{c-1} + g_c) + (g_c + g_1) = \underbrace{1 + 1 + \cdots + 1}_{c \text{ times}}. \tag{21}$$

The left-hand side of (21) is 0 since each g_i occurs twice. But the right-hand side is 1 because c is odd. This contradiction completes the proof. □

Lemma 2.11. *G^+ has no independent set of size $> n^+\beta^+/k$.*

Proof. Assume for contradiction that $X \subseteq G^+$ is an independent set with $|X| > n^+\beta^+/k$. Write $X = (X_0 \times \{0\}) \cup (X_1 \times \{1\})$, where $X_0, X_1 \subseteq G'$. Since $n^+ = 2nn'$, we may choose $g \in \mathbb{Z}_2$ such that

$$|X_g| > nn'\beta^+/k. \tag{22}$$

Let $Y \subseteq G'$ be a set consisting of one node from each edge in $\mathcal{E}$. By (18), $|Y| \leq |\mathcal{E}| < cn'\delta$. Let $Z = X_g \setminus Y$. By (22), (10), and (2), we have

$$\begin{aligned}
|Z| &\geq |X_g| - |Y| \\
&> nn'\beta^+/k - cn'\delta \\
&= nn'\beta^+/k - cn' \cdot n(\beta^+ - \beta)/2kc \\
&= (nn'/k)\Big(\beta^+ - \frac{\beta^+ - \beta}{2}\Big) \\
&= (nn'/k)\Big(\frac{\beta^+ + \beta}{2}\Big) \\
&= nn'\beta'/k.
\end{aligned} \tag{23}$$

But there are no edges from $\mathcal{E}$ inside Z, so by (20), Z is an independent set in G'. This contradicts the definition of G'. □

So we have constructed G^+ as required by the proposition. □

3. Boolean algebras with operators and inverse systems

There is an approximate equivalence between these two notions. Roughly, a BAO $\mathcal{A}$ yields an inverse system $\mathcal{P}_\mathcal{A}$ consisting of all its finite partitions of 1. Then the inverse limit $\lim_\leftarrow \mathcal{P}_\mathcal{A}$ is isomorphic to the ultrafilter frame $\mathcal{A}_+$ of $\mathcal{A}$. If $\mathcal{A}$ is locally finite, $\mathcal{P}_\mathcal{A}$ has a cofinal subsystem in which the maps are bounded morphisms. Conversely, let $\mathcal{D}$ be an inverse system of finite structures and bounded morphisms. The inverse limit $\mathcal{I} = \lim_\leftarrow \mathcal{D}$ is essentially a Stone space. Let $\mathcal{A}_\mathcal{D}$ be the (locally finite) subalgebra of the complex algebra of $\mathcal{I}$ consisting of all clopen sets in it. Then $\mathcal{D}$ is in effect cofinal in $\mathcal{P}_{\mathcal{A}_\mathcal{D}}$, and $(\mathcal{A}_\mathcal{D})_+ \cong \lim_\leftarrow(\mathcal{P}_{\mathcal{A}_\mathcal{D}}) \cong \lim_\leftarrow \mathcal{D} = \mathcal{I}$.

The aim of this section is to establish these technical results. With the exception of the results in subsection 3.2, all of our observations can easily be derived from, or are special cases of, results in section 11 of Goldblatt [11].

3.1. **Basics.** First we recall from [22] the definition of (normal) BAOs. For this section fix a functional signature L containing the boolean operators $+, -, 0, 1$. We continue to identify (notationally) a structure with its domain. We assume familiarity with basic aspects of boolean algebras, such as the boolean product operation $\cdot$, the ordering $\leq$, atoms and atomic algebras, and ultrafilters. For background information, see, e.g., [3, 15].

Definition 3.1 (BAOs)**.**

(1) A non-boolean function symbol of L (i.e., one other than $+, -, 0, 1$) is called an *operator symbol.* A *boolean algebra with operators (BAO)* is an L-structure $\mathcal{A}$ whose boolean reduct (the reduct to signature $\{+, -, 0, 1\}$) is a boolean algebra and in which the interpretation $f^\mathcal{A}$ of each n-ary operator symbol f is normal ($f^\mathcal{A}(a_1, \ldots, a_n) = 0$ whenever any a_i is 0) and additive in each argument (e.g., for binary f, $f^\mathcal{A}(a + b, c) = f^\mathcal{A}(a, c) + f^\mathcal{A}(b, c)$ and $f^\mathcal{A}(a, b + c) = f^\mathcal{A}(a, b) + f^\mathcal{A}(a, c)$ for all $a, b, c \in \mathcal{A}$). To reduce clutter, we will often write $f^\mathcal{A}$ simply as f where the meaning is clear from the context.

(2) Let L^a be the relational signature consisting of an $(n + 1)$-ary relation symbol R_f for each n-ary operator symbol $f \in L$.

(3) If M, N are L^a-structures and $\theta : M \to N$ is a map, we say that θ is a *bounded morphism* (or *p-morphism*) if whenever $f \in L$ is an n-ary operator symbol, $b_1, \ldots, b_n, b' \in N$, and $a' \in M$ satisfies $\theta(a') = b'$, we have $N \models R_f(b_1, \ldots, b_n, b')$ iff there are $a_1, \ldots, a_n \in M$ with $\theta(a_1) = b_1, \ldots, \theta(a_n) = b_n$, and $M \models R_f(a_1, \ldots, a_n, a')$. Note that such a θ is an L^a-homomorphism (i.e., it preserves L^a-relations forwards).

(4) If $\mathcal{A}$ is an atomic L-BAO, its *atom structure* $\operatorname{At}\mathcal{A}$ is the L^a-structure with domain the set of atoms of $\mathcal{A}$, and with $\operatorname{At}\mathcal{A} \models R_f(a_1, \ldots, a_n, b)$ iff $\mathcal{A} \models f(a_1, \ldots, a_n) \geq b$, for all atoms $a_1, \ldots, a_n, b$ of $\mathcal{A}$ and n-ary operator symbols $f \in L$.

(5) Given an L-BAO $\mathcal{A}$, we let $\mathcal{A}_+$ denote the 'ultrafilter frame' of $\mathcal{A}$: it is the L^a-structure with domain the set of all ultrafilters of (the boolean reduct of) $\mathcal{A}$, and with $\mathcal{A}_+ \models R_f(\mu_1, \ldots, \mu_n, \nu)$ iff $f(a_1, \ldots, a_n) \in \nu$ for all $a_1 \in \mu_1, \ldots, a_n \in \mu_n$, for each n-ary operator symbol $f \in L$ and all ultrafilters $\mu_1, \ldots, \mu_n, \nu$ of $\mathcal{A}$.

(6) Let M be an L^a-structure; then we can form its *complex algebra:* that is, the L-BAO M^+ with domain the power set $\wp(M)$ of M, with $+, -, 0, 1$

defined by $X + Y = X \cup Y$, $-X = M \setminus X$, $0 = \emptyset$, $1 = M$, and with operators defined by

$$f(X_1, \ldots, X_n) = \{b \in M : M \models R_f(a_1, \ldots, a_n, b) \text{ for some } a_1 \in X_1, \ldots, a_n \in X_n\},$$

for each n-ary operator symbol $f \in L$ and each $X_1, \ldots, X_n \subseteq M$.

(7) For a BAO $\mathcal{A}$, write $\mathcal{A}^\sigma$ for its *canonical extension* $(\mathcal{A}_+)^+$.

We will need the following basic duality facts (see, e.g., [3, theorem 5.47]).

Fact 3.2. *Let M, N be L^a-structures. If $\theta : M \to N$ is a surjective bounded morphism, then $\theta^+ : X \mapsto \theta^{-1}[X]$ is an L-embedding : $N^+ \to M^+$.*

If M, N are finite and $\xi : N^+ \to M^+$ is an L-embedding, then the map $\xi^- : M \to N$ given by '$\xi^-(a)$ is the unique $b \in N$ with $\xi(\{b\}) \geq \{a\}$ in M^+' is a surjective bounded morphism.

Definition 3.3 (inverse systems)**.**

(1) We say that a partially ordered set $(I, \leq)$ is *directed* if every finite subset of I has an upper bound in I.

(2) An *inverse system* of L^a-structures is a triple

$$\mathcal{D} = ((I, \leq), (S_i : i \in I), (\pi_{ji} : i, j \in I,\ i \leq j)),$$

where $(I, \leq)$ is a directed partially ordered set, each S_i $(i \in I)$ is an L^a-structure, and for each $i \leq j$ in I, $\pi_{ji} : S_j \to S_i$ is a surjective homomorphism, such that whenever $k \geq j \geq i$ in I then π_{ii} is the identity map on S_i and $\pi_{ki} = \pi_{ji} \circ \pi_{kj}$. Here and below, $\circ$ denotes composition of maps, so that $f \circ g(x) = f(g(x))$.

(3) We say that $\mathcal{D}$ is an *inverse system of finite structures* if each S_i is finite, and an *inverse system of bounded morphisms* if each π_{ji} is a bounded morphism.

(4) The *inverse limit* $\lim_{\leftarrow} \mathcal{D}$ of $\mathcal{D}$ is the substructure of $\prod_{i \in I} S_i$ with domain

$$\{\chi \in \prod_{i \in I} S_i : \pi_{ji}(\chi(j)) = \chi(i) \text{ whenever } j \geq i \text{ in } I\}.$$

In more detail, it is the L^a-structure whose domain is the set of all maps $\chi : I \to \bigcup_{i \in I} S_i$ such that $\chi(i) \in S_i$ for all $i \in I$ and $\pi_{ji}(\chi(j)) = \chi(i)$ whenever $j \geq i$ in I, defined as an L^a-structure by: for each n-ary operator symbol $f \in L$, let $\lim_{\leftarrow} \mathcal{D} \models R_f(\chi_1, \ldots, \chi_n, \psi)$ iff $S_i \models R_f(\chi_1(i), \ldots, \chi_n(i), \psi(i))$ for all $i \in I$.

(5) For any $i \in I$, the projection $\pi_i : \lim_{\leftarrow} \mathcal{D} \to S_i$ is defined by $\pi_i(\chi) = \chi(i)$.

We will need the following for (28) below.

Lemma 3.4. *Let $\mathcal{D}$ be an inverse system of finite structures, as above. Write $\mathcal{I}$ for* $\lim_{\leftarrow} \mathcal{D}$*. Let $i \in I$. Then:*

(1) *The projection $\pi_i : \mathcal{I} \to S_i$ of definition* 3.3(5) *is a surjective homomorphism of L^a-structures.*

(2) *If $\mathcal{D}$ is an inverse system of bounded morphisms, then π_i is a bounded morphism.*

Proof. We give a proof sketch of (2) only, since a more general version of this result appears as item 5 of Lemma 11.2 in [11], and (1) is proved in a similar way. Let

$f \in L$ be an n-ary operator symbol, and let $i \in I$, $a_1, \ldots, a_n \in S_i$, and $\psi \in \mathcal{I}$ be given. We require that

$$\begin{aligned} & S_i \models R_f(a_1, \ldots, a_n, \psi(i)) \\ & \quad \iff \exists \chi_1, \ldots, \chi_n \in \mathcal{I}\Big(\mathcal{I} \models R_f(\chi_1, \ldots, \chi_n, \psi) \wedge \bigwedge_{l \leq n} \chi_l(i) = a_l\Big). \end{aligned} \tag{24}$$

This is shown as follows. π_i is the restriction to $\mathcal{I}$ of the canonical projection from $\prod_{t \in I} \mathcal{S}_t$ to S_i. This projection is a homomorphism, so π_i is as well, whence '$\Leftarrow$' holds. Conversely, assume that $S_i \models R_f(a_1, \ldots, a_n, \psi(i))$. In order to prove that there are $\chi_1, \ldots, \chi_n \in \mathcal{I}$ meeting the above requirements, we will use a topological compactness argument. To this aim, we impose the discrete topology on each S_t ($t \in I$); these topologies are all compact since each S_t is finite. Thus by Tychonoff's theorem, the product space $\prod_t S_t$ is compact, and hence, so is its product $(\prod_t S_t)^n$.

Now define, for an arbitrary $j \geq i$, the set j^* as the collection of those elements $(\chi_1, \ldots, \chi_n)$ of $(\prod_t S_t)^n$ that satisfy (i) $R_f(\chi_1(j), \ldots, \chi_n(j), \psi(j))$, (ii) $\chi_l(k) = \pi_{jk}(\chi_l(j))$ for all $k \leq j$ and all $l \leq n$, and (iii) $\chi_l(i) = a_l$ for all $l \leq n$. Using the assumptions on $\mathcal{D}$, it is fairly straightforward to show that j^* is non-empty, and closed in the topology on $(\prod_t S_t)^n$.

It is also easy to see that $k \geq j$ implies $k^* \subseteq j^*$. But then it follows from the directedness of $(I, \leq)$ that the collection $\{j^* : j \geq i\}$ has the finite intersection property. By compactness we find that the collection has a non-empty intersection. We leave it for the reader to verify that any element of this intersection witnesses the right-hand side of (24). $\square$

Definition 3.5. Let $\mathcal{D} = ((I, \leq), (S_i : i \in I), (\pi_{ji} : i, j \in I,\ i \leq j))$ and $\mathcal{D}' = ((I', \leq'), (S'_i : i \in I'), (\pi'_{ji} : i, j \in I',\ i \leq' j))$ be inverse systems. We say that $\mathcal{D}'$ is a *sub-inverse system of* $\mathcal{D}$ if $I' \subseteq I$, $\leq'$ is $\leq$ restricted to I', $S'_i = S_i$ for all $i \in I'$, and $\pi'_{ji} = \pi_{ji}$ for all $i \leq' j$ in I'. A sub-inverse system $\mathcal{D}'$ of $\mathcal{D}$ is said to be *cofinal (in* $\mathcal{D}$*)* if for all $i \in I$ there is $j \in I'$ with $j \geq i$.

It is easily seen that if $\mathcal{D}'$ is a cofinal sub-inverse system of $\mathcal{D}$, then $\lim_{\leftarrow} \mathcal{D}' \cong \lim_{\leftarrow} \mathcal{D}$.

3.2. **From BAOs to inverse systems.** Fix an L-BAO $\mathcal{A}$ with at least two elements. A *partition* in $\mathcal{A}$ is a non-empty finite set $P \subseteq \mathcal{A} \setminus \{0\}$ such that $\sum P = 1$ and $a \cdot b = 0$ for distinct $a, b \in P$. Note that $\{1\}$ is the unique partition containing 1.

Let Π be the set of all partitions in $\mathcal{A}$, and for $P, Q \in \Pi$ write $P \leq Q$ if for all $a \in Q$ there is $b \in P$ with $b \geq a$ (i.e., 'Q refines P'). Note that $\leq$ partially orders Π, and $(\Pi, \leq)$ is directed: indeed, any two elements $P, P' \in \Pi$ have a unique least $\leq$-upper bound $Q \in \Pi$, where Q is the partition $\{a \cdot b : a \in P,\ b \in P'\} \setminus \{0\}$ obtained by combining P and P'.

Now for each $P \in \Pi$ we define an L^a-structure S_P with domain P, as follows:

$$S_P \models R_f(a_1, \ldots, a_n, b) \iff \mathcal{A} \models f(a_1, \ldots, a_n) \cdot b \neq 0, \tag{25}$$

for each $a_1, \ldots, a_n, b \in P$ and each n-ary operator symbol $f \in L$. For $Q \geq P$ in Π, let $\pi_{QP} : S_Q \to S_P$ be given by $\pi_{QP}(a) = b$ where b is the unique element of P with $a \leq b$. Then $\sum_{a \in Q} \pi_{QP}(a) \geq \sum Q = 1$, and it follows that π_{QP} is surjective. Clearly, π_{PP} is the identity on S_P, and $\pi_{RP} = \pi_{QP} \circ \pi_{RQ}$ if $R \geq Q \geq P$ in Π. As $f^{\mathcal{A}}$ is additive, it follows that every π_{QP} is a (surjective) L^a-homomorphism from

S_Q onto S_P. So

$$\mathcal{P}_{\mathcal{A}} = ((\Pi, \leq), (S_P : P \in \Pi), (\pi_{QP} : Q \geq P \text{ in } \Pi)) \tag{26}$$

is an inverse system of finite L^a-structures.

Lemma 3.6. $\lim_{\leftarrow} \mathcal{P}_{\mathcal{A}} \cong \mathcal{A}_+$.

Proof. Write $\mathcal{I}$ for $\lim_{\leftarrow} \mathcal{P}_{\mathcal{A}}$. For $\chi \in \mathcal{I}$, let

$$\hat{\chi} = \{\chi(P) : P \in \Pi\}. \tag{27}$$

One may check that $\hat{\chi}$ is an ultrafilter of $\mathcal{A}$. In particular, $\{a, -a\} \in \Pi$ for all $a \in \mathcal{A} \setminus \{0, 1\}$, so $a \in \hat{\chi}$ or $-a \in \hat{\chi}$. If μ is any ultrafilter of $\mathcal{A}$, define $\chi \in \prod_{P \in \Pi} S_P$ by letting $\chi(P)$ (for $P \in \Pi$) be the unique element of S_P in μ. Then $\chi \in \mathcal{I}$ and $\hat{\chi} = \mu$. It follows that the map $\chi \mapsto \hat{\chi}$ is a surjection : $\mathcal{I} \to \mathcal{A}_+$. It is also injective. For if $\chi, \psi \in \mathcal{I}$ and $\hat{\chi} = \hat{\psi}$, then for any $P \in \Pi$, we have $\chi(P) \in \hat{\chi}$, and $\psi(P) \in \hat{\psi} = \hat{\chi}$. As $\hat{\chi}$ is an ultrafilter, $\chi(P) \cdot \psi(P) \neq 0$. As P is a partition, $\chi(P) = \psi(P)$. This holds for all P, so that $\chi = \psi$.

Finally we check that $\chi \mapsto \hat{\chi}$ is an L^a-isomorphism. Let $\chi_1, \ldots, \chi_n, \psi \in \mathcal{I}$ and let $f \in L$ be n-ary. Then $\mathcal{A}_+ \models R_f(\hat{\chi}_1, \ldots, \hat{\chi}_n, \hat{\psi})$ iff $f(\chi_1(P_1), \ldots, \chi_n(P_n)) \in \hat{\psi}$ for all $P_1, \ldots, P_n \in \Pi$. Since $\hat{\psi}$ is an ultrafilter, this is clearly iff $f(\chi_1(P_1), \ldots, \chi_n(P_n)) \cdot \psi(Q) \neq 0$ for all $P_1, \ldots, P_n, Q \in \Pi$. But we may find $P \in \Pi$ with $P \geq P_1, \ldots, P_n, Q$, so by additivity of f this is iff $f(\chi_1(P), \ldots, \chi_n(P)) \cdot \psi(P) \neq 0$ for all $P \in \Pi$. By (25), this is iff $S_P \models R_f(\chi_1(P), \ldots, \chi_n(P), \psi(P))$ for all $P \in \Pi$, and this is iff $\mathcal{I} \models R_f(\chi_1, \ldots, \chi_n, \psi)$. □

Remark 3.7. $\mathcal{A}$ is *locally finite* if every finite set of elements of $\mathcal{A}$ is contained in a finite subalgebra of $\mathcal{A}$. If $\mathcal{A}$ is locally finite, then by the second part of fact 3.2, $\mathcal{P}_{\mathcal{A}}$ has a cofinal sub-inverse system of finite L^a-structures and bounded morphisms, consisting of all partitions of the form $\mathrm{At}\,\mathcal{B}$ for some finite subalgebra $\mathcal{B} \subseteq \mathcal{A}$. Note that (25) and definition 3.1(4) lead to the same L^a-structure on $\mathrm{At}\,\mathcal{B}$ in this case.

3.3. **From inverse systems to BAOs.** We have seen that the ultrafilter frame of an arbitrary BAO can be obtained (up to isomorphism) as the limit of an inverse system of finite structures. A converse to this can be proved for inverse systems of finite L^a-structures and *bounded morphisms*. Let $\mathcal{D} = ((I, \leq), (S_i : i \in I), (\pi_{ji} : i, j \in I,\ i \leq j))$ be one, and let $\mathcal{I} = \lim_{\leftarrow} \mathcal{D}$. $\mathcal{I}$ is an L^a-structure, so we can consider its complex algebra $\mathcal{I}^+$.

Definition 3.8. For $i \in I$, define $\pi_i^+ : S_i^+ \to \mathcal{I}^+$ as in fact 3.2, by $\pi_i^+(X) = \pi_i^{-1}[X]$, for $X \subseteq S_i$, where $\pi_i : \mathcal{I} \to S_i$ is the projection of definition 3.3(5).

By lemma 3.4, π_i is a surjective bounded morphism. So by fact 3.2, π_i^+ is an algebra embedding : $S_i^+ \to \mathcal{I}^+$, and its range $\pi_i^+(S_i^+)$ is (the domain of) a finite subalgebra of $\mathcal{I}^+$. By fact 3.2 again, if $i \leq j$ in I, then π_{ji}^+ embeds S_i^+ into S_j^+. Since $\pi_i = \pi_{ji} \circ \pi_j$, it follows that $\pi_i^+(S_i^+) \subseteq \pi_j^+(S_j^+)$. Since $(I, \leq)$ is directed, it follows that

$$\mathcal{A}_{\mathcal{D}} = \bigcup_{i \in I} \pi_i^+(S_i^+) \tag{28}$$

is a directed union of finite subalgebras of $\mathcal{I}^+$, and hence is a locally finite subalgebra of $\mathcal{I}^+$.

Lemma 3.9. $(\mathcal{A}_{\mathcal{D}})_+ \cong \lim_{\leftarrow} \mathcal{D} = \mathcal{I}$.

Proof (sketch). The lemma can be proved by mapping $\mathcal{D}$ to a cofinal sub-inverse system of the inverse system $\mathcal{P}_{\mathcal{A}_\mathcal{D}}$ of (26) and applying lemma 3.6, or by showing that $\mathcal{I}^+$ is a 'perfect extension' of $\mathcal{A}_\mathcal{D}$ [22, definition 2.14]. A more general result is proved in [11, Lemma 11.2(8)].

Here, we will adapt the proof of lemma 3.6. Only minor changes are needed. In the lemma, all partitions of $\mathcal{A}$ were available. Here, we only have those partitions of $\mathcal{A}_\mathcal{D}$ of the form $\pi_i^+(\mathrm{At}\, S_i^+)$ for $i \in I$. These are sufficient to carry through the proof because by (28), $\mathcal{A}_\mathcal{D}$ is the directed union of the $\pi_i^+(S_i^+)$.

Write $\mathcal{I}$ for $\lim_\leftarrow \mathcal{D}$. For an ultrafilter $\mu \in (\mathcal{A}_\mathcal{D})_+$, let $\check{\mu} \in \mathcal{I}$ be given by: for $i \in I$, $\check{\mu}(i)$ is the unique element $a \in S_i$ with $\pi_i^+(\{a\}) \in \mu$. Much as before, it can be checked that $\check{\mu}$ is a well-defined element of $\mathcal{I}$, and that the map $(\mu \mapsto \check{\mu}) : (\mathcal{A}_\mathcal{D})_+ \to \mathcal{I}$ is an L^a-isomorphism. We leave the details as an exercise. □

4. A simple example

Before turning to the variety of representable relation algebras, we first prove by a direct application of Theorem 2.3 that a fairly simple canonical variety of boolean algebras with operators has no canonical axiomatisation. The crucial point is that the graphs in the mentioned theorem form a bounded morphic inverse system with some interesting properties, as we shall now see.

Let $L = \{+, -, 0, 1, f\}$, where f is unary. We regard graphs as L^a-structures by interpreting R_f as the edge relation. Fix integers $k \geq m \geq 2$, and let $H_0, H_1, \ldots$ and $\pi_i : H_{i+1} \to H_i$ be graphs and homomorphisms as in theorem 2.3. Each H_i is finite, has chromatic number k, and has no odd cycles of length $\leq i$. It is clear from property (1) of the theorem that the π_i are bounded morphisms.

Now fix a complete graph K_m with m nodes, and for each $i < \omega$ let G_i be the disjoint union of H_i and K_m. For $i < j < \omega$ define ρ_{ii} to be the identity on G_i, and $\rho_{ji} : G_j \to G_i$ by

$$\rho_{ji}(x) = \begin{cases} \pi_i \circ \cdots \circ \pi_{j-1}(x), & \text{if } x \in H_j, \\ x, & \text{if } x \in K_m. \end{cases}$$

This is clearly a surjective bounded morphism. Let $\mathcal{D}_m^k$ be the inverse system

$$\mathcal{D}_m^k = ((\omega, \leq), (G_i : i < \omega), (\rho_{ji} : i \leq j < \omega)) \tag{29}$$

of finite L^a-structures and bounded morphisms. By fact 2.1(2), each G_i has chromatic number $\max(k, m) = k$.

Lemma 4.1. $\lim_\leftarrow \mathcal{D}_m^k$ *is a graph with chromatic number* m.

Proof. Write $\mathcal{I}$ for $\lim_\leftarrow \mathcal{D}_m^k$, and $\rho_i : \mathcal{I} \to G_i$ $(i < \omega)$ for the homomorphism of definition 3.3(5). For $x, y \in \mathcal{I}$, xy is an edge of $\mathcal{I}$ iff $\rho_i(x)\rho_i(y)$ is a edge of G_i for all $i < \omega$. So clearly, the edge relation on $\mathcal{I}$ is irreflexive and symmetric. Hence $\mathcal{I}$ is a graph.

Let $\mathcal{H}$ be the inverse limit of the inverse system $((\omega, \leq), (H_i : i < \omega), \rho_{ji} \restriction H_j : i \leq j < \omega))$. We claim that $\mathcal{H}$ has no odd cycles, and hence (by fact 2.1) is 2-colourable. Suppose for contradiction that $\mathcal{H}$ has a cycle of odd length c. Since $\rho_c \restriction \mathcal{H} : \mathcal{H} \to H_c$ is a homomorphism, by lemma 2.9 it must be that H_c has an odd cycle of length $\leq c$. This contradicts the choice of H_c, and proves the claim.

Now obviously $\mathcal{I}$ is isomorphic to the disjoint union of $\mathcal{H}$ and K_m. So by fact 2.1(1,2), we obtain $\chi(\mathcal{I}) = \max(\chi(\mathcal{H}), \chi(K_m)) = \max(2, m) = m$. □

Translating the above observation into an appropriate algebraic language is surprisingly easy, since the main concepts involved live on the level of complex algebras of graphs. The notion of a *partition* for an arbitrary boolean algebra with operators was already defined in section 3.2. Turning to the notion of independence, and noting that for a graph G and a set $X \in G^+$, $f(X)$ is the set of neighbours of nodes in X, we call an arbitrary element a of a BAO $\mathcal{A}$ of type L *independent* if $a \cdot fa = 0$.

We also generalize the notion of chromatic number to L-type boolean algebras with operators. The *chromatic number* of such an algebra $\mathcal{A}$ is defined to be the least $n < \omega$ such that $\mathcal{A}$ has a partition consisting of n independent elements, and ∞ if there is no such n. Clearly, the chromatic number of a graph G is equal to the chromatic number of the algebra G^+. Let χ_n $(n \geq 1)$ be the sentence

$$\forall x_0 \cdots x_{n-1} \Big(\sum_{i<n} x_i = 1 \ \rightarrow \ \bigvee_{i<n} x_i \cdot f x_i > 0 \Big).$$

It is straightforward to verify that an algebra has chromatic number $> n$ iff the formula χ_n holds in it.

The next result forms the technical pivot of our paper.

Theorem 4.2. *Let k, m be given, with $2 \leq m \leq k < \omega$. There is an L-algebra $\mathcal{A}$ which has chromatic number k and whose canonical extension $\mathcal{A}^\sigma$ has chromatic number m.*

Proof. Consider the inverse system $\mathcal{D} = \mathcal{D}^k_m$ of (29). Let $\mathcal{A}_\mathcal{D}$ be as in (28). Since all the G_i have chromatic number k, so does each G_i^+. By (28), $\mathcal{A}_\mathcal{D}$ is the directed union of the algebras G_i^+ $(i < \omega)$. So up to isomorphism, any partition in any G_i^+ is also a partition in $\mathcal{A}_\mathcal{D}$, and any partition in $\mathcal{A}_\mathcal{D}$ is a partition in all but finitely many G_i^+. It follows that $\mathcal{A}_\mathcal{D}$ also has chromatic number k.

On the other hand, by lemma 4.1 the inverse limit, $\lim_{\leftarrow} \mathcal{D}$, has chromatic number m. By lemma 3.9, $\lim_{\leftarrow} \mathcal{D} \cong (\mathcal{A}_\mathcal{D})_+$, so that $(\mathcal{A}_\mathcal{D})^\sigma \cong (\lim_{\leftarrow} \mathcal{D})^+$. So $(\mathcal{A}_\mathcal{D})^\sigma$ has chromatic number m as required. ☐

Corollary 4.3. *For every $n \geq m \geq 2$ there is an L-algebra $\mathcal{A}_{n,m}$ such that $\mathcal{A}_{n,m} \models \chi_n$ and $(\mathcal{A}_{n,m})^\sigma \models \chi_{m-1} \wedge \neg\chi_m$.*

Proof. By theorem 4.2, we may take $\mathcal{A}_{n,m}$ to be an algebra of chromatic number $n+1$, with canonical extension of chromatic number m. Then $\mathcal{A}_{n,m} \models \chi_n$ and $(\mathcal{A}_{n,m})^\sigma \models \chi_{m-1}$, but $(\mathcal{A}_{n,m})^\sigma \not\models \chi_m$. So $\mathcal{A}_{n,m}$ is as required. ☐

In order to use this observation to derive results concerning varieties, it is convenient to expand the language with a so-called unary discriminator term (cf. Jipsen [19]). Let L_2 be the similarity type $\{+, -, 0, 1, f, c\}$, where f and c are unary. Let $\mathbf{K}$ be the class of algebras that satisfy $\{\chi_n : n \geq 1\}$ (i.e., they have infinite chromatic number), and on which $c(x)$ is a *unary discriminator term.* That is, every algebra in $\mathbf{K}$ satisfies

$$c(x) \ = \ \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases} \tag{30}$$

Finally, let $\mathbf{V}$ be the variety generated by $\mathbf{K}$; $\mathbf{V}$ is thus a discriminator variety.

Part of the attraction of discriminator terms lies in the fact that there is a simple, effective procedure rewriting a universal formula ψ into an equation ψ' such that ψ and ψ' are equivalent on every algebra satisfying (30). Another straightforward application of the theory of discriminator varieties reveals that $\mathbf{V}$ can be axiomatised by the set $\{\chi'_n : n \geq 1\}$ of equational translations of the colouring formulas, together with some finite set Δ. (Δ contains axioms ensuring that the boolean reduct of the algebra at stake is indeed a boolean algebra, axioms stating that the $\{+, -, 0, 1, c\}$-reduct is a so-called monadic algebra, and the axiom $f(x) \leq c(x)$. Readers interested in further details are referred to [19].)

Definition 4.4. Let L be a signature of BAOs. An L-equation or first-order L-sentence φ is said to be *canonical* if for every L-BAO $\mathcal{A}$, if $\mathcal{A} \models \varphi$, then $\mathcal{A}^\sigma \models \varphi$.

The aim of this section is to prove the following result.

Theorem 4.5. *The variety* $\mathbf{V}$ *is a canonical variety which cannot be axiomatised by any set of equations that contains only finitely many non-canonical equations. The same holds for first-order sentences instead of equations.*

Proof. For the first part of the theorem, let $\mathbf{R}$ be the class of all L_2^a-structures (G, R_f, R_c) such that (i) there is at least one R_f-reflexive point, and (ii) R_c is universal (i.e., $R_c = G \times G$). It is fairly straightforward to prove that for all algebras $\mathcal{A}$ in $\mathbf{K}$, the ultrafilter structure $\mathcal{A}_+$ belongs to $\mathbf{R}$ (simply check that the set $\{-a : a \text{ an independent element of } \mathcal{A}\}$ extends to an R_f-reflexive ultrafilter of $\mathcal{A}$; see Hughes [18] or Venema [38] for very similar proofs). Conversely, it is easily seen that the complex algebra of any $\mathbf{R}$-structure belongs to $\mathbf{K}$. This means that $\mathbf{V}$ is generated by the complex algebras of structures in $\mathbf{R}$. Since $\mathbf{R}$ is an elementary class, it then follows from the results in Goldblatt [12] that $\mathbf{V}$ is canonical.

Now suppose for contradiction that $\mathbf{V}$ is axiomatised by $\mathcal{C} \cup \mathcal{F}$, where $\mathcal{C}$ is a set of canonical equations — or indeed, canonical first-order sentences — and $\mathcal{F}$ is a finite set of arbitrary equations (or first-order sentences). Then $\mathcal{C} \cup \mathcal{F}$ and $\{\chi'_l : l \geq 1\} \cup \Delta$ are logically equivalent. By compactness, and the fact that on algebras satisfying Δ and (30) χ'_k implies χ'_l if $k > l$, there is $m \geq 2$ such that any algebra satisfying $\Delta \cup \{\chi'_{m-1}\}$ and (30) also satisfies $\mathcal{F}$. Then from $\mathcal{C} \cup \mathcal{F} \models \chi'_m$ it follows, again by compactness, that there is a finite subset C of $\mathcal{C}$ such that $C \cup \mathcal{F} \models \chi'_m$. But then, by compactness, there must be a single χ'_n such that C is valid in any algebra satisfying $\Delta \cup \{\chi'_n\}$ and (30). We may suppose that $n \geq m$.

To see that this cannot be the case, consider the algebra $\mathcal{A} = \mathcal{A}_{n,m}$ of corollary 4.3. Note that $\mathcal{A}$ does not have the right type; to remedy this, define, for an arbitrary L-type BAO $\mathcal{A}$, the algebra $\mathcal{A}'$ as the L_2-expansion of $\mathcal{A}$ in which c is defined by (30). It is not difficult to prove that this operation commutes with taking canonical extensions. That is, $\mathcal{A}'^\sigma \cong (\mathcal{A}^\sigma)'$.

It is clear that the equations in Δ hold in $\mathcal{A}'$; but since c is defined to satisfy (30) it follows from $\mathcal{A} \models \chi_n$ that $\mathcal{A}' \models \chi'_n$. Since all the equations in $\Delta \cup \{\chi'_n\}$ hold on $\mathcal{A}'$, so must the set C. Now C consists of canonical equations, so C must be valid on $\mathcal{A}'^\sigma \cong (\mathcal{A}^\sigma)'$ as well. Moreover, by corollary 4.3, $\mathcal{A}^\sigma \models \chi_{m-1}$, and since $(\mathcal{A}^\sigma)'$ satisfies (30), we have $(\mathcal{A}^\sigma)' \models \chi'_{m-1}$. It is clear that $(\mathcal{A}^\sigma)' \models \Delta$; so we obtain $(\mathcal{A}^\sigma)' \models \mathcal{F}$. So $(\mathcal{A}^\sigma)' \models C \cup \mathcal{F}$, whence $(\mathcal{A}^\sigma)' \models \chi'_m$. But χ_m and χ'_m are equivalent on $(\mathcal{A}^\sigma)'$ as well. This implies that $\mathcal{A}^\sigma \models \chi_m$, which clearly contradicts the statement in corollary 4.3 concerning the properties of $\mathcal{A}^\sigma$. □

5. Games on relation algebras

We now aim to extend the results of the preceding section to relation algebras. It will be necessary to have to hand axioms for the representable relation algebras, and here we recall the axioms obtained by games. We assume familiarity with basic aspects of relation algebras; see, e.g., [29, 15] for background.

Let $\mathcal{A} = (A, +, -, 0, 1, 1\text{'}, \breve{}, ;)$ be a relation algebra. The 'pieces' used in the games played on $\mathcal{A}$ are called ($\mathcal{A}$-)*pre-networks* — i.e., pairs $N = (N_1, N_2)$ where N_1 is a finite non-empty set of 'nodes' and $N_2 : N_1 \times N_1 \to \mathcal{A}$ is a map. (Recall that notationally we identify $\mathcal{A}$ and A.) For pre-networks $N = (N_1, N_2)$ and $N' = (N'_1, N'_2)$, we write $N \subseteq N'$ if $N_1 \subseteq N'_1$ and $N'_2(x, y) \leq N_2(x, y)$ for all $x, y \in N_1$. We will denote N, N_1, N_2 simply by N, leaving the mental addition of the indices to the reader; but sometimes we write nodes(N) for N_1.

Let N be a pre-network. A triple (x, y, z) of nodes of N is said to be *consistent* if $(N(x, z) ; N(z, y)) \cdot N(x, y) \neq 0$. N is called a *network* if $N(x, x) \leq 1\text{'}$ for all $x \in N$, and every triple (x, y, z) of nodes of N is consistent. It is easily checked (cf. [15, lemma 7.2]) that if N is a network, then for any $x, y \in N$ we have

$$N(x, y) \neq 0, \text{ and } N(x, y)\breve{} \cdot N(y, x) \neq 0. \tag{31}$$

The following definition is from [15, definition 7.12] (see also [14]; here we have changed the notation slightly).

Definition 5.1. The game $\mathcal{G}_n(\mathcal{A})$ (for $n < \omega$) is played on an arbitrary relation algebra $\mathcal{A} = (A, +, -, 0, 1, 1\text{'}, \breve{}, ;)$. There are two players, called $\forall$ (male) and $\exists$ (female), and $n + 1$ rounds. In a preliminary (unnumbered) round, $\forall$ picks any $a \in \mathcal{A} \setminus \{0\}$ and $\exists$ must respond with the network N_0 with just two nodes — x, y, say — given by $N_0(x, y) = a$, $N_0(x, x) = N_0(y, y) = 1\text{'}$, and $N_0(y, x) = 1$. The remaining rounds are numbered $0, 1, \ldots, n - 1$, and play constructs a chain $N_0 \subseteq \cdots \subseteq N_n$ of pre-networks. In round t, assuming play so far has built the pre-network N_t, $\forall$ picks nodes $x, y \in N_t$ and elements $a, b \in \mathcal{A}$. $\exists$ may either

- **accept:** playing the pre-network $N_{t+1} \supseteq N_t$ with nodes those of N_t plus a new one, z, and with $N_{t+1}(x, y) = N_t(x, y) \cdot (a ; b)$, $N_{t+1}(x, z) = a$, $N_{t+1}(z, y) = b$, $N_{t+1}(z, z) = 1\text{'}$, $N_{t+1}(u, v) = N_t(u, v)$ for all $u, v \in N_t$ with $(u, v) \neq (x, y)$, and for all pairs (u, v) not yet mentioned, $N_{t+1}(u, v) = 1$.
- **reject:** in which case N_{t+1} is the same as N_t except that $N_{t+1}(x, y) = N_t(x, y) \cdot -(a ; b)$. Again, $N_{t+1} \supseteq N_t$.

$\exists$ wins iff every N_t she plays is a network. She also wins if $\forall$ cannot move in the preliminary round — this can only happen if $\mathcal{A}$ is degenerate (i.e., $|A| = 1$).

A *strategy* for a player in $\mathcal{G}_n(\mathcal{A})$ is a set of rules giving that player a non-empty set of permissible moves in any situation. A strategy is said to be *winning* if its owner wins any play of $\mathcal{G}_n(\mathcal{A})$ in which the strategy is used, and *deterministic* if it provides exactly one permissible move in each situation.

Clearly, any winning strategy may be refined to a deterministic winning strategy. Also clearly, for any n, a winning strategy for $\exists$ in $\mathcal{G}_{n+1}(\mathcal{A})$ yields by restriction a winning strategy for her in $\mathcal{G}_n(\mathcal{A})$. We collect the remaining salient facts about winning strategies in the following propositions.

Proposition 5.2.

(1) [15, theorem 7.19, proposition 7.24] *For any relation algebra $\mathcal{A}$, $\mathcal{A}$ is representable iff $\exists$ has a winning strategy in $\mathcal{G}_n(\mathcal{A})$ for all $n < \omega$.*

(2) [15, proposition 8.2] *For each $n < \omega$ there is a universal first-order sentence σ_n of the signature of relation algebras, such that for any relation algebra $\mathcal{A}$, $\exists$ has a winning strategy in $\mathcal{G}_n(\mathcal{A})$ iff $\mathcal{A} \models \sigma_n$.*

(3) *(Hence) the class* **RRA** *of representable relation algebras is axiomatised by Tarski's finite set of equations defining relation algebras* [34]*, plus the* σ_n *for all* $n < \omega$.

Proposition 5.3. *If* $\mathcal{A}$ *is a relation algebra,* $n < \omega$*, and* $\forall$ *has a winning strategy in* $\mathcal{G}_n(\mathcal{A})$*, then he has a winning strategy such that for some subset* $W \subseteq \mathcal{A}$ *with* $|W| < 2^{n+1}$*, the elements chosen by* $\forall$ *in any play of* $\mathcal{G}_n(\mathcal{A})$ *in which he uses the strategy all lie in* W.

Proof. Cf. [15, lemma 7.17]. We may fix a deterministic winning strategy for $\forall$. If he uses it, then since $\exists$ has two choices in each of the n numbered rounds, the positions in which $\forall$ is to move, and that are attainable in numbered rounds of plays of $\mathcal{G}_n(\mathcal{A})$, correspond in a one-to-one fashion to the sequences of 0s and 1s of length $\leq n-1$. So there are $2^n - 1$ such positions. In each of them, $\forall$'s strategy will pick at most two elements of $\mathcal{A}$. He also picks one element in the preliminary round. The total is at most $2(2^n-1)+1 < 2^{n+1}$ elements of $\mathcal{A}$. □

We saw in §1 that **RRA** is canonical: if $\mathcal{A} \in$ **RRA**, then $\mathcal{A}^\sigma \in$ **RRA**. So by proposition 5.2, if $\exists$ has a winning strategy in $\mathcal{G}_n(\mathcal{A})$ for all $n < \omega$, then she has a winning strategy in $\mathcal{G}_n(\mathcal{A}^\sigma)$ for all $n < \omega$, too. If **RRA** has an axiomatisation using at most finitely many non-canonical sentences, then a rough 'level by level' analogue of this 'eventually' holds:

Lemma 5.4. *Suppose that* **RRA** *has a first-order axiomatisation using only finitely many non-canonical first-order sentences. Then there is* $n_0 < \omega$ *such that for any* $n < \omega$ *there is* $n^* < \omega$ *such that for any relation algebra* $\mathcal{A}$*, if* $\exists$ *has winning strategies in* $\mathcal{G}_{n^*}(\mathcal{A})$ *and* $\mathcal{G}_{n_0}(\mathcal{A}^\sigma)$*, then she has a winning strategy in* $\mathcal{G}_n(\mathcal{A}^\sigma)$.

Proof. Assume that $\mathcal{C}$ is a set of canonical first-order sentences and $\mathcal{F}$ a finite set of sentences, both in the signature of relation algebras, such that $\mathcal{C} \cup \mathcal{F}$ axiomatises **RRA**. Let Δ denote the finite set of equations defining relation algebras, as given in [34]. It was shown in [22] that these equations are canonical (they are Sahlqvist equations). By proposition 5.2(3), the theories $\mathcal{C} \cup \mathcal{F}$ and $\Delta \cup \{\sigma_n : n < \omega\}$ have the same models. So by compactness, there is $n_0 < \omega$ such that $\Delta \cup \{\sigma_{n_0}\} \models \mathcal{F}$. (Note here and below that by proposition 5.2(2), $\Delta \cup \{\sigma_{m+1}\} \models \sigma_m$, for all m.)

Let $n < \omega$. Again by compactness, σ_n is logically implied by $C \cup \mathcal{F}$ for some finite subset $C \subseteq \mathcal{C}$, and there is $n^* < \omega$ such that $\Delta \cup \{\sigma_{n^*}\} \models C$. Suppose that $\mathcal{A}$ is a relation algebra and $\exists$ has a winning strategy in $\mathcal{G}_{n^*}(\mathcal{A})$ and in $\mathcal{G}_{n_0}(\mathcal{A}^\sigma)$. By proposition 5.2(2), $\mathcal{A} \models \sigma_{n^*}$ and $\mathcal{A}^\sigma \models \sigma_{n_0}$. Then $\mathcal{A} \models C \cup \Delta$, so as the sentences in $C \cup \Delta$ are canonical, $\mathcal{A}^\sigma \models C \cup \Delta$. So $\mathcal{A}^\sigma \models C \cup \mathcal{F}$, whence $\mathcal{A}^\sigma \models \sigma_n$. Hence, $\exists$ has a winning strategy in $\mathcal{G}_n(\mathcal{A}^\sigma)$. □

A similar argument shows that if **RRA** has a canonical axiomatisation, then the 'eventually' part falls away: $\exists$ does not need to have a winning strategy in $\mathcal{G}_{n_0}(\mathcal{A}^\sigma)$. See [15, exercise 8.1(1)].

At first sight, it might seem plausible that the conclusion of the lemma is in fact a true property of relation algebras. However, in the next section we will show that it fails, so that every axiomatisation of **RRA** has infinitely many non-canonical sentences.

6. **RRA** has no canonical axiomatisation

Our proof that **RRA** cannot be axiomatised by canonical equations, and indeed that any axiomatisation needs infinitely many non-canonical sentences, is very similar to the one given in section 4. The main differences are that the switch in similarity type is more involved in the relation algebraic case, and that the correspondence of axioms to chromatic number is not so close.

6.1. **Relation algebras from graphs.** Let us first show how to obtain a relation algebra from a graph. We take the same approach as in [16] and [15, definition 14.10]. We let L_{RA} denote the signature $\{+, -, 0, 1, 1', \breve{}, ;\}$ of relation algebras. So $L^a_{RA} = \{R_{1'}, R_{\smile}, R_;\}$.

Definition 6.1. Let G be a graph.

(1) We define an L^a_{RA}-structure $\alpha(G)$. The elements of $\alpha(G)$ are 1' plus red, blue, and yellow copies $\mathsf{r}_x, \mathsf{b}_x, \mathsf{y}_x$ of each node $x \in G$. The relations of L^a_{RA} are defined on $\alpha(G)$ as follows: $R_{1'}(a)$ holds iff $a = 1'$, $R_{\smile}(a, b)$ holds iff $a = b$, and $R_;$ holds on all triples (a, b, c) of elements of $\alpha(G)$ except:
   - $(1', a, b)$, $(a, 1', b)$, and $(a, b, 1')$ when $a \neq b$,
   - $(\mathsf{r}_x, \mathsf{r}_y, \mathsf{r}_z)$, $(\mathsf{b}_x, \mathsf{b}_y, \mathsf{b}_z)$, and $(\mathsf{y}_x, \mathsf{y}_y, \mathsf{y}_z)$, when $\{x, y, z\}$ is an independent set in G.

(2) For $X \subseteq G$ let $\mathsf{R}_X = \{\mathsf{r}_x : x \in X\}$, and define $\mathsf{B}_X, \mathsf{Y}_X$ similarly. These are in $(\alpha(G))^+$.

(3) An element $a \in \alpha(G)^+$ is said to be *monochromatic* if $a \leq 1'$, $a \leq \mathsf{R}_G$, $a \leq \mathsf{B}_G$, or $a \leq \mathsf{Y}_G$.

Lemma 6.2. *For any graph G, the complex algebra $\alpha(G)^+$ is a relation algebra.*

Proof. It is readily checked that $\alpha(G)$ satisfies:

(1) $\forall xy(x = y \leftrightarrow R_;(x, 1', y))$,
(2) $\forall xyz(R_;(x, y, z) \to R_;(x, z, y) \wedge R_;(y, x, z))$,
(3) $\forall xyzt\big(\exists u\big(R_;(x, y, u) \wedge R_;(u, z, t)\big) \leftrightarrow \exists v\big(R_;(y, z, v) \wedge R_;(x, v, t)\big)\big)$ — for example, in the case where $1' \notin \{x, y, z, t\}$, any $u \neq 1'$ such that $\{x, y, u\}$ and $\{u, z, t\}$ are not monochromatic witnesses the left-hand side, and the right-hand side can be similarly witnessed.

It is well known that these conditions are sufficient for $\alpha(G)^+$ to be a relation algebra — cf. [25] or [15, lemma 3.24]. $\square$

In the next two sections, we will show that there are functions $e, u : \omega \to \omega$ such that for any n and graph G, $\exists$ has a winning strategy in $\mathcal{G}_n(\alpha(G)^+)$ if $\chi(G) \geq e(n)$, and conversely that so long as G is large enough, $\forall$ has a winning strategy in $\mathcal{G}_{u(n)}(\alpha(G)^+)$ if $\chi(G) \leq n$. Armed with these results, the lack of canonical axiomatisation of **RRA** will follow easily from theorem 2.3 and lemma 5.4.

6.2. **$\exists$'s winning strategy.**

Definition 6.3. We define a map $e : \omega \to \omega$ by $e(n) = 2^{15 \cdot 4^n}$.

Proposition 6.4. *Let G be a graph and let $n < \omega$ be arbitrary. If $\chi(G) \geq e(n)$, then $\exists$ has a winning strategy in $\mathcal{G}_n(\alpha(G)^+)$.*

Proof. Cf. [16, theorem 11] or [15, theorem 14.13]. Write $\mathcal{A}$ for $\alpha(G)^+$. Trivially, $\exists$ has a winning strategy in $\mathcal{G}_0(\mathcal{A})$. Assume now that $n > 0$, and suppose, for contradiction, that $\exists$ has no winning strategy in $\mathcal{G}_n(\mathcal{A})$. Since finite-length games are determined [9], $\forall$ must have a winning strategy in this game. By proposition 5.3, for some set W of $< 2^{n+1}$ elements of $\mathcal{A}$, $\forall$ has a winning strategy that only ever directs him to choose elements in W.

Let

$$W^+ = W \cup \{a\,;b : a, b \in W\} \subseteq \mathcal{A}.$$

Then $|W^+| < 2 \cdot 2^n + 4 \cdot 4^n$. Since $n \geq 1$, we have $2 \cdot 2^n \leq 4^n$, so $|W^+| < 5 \cdot 4^n$. Define an equivalence relation $\sim$ on G by: $x \sim y$ iff for all $w \in W^+$ we have $\mathsf{r}_x \leq w \iff \mathsf{r}_y \leq w$, $\mathsf{b}_x \leq w \iff \mathsf{b}_y \leq w$, and $\mathsf{y}_x \leq w \iff \mathsf{y}_y \leq w$. Let $G/{\sim}$ denote the set of $\sim$-classes. Then

$$|G/{\sim}| \leq 2^{3|W^+|} < 2^{15 \cdot 4^n} = e(n) \leq \chi(G).$$

So there must be some $\sim$-class $E \subseteq G$ such that xy is an edge of G for some $x, y \in E$. Then by definition 6.1, $\alpha(G) \models R_;(\mathsf{r}_x, \mathsf{r}_y, \mathsf{r}_z) \wedge R_;(\mathsf{r}_x, \mathsf{r}_y, \mathsf{b}_z) \wedge R_;(\mathsf{r}_x, \mathsf{r}_y, \mathsf{y}_z)$ for any $z \in G$, and $\alpha(G) \models R_;(\mathsf{r}_x, \mathsf{r}_x, 1\text{'})$. The same holds with $\mathsf{b}_x, \mathsf{b}_y$, and with $\mathsf{y}_x, \mathsf{y}_y$. Hence, by definition of complex algebras, in $\mathcal{A}$ we have

$$\mathsf{R_E}\,;\mathsf{R_E} = \mathsf{B_E}\,;\mathsf{B_E} = \mathsf{Y_E}\,;\mathsf{Y_E} = 1. \tag{32}$$

To obtain our contradiction, we will show $\exists$ how to win $\mathcal{G}_n(\mathcal{A})$ when $\forall$ uses his supposedly winning strategy. Let $\mathcal{B}$ be the boolean subalgebra of the boolean reduct of $\mathcal{A}$ generated by $\{1\text{'}, \mathsf{R_X}, \mathsf{B_X}, \mathsf{Y}_X : X \in G/{\sim}\}$. This latter set is precisely the set $\operatorname{At}\mathcal{B}$ of atoms of $\mathcal{B}$. Observe that

- $W^+ \cup \{1\text{'}\} \subseteq \mathcal{B}$,
- each atom of $\mathcal{B}$ is monochromatic,
- for any $X \subseteq G$, the conditions $\mathsf{R_X} \in \mathcal{B}$, $\mathsf{B_X} \in \mathcal{B}$, and $\mathsf{Y}_X \in \mathcal{B}$ are equivalent.

To help her win, $\exists$ will maintain as a guide an $\mathcal{A}$-network M_t (for each $t \leq n$) satisfying the conditions:

(I) for all $x, y \in M_t$, $M_t(x, y) \in \operatorname{At}\mathcal{B}$, and $M_t(x, y) = 1\text{'}$ iff $x = y$,
(II) there is a map $'$: nodes$(N_t) \to$ nodes(M_t) such that $M_t(x', y') \leq N_t(x, y)$ for all $x, y \in$ nodes(N_t),

where N_t is the pre-network in play just before round t, the round creating N_{t+1} from it.

In the preliminary round in $\mathcal{G}_n(\mathcal{A})$, $\forall$ chooses some non-zero element $a_0 \in \mathcal{B}$. $\exists$ must respond with a two-node network N_0 with distinct nodes x_0, y_0, say, with $N_0(x_0, y_0) = a_0$, $N_0(y_0, x_0) = 1$, and $N_0(x_0, x_0) = N_0(y_0, y_0) = 1\text{'}$. To define M_0, she chooses arbitrary $a^* \in \operatorname{At}\mathcal{B}$ with $a^* \leq a_0$. If $a^* = 1\text{'}$, she defines M_0 to be the network with a single node, say x_0, with $M_0(x_0, x_0) = 1\text{'}$. She also puts $x_0' = y_0' = x_0$. Otherwise, since $1\text{'} \in \operatorname{At}\mathcal{B}$, we must have $a^* \leq -1\text{'}$. $\exists$ then defines M_0 to have nodes x_0, y_0 only, with $M_0(x_0, y_0) = M_0(y_0, x_0) = a^*$, and $M_0(x_0, x_0) = M_0(y_0, y_0) = 1\text{'}$. She puts $x_0' = x_0$ and $y_0' = y_0$. Clearly, M_0 is a network, and conditions (I) and (II) are met.

Let $t < n$, and assume inductively that $\exists$ has defined M_t for N_t. We will show $\exists$ how to respond in the next round of the game, and how to construct a new network M_{t+1} meeting the above two conditions with respect to N_{t+1}.

$\forall$ plays round t by choosing nodes $x, y \in N_t$ and elements $a, b \in \mathcal{A}$ according to his winning strategy. Note that $a, b, a\,;b \in \mathcal{B}$. $\exists$ uses the value of $M_t(x', y')$ to decide her response. If $M_t(x', y') \le -(a ; b)$, then she rejects, and plays N_{t+1} as specified in definition 5.1. She defines $M_{t+1} = M_t$; the map $'$ is unchanged. This preserves conditions (I) and (II).

Otherwise, since $M_t(x', y') \in \mathrm{At}\,\mathcal{B}$, we have $M_t(x', y') \le a ; b$. $\exists$ then accepts $\forall$'s move, playing a labelled graph N_{t+1} with a new node z and labelling it as specified in definition 5.1. She now defines M_{t+1} meeting the conditions, as follows.

If there is already a node $p \in M_t$ with $M_t(x', p) \le a$ and $M_t(p, y') \le b$, then $\exists$ can set $M_{t+1} = M_t$ and $z' = p$. This meets the required conditions on N_{t+1}, M_{t+1}.

Otherwise, since $a ; b \ge M_t(x', y') \in \mathrm{At}\,\mathcal{B}$ and $a, b \in \mathcal{B}$, by additivity of composition in $\mathcal{A}$ there must be $a^*, b^* \in \mathrm{At}\,\mathcal{B}$ with $a^* \le a$, $b^* \le b$, and $(a^* ; b^*) \cdot M_t(x', y') \ne 0$. $\exists$ chooses such a^*, b^*; it can be checked using standard relation algebra properties that $a^*, b^* \ne 1'$, and that if $x' = y'$ then $a^* = b^*$. Note that a^*, b^* are monochromatic. $\exists$ chooses a colour C (from R, B, Y) different from the colours of (atoms in) a^*, b^*. She then 'refines' M_t to $M_{t+1} \supseteq M_t$ by adding a new node, defining z' to be that new node, and labelling edges of M_{t+1} with atoms of $\mathcal{B}$ as follows (see figure 1):

- $M_{t+1}(p, q) = M_t(p, q)$ for all $p, q \in M_t$.
- $M_{t+1}(x', z') = M_{t+1}(z', x') = a^*$, $M_{t+1}(z', z') = 1'$, and $M_{t+1}(z', y') = M_{t+1}(y', z') = b^*$. This is well-defined if $x' = y'$, as $a^* = b^*$ in that case.
- For $p \in \mathrm{nodes}(M_t) \setminus \{x', y'\}$, $\exists$ labels $M_{t+1}(p, z') = M_{t+1}(z', p) = \mathsf{C}_E$, where E is as in (32).

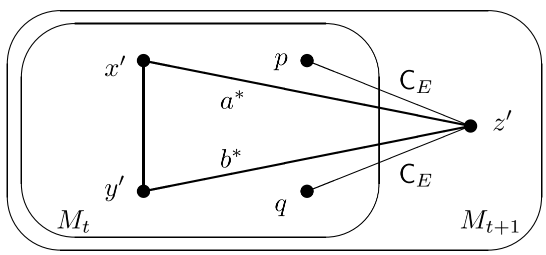

FIGURE 1. The network M_{t+1}

We claim that every triple (p, q, r) of nodes of M_{t+1} is consistent — that is, in $\mathcal{A}$ we have $M_{t+1}(p, q) \cdot \big(M_{t+1}(p, r) ; M_{t+1}(r, q)\big) \ne 0$. If $p, q, r \in M_t$, this is clear inductively, if $|\{p, q, r\}| < 3$ it is trivial, and by definition of $\alpha(G)$ and $\mathcal{A}$, the order of p, q, r is not significant, so we need only consider triples of distinct nodes of the form (p, q, z') for $p, q \in M_t$. We already know (x', y', z') to be consistent, since $(a^* ; b^*) \cdot M_t(x', y') \ne 0$. The labels in any triple of the form (x', q, z') with $q \ne x', y'$ are c, C_E, a^* for some $c \in \mathrm{At}\,\mathcal{B} \setminus \{1'\}$; because C is not the colour of a^*, the triple must be consistent. The case of (y', q, z') is similar. Finally, if (p, q, z') is a triple with $p, q \in M_t \setminus \{x, y\}$, we have $M_{t+1}(p, z') = M_{t+1}(q, z') = \mathsf{C}_E$ and $M_{t+1}(p, q) = c$ for some $c \in \mathrm{At}\,\mathcal{B}$. By (32) above, $c \le \mathsf{C}_E ; \mathsf{C}_E$, whence (p, q, z') is consistent. This proves the claim.

By the claim, M_{t+1} is a network. We have arranged that every edge of M_{t+1} is labelled by an element of $\mathrm{At}\,\mathcal{B}$ beneath the corresponding label in N_{t+1}, and is labelled by 1' iff it is a reflexive edge. Thus, conditions (I) and (II) on M_t are maintained for another round.

It follows that each labelled graph N_t ($t \leq n$) is a network. To see this, let $x, y, z \in N_t$. Then $N_t(x,z) ; N_t(z,y) \cdot N_t(x,y) \geq M_t(x',z') ; M_t(z',y') \cdot M_t(x',y') \neq 0$. Thus, $\exists$ never loses, which is a contradiction. $\square$

6.3. $\forall$**'s winning strategy.**

Definition 6.5.

(1) For each $n < \omega$, choose $n' < \omega$ so large that any colouring, using $3n$ colours, of the edges of a complete graph with n' nodes has a monochromatic triangle.
(2) We define a map $u : \omega \to \omega$ by $u(n) = n' - 2 + n'(n'-1)(3n+1)$.

The finite Ramsey theorem shows that n' exists. For example, define $\rho(0) = 1$ and $\rho(k+1) = 1 + (k+1) \cdot \rho(k)$ for each $k \geq 0$. Then $n' = 1 + \rho(3n)$ suffices; see [13, corollary 3]. It is not essential later, but under these definitions it can be checked that $2' = 1958$, $u(2) = 26{,}824{,}598$, and $u(n) \leq 2^{20n^2}$ for all $n \geq 1$.

Proposition 6.6. *If G is a graph with $\chi(G) \leq n < \infty$ and $|G| \geq n' - 1$ (with n' as in definition 6.5), then $\forall$ has a winning strategy in $\mathcal{G}_{u(n)}(\alpha(G)^+)$.*

Proof. Cf. [16, theorem 10] or [15, theorem 14.12]. Let $I_0, \ldots, I_{n-1} \subseteq G$ be a partition of G into independent sets. Again, we write $\mathcal{A}$ for $\alpha(G)^+$, and for $X \subseteq G$, we continue to let $\mathsf{R}_X = \{\mathsf{r}_x : x \in X\} \in \mathcal{A}$, and similarly for $\mathsf{B}_X, \mathsf{Y}_X$. Let

$$\Pi = \{1\text{'}, \mathsf{R}_{I_j}, \mathsf{B}_{I_j}, \mathsf{Y}_{I_j} : j < n\} \subseteq \mathcal{A}. \tag{33}$$

Note that

$$(a ; a) \cdot a = 0 \text{ for all } a \in \Pi \setminus \{1\text{'}\}. \tag{34}$$

Let n' be as in definition 6.5. Let $\forall$ play the first rounds of $\mathcal{G}_{u(n)}(\mathcal{A})$ so as to create a pre-network N with n' nodes which are 'logically distinct'. He does this as follows. He first picks distinct nodes $\gamma_1, \ldots, \gamma_{n'-1} \in G$. We do need $|G| \geq n' - 1$ for this to be possible. In the preliminary round, he plays r_{γ_1} (here, we identify the atom $\{\mathsf{r}_{\gamma_1}\} \in \mathcal{A}$ with r_{γ_1}). $\exists$ must respond with a pre-network N_0 with nodes x_0, x_1, say, with $N_0(x_0, x_1) = \mathsf{r}_{\gamma_1}$. In subsequent rounds $t = 0, 1, \ldots, n' - 3$, he picks the nodes x_0, x_0 of the current pre-network N_t, and elements $\mathsf{r}_{\gamma_{t+2}}, \mathsf{r}_{\gamma_{t+2}}$. Since $N_t(x_0, x_0) = 1\text{'} \leq \mathsf{r}_{\gamma_{t+2}} ; \mathsf{r}_{\gamma_{t+2}}$, if $\exists$ is to avoid losing she has no choice but to accept and add a new node x_{t+2}, so that $N_{t+1}(x_0, x_{t+2}) = \mathsf{r}_{\gamma_{t+2}}$. After these $n' - 2$ rounds, $\forall$ has forced a pre-network $N = N_{n'-2}$ with nodes $\{x_0, \ldots, x_{n'-1}\}$ and with $N(x_0, x_i) = \mathsf{r}_{\gamma_i}$ for all $i \geq 1$.

$\forall$ continues $\mathcal{G}_{u(n)}(\mathcal{A})$ by forcing $\exists$ to decide which element of Π should hold on each edge of N, as follows. Suppose in a subsequent round, the current pre-network is $N' \supseteq N$. He picks a pair x, y of distinct nodes of N, and the algebra elements $a, 1\text{'} \in \mathcal{A}$, for some $a \in \Pi$. We denote such a move by (x, y, a). $\exists$ must respond with a new pre-network $N'' \supseteq N'$ with either $N''(x,y) \leq a$ or $N''(x,y) \leq -a$. Note that when he plays $(x_i, x_j, 1\text{'})$ with distinct $i, j > 0$, $\exists$ must reject or lose, since if she accepted we would have $N''(x_i, x_j) \leq 1\text{'}$, so $(N''(x_0, x_i) ; N''(x_i, x_j)) \cdot N''(x_0, x_j) \leq \mathsf{r}_{\gamma_i} \cdot \mathsf{r}_{\gamma_j} = 0$, and by (31), N'' would not be a network. So $N''(x_i, x_j) \leq -1\text{'}$. $\forall$ continues in this way until he has made every move (x, y, a), for all $a \in \Pi$ and all distinct nodes $x, y \in N$. It takes $n'(n'-1)(3n+1)$ rounds. So $\forall$ can complete these actions within the length of $\mathcal{G}_{u(n)}(\mathcal{A})$.

Let M be the pre-network resulting at the end of these moves. We have $N \subseteq M$, and $M(x,y) \leq -1$' and either $M(x,y) \leq a$ or $M(x,y) \leq -a$ for all distinct $x, y \in N$ and all $a \in \Pi$.

We claim that M is not an $\mathcal{A}$-network, so that $\exists$ has lost. Assume for contradiction that M is a network. By (31), $M(x,y) \neq 0$ for all $x, y \in M$. Since the elements of Π partition 1 in $\mathcal{A}$, it follows that for each distinct $x, y \in N$ there is unique $\pi(x,y) \in \Pi \setminus \{1'\}$ with $M(x,y) \leq \pi(x,y)$. By (31) and the definition of $R_\smile$ in $\alpha(G)$, $M(x,y) \cdot M(y,x) = M(x,y) \cdot M(y,x)^\smile \neq 0$. So $\pi(x,y) = \pi(y,x)$. By choice of n', there are distinct $x, y, z \in N$ such that $\pi(x,y) = \pi(x,z) = \pi(z,y) = a$, say. But now, by (34), we have

$$(M(x,z)\,;M(z,y)) \cdot M(x,y) \leq (a\,;a) \cdot a = 0, \tag{35}$$

contradicting the definition of 'network'. Thus, we have described a winning strategy for $\forall$ in $\mathcal{G}_{u(n)}(\mathcal{A})$. $\square$

The bound $u(n)$ in the proposition is not optimal. For example, we saw that $u(2)$ can be taken to be about 27 million, but $\forall$ can win in fewer than 6 million rounds by using a 'binary chop' strategy and oriented colourings in the second phase of the game. Further, since $\mathcal{A}$ has three 'colours' of atoms, $|G| \geq (n'-1)/3$ would suffice.

6.4. **RRA has no canonical axiomatisation.** We continue to regard graphs as R_f-structures, as in section 4: so the notion of a bounded morphism of graphs makes sense. Suppose that G, H are graphs and $\rho : G \to H$ is a map. Define $\alpha^\rho : \alpha(G) \to \alpha(H)$ by $\alpha^\rho(1') = 1'$, and $\alpha^\rho(\mathsf{r}_x) = \mathsf{r}_{\rho(x)}$ and similarly for $\mathsf{b}_x, \mathsf{y}_x$. The following is easy to prove.

Lemma 6.7. *If ρ is a homomorphism of graphs, then α^ρ is an L^a_{RA}-homomorphism. If ρ is a surjective bounded morphism of graphs then α^ρ is a surjective bounded morphism of L^a_{RA}-structures.*

We can now prove the theorem we wanted. Equational axiomatisations are a special case.

Theorem 6.8. RRA *has no first-order axiomatisation that contains only finitely many non-canonical first-order sentences.*

Proof. Assume for contradiction that **RRA** has such an axiomatisation, so that lemma 5.4 applies. Let n_0 be as in that lemma, and put

$$\begin{array}{rcl} m &=& e(n_0), \\ n &=& u(m), \end{array} \tag{36}$$

where e, u are as in definitions 6.3 and 6.5. Bearing in mind that finite-length games are determined, lemma 5.4 tells us that there is $n^* < \omega$ such that for any relation algebra $\mathcal{A}$ such that $\exists$ has a winning strategy in $G_{n_0}(\mathcal{A}^\sigma)$, if $\forall$ has a winning strategy in $\mathcal{G}_n(\mathcal{A}^\sigma)$, then he has a winning strategy in $\mathcal{G}_{n^*}(\mathcal{A})$. Define

$$k = e(n^*). \tag{37}$$

Let $\mathcal{D} = \mathcal{D}^k_m = ((\omega, \leq), (G_i : i < \omega), (\rho_{ji} : i \leq j < \omega))$ be the inverse system of (29), and let $G = \lim_\leftarrow \mathcal{D}$. We have $\chi(G_i) = k$ for all i, and by lemma 4.1, $\chi(G) = m$. By lemma 6.7, $\alpha(\mathcal{D}) = ((\omega, \leq), (\alpha(G_i) : i < \omega), (\alpha^{\rho_{ji}} : i \leq j < \omega))$ is an inverse system of finite L^a_{RA}-structures and bounded morphisms. By lemma 6.2, each $\alpha(G_i)^+$ is a relation algebra. Write $\mathcal{A}$ for the L_{RA}-BAO $\mathcal{A}_{\alpha(\mathcal{D})}$ as in (28). By (28), up to isomorphism we have

$$\alpha(G_i)^+ \subseteq \mathcal{A} \text{ for all } i < \omega, \text{ and } \mathcal{A} \text{ is the directed union } \bigcup_{i<\omega} \alpha(G_i)^+. \tag{38}$$

So since the class of relation algebras is defined by equations, $\mathcal{A}$ is also a relation algebra. Finally, it is easily seen that $\alpha(\lim_{\leftarrow} \mathcal{D}) \cong \lim_{\leftarrow} \alpha(\mathcal{D})$, so by lemma 3.9,

$$\mathcal{A}^\sigma \cong \alpha(G)^+. \tag{39}$$

Now G has chromatic number m and (plainly) is infinite. So using (36) and (39), we see by propositions 6.4 and 6.6 that $\exists$ has a winning strategy in $\mathcal{G}_{n_0}(\mathcal{A}^\sigma)$, while $\forall$ has a winning strategy in $\mathcal{G}_n(\mathcal{A}^\sigma)$. By choice of n^*, $\forall$ also has a winning strategy in $\mathcal{G}_{n^*}(\mathcal{A})$. By proposition 5.3, he has a winning strategy that only uses elements of some finite set $W \subseteq \mathcal{A}$. According to (38), we may choose $i < \omega$ such that $W \subseteq \alpha(G_i)^+$. Since $\alpha(G_i)^+$ is a subalgebra of $\mathcal{A}$, consideration of the winning conditions for $\mathcal{G}_{n^*}$ shows that this latter strategy is a winning strategy for $\forall$ in $\mathcal{G}_{n^*}(\alpha(G_i)^+)$.

But $\chi(G_i) = e(n^*)$, so by proposition 6.4, $\exists$ has a winning strategy in this same game $\mathcal{G}_{n^*}(\alpha(G_i)^+)$. This contradiction completes the proof. $\square$

We remark that the theorem implies Monk's result of [31] that **RRA** is not finitely axiomatisable, and the result of [37] that **RRA** has no Sahlqvist axiomatisation.

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